

# Hartshorne Exercise Solution

Yuheng Shi

January 2, 2023

In this document,  $\subset$  means strict set containment, and  $\subseteq$  is any set containment. “WLOG” means “without loss of generality”. “iff” means “if and only if”.  $K(R)$  means field of fraction of the integral domain  $R$ . Unless stated otherwise,  $A = k[x_1, \dots, x_n]$  and  $S = k[x_0, \dots, x_n]$ .  $\langle S \rangle$  denotes ideal generated by the set  $S$ . Homogeneous coordinate is represented by  $(a_0 : \dots : a_n)$ . All references like “Theorem” and “Proposition” refer to Hartshorne’s Algebraic Geometry.

## 1 Affine Varieties

**Ex.1.1.** We see  $y - x^2$  and  $xy - 1$  are irreducible polynomials by viewing them as polynomials in  $y$  with coefficients in  $k[x]$ , then look at degree (in  $y$ ) of possible factors.

(a)  $A(Y) = k[x, y]/(y - x^2) \cong k[t]$  where the isomorphism is induced by  $k[x, y] \rightarrow k[t]$  defined by  $x \mapsto t, y \mapsto t^2$ .

(b)  $A(Z) = k[x, y]/(xy - 1) \cong k[t, t^{-1}]$  where the isomorphism is induced by  $k[x, y] \rightarrow k[t, t^{-1}]$  defined by  $x \mapsto t, y \mapsto t^{-1}$ . Because  $t, t^{-1}$  are units in  $k[t, t^{-1}]$ , any ring homomorphism  $k[t, t^{-1}] \rightarrow k[t]$  must send  $t$  and  $t^{-1}$  to units, so the image of this ring homomorphism is  $k$ , so in particular  $k[t] \not\cong k[t, t^{-1}]$ .

(c) Write  $f = ax^2 + bxy + cy^2 + dx + ey + g$  irreducible. If  $a = c = 0$ , then  $b \neq 0$ . Since  $A(W) = k[x, y]/(f)$  is invariant under scaling  $f$  by a unit, we can assume  $b = 1$ . Then  $f = (x + e)(y + d) + (g - ed)$ . Do a change of variable  $k[x', y'] \rightarrow k[x, y]$  by  $x' \mapsto x + e, y' \mapsto y + d$ , then this isomorphism  $k[x', y'] \cong k[x, y]$  induces isomorphism  $\frac{k[x', y']}{(x'y' + (g - ed))} \cong A(W)$ . Note  $g - ed \neq 0$  because  $f$  is irreducible. Then we have

$\frac{k[x', y']}{(x'y' + (g - ed))} \cong k[t, t^{-1}]$  by the argument in part (b).

If  $a \neq 0$  or  $c \neq 0$ , WLOG assume  $c \neq 0$ , then we can assume  $c = 1$ . Temporarily working in fractional field of  $k[x, y]$ , we have  $ax^2 + bxy + y^2 = x^2(a + b\frac{y}{x} + (\frac{y}{x})^2) = x^2(\frac{y}{x} - u_1)(\frac{y}{x} - u_2) = (y - u_1x)(y - u_2x)$  where  $u_1$  and  $u_2$  are two roots of the polynomial  $z^2 + bz + a$ . Our classification of  $A(W)$  will depend on whether  $u_1 = u_2$ .

If  $u_1 \neq u_2$ , then we let  $x' = y - u_1x, y' = y - u_2x$ , then  $k[x', y'] \cong k[x, y]$  because we can inverse the equations and write  $x, y$  in terms of  $x', y'$ . Under this isomorphism  $f$  becomes  $x'y' + c_1x' + c_2y' + c_3 = (x' + c_2)(y' + c_1) + (c_3 - c_1c_2)$  for some  $c_i \in k$ , where  $c_3 - c_1c_2 \neq 0$  because  $f$  is irreducible. This isomorphism induces  $A(W) \cong \frac{k[x', y']}{(x' + c_2)(y' + c_1) + (c_3 - c_1c_2)} \cong \frac{k[x'', y'']}{x''y'' + (c_3 - c_1c_2)} \cong k[t, t^{-1}]$  where we have done another change of variables in the second step.

If  $u_1 = u_2$ , then let  $u$  be  $u = u_1 = u_2$ , and  $f = (y - ux)^2 + dx + ey + g$ . Let  $x' = y - ux, y' = -(dx + ey)$ , then  $f = x'^2 - y' + g$ . We note that  $x$  and  $y$  can be expressed in linear homogeneous polynomial of  $x'$  and  $y'$  because  $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -u & 1 \\ -d & -e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$  and  $\begin{pmatrix} -u & 1 \\ -d & -e \end{pmatrix}$  is invertible, since if its determinant  $eu + d = 0$ ,  $f = (y - ux)^2 + e(y - ux) + g$  is reducible. Thus we have  $k[x, y] \cong k[x', y']$ , which induces  $A(W) \cong \frac{k[x', y']}{(x'^2 - y' + g)} \cong k[x']$  where the last isomorphism is induced by  $x' \mapsto x', y' \mapsto x'^2 + g$ .

Thus we see  $A(W)$  is isomorphic of  $A(Y)$  or  $A(Z)$ , and the specific case conditions are described in the course of this proof.

**Ex.1.2.** We can verify  $Y = Z(y - x^2, z - x^3)$ . The ideal  $(y - x^2, z - x^3)$  is prime, because it is kernel of the map  $k[x, y, z] \rightarrow k[t]$  given by identity on  $k$ ,  $x \mapsto t, y \mapsto t^2, z \mapsto t^3$ . Then  $Y$  is affine variety, and

$\dim(Y) = \dim A(Y) = \dim k[t] = 1$ .

**Ex.1.3.** By assumption  $Y = Z(x^2 - yz, xz - x)$ . We have  $Y = Z(x, y) \cup Z(x, z) \cup Z(z - 1, x^2 - y)$ . Since  $(x, y)$ ,  $(x, z)$ ,  $(z - 1, x^2 - y)$  are prime ideals none of which contain another,  $Z(x, y)$ ,  $Z(x, z)$ ,  $Z(z - 1, x^2 - y)$  are irreducible components of  $Y$ . Their ideals are  $(x, y)$ ,  $(x, z)$ ,  $(z - 1, x^2 - y)$ .

**Ex.1.4.** Let  $(x, y)$  be affine coordinates on  $\mathbb{A}^2$ . Then  $Z(x - y)$  is closed in  $\mathbb{A}^2$ . FSOC, suppose  $Z(x - y)$  is closed in  $\mathbb{A}^1 \times \mathbb{A}^1$  with product topology. Then there is a basic open set  $U \times V$  such that  $U \times V \subseteq Z(x - y)^c$ . Proper closed sets in  $\mathbb{A}^1$  are finite sets, so  $U = \mathbb{A}^1 - Y_1$ ,  $V = \mathbb{A}^1 - Y_2$  where  $Y_1, Y_2$  are finite sets in  $\mathbb{A}^1$ . But  $k$  is algebraically closed, so  $k$  is infinite, so there exists  $a \in \mathbb{A}^1$  such that  $a \in U \cap V$ . Then  $(a, a) \in U \times V \subseteq Z(x - y)^c$  implies  $a \neq a$ , contradiction. So the topology on  $\mathbb{A}^2$  is different from the topology on  $\mathbb{A}^1 \times \mathbb{A}^1$ .

**Ex.1.5.** ( $\implies$ ) Suppose  $B \cong \frac{k[x_1, \dots, x_n]}{I(Y)}$  as  $k$ -algebra, where  $Y \subseteq \mathbb{A}^n$  is algebraic set. Then  $B$  is finitely generated by  $x_1, \dots, x_n$  as  $k$ -algebra.  $I(Y)$  is radical ideal, because if  $f^n \in I(Y)$ , then  $f \in I(Y)$ . This implies  $B$  has no (nonzero) nilpotent elements.

( $\impliedby$ ) Let  $\pi : k[x_1, \dots, x_n] \rightarrow B$  be a projection. Then  $\ker \pi$  is a radical ideal because  $B$  has no nonzero nilpotent elements. Let  $Y = Z(\ker \pi)$ , then  $A(Y) = \frac{k[x_1, \dots, x_n]}{I(Z(\ker \pi))} = \frac{k[x_1, \dots, x_n]}{\ker \pi} = B$ .

**Ex.1.6.** Let  $Y \subseteq X$  be a nonempty open subset of irreducible topological space. If  $Y$  is not dense in  $X$ , then  $X = \bar{Y} \cup Y^c$  is union of two proper closed subsets, contradiction. So  $Y$  is dense in  $X$ . FSOC, suppose  $Y$  is not irreducible, then  $Y = C_1 \cup C_2$  where  $C_1, C_2$  are proper closed in  $Y$ . Then  $C_1 = C'_1 \cap Y$ ,  $C_2 = C'_2 \cap Y$  where  $C'_1$  and  $C'_2$  are closed in  $X$ , and  $C_1 \cup C_2$  is proper closed subset of  $X$  because  $X$  is irreducible. But then  $X = (C_1 \cup C_2) \cup Y^c$  is union of two proper closed subset, contradiction. Also note  $Y$  is nonempty by assumption. So  $Y$  is irreducible with induced topology.

Next suppose  $Y \subseteq X$  is irreducible with induced topology. Then  $\bar{Y}$  is nonempty, and if  $\bar{Y}$  is not irreducible,  $\bar{Y} = Y_1 \cup Y_2$  where  $Y_1, Y_2$  are proper closed in  $\bar{Y}$ . Then  $Y = (Y_1 \cap Y) \cup (Y_2 \cap Y)$  is union of two closed subsets, so one of them must be  $Y$ . WLOG, say  $Y_1 \cap Y = Y$ . But note  $Y_1$  is closed in  $X$  as well, so  $\bar{Y} \subseteq Y_1$  by definition of closure. So  $\bar{Y} = Y_1$ , contradiction. So  $\bar{Y}$  is irreducible.

**Ex.1.7.**(a) (i) $\implies$ (ii): If there is no minimal element, then we can construct a descending chain that never stabilizes.

(ii) $\implies$ (iii): Obvious.

(iii) $\implies$ (iv): Same argument as in (i) $\implies$ (ii).

(iv) $\implies$ (i): First we have (ii). Then apply (ii) to any descending chain of closed sets.

(b) If an open covering has no finite subcover, then we can construct an ascending chain of open sets which never stabilizes by choosing appropriate open sets (then taking union) from the open sets of the open covering.

(c) Let  $X$  be noetherian topological space,  $Y \subseteq X$  be a subset with induced topology. Let  $D_1 \supseteq D_2 \supseteq \dots$  be a descending chain of closed sets in  $Y$ . Then  $D_i = D'_i \cap Y$  where  $D'_i$  is closed in  $X$ . For  $i \in \mathbb{Z}_+$ , let  $S_i = \cap_{k=1}^i D'_k$ , then  $S_i$  is a descending chain of closed sets in  $X$ , so there exists  $n$  such that  $S_i = S_n$  for all  $i \geq n$ . Then  $D'_i \supseteq \cap_{k=1}^n D'_k$  for all  $i \geq n$ . Then for all  $i \geq n$ ,  $D'_i \cap Y \supseteq (\cap_{k=1}^n D'_k) \cap Y$ , which implies  $D_i \supseteq D_n$  for all  $i \geq n$ , so  $Y$  is noetherian.

(d) Let  $X$  be noetherian and Hausdorff. FSOC, suppose  $X$  is infinite. Then we can pick  $x, y \in X$ ,  $x \neq y$ . Because  $X$  is Hausdorff,  $\exists U, V$  open sets such that  $x \in U$ ,  $y \in V$ ,  $U \cap V = \emptyset$ . Then  $X = U^c \cup V^c$ . Because  $X$  is infinite,  $U^c$  or  $V^c$  is infinite. WLOG, suppose  $U^c$  is infinite. By part (c),  $U^c$  with its induced topology is noetherian. It is also straightforward to verify that  $U^c$  with its induced topology is Hausdorff. Then we can find an infinite closed subset of  $U^c$ . Continue this process, we get a descending chain of closed subsets in  $X$  which does not stabilize. Contradiction. So  $X$  must be finite. Because in a Hausdorff space, each point is closed, and because finite union of closed sets is closed set, we conclude that  $X$  has discrete topology.

