COMM8102 Econometric Analysis
Assignment 3

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Exercise 7.1

Let $\hat{\beta}_1$ be the estimator of β_1 by only regressing Y on X_1.

$$
\begin{aligned}
\hat{\beta}_1 &= \left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{1i}'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}Y_i\right) \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{1i}'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}\left(X_{1i}'\beta_1 + X_{2i}'\beta_2 + e_i\right)\right) \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{1i}'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{1i}'\beta_1 + \frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{2i}'\beta_2 + \frac{1}{n}\sum_{i=1}^{n} X_{1i}e_i\right) \\
&= \beta_1 + \left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{1i}'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{2i}'\beta_2\right) + \left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}X_{1i}'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_{1i}e_i\right) \\
&\xrightarrow{p} \beta_1 + E\left[X_1X_1'\right]^{-1}E\left[X_1X_2'\right]\beta_2 + E\left[X_1X_1'\right]^{-1}E\left[X_1e\right] \\
&= \beta_1 + E\left[X_1X_1'\right]^{-1}E\left[X_1X_2'\right]\beta_2 + 0 \\
&= \beta_1 + E\left[X_1X_1'\right]^{-1}E\left[X_1X_2'\right]\beta_2
\end{aligned}
$$

by WLLN and CMT. $E[X_1e] = 0$, since $E[Xe] = 0$. Because $\hat{\beta}_1 \xrightarrow{p} \beta_1 + E[X_1X_1']^{-1}E[X_1X_2']\beta_2$, $\hat{\beta}_1$ is not consistent for β_1. When $E[X_1X_2'] = \mathbf{0}$ or $\beta_2 = \mathbf{0}$, $\hat{\beta}_1 \xrightarrow{p} \beta_1$.

Exercise 7.15

$$
\begin{aligned}
\sqrt{n}\left(\hat{\beta}-\beta\right) &= \sqrt{n}\left(\frac{\sum_{i=1}^{n} X_i^3 Y_i}{\sum_{i=1}^{n} X_i^4}-\beta\right) \\
&= \sqrt{n}\left(\frac{\sum_{i=1}^{n} X_i^3 \left(X_i\beta+e_i\right)}{\sum_{i=1}^{n} X_i^4}-\beta\right) \\
&= \sqrt{n}\left(\beta+\frac{\sum_{i=1}^{n} X_i^3 e_i}{\sum_{i=1}^{n} X_i^4}-\beta\right) \\
&= \frac{\sqrt{n}\left(\sum_{i=1}^{n} X_i^3 e_i-\underbrace{E\left[X_i^3 e\right]}_{0}\right)}{\sum_{i=1}^{n} X_i^4} \\
&\xrightarrow{d} \frac{\mathcal{N}\left(0,E\left[X_i^6 e_i^2\right]\right)}{E\left[X_i^4\right]} \\
&= \mathcal{N}\left(0,\frac{E\left[X_i^6 e_i^2\right]}{\left(E\left[X_i^4\right]\right)^2}\right)
\end{aligned}
$$

by CLT and CMT.

$$
\begin{aligned}
E\left[X_i^3 e\right] &= E\left[E\left[\left.X_i^3 e\right| X_i\right]\right] \\
&= E\left[X_i^3 E\left[\left.e\right| X_i\right]\right] \\
&= 0
\end{aligned}
$$

Exercise 9.4

(a)

The size of a test is the probability of a false rejection of the (true) null hypothesis:

$$
\begin{aligned}
P\left(\text{Reject } H_0 \middle| H_0 \text{ is true}\right) &= P\left(\text{Reject } H_0 \middle| H_0\right) \\
&= P\left(W<c_1 \text{ or } W>c_2 \middle| H_0\right)
\end{aligned}
$$

Under H_0, $W \xrightarrow{d} \chi_q^2$, $P\left(W<c_1 \text{ or } W>c_2 \middle| H_0\right)=P\left(W<c_1 \middle| H_0\right)+P\left(W>c_2 \middle| H_0\right)=\frac{\alpha}{2}+1-(1-\frac{\alpha}{2})=\alpha$.

(b)

This is not a good test of H_0 vs H_1.Wald test is the extension of t-test to multivariate case. Intuitively, Wald test statistic is the sqaured version of t statistic in the scalar case. Only the right tail captures the extreme deviation from 0 (both positive and negative). This can also be extended to the multivarite case. We do not need to look at the left tail of the χ_q^2 distribution. A one-side test is enough.

Exercise 9.12

This interpretation is not correct.

The power of a hypothesis test is the probability that reject H_0 when the H_1 is true. p value is a quantity depending on the value of test statistic and the asymptotic null distribution G. The power depends on the critical value and the distribution of test statistic under the alternative hypothesis.

