

COMM8102 ECONOMETRIC ANALYSIS

ASSIGNMENT 3

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Exercise 7.1

Let $\hat{\beta}_1$ be the estimator of β_1 by only regressing Y on X_1 .

$$\begin{aligned}\hat{\beta}_1 &= \left(\frac{1}{n} \sum_{i=1}^n X_{1i} X'_{1i} \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n X_{1i} Y_i \right) \\&= \left(\frac{1}{n} \sum_{i=1}^n X_{1i} X'_{1i} \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n X_{1i} (X'_{1i} \beta_1 + X'_{2i} \beta_2 + e_i) \right) \\&= \left(\frac{1}{n} \sum_{i=1}^n X_{1i} X'_{1i} \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n X_{1i} X'_{1i} \beta_1 + \frac{1}{n} \sum_{i=1}^n X_{1i} X'_{2i} \beta_2 + \frac{1}{n} \sum_{i=1}^n X_{1i} e_i \right) \\&= \beta_1 + \left(\frac{1}{n} \sum_{i=1}^n X_{1i} X'_{1i} \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n X_{1i} X'_{2i} \beta_2 \right) + \left(\frac{1}{n} \sum_{i=1}^n X_{1i} X'_{1i} \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n X_{1i} e_i \right) \\&\xrightarrow{p} \beta_1 + E[X_1 X'_1]^{-1} E[X_1 X'_2] \beta_2 + E[X_1 X'_1]^{-1} E[X_1 e] \\&= \beta_1 + E[X_1 X'_1]^{-1} E[X_1 X'_2] \beta_2 + 0 \\&= \beta_1 + E[X_1 X'_1]^{-1} E[X_1 X'_2] \beta_2\end{aligned}$$

by WLLN and CMT. $E[X_1 e] = 0$, since $E[Xe] = 0$. Because $\hat{\beta}_1 \xrightarrow{p} \beta_1 + E[X_1 X'_1]^{-1} E[X_1 X'_2] \beta_2$, $\hat{\beta}_1$ is not consistent for β_1 . When $E[X_1 X'_2] = \mathbf{0}$ or $\beta_2 = \mathbf{0}$, $\hat{\beta}_1 \xrightarrow{p} \beta_1$.

Exercise 7.15

$$\begin{aligned}
 \sqrt{n}(\hat{\beta} - \beta) &= \sqrt{n} \left(\frac{\sum_{i=1}^n X_i^3 Y_i}{\sum_{i=1}^n X_i^4} - \beta \right) \\
 &= \sqrt{n} \left(\frac{\sum_{i=1}^n X_i^3 (X_i \beta + e_i)}{\sum_{i=1}^n X_i^4} - \beta \right) \\
 &= \sqrt{n} \left(\beta + \frac{\sum_{i=1}^n X_i^3 e_i}{\sum_{i=1}^n X_i^4} - \beta \right) \\
 &= \frac{\sqrt{n} \left(\sum_{i=1}^n X_i^3 e_i - \underbrace{E[X_i^3 e]}_0 \right)}{\sum_{i=1}^n X_i^4} \\
 &\xrightarrow{d} \frac{\mathcal{N}(0, E[X_i^6 e_i^2])}{E[X_i^4]} \\
 &= \mathcal{N} \left(0, \frac{E[X_i^6 e_i^2]}{(E[X_i^4])^2} \right)
 \end{aligned}$$

by CLT and CMT.

$$\begin{aligned}
 E[X_i^3 e] &= E[E[X_i^3 e | X_i]] \\
 &= E[X_i^3 E[e | X_i]] \\
 &= 0
 \end{aligned}$$

Exercise 9.4

(a)

The size of a test is the probability of a false rejection of the (true) null hypothesis:

$$\begin{aligned}
 P(\text{Reject } H_0 | H_0 \text{ is true}) &= P(\text{Reject } H_0 | H_0) \\
 &= P(W < c_1 \text{ or } W > c_2 | H_0)
 \end{aligned}$$

Under H_0 , $W \xrightarrow{d} \chi_q^2$, $P(W < c_1 \text{ or } W > c_2 | H_0) = P(W < c_1 | H_0) + P(W > c_2 | H_0) = \frac{\alpha}{2} + 1 - (1 - \frac{\alpha}{2}) = \alpha$.