**Ex.1.8.** Suppose  $H = Z(f)$  where  $f$  is irreducible and has positive degree. We have  $A = k[x_1, \dots, x_n] \supseteq I(Y) + (f) \supset I(Y)$ , where the second inclusion is strict because  $H \supsetneq Y$ . Because ideals of  $A$  containing  $Y$  naturally correspond to ideals of  $A(Y)$ , it is quick to verify that  $I(Y) + (f) \subseteq A$  correspond to  $(\bar{f}) \subseteq A(Y)$  where  $\bar{f}$  denotes equivalence class of  $f$  in  $A(Y)$ .  $\bar{f} \neq 0$  because  $I(Y) + (f) \neq I(Y)$ . Because

$A(Y)$  is integral domain,  $\bar{f}$  is not a zero-divisor. If  $\bar{f}$  is a unit, then  $(\bar{f}) = A(Y)$ , so  $I(Y) + (f) = A$ , so  $Y \cap H = Z(I(Y) + (f)) = Z(A) = \emptyset$ , then we are trivially done. So suppose  $\bar{f}$  is not a unit in  $A(Y)$ . Let  $S_i$  be any irreducible component of  $Y \cap H$ . Then  $I(S_i)/I(Y) \subseteq A(Y)$  is a minimal prime ideal containing  $(\bar{f})$ . By Krull's Hauptidealsatz (Theorem 1.11A),  $I(S_i)/I(Y)$  has height 1. By Theorem 1.8A, we have  $\dim \frac{A(Y)}{I(S_i)/I(Y)} = \dim(A(Y)) - 1 = r - 1$ . We also have  $\frac{A(Y)}{I(S_i)/I(Y)} \cong A/I(S_i) = A(S_i)$ , so  $\dim S_i = \dim A(S_i) = r - 1$ , and we are done.

**Ex.1.9.** Induction on  $r$ . The case  $r = 1$  is easy to verify. For a general  $r > 1$ , suppose  $\mathbf{a} = (f_1, \dots, f_r)$ . Let  $Y_1, \dots, Y_m$  be irreducible components of  $Z(f_1, \dots, f_{r-1})$ , then because  $Z(\mathbf{a}) = \cup_{i=1}^m (Y_i \cap Z(f_r))$ , each irreducible component of  $Z(\mathbf{a})$  is irreducible component of some  $Y_i \cap Z(f_r)$ . Suppose  $Z(f_r) = \cup_{j=1}^k H_j$  where  $H_j$ 's are hypersurfaces. Then by a similar argument, each irreducible component of  $Y_i \cap Z(f_r)$  is irreducible component of some  $Y_i \cap H_j$ . By Ex.1.8 and induction hypothesis, dimension of each irreducible component of  $Y_i \cap H_j$  is at least  $n - r$ , so we are done.

**Ex.1.10.** (a) Given any ascending chain of irreducible closed distinct sets in  $Y$ , we can get a corresponding ascending chain of irreducible closed distinct sets in  $X$  by taking closure.

(b) Intersection of any ascending chain of irreducible closed distinct sets  $X_0 \subset X_1 \subset X_2 \dots$  with an  $U_i$  where  $U_i \cap X_0 \neq \emptyset$  gives an ascending chain of irreducible closed distinct subsets of  $U_i$ .

(c) Let  $X = \{1, 2\}$  with closed sets:  $X$ ,  $\{1\}$ , and  $\emptyset$ . Let  $U = \{2\}$ .

(d) If  $Y \neq X$ , take an ascending chain of closed irreducible distinct subsets of  $Y$  of maximum length. Append  $X$  to it, and we see contradiction to  $\dim Y = \dim X$ .

(e) Let  $X = \mathbb{N}$  with closed sets  $\{0, 1, \dots, n\}$  where  $n \in \mathbb{N}$ .

**Ex.1.11.** It's straightforward to verify that  $Y = Z(y^3 - x^4, z^3 - x^5, z^4 - y^5)$ , so  $Y$  is closed in  $\mathbb{A}^3$ . Let  $\varphi : k[x, y, z] \rightarrow k[t]$  be identity on  $k$  and  $x \mapsto t^3, y \mapsto t^4, z \mapsto t^5$ . Then  $I(Y) = \ker \varphi$ . Indeed, if  $f \in I(Y)$ , then  $\varphi(f)(a) = f(a^3, a^4, a^5) = 0$  for all  $a \in k$ .  $k$  is infinite, so  $\varphi(f) = 0$ , so  $f \in \ker \varphi$ . Conversely if  $f \in \ker \varphi$ , then for all  $(a^3, a^4, a^5) \in Y$ ,  $f(a^3, a^4, a^5) = \varphi(f)(a) = 0$ . so  $f \in I(Y)$ . Thus  $I(Y) = \ker \varphi$  is a prime ideal. height  $I(Y) = \dim k[x, y, z] - \dim \frac{k[x, y, z]}{I(Y)} = 3 - \dim k[t] = 2$ . By my answer here: <https://math.stackexchange.com/questions/4365408>,  $I(Y)$  cannot be generated by 2 elements, and in fact  $I(Y) = (zx - y^2, yz - x^3, yx^2 - z^2)$ .

**Ex.1.12.** Let  $f(x) = x^2 + 1$ , then  $Z(f) = \emptyset$ , which is by definition not irreducible.

## 2 Projective Varieties

**Ex.2.1.** If  $\mathbf{a} = S$  then the statement is trivial. Suppose  $\mathbf{a} \neq S$ , then  $0 \in Z_{\mathbb{A}^{n+1}}(\mathbf{a})$ , where  $Z_{\mathbb{A}^{n+1}}(\mathbf{a})$  means the zero set of  $\mathbf{a}$  in the affine space. Let  $\psi : \mathbb{A}^{n+1} - \{0\} \rightarrow \mathbb{P}^n$  be the canonical projection map. Then it is easy to verify  $\psi^{-1}(Z_{\mathbb{P}^n}(\mathbf{a})) \cup \{0\} = Z_{\mathbb{A}^{n+1}}(\mathbf{a})$ . Then since  $\deg f > 0$  and  $f$  is homogeneous,  $f(0) = 0$ , and then  $f(Z_{\mathbb{P}^n}(\mathbf{a})) = 0$  implies  $f(Z_{\mathbb{A}^{n+1}}(\mathbf{a})) = 0$ , and we are done by Hilbert's Nullstellensatz for affine case (Theorem 1.3A).

**Ex.2.2.** (i $\Rightarrow$ ii) Use 2.1. (ii $\Rightarrow$ iii) Since  $\{x_0, \dots, x_n\} \subseteq \sqrt{\mathbf{a}}, \exists k_0, \dots, k_n$  such that  $\forall i, x_i^{k_i} \in \mathbf{a}$ . Let  $d = \sum_{i=0}^n k_i$ . Then  $\mathbf{a} \supseteq S_d$  by pigeon hole principle. (iii $\Rightarrow$ i)  $\mathbf{a} \supseteq \{x_0^d, x_1^d, \dots, x_n^d\}$ , so  $Z(\mathbf{a}) = \emptyset$ .

**Ex.2.3.** These are similar to the affine case and easy to verify. Note we need Ex.2.1 for the " $\subseteq$ " direction of (d).

**Ex.2.4.** (a) By 2.3(d),  $I$  sends algebraic sets to homogeneous radical ideals. If  $\exists$  algebraic set  $Y$  such that  $I(Y) = S_+$ , then applying  $IZ$  to both sides we see  $I(Y) = S$ , contradiction. By 2.3(e),  $ZI(Y) = Y$  when  $Y$  is algebraic set. For  $\mathbf{a}$  a homogeneous radical ideal not equal to  $S_+$ , If  $Z(\mathbf{a}) = \emptyset$ , then by 2.2,  $\mathbf{a} = S$ , then  $IZ(\mathbf{a}) = \mathbf{a}$ . If  $Z(\mathbf{a}) \neq \emptyset$ , then  $IZ(\mathbf{a}) = \mathbf{a}$  by 2.3(d).

(b) ( $\Rightarrow$ ) Pick  $f, g \in S$  such that  $fg \in I(Y)$ . Let  $\{f_0, \dots, f_n\}$  and  $\{g_0, \dots, g_m\}$  be homogeneous parts of  $f$  and  $g$  respectively, then  $Z(f_0, \dots, f_n) \cup Z(g_0, \dots, g_m) \supseteq Y$ . Indeed,  $\forall y \in Y$ , because  $I(Y)$  is homogeneous,  $\forall i, (fg)_i \in I(Y)$ , so  $(fg)_i(y) = 0$  where  $(fg)_i$  means homogeneous component of degree  $i$  in  $fg$ . But each  $(fg)_i$  can also be written as a finite sum of products of components of  $f$  and  $g$ , so if  $y \notin Z(g_0, \dots, g_m)$ ,

then pick the smallest  $i$  such that  $g_i(y) \neq 0$ , and consider  $(fg)_i(y) = 0, (fg)_{i+1}(y) = 0, \dots$ , we see that  $y \in Z(f_0, \dots, f_n)$ .

Then because  $Y$  is irreducible, we can suppose  $Z(f_0, \dots, f_n) \supseteq Y$ . Then each  $f_i \in I(Y)$  so  $f \in I(Y)$ .

( $\Leftarrow$ )  $Y \neq \emptyset$  because otherwise  $I(Y) = S$ , not a prime ideal. Let  $C_1, C_2$  be two closed subsets of  $Y$  such that  $Y = C_1 \cup C_2$ . Then by 2.3(c)  $I(Y) = I(C_1) \cap I(C_2)$ . Since  $I(Y)$  is prime, WLOG we have  $I(C_1) = I(Y)$ . Apply  $Z$  to both sides and we are done.

(c)  $I(\mathbb{P}^n) = (0)$  because any homogeneous polynomial vanishing on  $\mathbb{P}^n$  also vanishes on  $\mathbb{A}^{n+1}$ , and  $I(\mathbb{A}^{n+1}) = (0)$ . Then use (b).

**Ex.2.5.** (a) It follows from  $S$  is noetherian, the inclusion-reversing function  $I$  in 2.4(a), and 2.3(e).

(b) It follows from part(a) and proposition 1.5.

**Ex.2.6.** To work with  $\dim S(Y)$  we need to work with its prime ideals, but all the theory we know is about homogeneous ideals. So the first step is to convert to the more familiar affine case.

Using 1.10(b)  $\exists U_i$  such that  $\dim Y = \dim U_i \cap Y$ . Let  $Y_i = \varphi_i(Y \cap U_i)$ , then by Proposition 1.7  $\dim Y = \dim A(Y_i)$ . For convenience of notation, assume  $i = 0$ . The map  $k[x_1, \dots, x_n] \rightarrow S(Y)_{x_0}$  defined by  $x_i \mapsto \frac{x_i}{x_0}$  and identity on  $k$  induces an embedding  $A(Y_0) \hookrightarrow S(Y)_{x_0}$ . This furthermore induces isomorphism

$$A(Y_0)[x_0, x_0^{-1}] \cong S(Y)_{x_0} \quad (1)$$

where  $x_0 \mapsto \frac{x_0}{1}$ .

For any integral domain  $R$ ,  $K(R[x, x^{-1}]) = K(R[x])$  by canonically embedding  $K(R[x])$  into  $K(R[x, x^{-1}])$ . Thus, when  $R$  is both integral domain and finitely generated  $k$ -algebra and  $\dim R$  is finite, using Proposition 1.8A(a) we have  $\dim R + 1 = \dim R[x, x^{-1}]$ .

Also, for any integral domain and finitely generated  $k$ -algebra  $R$ ,  $\forall x \in R \setminus \{0\}$ , denote localization of  $R$  at  $x$  by  $R_x$ . Then  $\dim R = \dim R_x$ , by considering transcendental basis using Proposition 1.8A(a).

Applying the previous two paragraphs to equation (1) we conclude  $\dim S(Y) = \dim(Y) + 1$ .

Remark:  $\forall i$  such that  $Y_i \neq \emptyset$ , equation (1) holds when 0 is replaced by  $i$ , so  $\dim Y_i = \dim Y$ .

**Ex.2.7.**(a) Use 2.6.

(b) Note that for any quasi-affine (quasi-projective) variety  $Y$ , there can be only one affine (projective) variety containing  $Y$  as an open subset, and it is  $\bar{Y}$ . By 1.10(b),  $\exists U_i$  such that  $\dim U_i \cap Y = \dim Y$ . Then  $\dim Y = \dim \varphi_i(U_i \cap Y) = \dim \overline{\varphi_i(U_i \cap Y)} = \dim \varphi_i(U_i \cap \bar{Y}) = \dim U_i \cap \bar{Y} = \dim \bar{Y}$ , where we used Proposition 1.10 in the second step and used the remark at the end of 2.6 in the last step.

