COMM8102 Econometric Analysis
Assignment 4

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Q1 Exercise 12.9

(a)

To see if $\hat{\beta}_{\mathrm{IV}}$ is unbiased, check whether the expectation of $\hat{\beta}_{\mathrm{IV}}$ is equal to β.

$$
\begin{aligned}
E\left[\hat{\beta}_{\mathrm{IV}}\right] &= E\left[\left(\frac{1}{n}\sum_{i=1}^{n} Z_i X_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i Y_i\right)\right] \\
&= E\left[\left(\sum_{i=1}^{n} Z_i X_i'\right)^{-1}\left(\sum_{i=1}^{n} Z_i\left(X_i'\beta + e_i\right)\right)\right] \\
&= E\left[\left(\sum_{i=1}^{n} Z_i X_i'\right)^{-1}\sum_{i=1}^{n} Z_i X_i'\beta + \left(\sum_{i=1}^{n} Z_i X_i'\right)^{-1}\sum_{i=1}^{n} Z_i e_i\right] \\
&= E\left[I\beta + \left(\sum_{i=1}^{n} Z_i X_i'\right)^{-1}\sum_{i=1}^{n} Z_i e_i\right] \\
&= \beta + E\left[E\left[\left(\sum_{i=1}^{n} Z_i X_i'\right)^{-1}\sum_{i=1}^{n} Z_i e_i \middle| Z, X\right]\right] \quad \text{by iterated conditional expectation} \\
&= \beta + E\left[\left(\sum_{i=1}^{n} Z_i X_i'\right)^{-1}\sum_{i=1}^{n} Z_i \underbrace{E\left[e_i \middle| Z, X\right]}_{0}\right] \quad \text{since they are random sample (i.i.d.)} \\
&= \beta
\end{aligned}
$$

Therefore, the IV estimator is unbiased.

(b)

$$\begin{aligned}
Var\left[\hat{\beta}_{\text{IV}}\middle| X, Z\right] &= Var\left[\hat{\beta}_{\text{IV}}\middle| X, Z\right] \\
&= Var\left[\hat{\beta}_{\text{IV}} - \beta\middle| X, Z\right] \\
&= Var\left[\left(\sum_{i=1}^{n} Z_i X_i'\right)^{-1} \sum_{i=1}^{n} Z_i e_i \middle| X, Z\right] \\
&= \left(\sum_{i=1}^{n} Z_i X_i'\right)^{-1} Var\left[\sum_{i=1}^{n} Z_i e_i \middle| X, Z\right] \left(\sum_{i=1}^{n} X_i Z_i'\right)^{-1} \\
&= \left(\sum_{i=1}^{n} Z_i X_i'\right)^{-1} \sum_{i=1}^{n} Z_i Z_i' Var\left[e_i\middle| X, Z\right] \left(\sum_{i=1}^{n} X_i Z_i'\right)^{-1} \\
&= \left(\sum_{i=1}^{n} Z_i X_i'\right)^{-1} \sum_{i=1}^{n} Z_i Z_i' E\left[e_i^2\middle| X, Z\right] \left(\sum_{i=1}^{n} X_i Z_i'\right)^{-1}
\end{aligned}$$

Q2 Exercise 12.12

(a)

It is easy to see that $\hat{\gamma} = \frac{\sum_{i=1}^{n} Z_i X_i}{\sum_{i=1}^{n} Z_i^2}$. Then $\hat{X}_i^2 = \left(\frac{\sum_{i=1}^{n} Z_i X_i}{\sum_{i=1}^{n} Z_i^2}\right)^2 \cdot Z_i^2$. By regressing Y on $\hat{X}^2$, we have

$$\begin{aligned}
\hat{\beta} &= \frac{\sum_{j=1}^{n} \hat{X}_j^2 Y_j}{\sum_{j=1}^{n} \left(\hat{X}_j^2\right)^2} \\
&= \frac{\sum_{j=1}^{n} \left(\frac{\sum_{i=1}^{n} Z_i X_i}{\sum_{i=1}^{n} Z_i^2}\right)^2 Z_j^2 Y_j}{\sum_{j=1}^{n} \left(\frac{\sum_{i=1}^{n} Z_i X_i}{\sum_{i=1}^{n} Z_i^2}\right)^4 Z_j^4} \\
&= \left(\frac{\sum_{i=1}^{n} Z_i^2}{\sum_{i=1}^{n} Z_i X_i}\right)^2 \frac{\sum_{j=1}^{n} Z_j^2 Y_j}{\sum_{j=1}^{n} Z_j^4}
\end{aligned}$$

