Solutions to Problem Set #3

Q1. Exercise 7.1

Let $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{Y}$ be $n \times k_1$, $n \times k_2$, and $n \times 1$ matrices, respectively. The least squares estimator of β_1 by regressing Y on X_1 only is

$$\widetilde{\beta}_1 = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1\mathbf{Y}.$$

By replacing $\mathbf{Y}$ for $\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \mathbf{e}$ and taking $n \to \infty$,

$$\begin{aligned}
\widetilde{\beta}_1 =& (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1(\mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \mathbf{e}) \\
=& \beta_1 + \left(\frac{1}{n}\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\frac{1}{n}\mathbf{X}_1\mathbf{X}_2\beta_2 + \left(\frac{1}{n}\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\frac{1}{n}\mathbf{X}_1\mathbf{e} \\
=& \beta_1 + \left(\frac{1}{n}\sum_{i=1}^n X_{1i}X_{1i}'\right)^{-1}\frac{1}{n}\sum_{i=1}^n X_{1i}X_{2i}'\beta_2 + \left(\frac{1}{n}\sum_{i=1}^n X_{1i}X_{1i}'\right)^{-1}\frac{1}{n}\sum_{i=1}^n X_{1i}e_i \\
\xrightarrow{p}& \beta_1 + (E[X_{1i}X_{1i}'])^{-1}E[X_{1i}X_{2i}']\beta_2 + (E[X_{1i}X_{1i}'])^{-1}E[X_{1i}e_i],
\end{aligned}$$

where X_{1i} and X_{2i} are $k_1 \times 1$ and $k_2 \times 1$ vectors, respectively. We assume that $E[X_{1i}X_{1i}']$ is invertible. The last term on the right-hand side (RHS) equals to zero by the assumption ($E[Xe] = 0$). In general, the second term is not zero. Therefore, $\widetilde{\beta}_1$ is not consistent for β_1 in general. This is the omitted variable bias. However, if $E[X_{1i}X_{2i}'] = 0$ or $\beta_2 = 0$, then the omitted variable bias disappears and $\widetilde{\beta}_1$ is consistent for β_1. In words, if the omitted variables are not correlated with the included explanatory variables or Y, then the omission does not affect consistency.

Q2. Exercise 7.15

First we show consistency. By replacing Y_i with $Y_i = X_i\beta + e_i$,

$$\widehat{\beta} = \frac{\sum_{i=1}^n X_i^3(X_i\beta + e_i)}{\sum_{i=1}^n X_i^4} = \beta + \frac{\frac{1}{n}\sum_{i=1}^n X_i^3 e_i}{\frac{1}{n}\sum_{i=1}^n X_i^4} \xrightarrow{p} \beta + \frac{E[X_i^3 e_i]}{E[X_i^4]}.$$

But by the law of iterated expectations,

$$E[X_i^3 e_i] = E[E[X_i^3 e_i|X_i]] = E[X_i^3 E[e_i|X_i]] = 0.$$

Thus, $\widehat{\beta} \xrightarrow{p} \beta$.

Next, we show the asymptotic distribution. By multiplying $\sqrt{n}$,

$$\sqrt{n}(\widehat{\beta} - \beta) = \frac{\sqrt{n}\frac{1}{n}\sum_{i=1}^n X_i^3 e_i}{\frac{1}{n}\sum_{i=1}^n X_i^4} \xrightarrow{d} N\left(0, \frac{E[X_i^6 e_i^2]}{(E[X_i^4])^2}\right).$$

Note that no additional information is given about the conditional variance of e_i, so the asymptotic variance term cannot be further simplified.

Q3. Exercise 9.4

(a) Let $X \sim \chi_q^2$ (r.v. distributed as the chi-squared distribution with the degrees of freedom q). Then $W \xrightarrow{d} X$ as $n \to \infty$. Since $c_1 < c_2$, the asymptotic size of the test is

$$P(X < c_1 \text{ or } X > c_2) = P(X < c_1) + P(X > c_2) = \alpha/2 + \alpha/2 = \alpha.$$

(b) This is not a good test. Suppose that $\widehat{\theta} = 0$ so that the null hypothesis is exactly satisfied. But in this case, $W = 0 < c_1$ so we would reject the null hypothesis, which is not desirable. The correct test would be a one-sided test which rejects if $W > c_3$ where c_3 is the $1 - \alpha$th quantile of the χ_q^2 distribution.

Q4. Exercise 9.12

The statement is not necessarily true. It is true that the power of the test increases with the sample size. The power is the probability of rejecting the incorrect null hypothesis, so we would be able to reject the null with a higher probability. On the other hand, if the null hypothesis is true, then it is more likely that we would not reject the null..