**Ex.2.8.**( $\Rightarrow$ )  $\dim S(Y) = n$  by 2.6. Then by Theorem 1.8A(b), height  $I(Y) = 1$ . By Proposition 1.12A,  $I(Y)$  is principal, say  $I(Y) = (f)$ . Then  $Y = Z(f)$ . Because  $(f)$  is prime, and because in integral domain non zero prime element is irreducible,  $f$  is irreducible (thus  $f$  also has positive degree).  $f$  is homogeneous, otherwise  $(f)$  is not homogeneous.

( $\Leftarrow$ )  $\dim S(Y) = \dim k[x_0, \dots, x_n]/(f) = \dim k[x_0, \dots, x_n] - \text{height}(f) = (n+1) - 1 = n$ , where  $\text{height}(f) = 1$  by Theorem 1.11A. Then by 2.6,  $\dim Y = n - 1$ .

**Ex.2.9.** (a)  $\alpha$  and  $\beta$  will denote functions between  $A$  and  $S$  used in proof of Proposition (2.2). For  $\subseteq$  direction, let  $f \in S$  be a homogeneous polynomial killing  $\bar{Y}$ . Then  $\forall y \in Y$ ,  $\alpha(f)(y) = f(\varphi_0^{-1}(y)) = 0$ , so  $\alpha f \in I(Y)$ . Then since  $f = \beta \alpha(f) \cdot x_0^k$  for some  $k \geq 0$ ,  $f \in \langle \beta(I(Y)) \rangle$ . For  $\supseteq$  direction, take  $f \in A$  such that  $f$  kills  $Y$ . Then  $\forall y' \in \varphi_0^{-1}(Y)$ ,  $y' = \varphi_0^{-1}(y)$  for some  $y \in Y$ , and  $\beta(f)(y') = \beta(f)(\varphi_0^{-1}(y)) = f(y) = 0$ . So  $Z(\beta(f))$  is a closed set containing  $\varphi_0^{-1}(Y)$ , so  $\beta(f)$  kills  $\bar{Y}$ .

Remark: Exactly the same arguments show  $I(\varphi_0^{-1}(Y)) = I(\bar{Y})$ , and this exercise gives a good description of the ideal generated by projective closure of an affine variety.

(b) By Ex1.2,  $I(Y) = (x_2 - x_1^2, x_3 - x_1^3)$ . It is tempting to use part (a) to conclude generators for  $I(\bar{Y})$  using generators of  $I(Y)$ , but as the problem statement emphasizes, this is not the case. Instead, we try to first describe  $\bar{Y}$ . One obvious description of  $\bar{Y}$  is  $\{(1 : t : t^2 : t^3) | t \in k\}$ , which follows from parametric representation of the twisted cubic in  $\mathbb{A}^3$ . We have  $\bar{Y} = Z(x_0 x_2^3 - x_3^2, x_1^2 - x_0 x_2, x_1^3 - x_0^2 x_3)$ . The  $\subseteq$  direction is obvious. For  $\supseteq$  direction, take any  $(a_0 : a_1 : a_2 : a_3)$  which satisfies the equations. If  $a_0 \neq 0$ ,  $(a_0 : a_1 : a_2 : a_3) = (1 : \frac{a_1}{a_0} : \frac{a_2}{a_0} : \frac{a_3}{a_0})$  is in  $Y$ . Otherwise,  $a_0 = a_1 = a_2 = 0$ , and we are left with  $(a_0 : a_1 : a_2 : a_3) = (0 : 0 : 0 : a_3)$  which is not in  $Y$  but in  $\bar{Y}$ . Indeed, any homogeneous polynomial  $f$

which kills all points of  $Y$  must be zero polynomial after the substitution  $x_0 \mapsto 1, x_1 \mapsto t, x_2 \mapsto t^2, x_3 \mapsto t^3$  where  $t$  is a variable. Because it is homogeneous, there cannot be a monomial only in  $x_3$  in  $f$  (otherwise the polynomial after  $t$ -substitution is nontrivial). Thus all  $x_3$  in  $f$  is in a product with other variables  $x_i$ , so  $f(0 : 0 : 0 : a_3) = 0$ , so  $(0 : 0 : 0 : a_3) \in \bar{Y}$ .

We also know that  $\bar{Y}$  is irreducible, because  $Y$  is irreducible in the affine space, and closure of irreducible space is irreducible. Therefore  $I(\bar{Y}) = \sqrt{(x_0x_3^2 - x_2^3, x_1^3 - x_0x_2, x_1^3 - x_0^2x_3)}$  is a prime ideal. I guess that  $(x_0x_3^2 - x_2^3, x_1^3 - x_0x_2, x_1^3 - x_0^2x_3)$  is a prime ideal (which will allow me to conclude the problem), but I do not know how to prove that.

**Ex.2.10.**(a)  $C(Y)$  is zero set of the same set of polynomials whose zero set in projective space is  $Y$ . It follows that  $Y$  and  $C(Y)$  has the same ideal.

(b)  $C(Y)$  is irreducible if and only if  $I(C(Y))$  is prime ideal if and only if  $I(Y)$  is prime ideal if and only if  $Y$  is irreducible.

(c) I don't know if the result holds for a general projective algebraic set, but if we assume  $Y$  is irreducible, then  $\dim C(Y) = \dim A(C(Y)) = \dim S(Y) = \dim Y + 1$  where the second step is true by part (a) and the last step is true by Ex2.6.

**Ex.2.11.**(a) (i $\implies$ ii) Say  $I(Y) = \langle S \rangle$  where the  $S$  is a set of linear polynomials. (We can assume  $S$  is finite because  $k[x_0, \dots, x_n]$  is noetherian,  $\langle S \rangle$  is a finitely generated. Each of the generator can be written as a finite sum of products of some element from  $S$  with some element from  $k[x_0, \dots, k_n]$ . Put all elements from  $S$  which generate all of these generators together, we get a finite set of generators of the ideal  $\langle S \rangle$  where each generator is in  $S$ .) Then from Ex.2.3(e),  $Y = \bar{Y} = ZI(Y) = Z(\langle S \rangle) = Z(S) = \cap_{f \in S} Z(f)$  is an intersection of hyperplanes.

(ii $\implies$ i) If  $Y = \cap_{i \in I} Z(f_i)$  where  $f_i$  are linear polynomials, then  $Y = Z(\cup_{i \in I} f_i) = Z(J)$  where  $J$  is the ideal generated by  $f_i$  for all  $i \in I$ . Then  $I(Y) = IZ(J) = \sqrt{J}$ . As explained in part (a),  $J$  is finitely generated by linear polynomials. Thus it suffices to prove that  $J$  is prime.

Thus we will prove the following:

**Lemma 1:** Any ideal  $J \subseteq k[x_0, \dots, x_n]$  finitely generated by linear polynomials is prime ideal.

**Proof of Lemma 1:** Note for each polynomial, the  $(n+1)$ -tuple of its coefficients can be viewed as an element of the  $k$ -vector space  $k^{n+1}$ . Let  $S$  be the set of these vectors (From now on by "vector" I will mean either the element in  $k^{n+1}$  or the corresponding polynomial whose coefficients form this vector, depending on the situation). Eliminate some vectors until they become linearly independent. Then the ideal generated by these vectors is the same as before. Let each vector form a row of a matrix  $M$ , and do Gaussian Elimination on the matrix. Call the new matrix  $M'$ . Then rows of  $M'$  generate the same ideal as rows of  $M$  do, because  $(f, g) = (f + ag)$  for all  $f, g \in k[x_0, \dots, x_n]$  and  $a \in k - \{0\}$ . If  $M'$  has rank  $n+1$ , then the ideal generated by rows is  $(x_0, \dots, x_n)$ , which is maximal ideal. Otherwise, by renaming the variables  $x_i$  if necessary, we can assume the matrix is  $(I|A)$  where  $I$  is identity matrix of size less than  $n+1$ , and  $A$  is any matrix. Thus, the corresponding ideal  $J$  is generated by  $m+1$  polynomials  $(p_i)_{0 \leq i \leq m}$  where  $m < n$  and coefficient of  $x_i$  in  $p_i$  is 1, and coefficient of  $x_j$  in  $p_i$  is 0 for  $j$  from 0 to  $m$ ,  $j \neq i$ .

Next we use polynomial division to prove  $J$  is prime. Suppose  $fg \in J$ . Divide  $f$  by  $p_0$  as polynomials in  $x_0$ , call the remainder  $r_0$ , which is a polynomial involving no  $x_0$ . then divide  $r_0$  by  $p_1$  as polynomials in  $x_1$ , call the remainder  $r_1$ , which is a polynomial involving no  $x_0, x_1$ . Continue this process, we get the final remainder  $r_m$ , which involves no  $x_0, x_1, \dots, x_m$ . Do the same for  $g$  and call the final remainder  $r'_m$ . Then multiply  $f$  and  $g$  using this expansion. Because  $J$  is generated by all the  $p_i$  and  $fg \in J$ , we conclude  $r'_m r_m \in J$ . But  $r_m$  and  $r'_m$  are polynomials involving no  $x_0, x_1, \dots, x_m$ , and  $J$  is generated by all the  $p_i$  which each has a coefficient equal to 1 at  $x_i$ . Thus  $r'_m r_m = 0$  (In more detail, we conclude  $r'_m r_m = 0$  by viewing  $r'_m r_m$  as a polynomial in  $x_i$  and consider its degree.) Then WLOG  $r_m = 0$ , then  $f \in J$ .  $\square$

(b) It suffices to prove that for any  $Y$  a projective variety, if  $I(Y)$  is generated by  $s$  linear polynomials, then  $\dim Y \geq n - s$ . We have  $Y = ZI(Y) = \cap_{i=1}^s Z(f_i)$  where  $f_i$  are linear generators of  $I(Y)$ . By Ex.1.10(b),  $\exists i$  such that  $\dim Y = \dim Y_i$  where  $Y_i = Y \cap U_i$ . For sake of convenience of notation, assume  $i = 0$ . Then  $\varphi_0(Y_0) \subseteq \mathbb{A}^n$  is an affine variety of dimension equal to  $\dim Y$ . We claim  $\varphi_0(Y_0) = Z(\alpha(f_1), \dots, \alpha(f_s))$  where  $\alpha$  is defined in proof of Proposition 2.2. The proof is indeed just using definitions and is easy to check. Then by Ex1.9,  $\dim \varphi_0(Y_0) \geq n - s$ , therefore  $\dim Y \geq n - s$ .

Remark: I think this problem shows that intersection theory works easier in the affine case than in the

projective case (maybe because we have extra details and more complicated structure to worry about in projective geometry than in affine geometry), thus it is a good idea to convert problems in projective space to problems in affine space.

Remark: Extension to any projective variety? Algebraic set?

(c) Before anything, we first prove one important observation for this problem, which is beautiful in itself:

**Lemma 2:** View  $\mathbb{A}^n$  as a  $k$ -vector space, then any nonempty  $k$ -linear subspace  $V$  of  $\mathbb{A}^n$  is affine variety, and  $\dim V = \dim_k V$ , where the first "dim" means dimension of topological space (with induced Zariski topology), and the second "dim" means dimension of vector space.

**Proof of Lemma 2:** Let  $v_1, \dots, v_m$  be a basis of  $V$  as  $k$ -vector space. Then  $m \leq n$ . We can also assume  $m < n$ , otherwise there is nothing to prove. Let  $M$  be a  $m$ -by- $n$  matrix where  $i$ -th row is  $v_i$ . Let  $\varphi : k^n \rightarrow k^m$  be the linear map represented by  $M$  (with respect to the standard basis).  $\text{rank} M = m$ , so  $\varphi$  is surjective. From linear algebra,  $\dim \ker \varphi + \dim k^m = \dim k^n$ , so  $\dim \ker \varphi = n - m$ . Let  $w_1, \dots, w_{n-m}$  be a basis of  $\ker \varphi$ . Let  $f_1, \dots, f_{n-m} \in k[x_1, \dots, x_n]$  be linear polynomials where coefficient of  $f_i$  is the vector  $w_i$ . Then we claim

$$\text{span}_k(v_1, \dots, v_m) = Z(f_1, \dots, f_{n-m}). \quad (2)$$

To prove (2), first note because  $f_i$  are linear polynomials, the RHS of (2) is a  $k$ -subspace of  $\mathbb{A}^n$ . The  $\subseteq$  direction is true because each  $v_i$  is killed by all the  $f$ 's as the coefficients of  $f$ 's are in  $\ker \varphi$ . To prove  $\supseteq$ , note RHS is exactly kernel of the linear map represented by an  $n - m$ -by- $n$  matrix whose rows are  $w_i$ , and this linear map is surjective because the matrix has full rank. Thus, dimension of this kernel (as  $k$ -vector space) is equal to  $\dim k^n - \dim k^{n-m} = m$ . Because both LHS and RHS of (2) as dimension  $m$  and LHS is a subspace of RHS, by linear algebra conclude the equality.

