COMM8102 Econometric Analysis
Assignment 3

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1 Exercise 7.1

$$
\begin{aligned}
\hat{\beta}_1 &= \left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{1i}'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}Y_i\right) \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{1i}'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}\left(X_{1i}'\beta_1 + X_{2i}'\beta_2 + e_i\right)\right) \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{1i}'\right)^{-1}\left(\beta_1\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{1i}' + \beta_2\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{2i}' + \frac{1}{n}\sum_{i=1}^{n} X_{1i}e_i\right) \\
&\xrightarrow{p} E\left[X_1X_1'\right]^{-1}\left(E\left[X_1X_1'\right]\beta_1 + E\left[X_1X_2'\right]\beta_2 + E\left[X_1e\right]\right) \\
&= \beta_1 + E\left[X_1X_1'\right]^{-1}E\left[X_1X_2'\right]\beta_2 + E\left[X_1X_1'\right]^{-1}E\left[X_1e\right] \\
&= \beta_1 + E\left[X_1X_1'\right]^{-1}E\left[X_1X_2'\right]\beta_2 + 0 \\
&= \beta_1 + E\left[X_1X_1'\right]^{-1}E\left[X_1X_2'\right]\beta_2
\end{aligned}
$$

by WLLN and CMT. $E\left[X_1e\right] = 0$, since $E\left[Xe\right] = 0$. Since $\hat{\beta}_1 \xrightarrow{p} \beta_1 + E\left[X_1X_1'\right]^{-1}E\left[X_1X_2'\right]\beta_2$, it is not consistent for β_1. When $E\left[X_1X_2'\right] = \mathbf{0}$ or $\beta_2 = \mathbf{0}$, $\hat{\beta}_1 \xrightarrow{p} \beta_1$.

2 Exercise 7.15

$$
\begin{aligned}
\sqrt{n}\left(\hat{\beta}-\beta\right) &= \sqrt{n}\left(\frac{\sum_{i=1}^{n} X_i^3 Y_i}{\sum_{i=1}^{n} X_i^4}-\beta\right) \\
&= \frac{\sqrt{n}\frac{1}{n}\sum_{i=1}^{n} X_i^3 Y_i}{\frac{1}{n}\sum_{i=1}^{n} X_i^4}-\sqrt{n}\beta \\
&= \frac{\sqrt{n}\frac{1}{n}\sum_{i=1}^{n} X_i^3 \left(X_i\beta+e_i\right)}{\frac{1}{n}\sum_{i=1}^{n} X_i^4}-\sqrt{n}\beta \\
&= \frac{\sqrt{n}\frac{1}{n}\sum_{i=1}^{n} X_i^4 \beta}{\frac{1}{n}\sum_{i=1}^{n} X_i^4}+\frac{\sqrt{n}\frac{1}{n}\sum_{i=1}^{n} X_i^3 e_i}{\frac{1}{n}\sum_{i=1}^{n} X_i^4}-\sqrt{n}\beta \\
&= \underbrace{\frac{\beta\sqrt{n}\frac{1}{n}\sum_{i=1}^{n} X_i^4}{\frac{1}{n}\sum_{i=1}^{n} X_i^4}-\sqrt{n}\beta}_{0}+\frac{\sqrt{n}\frac{1}{n}\sum_{i=1}^{n} X_i^3 e_i}{\frac{1}{n}\sum_{i=1}^{n} X_i^4} \\
&= \frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i^3 e_i-\underbrace{E\left[X_i^3 e\right]}_{0}\right)\right)}{\frac{1}{n}\sum_{i=1}^{n} X_i^4} \\
&\xrightarrow{d} \frac{\mathcal{N}\left(0, E\left[X_i^6 e_i^2\right]\right)}{E\left[X_i^4\right]} \\
&= \mathcal{N}\left(0, \frac{E\left[X_i^6 e_i^2\right]}{\left(E\left[X_i^4\right]\right)^2}\right)
\end{aligned}
$$

3 Exercise 9.4

3.1 (a)

The size of a test is the type I error.

$$
\begin{aligned}
P\left(\text{Reject } H_0 \middle| H_0 \text{ is true}\right) &= P\left(\text{Reject } H_0 \middle| H_0\right) \\
&= P\left(W < c_1 \text{ or } W > c_2 \middle| H_0\right)
\end{aligned}
$$

Under H_0, $W \underset{d}{\rightarrow} \chi_q^2$, $P\left(W < c_1 \text{ or } W > c_2 \middle| H_0\right) = P\left(W < c_1 \middle| H_0\right) + P\left(W > c_2 \middle| H_0\right) = \frac{\alpha}{2} + 1 - (1 - \frac{\alpha}{2}) = \alpha$.