Q5. Exercise 9.16

(a) A consistent estimator would be

$$\widehat{\theta} = \frac{1}{n}\sum_{i=1}^{n} e_{1i}^2 - \frac{1}{n}\sum_{i=1}^{n} e_{2i}^2.$$

Note that the consistency holds by the WLLN.

(b) Let $\widehat{\sigma}_1^2 = \frac{1}{n}\sum_{i=1}^{n} e_{1i}^2$ and $\widehat{\sigma}_2^2 = \frac{1}{n}\sum_{i=1}^{n} e_{2i}^2$. To find the asymptotic distribution, we first need to derive the joint distribution of $\widehat{\sigma}_1^2$ and $\widehat{\sigma}_2^2$. By the CLT (what would be the conditions under which the CLT holds?),

$$\sqrt{n}\begin{pmatrix} \widehat{\sigma}_1^2 - \sigma_1^2 \\ \widehat{\sigma}_2^2 - \sigma_2^2 \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} E[(e_{1i}^2 - \sigma_1^2)^2] & E[(e_{1i}^2 - \sigma_1^2)(e_{2i}^2 - \sigma_2^2)] \\ E[(e_{1i}^2 - \sigma_1^2)(e_{2i}^2 - \sigma_2^2)] & E[(e_{2i}^2 - \sigma_2^2)^2] \end{bmatrix}\right).$$

Now we use the delta method. Consider a function $g(a, b) = a - b$. The derivative with respect to (a, b) is $g'(a, b) = (1, -1)$. Then

$$\sqrt{n} g\left(\widehat{\sigma}_1^2 - \sigma_1^2, \widehat{\sigma}_2^2 - \sigma_2^2\right) = \sqrt{n}(\widehat{\theta} - \theta) \xrightarrow{d} N(0, \Omega)$$

where $\Omega = E[(e_{1i}^2 - \sigma_1^2)^2] + E[(e_{2i}^2 - \sigma_2^2)^2] - 2E[(e_{1i}^2 - \sigma_1^2)(e_{2i}^2 - \sigma_2^2)]$.

(c) By replacing E with the sample mean, a consistent estimator of Ω is

$$\widehat{\Omega} = \frac{1}{n}\sum_{i=1}^{n}(e_{1i}^2 - \widehat{\sigma}_1^2)^2 + \frac{1}{n}\sum_{i=1}^{n}(e_{2i}^2 - \widehat{\sigma}_2^2)^2 - 2\frac{1}{n}\sum_{i=1}^{n}(e_{1i}^2 - \widehat{\sigma}_1^2)(e_{2i}^2 - \widehat{\sigma}_2^2).$$

(d) We can use the t test. For the null hypothesis of $H_0 : \theta = 0$, The test statistic is

$$t_n = \frac{\widehat{\theta}}{\sqrt{\widehat{\Omega}/n}}.$$

We reject the null if $|t_n| > c_{1-\alpha/2}$ where $c_{1-\alpha/2}$ is the 1-$\alpha/2$th quantile of the standard normal distribution.

(e) If the null is not rejected, we conclude that there is not enough evidence that one model fits better than the other.

Q6. (a) The estimated equation is as follows. The standard errors are heteroskedasticity-robust, with a finite sample adjustment (HC1 formula).

$$\widehat{\log(\mathrm{C})} = \underset{(1.72)}{-3.53} + \underset{0.03}{0.72}\ \log(\mathrm{Q}) + \underset{(0.25)}{0.44}\ \log(\mathrm{PL}) - \underset{(0.32)}{0.22}\ \log(\mathrm{PK}) + \underset{(0.08)}{0.43}\ \log(\mathrm{PF})$$

(b) The restriction means that the Cobb-Douglas production function has constant returns to scale.

(e) We test $H_0 : \beta_3 + \beta_4 + \beta_5 = 1$ using the Wald statistic at the 5% significance level. The number of restrictions is 1 (the dimension of the null hypothesis). Since the Wald statistic is asymptotically $\chi^2_{dof=1}$ distributed, we use the 0.95th quantile of the $\chi^2_{dof=1}$ distribution as the critical value, which is 3.84. The statistic is constructed as,

$$\mathcal{W} = (\widehat{\beta}_3 + \widehat{\beta}_4 + \widehat{\beta}_5 - 1)'(G'\widehat{V}_{\hat{\beta}}G)^{-1}(\widehat{\beta}_3 + \widehat{\beta}_4 + \widehat{\beta}_5 - 1) = 0.65$$

where $G' = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \end{pmatrix}$ and $\widehat{V}_{\hat{\beta}}$ is the covariance matrix estimator obtained in Part (a). Note that G is the derivative of the function $g(a, b, c, d, e) = c + d + e$ with respect to the parameters. Since the statistic is less than the critical value, we do not reject the null hypothesis. The p-value of the statistic is 0.42, so for any reasonable significance level, we would not reject the null.