As proved in (b),  $(f_1, \dots, f_{n-m})$  is prime ideal, so  $V = \text{span}_k(v_1, \dots, v_m) = Z(f_1, \dots, f_{n-m})$  is an affine variety. Therefore  $\{0\} \subset \text{span}_k(v_1) \subset \text{span}_k(v_1, v_2) \subset \dots \subset \text{span}_k(v_1, \dots, v_m)$  is a chain of distinct irreducible closed subsets of  $V$ , so  $\dim V \geq \dim_k V$ . For the other inequality, note that using Theorem 1.8A(b) and  $\text{height}(f_1, \dots, f_{n-m}) \geq n - m$  (as a result of Lemma 1) we have  $\dim V = \dim k[x_1, \dots, x_n]/(f_1, \dots, f_{n-m}) = n - \text{height}(f_1, \dots, f_{n-m}) \leq n - (n - m) = m = \dim_k V$ . Above all,  $\dim V = \dim_k V$ .  $\square$

Now we prove part(c). Assume  $Y$  and  $Z$  are linear varieties,  $\dim Y = r$ ,  $\dim Z = s$ , and  $r + s - n \geq 0$ . By Ex.2.10,  $C(Y)$  and  $C(Z)$  are affine varieties,  $\dim C(Y) = r + 1$  and  $\dim C(Z) = s + 1$ . Thus  $\dim C(Y) + \dim C(Z) \geq \dim \mathbb{A}^{n+1} + 1$ . Also by Ex.2.10,  $C(Y)$  and  $C(Z)$  are zeros sets of linear polynomials (the same polynomials as those which define  $Y$  and  $Z$ ). Thus  $C(Y)$  and  $C(Z)$  are linear  $k$ -subspaces of  $\mathbb{A}^{n+1}$ . Apply Lemma 2, we have  $\dim_k C(Y) + \dim_k C(Z) \geq \dim_k k^{n+1} + 1$ . If  $C(Y) \cap C(Z) = \{0\}$ , then  $\dim_k C(Y) + \dim_k C(Z) = \dim_k C(Y) \oplus C(Z) \leq \dim_k k^{n+1} = n + 1$ , so  $C(Y) \cap C(Z) \supset \{0\}$ . This implies  $Y \cap Z \neq \emptyset$ . Furthermore, if  $Y \cap Z \neq \emptyset$ , then first  $Y \cap Z$  is a projective variety because it is the zero set of a finite set of linear polynomials (finiteness comes from explanation in part (a)) and the ideal generated by finitely many linear polynomials is prime ideal by Lemma 1. Finally,  $\dim Y \cap Z = \dim C(Y \cap Z) - 1 = \dim C(Y) \cap C(Z) - 1 = \dim_k C(Y) \cap C(Z) - 1 = (\dim_k C(Y) + \dim_k C(Z) - \dim_k(C(Y) + C(Z))) - 1 \geq (r + 1) + (s + 1) - (n + 1) - 1 = r + s - n$ , where we have used the fact from linear algebra that for  $U, W$   $k$ -sub-vector spaces of  $V$  and  $U, W$  finite-dimensional,  $\dim_k(U + W) = \dim_k U + \dim_k W - \dim_k U \cap W$ .

**Ex.2.12.** As an aside, the example of "conic" in problem statement is equal to  $Z(x_1^2 - x_0x_2)$ .  $x_1^2 - x_0x_2$  is irreducible, so the conic is a projective variety. Using Theorem 1.11A and Ex.2.6, dimension of the conic is 1.

(a) (Note the definition of  $\theta$  is "reverse" to  $\rho_d$ .)  $\mathfrak{a}$  is prime because it is kernel of a map to a integral domain. For any  $f \in \mathfrak{a}$  and for any  $k \geq 0$ , the homogeneous component of  $f$  of degree  $k$  is sent to the homogeneous part of  $\theta(f)$  of degree  $dk$ . Because  $\theta(f) = 0$ , the homogeneous component of any degree of  $\theta(f)$  must be 0 (as a result of the graded ring structure, 0 has only one representation as a sum of homogeneous elements). In particular, each homogeneous component of  $f$  is sent to 0, so each homogeneous component of  $f$  is in  $\mathfrak{a}$ , so  $\mathfrak{a}$  is homogeneous.

(b) For  $\text{im } \rho_d \subseteq Z(\mathfrak{a})$ , take any  $(a_0 : \dots : a_n) \in \mathbb{P}^n$  and any homogeneous polynomial  $f \in \mathfrak{a}$ , then  $f(\rho_d(a_0 : \dots : a_n)) = \theta(f)(a_0 : \dots : a_n) = 0$ , so  $\text{im } \rho_d \subseteq Z(\mathfrak{a})$ . For  $\text{im } \rho_d \supseteq Z(\mathfrak{a})$ , take any  $b = (b_0 : \dots : b_N) \in Z(\mathfrak{a})$ .  $\forall 0 \leq i \leq n$ , let  $s_i$  be an integer from 0 to  $N$  such that the  $s_i$ -th component of  $\rho_d$  is  $a_i^d$ . Then  $\exists i$  such that  $b_{s_i} \neq 0$ , because if  $b_{s_i} = 0$  for all  $i$ , then  $\forall 0 \leq m \leq N$ ,  $b_m^d$  can be written as a product of powers of  $b_{s_i}$  by  $b \in Z(\mathfrak{a})$  (we can imagine as if  $b$  is the image of  $(a_0 : \dots : a_n)$ , then components of  $b$  satisfy equations

they should satisfy when they are written in products of  $a_i$ ), then  $b_m = 0$ , then all coordinates of  $b$  are 0, which is impossible. Now fix  $i$  where  $b_{s_i} \neq 0$ , that is, the  $s_i$ -th component of  $\rho_d$  is  $a_i^d$ . Now let  $(r_j)_{0 \leq j \leq n}$  be a set of integers from 0 to  $N$  such that the  $r_i$ -th component of  $\rho_d$  is  $a_i^d$  (i.e.  $r_i = s_i$ ), and for  $j \neq i$ , the  $r_j$ -th component of  $\rho_d$  is  $a_i^{d-1}a_j$ . Now we claim  $\rho_d(b_{r_0} : b_{r_1} : \dots : b_{r_n}) = b$ .  $\forall 0 \leq m \leq N$ , suppose the  $m$ -th coordinate of  $\rho_d(b_{r_0} : b_{r_1} : \dots : b_{r_n})$  is  $b_{r_0}^{c_0} \dots b_{r_n}^{c_n}$ , we claim  $b_{r_0}^{c_0} \dots b_{r_n}^{c_n} = b_{r_i}^{d-1}b_m$ . To see this, let's count the "power" of  $a_j$  in  $b_{r_0}^{c_0} \dots b_{r_n}^{c_n}$  and in  $b_{r_i}^{d-1}b_m$  (This is valid, because  $b \in Z(\mathfrak{a})$ ). When  $j = i$ , the power of  $a_j$  in  $b_{r_0}^{c_0} \dots b_{r_n}^{c_n}$  is  $c_i d + \sum_{j \neq i} c_j(d-1) = d(d-1) + c_i$  because  $\sum c_i = d$ . The power of  $a_j$  in  $b_{r_i}^{d-1}b_m$  is  $(d-1)d + c_i$ , so the power of  $a_i$  is the same. For  $j \neq i$ , the power of  $a_j$  in  $b_{r_0}^{c_0} \dots b_{r_n}^{c_n}$  is  $c_j$ , and the power of  $a_j$  in  $b_{r_i}^{d-1}b_m$  is  $c_j$ , so again they are equal. So  $b_{r_0}^{c_0} \dots b_{r_n}^{c_n} = b_{r_i}^{d-1}b_m$ . So  $\rho_d(b_{r_0} : b_{r_1} : \dots : b_{r_n}) = (b_{r_i}^{d-1}b_0 : b_{r_i}^{d-1}b_1 : b_{r_i}^{d-1}b_2 : \dots : b_{r_i}^{d-1}b_N) = (b_0 : b_1 : b_2 : \dots : b_N) = b$  where the second to last step is true because  $b_{r_i} = b_{s_i} \neq 0$ .

(c) First we show  $\rho_d$  is injective. Suppose  $\rho_d(a_0 : \dots : a_n) = \rho_d(b_0 : \dots : b_n)$ . WLOG, suppose  $a_0 \neq 0$ , then  $b_0 \neq 0$ . Then  $\exists \lambda \in k^\times$  such that  $a_0^d = \lambda b_0^d$ , and  $\forall 1 \leq i \leq n$ ,  $a_0^{d-1}a_i = \lambda b_0^{d-1}b_i$ . Then  $\forall 1 \leq i \leq n$ , if  $a_i = 0$  then  $b_i = 0$ , and  $a_i = \frac{a_0}{b_0}b_i$ . Otherwise  $a_i \neq 0$ , then  $b_i \neq 0$ , then  $\frac{a_i}{b_i} = \lambda(\frac{b_0}{a_0})^{d-1} = (\frac{a_0}{b_0})^d(\frac{b_0}{a_0})^{d-1} = \frac{a_0}{b_0}$ , so  $\forall 0 \leq i \leq n$ ,  $a_i = \frac{a_0}{b_0}b_i$ , so  $\rho_d$  is injective.

Next we prove  $\rho_d$  takes closed set to closed set. It suffices to take any homogeneous  $f \in k[x_0, \dots, x_n]$  and prove  $\rho_d(Z(f))$  is closed in  $Z(\mathfrak{a})$ . Note  $f^d \in \text{im } \theta$ . We can take  $\tilde{f} \in k[y_0, \dots, y_n]$  such that  $\theta(\tilde{f}) = f^d$  and  $\tilde{f}$  is homogeneous, and we claim  $\rho_d(Z(f)) = Z(\mathfrak{a}) \cap Z(\tilde{f})$ . For  $\subseteq$ ,  $\forall p \in \mathbb{P}^n$  we have  $\tilde{f}(\rho_d(p)) = \theta(\tilde{f})(p) = f^d(p)$ , so when  $p \in Z(f)$ ,  $\tilde{f}(\rho_d(p)) = 0$ . For  $\supseteq$ , take any  $q \in Z(\mathfrak{a}) \cap Z(\tilde{f})$ , then because  $Z(\mathfrak{a}) = \text{im } \rho_d$ ,  $\exists p \in \mathbb{P}^n$  such that  $\rho_d(p) = q$  (and such  $p$  is unique because  $\rho_d$  is injective). Then  $f^d(p) = \theta(\tilde{f})(p) = \tilde{f}(\rho_d(p)) = \tilde{f}(q) = 0$ , so  $f(p) = 0$ . Thus  $\rho_d$  takes closed set to closed set.

Next we prove preimage of closed subsets of  $Z(\mathfrak{a})$  under  $\rho_d$  is closed in  $\mathbb{P}^n$ . It suffices to take any homogeneous  $f \in k[y_0, \dots, y_n]$  and prove  $\rho_d^{-1}(Z(f) \cap Z(\mathfrak{a}))$  is closed in  $\mathbb{P}^n$ . We claim  $\rho_d^{-1}(Z(f) \cap Z(\mathfrak{a})) = Z(\theta(f))$ . Both directions follow from  $\theta(f)(p) = f(\rho_d(p))$ . Therefore,  $\rho_d$  is a homeomorphism from  $\mathbb{P}^n$  to  $Z(\mathfrak{a})$ .