Although a larger sample will make the test have more power, it does not necessarily reduce the p value. p value depends on both of the test statistic and the asymptotic null distribution G. The extra samples may make the value of test statistic increase, but it may also make the value of test statistic decrease. Therefore, it cannot be guranteed that a larger sample will make the null rejected.

Exercise 9.16

(a)

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} e_{1i}^2 - \frac{1}{n}\sum_{i=1}^{n} e_{2i}^2$$

(b)

By central limit theorem,

$$\begin{pmatrix} \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} e_{1i}^2 - \sigma_1^2\right) \\ \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} e_{2i}^2 - \sigma_2^2\right) \end{pmatrix} \xrightarrow{d} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, V\right),$$

where

$$V = \begin{pmatrix} E\left[\left(e_1^2 - \sigma_1^2\right)^2\right] & E\left[\left(e_1^2 - \sigma_1^2\right)\left(e_2^2 - \sigma_2^2\right)\right] \\ E\left[\left(e_1^2 - \sigma_1^2\right)\left(e_2^2 - \sigma_2^2\right)\right] & E\left[\left(e_2^2 - \sigma_2^2\right)^2\right] \end{pmatrix}.$$

Define a function $r : \mathbb{R}^2 \to \mathbb{R}$, such that $r\left(\sigma_1^2, \sigma_2^2\right) = \sigma_1^2 - \sigma_2^2$. Then we have $\theta = r\left(\sigma_1^2, \sigma_2^2\right), \hat{\theta} = r\left(\frac{1}{n}\sum_{i=1}^{n} e_{1i}^2, \frac{1}{n}\sum_{i=1}^{n} e_{2i}^2\right)$. By delta method,

$$\begin{aligned} \sqrt{n}\left(\hat{\theta} - \theta\right) &= \sqrt{n}\left(r\left(\frac{1}{n}\sum_{i=1}^{n} e_{1i}^2, \frac{1}{n}\sum_{i=1}^{n} e_{2i}^2\right) - r\left(\sigma_1^2, \sigma_2^2\right)\right), \\ &\xrightarrow{d} \mathcal{N}\left(0, R'VR\right), \end{aligned}$$

where

$$\begin{aligned} R &= \begin{pmatrix} \frac{\partial}{\partial \sigma_1^2} r\left(\sigma_1^2, \sigma_2^2\right) \\ \frac{\partial}{\partial \sigma_2^2} r\left(\sigma_1^2, \sigma_2^2\right) \end{pmatrix}, \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Then we have

$$\begin{aligned} R'VR &= E\left[\left(e_1^2 - \sigma_1^2\right)^2\right] - 2 \cdot E\left[\left(e_1^2 - \sigma_1^2\right)\left(e_2^2 - \sigma_2^2\right)\right] + E\left[\left(e_2^2 - \sigma_2^2\right)^2\right], \\ &= E\left[\left(\left(e_1^2 - \sigma_1^2\right) - \left(e_2^2 - \sigma_2^2\right)\right)^2\right], \\ &= E\left[\left(e_1^2 - e_2^2 - \left(\sigma_1^2 - \sigma_2^2\right)\right)^2\right]. \end{aligned}$$

Finally,

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, E\left[\left(e_1^2 - e_2^2 - \left(\sigma_1^2 - \sigma_2^2\right)\right)^2\right]\right).$$

(c)

Since $\hat{\theta} = \theta + \frac{\sqrt{n}(\hat{\theta}-\theta)}{\sqrt{n}}$, and $\sqrt{n}\left(\hat{\theta}-\theta\right) \xrightarrow{d} \mathcal{N}\left(0, E\left[\left(e_1^2 - e_2^2 - \left(\sigma_1^2 - \sigma_2^2\right)\right)^2\right]\right)$,

$$\hat{\theta} \xrightarrow{d} \mathcal{N}\left(\theta, \frac{E\left[\left(e_1^2 - e_2^2 - \left(\sigma_1^2 - \sigma_2^2\right)\right)^2\right]}{n}\right)$$

An estimator of the asymptotic variance of $\hat{\theta}$ is

$$\hat{V}_{\hat{\theta}} = \frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left(e_{1i}^2 - e_{2i}^2 - \left(\sigma_1^2 - \sigma_2^2\right)\right)^2\right).$$

(d)

H_0 is equivalent to $\theta = 0$, and H_1 is equivalent to $\theta \neq 0$. Under H_0 the test statistic is

$$\begin{aligned} T_n &= \frac{\hat{\theta} - 0}{\sqrt{\hat{V}_{\hat{\theta}}}} \\ &\sim \mathcal{N}(0,1). \end{aligned}$$

A test of asymptotic size α can be that we reject the null hypothesis if $T_n < a_1$ or $T_n > a_2$, where a_1 is the $\frac{\alpha}{2}$ quantile of $\mathcal{N}(0,1)$ and a_2 is the $1 - \frac{\alpha}{2}$ of $\mathcal{N}(0,1)$.