3.2 (b)

This is not a good test of H_0 vs H_1.Wald test is the extension of t-test to multivariate case. Wald test statistics is the sqaured version of t statistics in the scalar case. Only the right tail captures the extreme deviation from 0 (both positive and negative). We do not need to look at the left tail of the χ_q^2 distribution.

4 Exercise 9.12

This interpretation is not correct.

The power of a hypothesis test is the probability that reject H_0 when the H_1 is true. p value is a quantity depending on the value of test statistics and the asymptotic null distribution G. The power depends on the critical value and the distribution of test statistic under the alternative hypothesis.

Although a larger sample will make the test have more power, it does not necessarily reduce the p value. p value depends on both of the test statistic and the asymptotic null distribution G. The extra samples can make the value of test statistic increase, but it can also make the value of test statistic decrease. Therefore, it cannot be guranteed that a larger sample will make the test reject the null.

5 Exercise 9.16

5.1 (a)

$$\hat{\theta} = \frac{1}{n}\left(\sum_{i=1}^{n} e_{1i}^2 - e_{2i}^2\right)$$

5.2 (b)

By central limit theorem,

$$\begin{pmatrix} \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} e_{1i}^2 - \sigma_1^2\right) \\ \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} e_{2i}^2 - \sigma_2^2\right) \end{pmatrix} \xrightarrow{d} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, V\right),$$

where

$$V = \begin{pmatrix} E\left[\left(e_1^2 - \sigma_1^2\right)^2\right] & E\left[\left(e_1^2 - \sigma_1^2\right)\left(e_2^2 - \sigma_2^2\right)\right] \\ E\left[\left(e_1^2 - \sigma_1^2\right)\left(e_2^2 - \sigma_2^2\right)\right] & E\left[\left(e_2^2 - \sigma_2^2\right)^2\right] \end{pmatrix}.$$

Define a function $r : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \longmapsto x - y$. Then we have $\theta = r\left(\sigma_1^2, \sigma_2^2\right), \hat{\theta} = r\left(\frac{1}{n}\sum_{i=1}^{n} e_{1i}^2, \frac{1}{n}\sum_{i=1}^{n} e_{2i}^2\right)$. By delta method,

$$\begin{aligned} \sqrt{n}\left(\hat{\theta} - \theta\right) &= \sqrt{n}\left(r\left(\frac{1}{n}\sum_{i=1}^{n} e_{1i}^2, \frac{1}{n}\sum_{i=1}^{n} e_{2i}^2\right) - r\left(\sigma_1^2, \sigma_2^2\right)\right) \\ &\xrightarrow{d} \mathcal{N}\left(0, R'VR\right), \end{aligned}$$

where

$$\begin{aligned} R &= \begin{pmatrix} \frac{\partial}{\partial \sigma_1^2} r\left(\sigma_1^2, \sigma_2^2\right) \\ \frac{\partial}{\partial \sigma_2^2} r\left(\sigma_1^2, \sigma_2^2\right) \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Finally, we have

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, V_\theta\right),$$

with

$$\begin{aligned} V_\theta &= E\left[\left(e_1^2 - \sigma_1^2\right)^2\right] - 2 \cdot E\left[\left(e_1^2 - \sigma_1^2\right)\left(e_2^2 - \sigma_2^2\right)\right] + E\left[\left(e_2^2 - \sigma_2^2\right)^2\right] \\ &= E\left[\left(\left(e_1^2 - \sigma_1^2\right) - \left(e_2^2 - \sigma_2^2\right)\right)^2\right] \\ &= E\left[\left(e_1^2 - e_2^2 - \left(\sigma_1^2 - \sigma_2^2\right)\right)^2\right] \end{aligned}$$