(b)

This is not a good test of H_0 vs H_1 . Wald test is the extension of t-test to multivariate case. Intuitively, Wald test statistic is the squared version of t statistic in the scalar case. Only the right tail captures the extreme deviation from 0 (both positive and negative). This can also be extended to the multivariate case. We do not need to look at the left tail of the χ_q^2 distribution. A one-side test is enough.

Exercise 9.12

This interpretation is not correct.

The power of a hypothesis test is the probability that reject H_0 when the H_1 is true. p value is a quantity depending on the value of test statistic and the asymptotic null distribution G . The power depends on the critical value and the distribution of test statistic under the alternative hypothesis.

Although a larger sample will make the test have more power, it does not necessarily reduce the p value. p value depends on both of the test statistic and the asymptotic null distribution G . The extra samples may make the value of test statistic increase, but it may also make the value of test statistic decrease. Therefore, it cannot be guaranteed that a larger sample will make the null rejected.

Exercise 9.16

(a)

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^n e_{1i}^2 - \frac{1}{n} \sum_{i=1}^n e_{2i}^2$$

(b)

By central limit theorem,

$$\begin{pmatrix} \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n e_{1i}^2 - \sigma_1^2 \right) \\ \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n e_{2i}^2 - \sigma_2^2 \right) \end{pmatrix} \xrightarrow{d} \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, V \right),$$

where

$$V = \begin{pmatrix} E[(e_1^2 - \sigma_1^2)^2] & E[(e_1^2 - \sigma_1^2)(e_2^2 - \sigma_2^2)] \\ E[(e_1^2 - \sigma_1^2)(e_2^2 - \sigma_2^2)] & E[(e_2^2 - \sigma_2^2)^2] \end{pmatrix}.$$

Define a function $r : \mathbb{R}^2 \rightarrow \mathbb{R}$, such that $r(\sigma_1^2, \sigma_2^2) = \sigma_1^2 - \sigma_2^2$. Then we have $\theta = r(\sigma_1^2, \sigma_2^2)$, $\hat{\theta} = r\left(\frac{1}{n} \sum_{i=1}^n e_{1i}^2, \frac{1}{n} \sum_{i=1}^n e_{2i}^2\right)$. By delta method,

$$\begin{aligned} \sqrt{n}(\hat{\theta} - \theta) &= \sqrt{n} \left(r \left(\frac{1}{n} \sum_{i=1}^n e_{1i}^2, \frac{1}{n} \sum_{i=1}^n e_{2i}^2 \right) - r(\sigma_1^2, \sigma_2^2) \right), \\ &\xrightarrow{d} \mathcal{N}(0, R'VR), \end{aligned}$$

where

$$\begin{aligned} R &= \begin{pmatrix} \frac{\partial}{\partial \sigma_1^2} r(\sigma_1^2, \sigma_2^2) \\ \frac{\partial}{\partial \sigma_2^2} r(\sigma_1^2, \sigma_2^2) \end{pmatrix}, \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Then we have

$$\begin{aligned} R'VR &= E[(e_1^2 - \sigma_1^2)^2] - 2 \cdot E[(e_1^2 - \sigma_1^2)(e_2^2 - \sigma_2^2)] + E[(e_2^2 - \sigma_2^2)^2], \\ &= E[((e_1^2 - \sigma_1^2) - (e_2^2 - \sigma_2^2))^2], \\ &= E[(e_1^2 - e_2^2 - (\sigma_1^2 - \sigma_2^2))^2]. \end{aligned}$$

Finally,

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N} \left(0, E[(e_1^2 - e_2^2 - (\sigma_1^2 - \sigma_2^2))^2] \right).$$

(c)