(d) Denote the twisted cubic curve by  $C$ . From Ex.2.9(c) we know that  $C = Z(y_0y_3^2 - y_2^3, y_1^2 - y_0y_2, y_1^3 - y_0^2y_3)$ . Another description for  $C$  is  $C = \overline{\{(1 : t : t^2 : t^3) | t \in k\}}$ . On the other hand, we also have two descriptions for the 3-uple embedding of  $\mathbb{P}^1$  to  $\mathbb{P}^3$ :  $Z(\mathfrak{a})$  and  $\text{im } \rho_d = \{(a_0^3 : a_0^2a_1 : a_0a_1^2 : a_1^3) | a_0 \neq 0 \text{ or } a_1 \neq 0\}$ . Note  $(y_0y_3^2 - y_2^3, y_1^2 - y_0y_2, y_1^3 - y_0^2y_3) \subseteq \mathfrak{a}$ , so  $C = Z(y_0y_3^2 - y_2^3, y_1^2 - y_0y_2, y_1^3 - y_0^2y_3) \supseteq Z(\mathfrak{a})$ . Also,  $\{(1 : t : t^2 : t^3) | t \in k\} \subseteq \{(a_0^3 : a_0^2a_1 : a_0a_1^2 : a_1^3) | a_0 \neq 0 \text{ or } a_1 \neq 0\}$  by setting  $a_0 = 1$  and  $a_1 = t$ . Because the RHS is closed, taking closure on both sides we see  $C \subseteq \text{im } \rho_d$ . Therefore, the twisted cubic curve is the same as the 3-uple embedding of  $\mathbb{P}^1$  to  $\mathbb{P}^3$ .

**Ex.2.13.** By Ex.2.12,  $\rho_d : \mathbb{P}^2 \rightarrow Y$  is a homeomorphism, so  $\rho_d^{-1}(Z)$  is projective variety in  $\mathbb{P}^2$  with dimension 1. By Ex.2.8,  $\rho_d^{-1}(Z) = Z(f)$  for some irreducible, homogeneous  $f \in k[x_0, x_1, x_2]$ . Let  $d \geq 1$  be the smallest integer such that there exists homogeneous  $g \in k[y_0, \dots, y_5]$ ,  $\theta(g) = f^d$  ( $d$  is at most 2). Then  $g$  is irreducible, because if  $g = g_1g_2$  for some  $g_1, g_2 \in k[y_0, \dots, y_5]$ , then  $g_1$  and  $g_2$  must be homogeneous because  $g$  is homogeneous. Then  $\theta(g) = f^d = \theta(g_1)\theta(g_2)$ .  $f$  is irreducible, so  $\theta(g_1) = f^i$  for some  $0 \leq i \leq d$  up to multiplication by unit. But by our choice of  $d$ ,  $i$  can only be 0 or  $d$ . WLOG, assume  $i = d$ . Then  $g_1$  and  $g$  must be homogeneous of the same degree, then  $g_2$  must have degree 0, so  $g_2$  is a unit, so  $g$  is irreducible. Then we can show  $\rho_d(Z(f)) = Y \cap Z(g)$  using the same proof as in Ex.2.12(c). Then  $Z = Y \cap Z(g)$  where  $Z(g)$  is a hypersurface by Ex.2.8.

**Ex.2.14.** Let  $\varphi : k[\{z_{ij}\}] \rightarrow k[x_0, \dots, x_r, y_0, \dots, y_s]$  be the ring homomorphism sending  $z_{ij}$  to  $x_iy_j$ . Then  $\ker \varphi$  is a homogeneous prime ideal for the same reason as explained in Ex.2.12(a), so  $Z(\ker \varphi)$ , if nonempty, is a projective variety. Next we show  $\text{im } \psi = Z(\ker \varphi)$ .  $\text{im } \psi \subseteq Z(\ker \varphi)$  because for all homogeneous  $f \in \ker \varphi$  and  $p \in \mathbb{P}^r \times \mathbb{P}^s$ ,  $f(\psi(p)) = \varphi(f)(p) = 0(p) = 0$ . To see  $\text{im } \psi \supseteq Z(\ker \varphi)$ , suppose  $(c_{ab}) \in Z(\ker \varphi)$ , then there exists  $0 \leq i \leq r$  and  $0 \leq j \leq s$  such that  $c_{ij} \neq 0$  by definition of projective space. Then  $\forall 0 \leq a \leq r, 0 \leq b \leq s$ , the  $ab$ -th coordinate of  $\psi((c_{0j} : c_{1j} : \dots : c_{rj}), (c_{i0} : c_{i1} : \dots : c_{is}))$  is  $c_{aj}c_{ib} = c_{ij}c_{ab}$  because  $(c_{ij}) \in \ker \varphi$ . Thus  $\psi((c_{0j} : c_{1j} : \dots : c_{rj}), (c_{i0} : c_{i1} : \dots : c_{is})) = (c_{ij}c_{00} : \dots : c_{ij}c_{ab} : \dots : c_{ij}c_{rs}) = (c_{ab})$  where we cancel  $c_{ij}$  because  $c_{ij} \neq 0$ . Thus  $\text{im } \psi = Z(\ker \varphi)$  is a projective variety.

**Ex.2.15.**(a) Let the homogeneous coordinates of  $\mathbb{P}^3$  be  $(x, z, w, y)$ . The Segre embedding of  $\mathbb{P}^1 \times \mathbb{P}^1$  in  $\mathbb{P}^3$  is  $S = \{(a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1) | \exists i, j, a_i b_j \neq 0\}$ , so obviously  $S \subseteq Q$ . To see  $S \supseteq Q$ , take any  $(x : z : w : y) \in Q$ . WLOG, suppose  $x \neq 0$ . Then  $\psi((x : w), (x : z)) = (x^2 : xz : xw : wz) = (x^2 : xz : xw : xy) = (x : z : w : y)$ , so  $(x : z : w : y) \in S$ , so  $S \supseteq Q$ . Thus  $S = Q$ .

(b) Fix  $t = (t_0 : t_1) \in \mathbb{P}^1$ . Let  $L_t = Z(xt_0 - zt_1, wt_0 - yt_1)$  and  $M_t = Z(xt_0 - wt_1, zt_0 - yt_1)$ . Note  $L_t$  and  $M_t$  are well defined, i.e. they do not depend on representative of  $t$ . It's easy to verify  $L_t$  and  $M_t$  are subsets of  $Q$ . By Lemma 1 we proved in Ex.2.11, the ideals defining  $L_t$  and  $M_t$  are prime ideals, and it's easy to verify  $L_t$  and  $M_t$  are nonempty, so they are linear varieties. They have dimension 1 by Ex.2.10(c) and Lemma 2 we proved in Ex.2.11. When  $L_t \neq L_u$ ,  $t \neq u$ , then  $L_t \cap L_u = \emptyset$  because the ratio of two coordinates of  $t$  is different from the ratio of two coordinates of  $u$  (the case for one coordinate being 0 is also easy to check). Similarly, when  $M_t \neq M_u$ ,  $M_t \cap M_u = \emptyset$ . For all  $t, u$ ,  $L_t \cap M_u =$  one point, because for any  $(x : z : w : y) \in L_t \cap M_u$ , the ratios of its coordinates are determined, thus there can be at most one solution. On the other hand,  $(u_1 t_1 : t_0 u_1 : u_0 t_1 : u_0 t_0)$  is one solution.

(c) Consider  $Y := Z(xy - zw, x - y) = Z(x^2 - zw, x - y)$ . We first prove  $Y$  is projective variety. It suffices to prove  $(x^2 - zw, x - y)$  is prime ideal. By polynomial division, for each  $f \in k[x, z, w, y]$ , there exists unique polynomials  $h_1 \in k[x, z, w, y]$ ,  $h_2 \in k[x, z, w]$ ,  $h_3, h_4 \in k[z, w]$  such that  $f = (x - y)h_1 + (x^2 - zw)h_2 + xh_3 + h_4$ . Suppose  $fg \in (x^2 - zw, x - y)$ . If  $g = (x - y)h'_1 + (x^2 - zw)h'_2 + xh'_3 + h'_4$  where  $h'_i$  satisfies the same condition as  $h_i$ , then  $fg = (x - y)h''_1 + (x^2 - zw)h''_2 + x(h_3h'_4 + h_4h'_3) + (h_4h'_4 + zwh_3h'_3)$  where  $h''_1 \in k[x, z, w, y]$ ,  $h''_2 \in k[x, z, w]$ . Now uniqueness of such expression and  $fg \in (x - y, x^2 - zw)$  imply  $h_3h'_4 + h_4h'_3 = 0$  and  $h_4h'_4 + zwh_3h'_3 = 0$ . If  $h'_4 = 0$  or  $h_4 = 0$ , then manipulating the previous two equations a little bit tells us  $f \in (x^2 - zw, x - y)$  or  $g \in (x^2 - zw, x - y)$ . Otherwise,  $h_3 = \frac{-h_4h'_3}{h'_4}$ , and substituting into the latter equation we get  $h_4'^2 = zwh_3'^2$ . Viewed as polynomials in one variable (either  $z$  or  $w$ ), the LHS has even degree, while the RHS has odd degree, so the only possibility is that both sides are 0. Thus  $g \in (x^2 - zw, x - y)$ , so  $(x^2 - zw, x - y)$  is prime ideal.  $Y = Z(x^2 - zw, x - y)$  is nonempty because  $(1 : 1 : 1 : 1)$  is in it, thus we conclude  $Y$  is a projective variety.

Using Ex.2.10 and its notation,  $C(Y)$  is an affine variety given by  $C(Y) = Z(xy - zw, x - y)$ . Because this is intersection of two hypersurfaces neither of which contains the other, by Ex.1.9 we have  $\dim C(Y) = 2$ . Then by Ex.2.10(c),  $\dim Y = 1$ , so  $Y$  is a curve in  $Q$ .  $Y$  is not any line of form  $L_t$ , because all points on  $L_t$  have fixed ratio between  $x$  and  $z$ , which is not the case in  $Y$ . Similarly  $Y$  is not any line of form  $M_t$ .

Now we prove that  $\psi$  is not a homeomorphism from  $\mathbb{P}^1 \times \mathbb{P}^1$  to  $Q$ . First note that  $\mathbb{P}^1$  has cofinite topology (obviously finite sets are closed. Conversely, if  $X$  is closed in  $\mathbb{P}^1$ , then we can assume  $X = Z(f)$  where  $f \in k[x_0, x_1]$  is homogeneous. If  $X$  is infinite, then there are infinitely many  $x_1 \in k$  such that  $f(1, x_1) = 0$ , this implies  $f$  is zero polynomial, then  $Z(f) = \mathbb{P}^1$ ). If  $\psi^{-1}(C(Y))$  is closed, then its complement is open in  $\mathbb{P}^1 \times \mathbb{P}^1$  with product topology. By definition of product topology, complement of  $\psi^{-1}(C(Y))$  is a union of basic open sets of form  $U \times V$  where  $U, V$  are open on  $\mathbb{P}^1$ . Pick any such  $U, V$ . Because  $U$  and  $V$  are complements of finite sets and  $k$  is infinite,  $\exists t \in k$  such that  $(1 : t) \in U, (t : 1) \in V$ . Then  $\psi((1 : t), (t : 1)) = (t : 1 : t^2 : t) \in C(Y)$ , so  $((1 : t), (t : 1)) \in \psi^{-1}(C(Y))$ . But  $((1 : t), (t : 1)) \in U \times V \subseteq (\mathbb{P}^1 \times \mathbb{P}^1) - \psi^{-1}(C(Y))$ , contradiction. Thus  $\psi$  is not a homeomorphism from  $\mathbb{P}^1 \times \mathbb{P}^1$  to  $Q$ .

**Ex.2.16.**(a) Let  $C = Z(wz^2 - y^3, x^2 - wy, x^3 - w^2z)$ , then  $C$  is a twisted cubic curve by Ex.2.9. Then we have

$$Z(wz^2 - y^3, x^2 - wy, x^3 - w^2z) \cup Z(w, x) = Z(x^2 - yw, xy - zw). \quad (3)$$

The equality is easy to verify. By Lemma 1,  $(w, x)$  is a prime ideal, so  $Z(w, x)$  is a variety. It is a linear variety of dimension 1 because  $C(Z(w, x))$  (cone over  $Z(w, x)$ ) has dimension 2 by Lemma 2, then by Ex.2.10(c),  $\dim(Z(w, x)) = 1$ . So  $Z(w, x)$  is a line.

(b) If  $x^2 - yz = 0$  and  $y = 0$  then  $x = 0$ , so  $C \cap L = \{(0 : 0 : a) | a \in k^\times\}$  is a point  $P$ . It's easy to verify  $I(C) = (x^2 - yz), I(L) = (y), I(P) = (x, y)$ . Then  $I(P) \supsetneq I(C) + I(L)$  because  $x \in I(P)$  but  $x \notin I(C) + I(L)$ .