5.3 (c)

Since $\hat{\theta} = \theta + \frac{\sqrt{n}(\hat{\theta}-\theta)}{\sqrt{n}}$, and $\sqrt{n}\left(\hat{\theta} - \theta\right) \xrightarrow{d} \mathcal{N}(0, V_\theta)$,

$$\hat{\theta} \xrightarrow{d} \mathcal{N}\left(\theta, \frac{V_\theta}{n}\right)$$

An estimator of the asymptotic variance of $\hat{\theta}$ is

$$\hat{V_{\hat{\theta}}} = \frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left(e_{1i}^2 - e_{2i}^2 - \left(\sigma_1^2 - \sigma_2^2\right)\right)^2\right).$$

5.4 (d)

H_0 is equivalent to $\theta = 0$, and H_1 is equivalent to $\theta \neq 0$. Under H_0 the test statistic is

$$\begin{aligned} T &= \frac{\hat{\theta} - 0}{\sqrt{\frac{V_\theta}{n}}} \\ &\sim \mathcal{N}(0, 1). \end{aligned}$$

A test of asymptotic size α can be, we reject the null hypothesis if $T < a_1$ or $T > a_2$, where a_1 is the $\frac{\alpha}{2}$ quantile of $\mathcal{N}(0,1)$ and a_2 is the $1 - \frac{\alpha}{2}$ of $\mathcal{N}(0,1)$.

5.5 (e)

Acceptance of the null hypothesis means these two models have same variance for the residuals, therefore they fit the data equally well.

6 Exercise 9.26

6.1 (a)

	Estimations	Standard errors $HC1$
$\hat{\beta}_1$	-3.527	1.7186
$\hat{\beta}_2$	0.720	0.0326
$\hat{\beta}_3$	0.436	0.2456
$\hat{\beta}_4$	-0.220	0.3238
$\hat{\beta}_5$	0.427	0.0755

6.2 (b)

$$\begin{aligned} \log C &= \beta_1 + \beta_2 \log Q + \beta_3 \log PL + \beta_4 \log PK + \beta_5 \log PF + e \\ &= \beta_1 + \log Q^{\beta_2} + \log PL^{\beta_3} + \log PK^{\beta_4} + \log PF^{\beta_5} + e \end{aligned}$$

Then we have

$$C = \exp\left(\beta_1 + e\right) \cdot Q^{\beta_2} \cdot PL^{\beta_3} \cdot PK^{\beta_4} \cdot PF^{\beta_5}$$

For $\beta_3 + \beta_4 + \beta_5 = 1$, this means the cost function has a constant return to scale for labor, capital and fuel.

6.3 (e)

Define a function $r : \mathbb{R}^5 \to \mathbb{R}$, such that $r(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) = \beta_3 + \beta_4 + \beta_5$. The we have $\nabla r\prime\prime = [0, 0, 1, 1, 1]'$. By Delta method and affine transform of normal distribution, we establish

$$\sqrt{n}\left(\hat{\beta}_3 + \hat{\beta}_4 + \hat{\beta}_5 - (\beta_3 + \beta_4 + \beta_5)\right) \xrightarrow{d} \mathcal{N}(0, V)$$

where $V = \nabla r' \mathbf{V}_\beta \nabla r$. $\mathbf{V}_\beta$ is the asymptotic covariance matrix of $\sqrt{n}\left(\hat{\beta} - \beta\right)$.
Then the Wald statistic is

$$\begin{aligned} \mathcal{W}_n &= \frac{\left(\hat{\beta}_3 + \hat{\beta}_4 + \hat{\beta}_5 - (\beta_3 + \beta_4 + \beta_5)\right)^2}{\hat{V}} \\ &= 0.645. \end{aligned}$$

Since $\mathcal{W}_n \xrightarrow{d} \chi_1^2$, the critical value $c = 3.84$ for $\alpha = 0.05$. We reject the null hypothesis.