Automorphic Forms in $L^2(\Gamma\backslash SL_2(\mathbb{R}))$

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Outline

1. Iwasawa Decomposition for $SL_2(\mathbb{R})$
2. $SL_2(\mathbb{R})$ unimodular
3. Right Regular Representation on $L^2(\Gamma\backslash SL_2(\mathbb{R}))$

$SL_2(\mathbb{R})$

Definition

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Theorem (Iwasawa Decomposition)

Let A be the group of all diagonal matrices in G with positive entries. Let N be matrices of the form $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ where $s \in \mathbb{R}$. Then $\psi : A \times N \times K \to G$ given by $(a, n, k) \mapsto ank$ is a homeomorphism.

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We can parametrize A,N,K by one single real variable, and we have $A \cong N \cong (\mathbb{R}, +)$. This fact will be important in proving $SL_2(\mathbb{R})$ is unimodular.

Radon Measure and Haar Measure

Definition

A measure μ defined on Borel σ-algebra of a locally compact Hausdorff space X is called Radon measure if:
(a) $\mu(K) < \infty$ for all compact set K.
(b) $\mu(A) = inf_{U \supset A} \mu(U)$ for A measurable, U open. (Outer regularity)
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Theorem

Let G be a locally compact group. Then there exists a Radon measure $\mu \neq 0$ on the Borel σ-algebra, which is left-invariant. This measure μ is uniquely determined up to scaling by positive numbers. It is called a Haar measure of G.

Modular Function

Take locally compact group G. Take a Haar measure μ on it. For any $x \in G$, $\mu_x(A) = \mu(Ax)$ is another Haar measure on G. So $\mu_x = \Delta(x)\mu$ for some $\Delta(x) > 0$. Such $\Delta(x)$ is well defined by uniqueness of Haar measure.

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Theorem

Δ is a continuous group homomorphism.

Examples

If G is abelian/compact/discrete, then G is unimodular.

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Definition

Fix natural number N. We call $\Gamma(N) = ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/n\mathbb{Z}))$ the principal congruence group of level N. A subgroup $\Gamma \subset SL_2(\mathbb{Z})$ is called congruence group if it contains a principal congruence group, i.e. $\Gamma(N) \subset \Gamma$ for some N.

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Theorem

Discrete subgroup of a locally compact group G is closed in G.

Later we will use these facts to give a Radon measure on $\Gamma\backslash SL_2(\mathbb{R})$.

A few theorems

Lemma

Let G be a locally compact group and let H be a closed subgroup. Then H is again a locally compact group and the quotient space G/H , equipped with the quotient topology, is a locally compact Hausdorff space.

Theorem

Let $H \subset G$ be a closed subgroup of the locally compact group G. On the locally compact space G/H there exists a non-trivial, G-invariant Radon measure if and only if $\Delta_G(h) = \Delta_H(h)$ holds for every $h \in H$. If it exists, the invariant measure is unique up to scaling.

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Using these facts, we get a non-trivial G-invariant $(G = SL_2(\mathbb{R}))$ Radon measure on G/Γ. We can switch between left cosets and right cosets, so we get a Radon measure on $\Gamma\backslash G$. Now we have $L^2(\Gamma\backslash G)$.

Right Regular Representation on $L^2(\Gamma\backslash G)$

Definition

Let $R : G \to Aut(L^2(\Gamma\backslash G))$ be $R_g\varphi(x) = \varphi(xg)$ for $g \in G$, $x \in \Gamma\backslash G$.

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Definition

An automorphic form is a function φ in $L^2(\Gamma\backslash G)$.