**Ex.2.17.**(a) By Ex.2.10,  $C(Y) = Z(\mathfrak{a})$  where  $\mathfrak{a}$  is viewed not as homogeneous ideal, but a general ideal in  $k[x_0, \dots, x_n]$ . By Ex.1.9,  $\dim(C(Y)) \geq n + 1 - q$ , so by Ex.2.10(c)  $\dim Y \geq n - q$ .

(b) Let's assume that in the definition of strict complete intersection, the ideal of  $Y$  is generated by  $n - r$  homogeneous polynomials. (This is not explicitly stated in Hartshorne, but is intuitional and consistent with Wikipedia.) Let  $f_1, \dots, f_{n-r}$  be homogeneous generators of  $I(Y)$ , then  $Y = ZI(Y) = Z(f_1, \dots, f_{n-r}) = \bigcap_{i=1}^{n-r} Z(f_i)$  is intersection of  $n - r$  hypersurfaces (Here I am being loose and allowing hypersurface to be given by any single homogeneous polynomial, not necessarily irreducible).

(c) By Ex.2.12,  $Y$  is  $Z(\mathfrak{a})$  where  $\mathfrak{a}$  is kernel of  $\theta : k[y_0, \dots, y_3] \rightarrow k[x_0, x_1]$  given by  $y_0 \mapsto x_0^3, y_1 \mapsto x_0^2x_1, y_2 \mapsto x_0x_1^2, y_3 \mapsto x_1^3$ . Since  $\mathfrak{a}$  is prime,  $I(Y) = I(Z(\mathfrak{a})) = \mathfrak{a}$ .  $\mathfrak{a}$  cannot be generated by two elements for a similar reason as in Ex.1.11. Specifically, we look at terms of  $f \in \mathfrak{a}$  which will be sent to  $x_0^4x_1^2, x_0^3x_1^3, x_0^2x_1^4$ .  $\theta(f) = 0$



implies coefficient of  $y_1^2$  in  $f$  is the negative of coefficient of  $y_0y_2$ , coefficient of  $y_0y_3$  in  $f$  is the negative of coefficient of  $y_1y_2$ , coefficient of  $y_2^2$  in  $f$  is the negative of coefficient of  $y_1y_3$ . Define a  $k$ -linear map  $\psi : \mathfrak{a} \rightarrow k^3$  sending a polynomial to its coefficients of  $y_1^2 - y_0y_2$ , coefficients of  $y_0y_3 - y_1y_2$ , coefficients of  $y_2^2 - y_1y_3$ . This map is surjective because  $y_1^2 - y_0y_2, y_0y_3 - y_1y_2, y_2^2 - y_1y_3 \in \mathfrak{a}$ . Now if  $\mathfrak{a}$  is generated by two elements, this would imply  $\text{im } \psi$  is  $k$ -subspace of  $k^3$  spanned by coefficients of  $y_1^2 - y_0y_2, y_0y_3 - y_1y_2, y_2^2 - y_1y_3$  of the two generators. Then  $\dim_k \text{im } \psi \leq 2$ , contradiction (more detailed explanation is in my solution to Ex.1.11). So  $\mathfrak{a}$  cannot be generated by two elements.

### 3 Morphisms

We first prove some results which will be used later possibly without mentioning.

**Lemma 3:** Let  $X$  and  $Y$  be any varieties and  $\varphi : X \rightarrow Y$  be a morphism. Let  $X' \subseteq X$  and  $Y' \subseteq Y$  be subsets which are also varieties and  $\varphi(X') \subseteq (Y')$ . Then  $\varphi|_{X'} : X' \rightarrow Y'$  is a morphism of varieties.

**Proof:**  $\varphi|_{X'}$  is continuous because restriction of continuous function to a subset of its domain is continuous (when all relevant spaces have induced topology). To see  $\varphi|_{X'}$  is a morphism, take any open subset  $U \subseteq Y'$  and regular function  $f : U \rightarrow k$ . We want to prove  $f \circ \varphi|_{X'} : \varphi|_{X'}^{-1}(U) \rightarrow k$  is regular. Take any  $p \in \varphi|_{X'}^{-1}(U)$ . Then  $\varphi(p) \in U$ , so there exists an open subset of  $U$  containing  $\varphi(p)$  such that  $f = g/h$  where  $g, h$  are polynomials on this open subset. Because  $U$  has topology induced from  $Y$ , we can write this open set as  $V \cap U$  where  $V$  is open in  $Y$ . The function  $g/h$  is regular on  $V - Z(h)$ , so  $(g/h) \circ \varphi : \varphi^{-1}(V - Z(h)) \rightarrow k$  is a regular function. Note  $V - Z(h) \supseteq V \cap U$  and  $(g/h) \circ \varphi$  agrees with  $f \circ \varphi|_{X'}$  on  $\varphi|_{X'}^{-1}(V \cap U)$ . Since  $p \in \varphi|_{X'}^{-1}(V \cap U)$ ,  $f \circ \varphi|_{X'}$  can be written as a rational function in some open set (of  $\varphi|_{X'}^{-1}(U)$ ) near  $p$  by first using  $(g/h) \circ \varphi$  to find such open set in  $\varphi^{-1}(V - Z(h))$  and then restrict it to  $\varphi|_{X'}^{-1}(U)$ .  $\square$

**Lemma 4:** Any automorphism of the underlying  $k$ -vector space of  $\mathbb{P}^n$  induces an automorphism of  $\mathbb{P}^n$ .

**Proof:** Let  $\varphi : k^{n+1} \rightarrow k^{n+1}$  be an isomorphism of  $k$ -vector space. It is obvious that  $\varphi$  induces a bijective function  $\tilde{\varphi} : \mathbb{P}^n \rightarrow \mathbb{P}^n$ . Let  $\varphi^* : k[x_0, \dots, x_n] \rightarrow k[x_0, \dots, x_n]$  be  $k$ -algebra homomorphism defined by  $\varphi^*(x_i) = \mathbf{x}^t(M^{-1})^t \mathbf{e}_i$  where  $\mathbf{x} = (x_0, \dots, x_n)$ ,  $M$  is the matrix of  $\varphi$  w.r.t. the standard basis,  $\mathbf{e}_i$  is the standard unit vector. For any closed set  $Z(f) \subseteq \mathbb{P}^n$ ,  $\tilde{\varphi}(Z(f)) = Z(\varphi^*(f))$ , because for any  $P = (a_0 : \dots : a_n) \in Z(f)$ , let  $\mathbf{a} = (a_0, \dots, a_n)^t$ , then  $\varphi^*(f)(\tilde{\varphi}(P)) = f(\varphi^*(x_0), \dots, \varphi^*(x_n))(\tilde{\varphi}(P))$  and  $\varphi^*(x_i)(\tilde{\varphi}(P)) = (M\mathbf{a})^t(M^{-1})^t \mathbf{e}_i = \mathbf{a}^t M^t(M^{-1})^t \mathbf{e}_i = a_i$ , so  $\varphi^*(f)(\tilde{\varphi}(P)) = f(P) = 0$ . Conversely if  $Q = (b_0 : \dots : b_n) \in Z(\varphi^*(f))$ , let  $\mathbf{b} = (b_0, \dots, b_n)^t$ , then  $0 = \varphi^*(f)(Q) = f(\varphi^*(x_0), \dots, \varphi^*(x_n))(Q)$  and  $\varphi^*(x_i)(b_0, \dots, b_n) = \mathbf{b}^t(M^{-1})^t \mathbf{e}_i = (M^{-1}\mathbf{b})^t \mathbf{e}_i = \varphi^{-1}(\mathbf{b})_i$ , so  $0 = f(\varphi^*(x_0), \dots, \varphi^*(x_n))(Q) = f(\varphi^{-1}(\mathbf{b})) = f(\tilde{\varphi}^{-1}(Q))$ . Therefore,  $\tilde{\varphi}(Z(f)) = Z(\varphi^*(f))$ . Thus we see  $\tilde{\varphi}$  is homeomorphism.  $\tilde{\varphi}$  is an isomorphism because the definition of  $\tilde{\varphi}$  uses polynomial expression at each coordinate.

**Ex.3.1.(a)** By a similar argument as the one I give for Ex.3.6,  $\mathcal{O}(\mathbb{A}^1 - \{0\}) \cong k[t, t^{-1}]$  as  $k$ -algebra. We also know  $A(\mathbb{A}^1) = k[t]$ . Let  $W$  be any conic. By Ex.1.1,  $A(W) \cong k[t]$  or  $A(W) \cong k[t, t^{-1}]$ . By Proposition 3.5,  $W \cong \mathbb{A}^1$  or  $W \cong \mathbb{A}^1 - \{0\}$ .

(b) Let  $U \subset \mathbb{A}^1$  be a proper open subset. FSOC, suppose  $\mathbb{A}^1 \cong U$ . Then  $k[t] \cong \mathcal{O}(U)$  as  $k$ -algebra. By a similar argument as the one I give for Ex.3.6, we get  $\mathcal{O}(U) = \{ \frac{f}{g} | f, g \in k[t], Z(g) \subseteq \mathbb{A}^1 - U \}$ , a sub- $k$ -algebra of  $k(t)$ . Let  $\varphi : \mathcal{O}(U) \rightarrow k[t]$  be a  $k$ -algebra isomorphism. Pick any  $P \notin U$ , then  $t - P$  is a unit in  $\mathcal{O}(U)$ , so  $\varphi(t - P)$  is a unit in  $k[t]$ , so  $\varphi(t) - P = \varphi(t - P) \in k$ , so  $\varphi(t) \in k$ , which contradicts with  $\varphi$  being surjective. Thus  $\mathbb{A}^1$  is not isomorphic to any proper open subset.

(c) By Ex.3.4, the 2-uple embedding  $\rho_2 : \mathbb{P}^1 \rightarrow \mathbb{P}^2$  is isomorphism. Since  $\text{im } \rho_2 \subseteq Z(xz - y^2)$  and both sides are projective curves, we have  $\text{im } \rho_2 = Z(xz - y^2)$ . So it remains to prove that any projective curve in  $\mathbb{P}^2$  given by  $C = Z(ax^2 + by^2 + cz^2 + dxy + exz + fyz)$  is isomorphic to  $Z(xz - y^2)$ . By Lemma 4 and its proof, we can do linear change of coordinates on  $\mathbb{P}^n$  as long as the corresponding matrix is invertible. The rest is just manipulation of algebra similar to what we did in Ex.1.1(c). We note  $ace \neq 0$ , so  $ax^2 + cz^2 + exz$  is a product of 2 linear factors.

In the first case, suppose the two linear factors are distinct, write them as (new)  $x$  and  $z$ , we get  $C = Z(xz + ay^2 + bxy + cxy)$  for some (new) coefficients  $a, b, c$ . Since  $xz + ay^2 + bxy + cxy = x(z + by) + y(ay + cz)$  and since

$z+by$  and  $ay+cz$  should be relatively prime (otherwise the curve is reducible), we let  $y := z+by, z := ay+cz$ , then we get  $C = Z(xy+ayz+bz^2)$  for some new coefficients  $a$  and  $b$ . Then because  $xy+ayz = y(x+az)$  we let  $x := x+az$  and get  $C = Z(xy+bz^2)$ . Note  $b \neq 0$  because  $C$  is irreducible. Another change of coordinates gives  $C = Z(xz-y^2)$ .

In the second case, suppose the two linear factors are not distinct. WLOG suppose the linear factor has nonzero coefficient of  $x$ , then let (new)  $x$  be the linear factor we get  $C = Z(x^2+ay^2+bxy+cyz)$  for some new coefficients  $a, b, c$ . Now  $x^2+ay^2+bxy$  is a product of 2 linear factors. If these two are distinct factors, then a change of coordinates gives  $C = Z(xy+axz+byz)$  for new nonzero coefficients  $a$  and  $b$ . Let  $y := y+az$  we get  $C = Z(xy+ayz+bz^2)$ . Then let  $x := x+az$  we get  $C = Z(xy+bz^2)$ . Another easy change of coordinates gives  $C = Z(xz-y^2)$ . If  $x^2+ay^2+bxy$  is a product of 2 same factors, such factor must have nonzero coefficient in  $x$ , so we let our new  $x$  be this factor and get  $C = Z(x^2+cyz)$ . Another easy change of coordinates gives  $C = Z(xz-y^2)$ .

Above all, all conics in  $\mathbb{P}^2$  are isomorphic to  $\mathbb{P}^1$ .

