COMM8102 Econometric Analysis

Assignment 1

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Excercise 2.4

$$
\begin{aligned}
\mathbb{E}\left[Y \mid X=0\right] &= 1 \cdot \mathbb{P}\left(Y=1 \mid X=0\right) + 0 \cdot \mathbb{P}\left(Y=0 \mid X=0\right) \\
&= 1 \cdot \frac{\mathbb{P}\left(Y=1, X=0\right)}{\mathbb{P}\left(X=0\right)} + 0 \cdot \frac{\mathbb{P}\left(Y=0, X=0\right)}{\mathbb{P}\left(X=0\right)} \\
&= 1 \cdot \frac{0.4}{0.4+0.1} + 0 \cdot \frac{0.1}{0.4+0.1} \\
&= 0.8
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[Y \mid X=1\right] &= 1 \cdot \mathbb{P}\left(Y=1 \mid X=1\right) + 0 \cdot \mathbb{P}\left(Y=0 \mid X=1\right) \\
&= 1 \cdot \frac{\mathbb{P}\left(Y=1, X=1\right)}{\mathbb{P}\left(X=1\right)} + 0 \cdot \frac{\mathbb{P}\left(Y=0, X=1\right)}{\mathbb{P}\left(X=1\right)} \\
&= 1 \cdot \frac{0.3}{0.3+0.2} + 0 \cdot \frac{0.2}{0.3+0.21} \\
&= 0.6
\end{aligned}
$$

$$\begin{aligned}
\mathbb{E}\left[\left.Y^2\right|X=0\right] &= 1^2\cdot\mathbb{P}\left(\left.Y=1\right|X=0\right)+0^2\cdot\mathbb{P}\left(\left.Y=0\right|X=0\right)\\
&= 1\cdot\frac{\mathbb{P}\left(Y=1,X=0\right)}{\mathbb{P}\left(X=0\right)}+0\cdot\frac{\mathbb{P}\left(Y=0,X=0\right)}{\mathbb{P}\left(X=0\right)}\\
&= 1\cdot\frac{0.4}{0.4+0.1}+0\cdot\frac{0.1}{0.4+0.1}\\
&= 0.8
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\left.Y^2\right|X=1\right] &= 1^2\cdot\mathbb{P}\left(\left.Y=1\right|X=1\right)+0^2\cdot\mathbb{P}\left(\left.Y=0\right|X=1\right),\\
&= 1\cdot\frac{\mathbb{P}\left(Y=1,X=1\right)}{\mathbb{P}\left(X=1\right)}+0\cdot\frac{\mathbb{P}\left(Y=0,X=1\right)}{\mathbb{P}\left(X=1\right)},\\
&= 1\cdot\frac{0.3}{0.3+0.2}+0\cdot\frac{0.2}{0.3+0.21}\\
&= 0.6
\end{aligned}$$

$$\begin{aligned}
Var\left[\left.Y\right|X=0\right] &= \mathbb{E}\left[\left.Y^2\right|X=0\right]-\left(\mathbb{E}\left[\left.Y\right|X=0\right]\right)^2,\\
&= 0.8-0.8^2\\
&= 0.16
\end{aligned}$$

$$\begin{aligned}
Var\left[\left.Y\right|X=1\right] &= \mathbb{E}\left[\left.Y^2\right|X=1\right]-\left(\mathbb{E}\left[\left.Y\right|X=1\right]\right)^2,\\
&= 0.6-0.6^2\\
&= 0.24
\end{aligned}$$

Excercise 2.7

$$\begin{aligned}
\sigma(X) &= Var[Y|X] \\
&= \mathbb{E}\left[(Y-\mathbb{E}[Y|X])^2\Big|X\right] \\
&= \mathbb{E}\left[Y^2+(\mathbb{E}[Y|X])^2-2\cdot Y\cdot\mathbb{E}[Y|X]\Big|X\right] \\
&= \mathbb{E}\left[Y^2\big|X\right]+\mathbb{E}\left[(\mathbb{E}[Y|X])^2\Big|X\right]-2\cdot\mathbb{E}[Y\cdot\mathbb{E}[Y|X]|X] \\
&= \mathbb{E}\left[Y^2\big|X\right]+(\mathbb{E}[Y|X])^2-2\cdot\mathbb{E}[Y|X]\cdot\mathbb{E}[Y|X] \\
&= \mathbb{E}\left[Y^2\big|X\right]-(\mathbb{E}[Y|X])^2
\end{aligned}$$