(b)

The estimator can be written as

$$\hat{\beta} = \left(\frac{\frac{1}{n}\sum_{i=1}^{n} Z_i^2}{\frac{1}{n}\sum_{i=1}^{n} Z_i X_i}\right)^2 \frac{\frac{1}{n}\sum_{j=1}^{n} Z_j^2 Y_j}{\frac{1}{n}\sum_{j=1}^{n} Z_j^4}.$$

By WLLN and CMT,

$$\hat{\beta} \xrightarrow{p} \left(\frac{E\left[Z^2\right]}{E\left[ZX\right]}\right)^2 \cdot \frac{E\left[Z^2 Y\right]}{E\left[Z^4\right]}.$$

(c)

Continue from part b, $\hat{\beta}$ converges in probability to

$$
\begin{aligned}
\left(\frac{E\left[Z^2\right]}{E\left[ZX\right]}\right)^2 \cdot \frac{E\left[Z^2Y\right]}{E\left[Z^4\right]} &= \left(\frac{E\left[Z^2\right]}{E\left[ZX\right]}\right)^2 \cdot \frac{E\left[Z^2\left(\beta X^2+e\right)\right]}{E\left[Z^4\right]} \\
&= \left(\frac{E\left[Z^2\right]}{E\left[Z\left(\gamma Z+u\right)\right]}\right)^2 \cdot \frac{E\left[Z^2\left(\beta\left(\gamma Z+u\right)^2+e\right)\right]}{E\left[Z^4\right]} \\
&= \left(\frac{E\left[Z^2\right]}{\gamma E\left[Z^2\right]+\underbrace{E\left[uZ\right]}_{0}}\right)^2 \cdot \frac{E\left[\beta\gamma^2Z^4+2\beta\gamma Z^3u+\beta u^2Z^2+eZ^2\right]}{E\left[Z^4\right]} \\
&= \frac{1}{\gamma^2}\cdot\frac{\beta\gamma^2E\left[Z^4\right]+2\beta\gamma E\left[Z^3u\right]+\beta E\left[u^2Z^2\right]+E\left[eZ^2\right]}{E\left[Z^4\right]} \\
&= \beta+\frac{2\beta E\left[Z^3u\right]}{\gamma E\left[Z^4\right]}+\frac{\beta E\left[u^2Z^2\right]+E\left[eZ^2\right]}{\gamma^2E\left[Z^4\right]}
\end{aligned}
$$

Therefore, $\hat{\beta}$ is not consistent for β in general. If we assume that

$$
\begin{aligned}
E\left[u|\,Z\right] &= 0, \\
E\left[u^2\middle|\,Z\right] &= 0, \\
E\left[e|\,Z\right] &= 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
E\left[uZ^3\right] &= E\left[E\left[uZ^3\middle|\,Z\right]\right] \\
&= E\left[Z^3E\left[u|\,Z\right]\right] \\
&= 0, \\
E\left[u^2Z^2\right] &= E\left[E\left[u^2Z^2\middle|\,Z\right]\right] \\
&= E\left[Z^2E\left[u^2\middle|\,Z\right]\right] \\
&= 0, \\
E\left[eZ^2\right] &= E\left[E\left[eZ^2\middle|\,Z\right]\right] \\
&= E\left[Z^2E\left[e|\,Z\right]\right] \\
&= 0.
\end{aligned}
$$

Otherwise, we can also just assume that $E\left[uZ^3\right], E\left[u^2Z^2\right]$ and $E\left[eZ^2\right]$ are 0.

Q3

	Combinations
Always-taker	$(1,0),(1,1)$
Compliers	$(1,1),(0,0)$
Never-taker	$(0,1),(0,0)$

	Types of individuals (defiers excluded)
$(1,1)$	**Always-taker** or **Compliers**
$(0,1)$	**Never-taker**
$(1,0)$	**Always-taker**
$(0,0)$	**Never-taker** or **Compliers**

Q4 Exercise 13.10

First, we need to find the moment condition for this problem. It is

$$E\left[Z\left(Y-m\left(X,\beta\right)\right)\right]=0.$$

The criterion function is

$$J_n\left(\beta\right)=n\cdot\left(\frac{1}{n}\sum_{i=1}^{n}Z_i\left(Y_i-m\left(X_i,\beta\right)\right)\right)'\mathbf{W}\left(\frac{1}{n}\sum_{i=1}^{n}Z_i\left(Y_i-m\left(X_i,\beta\right)\right)\right),$$

where $\mathbf{W}$ is an $\ell\times\ell$ positive-definite matrix. The GMM estimator $\hat{\beta}$ is the one minimizes J_n given $\mathbf{W}$.