**Remark:** This result is cleaner than the affine counterpart (part (a)).

(d) We first prove that any two curves in  $\mathbb{P}^2$  have nonempty intersection. Let  $C_1, C_2$  be two curves in  $\mathbb{P}^2$ , let  $C(C_1), C(C_2)$  be cones over these curves in  $\mathbb{A}^3$  (Ex.2.10). Then  $\dim C(C_1) = \dim C(C_2) = 2$ , and  $C(C_1) \cap C(C_2) \neq \emptyset$ . By Ex.1.8, either  $C(C_1) = C(C_2)$  or  $C(C_1) \cap C(C_2)$  is a finite union of at least 1 affine curves. Since  $C(C_1 \cap C_2) = C(C_1) \cap C(C_2)$ ,  $C_1 \cap C_2 \neq \emptyset$ . (Then since  $\mathbb{P}^n$  is noetherian, it is obvious that  $C_1 \cap C_2 = \{P_1, \dots, P_n\}$  for some points  $P_i$ .)

But  $\mathbb{A}^2$  can have infinitely many curves with pairwise empty intersection. For example  $C_t = Z(t), t \in k$  is such a family of curves. So  $\mathbb{A}^2$  is not homeomorphic to  $\mathbb{P}^2$ .

(e) Let  $X$  be an affine variety and  $Y$  be a projective variety. Then  $X \cong Y$  implies  $\mathcal{O}(X) \cong \mathcal{O}(Y)$  (under the natural map). Using Theorem 3.2 and 3.4,  $A(X) \cong k$ . Because  $A(X) = k[x_1, \dots, x_n]/I(X)$ ,  $I(X)$  is maximal ideal. By Hilbert's Nullstellensatz,  $I(X) = I(P)$  for some  $P \in \mathbb{A}^n$ . Then  $X = ZI(X) = P$ .

**Ex.3.2.**(a) Let  $C = Z(y^2 - x^3)$ , then obviously  $\text{im } \varphi \subseteq C$ . Let  $x_1, x_2$  denote coordinate functions on  $\mathbb{A}^2$ , then  $x_1 \circ \varphi(t) = t^2, x_2 \circ \varphi(t) = t^3$  are regular functions on  $\mathbb{A}^1$ , so by Lemma 3.6,  $\varphi$  is a morphism. Define  $\psi : C \rightarrow \mathbb{A}^1$  by  $(x, y) \mapsto \frac{y}{x}$  when  $x \neq 0$  and  $(x, y) \mapsto 0$  when  $x = 0$ .  $\psi \circ \varphi = \text{id}_{\mathbb{A}^1}$  is easy to verify. For any  $(x, y) \in C$ , if  $x = 0$ , then  $y^2 = x^3 = 0$  so  $y = 0$ , so  $\varphi \circ \psi(x, y) = (0, 0) = (x, y)$ . If  $x \neq 0$ , then  $\varphi \circ \psi(x, y) = (\frac{y^2}{x^2}, \frac{y^3}{x^3}) = (\frac{x^3}{x^2}, \frac{y^3}{y^2}) = (x, y)$ , so we see  $\varphi \circ \psi = \text{id}_C$ .  $\psi$  is continuous because any closed subset of  $\mathbb{A}^1$  is finite, so its inverse image under  $\psi$  is obviously closed in  $C$ . Now we have shown  $\varphi$  is a bijective bicontinuous morphism onto  $C$ , but  $\varphi$  is not isomorphism, because if it is isomorphism, then by Corollary 3.7,  $A(\mathbb{A}^1) \cong A(C)$ , but  $A(\mathbb{A}^1) = 0$  and  $A(C) = k[x, y]/(y^2 - x^3) \neq 0$ , contradiction.

(b) If  $\varphi(a) = \varphi(b)$ , then  $a^p = b^p$ , then  $(a-b)^p = a^p - b^p = 0$  because  $k$  has characteristic  $p$ . So  $a-b = 0$  and  $\varphi$  is injective.  $\forall a \in k, x^p - a \in k[x]$  splits in  $k[x]$  because  $k$  is algebraically closed. So  $\varphi$  is surjective.  $\varphi$  is bicontinuous because  $\mathbb{A}^1$  has cofinite topology. Suppose  $\varphi$  is isomorphism. Let  $f : \mathbb{A}^1 \rightarrow k$  be  $f(t) = t$ . Then  $f \circ \varphi^{-1} : \mathbb{A}^1 \rightarrow k$  should be regular function. By Theorem 3.2(a), regular function on an affine variety has a polynomial expression. Thus  $\exists g \in k[x]$  such that when we view  $g$  as a function,  $\forall t \in k, g(t) = f \circ \varphi^{-1}(t) = \sqrt[p]{t}$  where  $\sqrt[p]{t}$  denotes the unique element of  $k$  whose  $p$ -th power is  $t$ . Then  $(g(x))^p(t) = t$  for all  $t \in k$ , so as a polynomial,  $(g(x))^p = x$ . But this is impossible because degree of  $(g(x))^p$  should be a multiple of  $p$ .

**Ex.3.3.**(a)  $\varphi_P^*$  is defined by pullback using  $\varphi$ . By definition of a morphism, the pullback of regular function is still regular function. That this map is well-defined and is  $k$ -algebra homomorphism is straightforward to see.

(b) ( $\implies$ ) An isomorphism is automatically a homeomorphism. For all  $P \in X$ , we have  $\varphi_P^* : \mathcal{O}_{\varphi(P), Y} \rightarrow \mathcal{O}_{P, X}$  and  $(\varphi^{-1})_{\varphi(P)}^* : \mathcal{O}_{P, X} \rightarrow \mathcal{O}_{\varphi(P), Y}$ . It is straightforward to see that these two maps are inverses of each other, so  $\varphi_P^*$  is an isomorphism.

( $\impliedby$ ) It suffices to prove  $\varphi^{-1}$  is morphism. Take any nonempty open set  $U \subseteq X$  and  $f : U \rightarrow k$  a regular function. We want to prove  $f \circ \varphi^{-1} : \varphi(U) \rightarrow k$  is regular. Pick any  $Q \in \varphi(U)$ , let  $P = \varphi^{-1}(Q)$ . Since  $\varphi_P^* : \mathcal{O}_{Q, Y} \rightarrow \mathcal{O}_{P, X}$  is surjective, there exists  $[(V, g)] \in \mathcal{O}_{Q, Y}$  such that  $\varphi_P^*([(V, g)]) = [(U, f)]$ . Thus  $g \circ \varphi = f$  on  $\varphi^{-1}(V) \cap U$ . Then  $g = f \circ \varphi^{-1}$  on  $V \cap \varphi(U)$ . So  $f \circ \varphi^{-1}$  is regular on  $V \cap \varphi(U)$ . Since  $Q$  is arbitrary, we

conclude that  $f \circ \varphi^{-1}$  is regular on  $\varphi(U)$ . So  $\varphi^{-1}$  is morphism, and  $\varphi$  is isomorphism.

(c) First we prove that for any nonempty open subset  $V \subseteq X$ ,  $\varphi(V)$  is dense in  $Y$ . Let  $D \subseteq Y$  be a closed set containing  $\varphi(V)$ , then  $\varphi^{-1}(D) \supseteq \varphi^{-1}(\varphi(V)) \supseteq V$ .  $X$  is irreducible, and any nonempty open subset of irreducible space is dense, so  $V$  is dense in  $X$ .  $\varphi^{-1}(D)$  is closed in  $X$ , so  $\varphi^{-1}(D) = X$ . Then  $D \supseteq \varphi(\varphi^{-1}(D)) = \varphi(X)$  which is dense in  $Y$  by assumption. Thus  $D = Y$  and  $\varphi(V)$  is dense in  $Y$ .

Back to our problem, suppose  $\varphi_P^*([(U, f)]) = 0$  for some  $[(U, f)] \in \mathcal{O}_{\varphi(P), Y}$ . Then  $f \circ \varphi = 0$  on  $\varphi^{-1}(U)$ . Since  $f$  is continuous on  $U$ , the zero set of  $f$  contains the closure of  $\varphi(\varphi^{-1}(U))$  in  $U$ . Since  $\varphi(\varphi^{-1}(U))$  is dense in  $Y$ , it is also dense in  $U$ , so  $f \equiv 0$  on  $U$ . so  $\varphi_P^*$  is injective.

**Ex.3.4.** We have shown in Ex.2.12 that the  $d$ -uple embedding  $\rho_d : \mathbb{P}^n \rightarrow Z(\mathfrak{a}) \subseteq \mathbb{P}^N$  is a homeomorphism where  $\mathfrak{a}$  is kernel of  $\theta$  defined in that exercise.  $\rho_d$  is a morphism because if  $U \subset Z(\mathfrak{a})$  is any open set and  $\varphi : U \rightarrow k$  is regular function, then locally on some open  $V \subseteq U$ ,  $\varphi = \frac{f}{g}$  where  $f, g \in k[y_0, \dots, y_N]$  are homogeneous of same degree. Then  $\varphi \circ \rho_d : \rho_d^{-1}(U) \rightarrow k$  restricted to  $\rho_d^{-1}(V)$  is equal to  $\frac{\theta(f)}{\theta(g)}$ , so we see  $\rho_d$  is morphism.

Conversely, take any open  $U \subseteq \mathbb{P}^n$ ,  $\varphi : U \rightarrow k$  regular function, and take any  $p \in \rho_d(U)$ . First take open set  $V$  such that  $\rho_d^{-1}(p) \in V \subseteq U$  and  $\varphi = \frac{f}{g}$  on  $V$  where  $f, g \in k[x_0, \dots, x_n]$  are homogeneous of same degree. Then  $p \in \rho_d(V)$ . We know  $p = \rho_d(a_0 : \dots : a_n)$  for some  $(a_0 : \dots : a_n) \in \mathbb{P}^n$ . WLOG, suppose  $a_0 \neq 0$ , and let  $i$  be the integer such that the  $i$ -th coordinate of  $\rho_d$  is  $a_0^d$ . Then  $p \in \rho_d(V) - Z(x_i)$ . When we restrict  $\rho_d^{-1}$  to  $Z(\mathfrak{a}) - Z(x_i)$ , we have  $\rho_d^{-1}(c_0 : \dots : c_n) = (c_i : c_{i_1} : c_{i_2} : \dots : c_{i_n})$  where each  $i_k$  is an integer such that the  $i_k$ -th coordinate of  $\rho_d$  is  $a_0^{d-1}a_{i_k}$ . Therefore, the restriction of  $\varphi \circ \rho_d^{-1}$  to  $\rho_d(V) - Z(x_i)$  is a rational function. Because  $p$  is arbitrary,  $\varphi \circ \rho_d^{-1} : \rho_d(U) \rightarrow k$  is a regular function, so  $\rho_d^{-1}$  is a morphism. Above all,  $\rho_d$  is isomorphism between  $\mathbb{P}^n$  and  $Z(\mathfrak{a})$ .

**Ex.3.5.** First we prove a useful result.

**Lemma 5:**  $\mathbb{P}^n - H \cong \mathbb{A}^n$  for any  $n$  and hyperplane  $H$ .

**Proof:** Suppose  $H = Z(f)$  where  $f$  is a linear polynomial in  $k[x_0, \dots, x_n]$ . WLOG, suppose the coefficient of  $x_0$  in  $f$  is not zero. Define  $\varphi : \mathbb{P}^n - H \rightarrow \mathbb{A}^n$  by  $(a_0 : \dots : a_n) \mapsto (\frac{a_1}{f(a)}, \dots, \frac{a_n}{f(a)})$  where  $a = (a_0 : \dots : a_n)$ . Note this is well defined. Define  $\psi : \mathbb{A}^n \rightarrow \mathbb{P}^n - H$  by  $(b_1, \dots, b_n) \mapsto (b_0 : b_1 : \dots : b_n)$  where  $b_0$  is the unique element in  $k$  such that  $f(b_0, \dots, b_n) = 1$ . It's straightforward to verify that  $\varphi$  and  $\psi$  are inverse maps.  $\varphi$  is a morphism by Lemma 3.6.  $\psi$  is continuous because for any closed set  $Z(g) \cap (\mathbb{P}^n - H) \subseteq \mathbb{P}^n - H$ , we can view  $f$  as a variable, replace the variable  $x_0$  in  $g$  by  $(f - c_1x_1 - \dots - c_nx_n)/c_0$  where  $c_n$  is coefficient of  $x_n$  in  $f$ , then let  $\tilde{g}$  be the polynomial got from  $g$  by replacing  $f$  with 1. Then  $\varphi(Z(g) \cap (\mathbb{P}^n - H)) = Z(\tilde{g})$ .  $\psi$  is then a morphism because each coordinate of  $\psi$  has polynomial expression.  $\square$

Now we prove Ex.3.5. By Ex.2.8,  $H = Z(f)$  for some  $f \in k[x_0, \dots, x_n]$  homogeneous, irreducible, and of positive degree. Suppose  $d = \deg f > 0$ . Let  $\rho_d : \mathbb{P}^n \rightarrow \mathbb{P}^N$  be the  $d$ -uple embedding, then  $\mathbb{P}^n - H \cong \rho_d(\mathbb{P}^n - H)$  by Ex.3.4. Because of our choice of  $d$ , using notation of Ex.2.12, there exists linear polynomial  $\tilde{f} \in k[y_0, \dots, y_N]$  such that  $\theta(\tilde{f}) = f$ . We have  $\rho_d(H) = Z(\tilde{f}) \cap \rho_d(\mathbb{P}^n)$  because  $\forall p \in \mathbb{P}^n$ ,  $\tilde{f}(\rho_d(p)) = f(p)$ . Thus  $\rho_d(\mathbb{P}^n - H) = \rho_d(\mathbb{P}^n) - Z(\tilde{f})$  is an irreducible closed subset of  $\mathbb{P}^N - Z(\tilde{f})$ . Then by Lemma 3 we see that  $\rho_d(\mathbb{P}^n - H)$  is isomorphic to an affine variety, so  $\mathbb{P}^n - H$  is isomorphic to an affine variety.