1 Excercise 2.18

(a)

$$\begin{aligned}
\mathbf{Q}_{XX} &= \mathbb{E}[XX^\intercal] \\
&= \begin{pmatrix} \mathbb{E}[1] & \mathbb{E}[X_2] & \mathbb{E}[X_3] \\ \mathbb{E}[X_2] & \mathbb{E}\left[X_2^2\right] & \mathbb{E}[X_2X_3] \\ \mathbb{E}[X_3] & \mathbb{E}[X_2X_3] & \mathbb{E}\left[X_3^2\right] \end{pmatrix} \\
&= \begin{pmatrix} 1 & \mathbb{E}[X_2] & \mathbb{E}[\alpha_1+\alpha_2X_2] \\ \mathbb{E}[X_2] & \mathbb{E}\left[X_2^2\right] & \mathbb{E}\left[\alpha_1X_2+\alpha_2X_2^2\right] \\ \mathbb{E}[\alpha_1+\alpha_2X_2] & \mathbb{E}\left[\alpha_1X_2+\alpha_2X_2^2\right] & \mathbb{E}\left[\alpha_1^2+\alpha_2^2X_2^2+2\alpha_1\alpha_2X_2\right] \end{pmatrix} \\
&= \begin{pmatrix} 1 & \mathbb{E}[X_2] & \alpha_1+\alpha_2\mathbb{E}[X_2] \\ \mathbb{E}[X_2] & \mathbb{E}\left[X_2^2\right] & \alpha_1\mathbb{E}[X_2]+\alpha_2\mathbb{E}\left[X_2^2\right] \\ \alpha_1+\alpha_2\mathbb{E}[X_2] & \alpha_1\mathbb{E}[X_2]+\alpha_2\mathbb{E}\left[X_2^2\right] & \alpha_1^2+2\alpha_1\alpha_2\mathbb{E}[X_2]+\alpha_2^2\mathbb{E}\left[X_2^2\right] \end{pmatrix}.
\end{aligned}$$

Define

$$\begin{aligned}
\mathbf{v}_1 &= (1,\mathbb{E}[X_2],\alpha_1+\alpha_2\mathbb{E}[X_2])^\intercal, \\
\mathbf{v}_2 &= \left(\mathbb{E}[X_2],\mathbb{E}\left[X_2^2\right],\alpha_1\mathbb{E}[X_2]+\alpha_2\mathbb{E}\left[X_2^2\right]\right)^\intercal, \\
\mathbf{v}_3 &= \left(\alpha_1+\alpha_2\mathbb{E}[X_2],\alpha_1\mathbb{E}[X_2]+\alpha_2\mathbb{E}\left[X_2^2\right],\alpha_1^2+2\alpha_1\alpha_2\mathbb{E}[X_2]+\alpha_2^2\mathbb{E}\left[X_2^2\right]\right)^\intercal.
\end{aligned}$$

$\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are the first, second and third column of the matrix $\mathbf{Q}_{XX}$. We can observe that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 = \mathbf{v}_3.$$

By defintion, $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are not linearly independent. Therefore, $\mathbf{Q}_{XX}$ is not invertible.

(b)

Let $\mathcal{X} = (1, X_2)$, and $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.As a result, $AX = \mathcal{X}$.Since X_3 is a linear function of X_2, the best linear predictor $X^\intercal\beta_1 \equiv \mathcal{X}^\intercal\beta_2$ for different β_1 and β_2. We also have

$$\begin{aligned}
\beta_2 &= \left(\mathbb{E}\left[\mathcal{X}\mathcal{X}^\intercal\right]\right)^{-1}\mathbb{E}\left[\mathcal{X}Y\right] \\
&= \begin{bmatrix} \mathbb{E}[1] & \mathbb{E}[X_2] \\ \mathbb{E}[X_2] & \mathbb{E}\left[X_2^2\right] \end{bmatrix}^{-1} \begin{bmatrix} \mathbb{E}[Y] \\ \mathbb{E}[X_2Y] \end{bmatrix} \\
&= \frac{1}{\mathbb{E}[X_2^2] - (\mathbb{E}[X_2])^2} \begin{bmatrix} \mathbb{E}\left[X_2^2\right] & -\mathbb{E}[X_2] \\ -\mathbb{E}[X_2] & 1 \end{bmatrix} \begin{bmatrix} \mathbb{E}[Y] \\ \mathbb{E}[X_2Y] \end{bmatrix} \\
&= \frac{1}{\mathbb{E}[X_2^2] - (\mathbb{E}[X_2])^2} \begin{bmatrix} \mathbb{E}\left[X_2^2\right]\mathbb{E}[Y] - \mathbb{E}[X_2]\mathbb{E}[X_2Y] \\ \mathbb{E}[X_2Y] - \mathbb{E}[X_2]\mathbb{E}[Y] \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathbb{E}\left[X_2^2\right]\mathbb{E}[Y] - \mathbb{E}[X_2]\mathbb{E}[X_2Y]}{\mathbb{E}\left[X_2^2\right] - (\mathbb{E}[X_2])^2} \\ \frac{\mathbb{E}[X_2Y] - \mathbb{E}[X_2]\mathbb{E}[Y]}{\mathbb{E}\left[X_2^2\right] - (\mathbb{E}[X_2])^2} \end{bmatrix}
\end{aligned}$$