$$\hat{\beta}=\arg\min_{\beta}J_n\left(\beta\right)$$

We start with $\mathbf{W}_1=\mathbf{I}_{\ell\times\ell}$, an identity matrix. Then we get the consistent preliminary estimator $\hat{\beta}_1=\arg\min_{\beta}J_n\left(\beta\right)$ with $\mathbf{W}=\mathbf{I}_{\ell\times\ell}$. Then, for $s\geq 2$, let

$$\mathbf{W}_s=\left(\frac{1}{n}\sum_{i=1}^{n}Z_iZ_i'\left(Y_i-m\left(X_i,\hat{\beta}_{s-1}\right)\right)^2\right)^{-1}.$$

Then we get the iterated estimator $\hat{\beta}_s=\arg\min_{\beta}J_n\left(\beta\right)$ with $\mathbf{W}=\mathbf{W}_s$. For 2-step efficient GMM estimator, we stop at the second step. For the iterated GMM estimator, we stop the iteration when $\left\|\hat{\beta}_s-\hat{\beta}_{s-1}\right\|=0$ or $\left\|\hat{\beta}_s-\hat{\beta}_{s-1}\right\|$ is less than some tolerance level.

Q5 Exercise 13.24

(a)

The moment condition is $E\left[X\left(Y-\theta\right)\right]=0$. The criterion function is

$$J_n\left(\theta\right)=n\cdot\left(\frac{1}{n}\sum_{i=1}^{n}X_i\left(Y_i-\theta\right)\right)'\mathbf{W}\left(\frac{1}{n}\sum_{i=1}^{n}X_i\left(Y_i-\theta\right)\right).$$

To find the $\hat{\theta}$ that minimizes $J_n\left(\theta\right)$, we first look at the first-order condition:

$$\frac{\partial}{\partial\theta}J_n\left(\hat{\theta}\right)=n\cdot\left(\frac{1}{n}\sum_{i=1}^{n}-X_i\right)'\mathbf{W}\left(\frac{1}{n}\sum_{i=1}^{n}X_i\left(Y_i-\hat{\theta}\right)\right)=0.$$

Then, we have

$$\begin{aligned}
n\cdot\left(-\frac{1}{n}\sum_{i=1}^{n}X_i\right)'\mathbf{W}\left(\frac{1}{n}\sum_{i=1}^{n}X_iY_i-\hat{\theta}\frac{1}{n}\sum_{i=1}^{n}X_i\right) &= 0\\
\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)'\mathbf{W}\left(\frac{1}{n}\sum_{i=1}^{n}X_iY_i\right) &= \hat{\theta}\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)'\mathbf{W}\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right).
\end{aligned}$$

Finally, we have

$$\hat{\theta}=\left(\left(\frac{1}{n}\sum_{i=1}^{n}X_i'\right)\mathbf{W}\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}X_i'\right)\mathbf{W}\left(\frac{1}{n}\sum_{i=1}^{n}X_iY_i\right).$$

Then, like exercise 13.10, starting with $\mathbf{W} = \mathbf{I}_{k\times k}$, we get

$$\hat{\theta}_1 = \frac{\left(\sum_{i=1}^n X_i'\right)\left(\sum_{i=1}^n X_i Y_i\right)}{\left(\sum_{i=1}^n X_i'\right)\left(\sum_{i=1}^n X_i\right)}.$$

For $s \geq 2$, let

$$\mathbf{W}_s = \left(\frac{1}{n}\sum_{i=1}^n X_i X_i' \left(Y_i - \hat{\theta}_{s-1}\right)^2\right)^{-1},$$

and let

$$\hat{\theta}_s = \left(\left(\frac{1}{n}\sum_{i=1}^n X_i'\right)\mathbf{W}_s\left(\frac{1}{n}\sum_{i=1}^n X_i\right)\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n X_i'\right)\mathbf{W}_s\left(\frac{1}{n}\sum_{i=1}^n X_i Y_i\right).$$

If we want the two step efficient estimator, we stop at the second step. If we want iterated estimator, we finally stop at $\hat{\theta}_s$ such that $\left\|\hat{\theta}_s - \hat{\theta}_{s-1}\right\| = 0$ or less than some tolerance level.

(b)

If $k = 1$, this model is just-identified. If $k > 1$, this model is over-identified.