Since $\hat{\theta} = \theta + \frac{\sqrt{n}(\hat{\theta} - \theta)}{\sqrt{n}}$, and $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}\left(0, E\left[(e_1^2 - e_2^2 - (\sigma_1^2 - \sigma_2^2))^2\right]\right)$,

$$\hat{\theta} \xrightarrow{d} \mathcal{N}\left(\theta, \frac{E\left[(e_1^2 - e_2^2 - (\sigma_1^2 - \sigma_2^2))^2\right]}{n}\right)$$

An estimator of the asymptotic variance of $\hat{\theta}$ is

$$\hat{V}_{\hat{\theta}} = \frac{1}{n} \left(\frac{1}{n} \sum_{i=1}^n (e_{1i}^2 - e_{2i}^2 - (\sigma_1^2 - \sigma_2^2))^2 \right).$$

(d)

H_0 is equivalent to $\theta = 0$, and H_1 is equivalent to $\theta \neq 0$. Under H_0 the test statistic is

$$\begin{aligned} T_n &= \frac{\hat{\theta} - 0}{\sqrt{\hat{V}_{\hat{\theta}}}} \\ &\sim \mathcal{N}(0, 1). \end{aligned}$$

A test of asymptotic size α can be that we reject the null hypothesis if $T_n < a_1$ or $T_n > a_2$, where a_1 is the $\frac{\alpha}{2}$ quantile of $\mathcal{N}(0, 1)$ and a_2 is the $1 - \frac{\alpha}{2}$ of $\mathcal{N}(0, 1)$.

(e)

Acceptance of the null hypothesis means these two models have same variance for the residuals. therefore they fit the data equally well.

Exercise 9.26

(a)

	Estimations	Standard errors HC1
$\hat{\beta}_1$	-3.527	1.7186
$\hat{\beta}_2$	0.720	0.0326
$\hat{\beta}_3$	0.436	0.2456
$\hat{\beta}_4$	-0.220	0.3238
$\hat{\beta}_5$	0.427	0.0755

(b)

$$\begin{aligned} \log C &= \beta_1 + \beta_2 \log Q + \beta_3 \log PL + \beta_4 \log PK + \beta_5 \log PF + e \\ &= \beta_1 + \log Q^{\beta_2} + \log PL^{\beta_3} + \log PK^{\beta_4} + \log PF^{\beta_5} + e \end{aligned}$$

Then we have

$$C = \exp(\beta_1) \cdot \exp(e) \cdot Q^{\beta_2} \cdot PL^{\beta_3} \cdot PK^{\beta_4} \cdot PF^{\beta_5}$$

For $\beta_3 + \beta_4 + \beta_5 = 1$, this means the cost function has a constant return to scale for labor, capital and fuel.

(e)

Define a function $r : \mathbb{R}^5 \rightarrow \mathbb{R}$, such that $r(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) = \beta_3 + \beta_4 + \beta_5$. Then we have $\nabla r = [0, 0, 1, 1, 1]'$. By Delta method and affine transform of normal distribution, we establish

$$\sqrt{n} \left(\hat{\beta}_3 + \hat{\beta}_4 + \hat{\beta}_5 - (\beta_3 + \beta_4 + \beta_5) \right) \xrightarrow{d} \mathcal{N}(0, V_r)$$

where $V_r = \nabla r' \mathbf{V}_\beta \nabla r$. \mathbf{V}_β is the asymptotic covariance matrix of $\sqrt{n}(\hat{\beta} - \beta)$. Then the Wald statistic is

$$\begin{aligned} \mathcal{W}_n &= \frac{\left(\hat{\beta}_3 + \hat{\beta}_4 + \hat{\beta}_5 - (\beta_3 + \beta_4 + \beta_5) \right)^2}{\hat{V}_{\hat{r}}^{HC1}} \\ &= 0.645. \end{aligned}$$

Since $\mathcal{W}_n \xrightarrow{d} \chi_1^2$, the critical value $c = 3.84$ for $\alpha = 0.05$. We do not reject the null hypothesis at 5% significance level.