(e)

Acceptance of the null hypothesis means these two models have same variance for the residuals. therefore they fit the data equally well.

Exercise 9.26

(a)

	Estimations	Standard errors $HC1$
$\hat{\beta}_1$	-3.527	1.7186
$\hat{\beta}_2$	0.720	0.0326
$\hat{\beta}_3$	0.436	0.2456
$\hat{\beta}_4$	-0.220	0.3238
$\hat{\beta}_5$	0.427	0.0755

(b)

$$\begin{aligned} \log C &= \beta_1 + \beta_2 \log Q + \beta_3 \log PL + \beta_4 \log PK + \beta_5 \log PF + e \\ &= \beta_1 + \log Q^{\beta_2} + \log PL^{\beta_3} + \log PK^{\beta_4} + \log PF^{\beta_5} + e \end{aligned}$$

Then we have

$$C = \exp(\beta_1) \cdot \exp(e) \cdot Q^{\beta_2} \cdot PL^{\beta_3} \cdot PK^{\beta_4} \cdot PF^{\beta_5}$$

For $\beta_3 + \beta_4 + \beta_5 = 1$, this means the cost function has a constant return to scale for labor, capital and fuel.

(e)

Define a function $r : \mathbb{R}^5 \to \mathbb{R}$, such that $r(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) = \beta_3 + \beta_4 + \beta_5$. The we have $\nabla r = [0, 0, 1, 1, 1]'$. By Delta method and affine transform of normal distribution, we establish

$$\sqrt{n}\left(\hat{\beta}_3 + \hat{\beta}_4 + \hat{\beta}_5 - (\beta_3 + \beta_4 + \beta_5)\right) \xrightarrow{d} \mathcal{N}(0, V_r)$$

where $V_r = \nabla r' \mathbf{V}_\beta \nabla r$. $\mathbf{V}_\beta$ is the asymptotic covariance matrix of $\sqrt{n}\left(\hat{\beta} - \beta\right)$.

Then the Wald statistic is

$$\begin{aligned} \mathcal{W}_n &= \frac{\left(\hat{\beta}_3 + \hat{\beta}_4 + \hat{\beta}_5 - (\beta_3 + \beta_4 + \beta_5)\right)^2}{\hat{V}_{\hat{r}}^{HC1}} \\ &= 0.645. \end{aligned}$$

Since $\mathcal{W}_n \xrightarrow{d} \chi^2_1$, the critical value $c = 3.84$ for $\alpha = 0.05$. We do not reject the null hypothesis at 5% significance level.

Code

```
library(readxl)
```

```
## Warning: package 'readxl' was built under R version 3.6.2
```

```
data1 <- read_excel("Nerlove1963.xlsx")
data<-log(data1)
y<-as.matrix(data[,1])
x<-as.matrix(cbind(matrix(1,nrow(y),1),data[,2:5]))
beta<-solve(t(x)%*%x,t(x)%*%y)
#standard error
n <- nrow(y)
k <- ncol(x)
invx <- solve(t(x)%*%x)
a <- n/(n-k)
e <- y-x%*%beta
leverage <- rowSums(x*(x%*%invx))
sig2 <- (t(e) %*% e)/(n-k)
u1 <- x*(e%*%matrix(1,1,k))
u2 <- x*((e/sqrt(1-leverage))%*%matrix(1,1,k))
u3 <- x*((e/(1-leverage))%*%matrix(1,1,k))
v0 <- as.numeric(sig2)*invx
v1 <- invx %*% (t(u1)%*%u1) %*% invx
v1a <- a * invx %*% (t(u1)%*%u1) %*% invx
v2 <- invx %*% (t(u2)%*%u2) %*% invx
v3 <- invx %*% (t(u3)%*%u3) %*% invx
s0 <- sqrt(diag(v0)) # Homoskedastic formula
s1 <- sqrt(diag(v1)) # HC0
s1a <- sqrt(diag(v1a)) # HC1
s2 <- sqrt(diag(v2)) # HC2
s3 <- sqrt(diag(v3)) # HC3

#wald test
R<-matrix(c(0,0,1,1,1),5,1)

waldstats<-(sum(beta[3:5])-1)^2/(t(R)%*%v1a%*%R)

c<-qchisq(0.95, 1, ncp = 0, log = FALSE)
```