(c)

Under $H_0 : E[Xe] = 0$, the test statistic is:

$$J_n\left(\hat{\theta}\right) = n \cdot \left(\frac{1}{n}\sum_{i=1}^n X_i\left(Y_i - \hat{\theta}\right)\right)' \hat{\mathbf{W}}\left(\frac{1}{n}\sum_{i=1}^n X_i\left(Y_i - \hat{\theta}\right)\right),$$

where $\hat{\theta}$ is the efficient GMM estimator we get from part (a), and $\hat{\mathbf{W}} = \left(\frac{1}{n}\sum_{i=1}^n X_i X_i'\left(Y_i - \hat{\theta}\right)^2\right)^{-1}$.

Q6 Exercise 12.25

(a)

The result of the replication of the reduced form regression in the final column of Table 12.2 are as follow.

	Estimation	Standard error (HC1)
experience	−0.4133	0.03203
experience^2/100	0.0927	0.17077
black	−1.0063	0.08798
south	−0.2671	0.078665
urban	0.39975	0.08475
public	0.43035	0.086167
private	0.1226	0.10131

The replication of the 2SLS regression is:

	Estimation	Standard error (HC1)
education	0.16109	0.04052
experience	0.11931	0.01819
experience^2/100	−0.2305	0.03680
black	−0.1017	0.04403
south	−0.0950	0.02177
urban	0.1164	0.02630

(b)

In this part, we try a different reduced form model with the variable **near a 2-year college** added.

	Estimation	Standard error (HC1)
experience	-0.41279477	0.03200792
experience^2/100	0.08945688	0.17067503
black	-1.01060399	0.08795667
south	-0.26033369	0.07891951
urban	0.39039534	0.08555575
public	0.42162279	0.08650685
private	0.13011730	0.10171720
college-2	0.06773502	0.07436626

The t-statistic of the coefficient of the new variable **near a 2-year college** is $\frac{0.06773502}{0.07436626} = 0.911$, and the p-value is 0.26. We cannot reject the hypothesis that the coefficient of this variable is 0. In addition, the value of the coefficient is relatively small and the values of other coefficients do not change significantly. Even though it is valid, it does not play a significant role in impacting the education.

(c)

	Estimation	Standard error (HC1)
experience	-0.6717638	0.03164301
experience^2/100	0.5537512	0.16889943
black	-0.6877297	0.07516382
south	-0.2014957	0.06738587
urban	0.2652910	0.07499348
public	-28.9256454	2.83009269
private	0.1363548	0.10542939
public*age	1.4830942	0.20047303
public*age^2/100	-1.5536585	0.35108191

These two coefficients indicate that the relation between the age of an individual who grew up near a public college is not linear. When age is less than $\frac{1.4830942}{2 \cdot 1.5536585} \times 100 \approx 47.72$ years, on average, individuals who grew up near a public college with an older age have more education time.

(d)

	Estimation	Standard error (HC1)
education	0.08253858	0.00622812
experience	0.08709405	0.00706157
experience^2/100	-0.22472050	0.03202664
black	-0.18102150	0.01805731
south	-0.12194012	0.01543661
urban	0.15701776	0.01530355

We can see that the coefficient of **education** decreases with the expanded instrument set. The structural estimate of the return to schooling decreases.

Q7 Exercise 13.28

(a)

	Preliminary GMM	2-step efficient GMM	Iterated GMM
education	0.1610917	0.16151623	0.16152244
experience	0.1193108	0.11955527	0.11955859
experience^2/100	-0.2305416	-0.23151083	-0.23151660
black	-0.1017273	-0.10119968	-0.10119373
south	-0.0950355	-0.09535566	-0.09535359
urban	0.1164481	0.11502105	0.11501566

As we can see from the table, the preliminary GMM is the estimation from 2SLS, and these three results are very close.

(b)

	Preliminary GMM	2-step efficient GMM	Iterated GMM
education	0.08253858	0.08385250	0.08387334
experience	0.08709405	0.08763553	0.08763928
experience^2/100	-0.22472050	-0.22493052	-0.22489366
black	-0.18102150	-0.17749140	-0.17744527
south	-0.12194012	-0.12449040	-0.12450300
urban	0.15701776	0.15293796	0.15289824

As we can see from the table, the preliminary GMM is the estimation from 2SLS, and these three results are very close.

(c)

The J statistic for overidentification for part a is $J_a = 0.868$, and the J statistic for overidentification for part b is $J_b = 10.4$.