The best linear predictor of Y given X is

$$\begin{aligned}
X^\intercal\beta_1 &= \mathcal{X}^\intercal\beta_2 \\
&= (AX)^\intercal\beta_2 \\
&= X^\intercal A^\intercal \begin{bmatrix} \frac{\mathbb{E}\left[X_2^2\right]\mathbb{E}[Y] - \mathbb{E}[X_2]\mathbb{E}[X_2Y]}{\mathbb{E}\left[X_2^2\right] - (\mathbb{E}[X_2])^2} \\ \frac{\mathbb{E}[X_2Y] - \mathbb{E}[X_2]\mathbb{E}[Y]}{\mathbb{E}\left[X_2^2\right] - (\mathbb{E}[X_2])^2} \end{bmatrix} \\
&= X^\intercal \begin{bmatrix} \frac{\mathbb{E}\left[X_2^2\right]\mathbb{E}[Y] - \mathbb{E}[X_2]\mathbb{E}[X_2Y]}{\mathbb{E}\left[X_2^2\right] - (\mathbb{E}[X_2])^2} \\ \frac{\mathbb{E}[X_2Y] - \mathbb{E}[X_2]\mathbb{E}[Y]}{\mathbb{E}\left[X_2^2\right] - (\mathbb{E}[X_2])^2} \\ 0 \end{bmatrix}
\end{aligned}$$

Excercise 3.3

$$
\begin{aligned}
X^\intercal \hat{\mathbf{e}} &= X^\intercal \left(Y - X\hat{\beta}\right) \\
&= X^\intercal \left(Y - X\left(X^\intercal X\right)^{-1} X^\intercal Y\right) \\
&= X^\intercal Y - \underbrace{X^\intercal X\left(X^\intercal X\right)^{-1}}_{I} X^\intercal Y \\
&= X^\intercal Y - X^\intercal Y \\
&= 0
\end{aligned}
$$

Excercise 3.12

Only (3.54) and (3.53) can be estimated by OLS. Since $\mathbf{D}_1 + \mathbf{D}_2 = \mathbf{1}_n$, there are perfect collinearity in (3.52), which violates the assumption of OLS. (3.52) has regressors $\mathbf{D}_1, \mathbf{D}_2$ and $\mathbf{1}_n$. (3.53) has regressors $\mathbf{D}_1$ and $\mathbf{D}_2$. (3.53) has regressors $\mathbf{D}_1$ and $\mathbf{1}_n$.

$$
\begin{aligned}
a\mathbf{D}_1 + b\mathbf{D}_2 &= a\mathbf{D}_1 + b\left(\mathbf{1}_n - \mathbf{D}_1\right) \\
&= (a-b)\,\mathbf{D}_1 + b\mathbf{1}_n = 0 \quad iff a = b = 0
\end{aligned}
$$

$\mathbf{D}_1$ and $\mathbf{D}_2$ are linearly independnent, but $\mathbf{D}_1, \mathbf{D}_2$ and $\mathbf{1}_n$ are linearly dependent.

(a)

No. $\mathbf{D}_1$ alone gives the same information as $\mathbf{D}_1$ and $\mathbf{D}_2$.

For men, $\alpha_1 = \mu + \phi$. For women, $\alpha_2 = \mu$. So we have

$$
\begin{aligned}
\alpha_1 &= \mu + \phi \\
\alpha_2 &= \mu
\end{aligned}
$$

or

$$\begin{aligned}
\mu &= \alpha_2 \\
\phi &= \alpha_1 - \alpha_2
\end{aligned}$$

(b)

The number of non-zero elements in $\mathbf{D}_1$ is the number of men and The number of non-zero elements in $\mathbf{D}_2$ is the number of women.

$$\begin{aligned}
\mathbf{1}^\intercal \mathbf{D}_1 &= n_1 \\
\mathbf{1}^\intercal \mathbf{D}_2 &= n_2
\end{aligned}$$

Excercise 3.19

$$\begin{aligned}
\tilde{e}_i &= Y_i - \tilde{Y}_i \\
&= \frac{1}{1-h_{ii}} \hat{e}_i \\
&= \frac{1}{1 - X_i \left(\mathbf{X}^\intercal \mathbf{X}\right)^{-1} X_i} \hat{e}_i \\
&= \frac{1}{1-\frac{1}{n}} \left(Y_i - X_i \hat{\beta}\right) \\
&= \frac{n}{n-1} \left(Y_i - \underbrace{X_i}_{1} \left(\underbrace{\mathbf{X}^\intercal \mathbf{X}}_{n} \right)^{-1} \underbrace{\mathbf{X}^\intercal \mathbf{Y}}_{\sum Y_i} \right) \\
&= \frac{n}{n-1} \left(Y_i - \bar{Y}\right)
\end{aligned}$$