**Ex.3.6.** By the argument I give in <https://math.stackexchange.com/questions/373315>, the inclusion  $i : X \rightarrow \mathbb{A}^2$  induces an  $k$ -algebra isomorphism  $i_* : A(\mathbb{A}^2) \rightarrow O(X)$ . If  $X$  is affine, then by Corollary 3.8,  $i$  is an isomorphism, which is obviously not. So  $X$  is not affine.

**Ex.3.7.(a)** This is a special case of part (b). Or, we gave another argument in Ex.3.1(d).

(b) If  $Y \cap H = \emptyset$ , then  $Y \subseteq \mathbb{P}^n - H$ . By Ex.3.5,  $\mathbb{P}^n - H$  is isomorphic to an affine variety. Because  $Y$  is an irreducible closed set  $\mathbb{P}^n - H$ ,  $Y$  is also isomorphic to an affine variety. By Ex.3.1(e),  $Y$  is a single point, then  $\dim Y = 0$ , contradiction.

**Ex.3.8.** Let  $Y = \mathbb{P}^n - (H_i \cap H_j)$ . Because  $\overline{Y} = \mathbb{P}^n$ , any regular function on  $Y$  has form  $\frac{f}{g}$  where  $f, g$  are homogeneous polynomials of the same degree and  $g$  nonzero on  $Y$ . The reason is the same as the one I give here: <https://math.stackexchange.com/questions/373315>. But if  $\deg g > 0$ , then  $n - 1 = \dim Z(g) > \dim H_i \cap H_j = n - 2$ , then  $Z(g) \not\subseteq H_i \cap H_j$ , then  $g$  is not nonzero on  $Y$ , contradiction. So  $g$  has to be constant, and so is  $f$ .

**Remark:** We can replace  $H_i \cap H_j$  by any closed set with dimension  $\leq n - 2$ , and the conclusion still holds.

**Ex.3.9.** It is easy to use the definition of  $\rho_2 : \mathbb{P}^1 \rightarrow \mathbb{P}^2$  to show that  $Y = Z(y_0y_2 - y_1^2)$ .  $(y_0y_2 - y_1^2)$  is prime ideal because  $y_0y_2 - y_1^2$  is irreducible, so  $I(Y) = (y_0y_2 - y_1^2)$  and  $S(Y) = k[y_0, y_1, y_2]/(y_0y_2 - y_1^2)$ . In  $S(Y)$ ,  $\bar{y}_0$  (the class of  $y_0$ ) is nonzero, nonunit, and irreducible. The proof is to use existence and uniqueness of division by  $y_0y_2 - y_1^2$  where we view polynomials as in  $y_1$  with coefficients in  $k[y_0, y_2]$ . The process is tedious so I will omit it here. Similarly  $\bar{y}_1$  and  $\bar{y}_2$  are irreducible in  $S(Y)$ , and we can prove these  $\bar{y}_i$  are distinct irreducibles up to multiplication by units. Then since  $\bar{y}_0\bar{y}_2 = \bar{y}_1^2$  in  $S(Y)$ ,  $S(Y)$  is not UFD. But  $S(X) = k[x_0, x_1]$  is UFD, so  $S(X) \not\cong S(Y)$ .

**Ex.3.10.** By Lemma 3.

**Ex.3.11.** First assume  $X$  is affine variety. Closed subvarieties of  $X$  containing  $P$  are just the affine varieties contained in  $X$  containing  $P$ . Since  $\mathcal{O}_{P,X} \cong A(X)_{\mathfrak{m}_P}$ , prime ideals of  $\mathcal{O}_{P,X}$  correspond to prime ideals of  $A(X)$  contained in  $\mathfrak{m}_P$ , which correspond to prime ideals of  $A = k[x_1, \dots, x_n]$  between  $I(X)$  and  $\mathfrak{m}_P$ , which correspond exactly to affine varieties between  $P$  and  $X$ .

Next, assume  $X$  is quasi-affine, then  $\mathcal{O}_{P,X} \cong \mathcal{O}_{P,\bar{X}}$ , and closed subvarieties of  $X$  containing  $P$  correspond to closed subvarieties of  $\bar{X}$  containing  $P$  (One map is given by taking closure in  $\bar{X}$ , the reverse map is given by restriction to  $X$ . Arguments in general topology show these are indeed inverse maps).

Last, assume  $X$  is quasi-projective. Suppose  $P \in U_i$  where  $U_i = Z(x_i)^c$ . Let  $\varphi_i : U_i \rightarrow \mathbb{A}^n$  be the canonical isomorphism, then  $\mathcal{O}_{P,X} \cong \mathcal{O}_{\varphi_i(P), \varphi_i(U_i \cap X)}$ , closed subvarieties of  $\varphi_i(U_i \cap X)$  containing  $\varphi_i(P)$  correspond to closed subvarieties of  $U_i \cap X$  containing  $P$  (because  $\varphi_i$  is isomorphism), which correspond to closed subvarieties of  $X$  containing  $P$  (under the same maps defined at the end of previous paragraph).

**Ex.3.12.** If  $X$  is quasi-affine, then we have  $\mathcal{O}_{P,X} \cong \mathcal{O}_{P,\bar{X}}$  and  $\dim X = \dim \bar{X}$ , so we reduce to the affine case, which is true by Theorem 3.2(c). If  $X$  is quasi-projective, then there exists  $U_i$  such that  $P \in U_i$  where  $U_i = Z(x_i)^c$ . Let  $\varphi_i : U_i \rightarrow \mathbb{A}^n$  be the canonical isomorphism then  $\mathcal{O}_{P,X} \cong \mathcal{O}_{\varphi_i(P), \varphi_i(X \cap U_i)}$  and  $\dim X = \dim \varphi_i(X \cap U_i)$  (by Ex.2.7, two quasi-projective varieties have the same dimension if they have the same closure, so  $\dim X = \dim X \cap U_i$ ), so we have reduced to the quasi-affine case, which is true by the first sentence.

**Ex.3.13.** The relation defined in problem is an equivalence relation because any finite number of nonempty open subsets of a variety have nonempty intersection and regular function is continuous. The addition in  $\mathcal{O}_{Y,X}$  is defined by  $\langle U, f \rangle + \langle V, g \rangle = \langle U \cap V, f + g \rangle$ , and multiplication is defined by  $\langle U, f \rangle \cdot \langle V, g \rangle = \langle U \cap V, fg \rangle$ , and it's easy to verify that these are well defined operations, and that  $\mathcal{O}_{Y,X}$  is an integral domain. We claim that  $\mathfrak{m} := \{\langle U, f \rangle \mid f(a) = 0 \forall a \in U \cap Y\}$  is the unique maximal ideal. First note that  $\mathfrak{m}$  is well-defined as a set because if  $\langle U, f \rangle = \langle V, g \rangle$  where  $f(a) = 0$  for all  $a \in U \cap Y$ , then because  $Y$  is a variety,  $(U \cap Y) \cap (V \cap Y) \neq \emptyset$ , so  $g$  is zero on a nonempty open subset of  $V \cap Y$ , so by continuity of regular functions, we see that  $g$  vanishes on  $V \cap Y$ . It is straightforward to see that  $\mathfrak{m}$  is an ideal. To see  $\mathfrak{m}$  is the unique maximal ideal, it suffices to prove that any  $\langle U, f \rangle \notin \mathfrak{m}$  is a unit. Pick  $a \in U \cap Y$  such that  $f(a) \neq 0$ . Because  $f$  is regular, there exists open set  $V \subseteq X$  such that  $a \in V$  and  $f = \frac{g}{h}$  on  $V$  where  $g, h$  are polynomials,  $h$  nowhere zero on  $V$ . Then  $\langle U, f \rangle \cdot \langle V - Z(g), \frac{h}{g} \rangle = \langle X, 1 \rangle$ , the multiplicative identity in  $\mathcal{O}_{Y,X}$ , so  $\mathfrak{m}$  is the unique maximal ideal, and  $\mathcal{O}_{Y,X}$  is a local ring.

Define  $\varphi : \mathcal{O}_{Y,X} \rightarrow K(Y)$  by  $\varphi(\langle U, f \rangle) = \langle U \cap Y, f|_{U \cap Y} \rangle$ . It's easy to see that this is a well-defined ring homomorphism.  $\varphi$  is surjective because  $\forall \langle U, f \rangle \in K(Y)$ ,  $f = \frac{g}{h}$  on some open set  $V \subseteq U$  where  $g, h$  are polynomials and  $h$  is nowhere zero on  $V$ . Then  $\varphi(\langle X - Z(h), \frac{g}{h} \rangle) = \langle Y - Z(h), \frac{g}{h} \rangle = \langle V, f \rangle = \langle U, f \rangle$ . Obviously  $\ker \varphi = \mathfrak{m}$ . So we have  $\frac{\mathcal{O}_{Y,X}}{\mathfrak{m}} \cong K(Y)$ .

Finally we show  $\dim \mathcal{O}_{Y,X} = \dim \bar{X} - \dim Y$ . First assume  $X$  and  $Y$  are affine varieties. Then there is a natural map  $\psi : A(X)_{I(Y)} \rightarrow \mathcal{O}_{Y,X}$  given by  $\frac{[f]}{[g]} \mapsto \langle X - Z(g), \frac{f}{g} \rangle$ . Note by definition of localization,  $g \notin I(Y)$ , so  $(X - Z(g)) \cap Y \neq \emptyset$ . Obviously this is a well-defined ring homomorphism.  $\psi$  is surjective because  $\forall \langle U, h \rangle \in \mathcal{O}_{Y,X}$ ,  $h = \frac{f}{g}$  on some open subset of  $X$  which has nonempty intersection with  $Y$ . In particular  $g \notin I(Y)$ , so  $\psi(\frac{[f]}{[g]}) = \langle U, h \rangle$ . It's easy to see that  $\psi$  is also injective. Thus  $\psi$  is an isomorphism and  $\dim \mathcal{O}_{Y,X} = \dim A(X)_{I(Y)} = \text{height } I(Y)/I(X) = \dim A(X) - \dim \frac{A(X)}{I(Y)/I(X)} = \dim X - \dim A(Y) = \dim X - \dim Y$  where the second step is true by the general fact that dimension of localization of integral domain at prime ideal is equal to height of the ideal.

Next assume  $X, Y$  are any quasi-affine varieties. Note we have natural map  $\varphi : \mathcal{O}_{\bar{Y}, \bar{X}} \rightarrow \mathcal{O}_{Y, X}$  given by  $\langle U, f \rangle \mapsto \langle U \cap X, f|_{U \cap X} \rangle$ . This is well defined because if  $V$  is any open set in  $\mathbb{A}^n$  and  $V \cap \bar{X} \neq \emptyset$ , then  $V \cap X \neq \emptyset$ . It's easy to see  $\varphi$  is ring homomorphism.  $\varphi$  is surjective because  $\forall \langle U, f \rangle \in \mathcal{O}_{Y, X}$ , there exists some  $V$  open in  $X$  such that  $V \cap Y \neq \emptyset$  and  $f = \frac{g}{h}$  on  $V$  where  $g, h$  are polynomials,  $h$  nonzero on  $V$ . Then  $\varphi(\langle \bar{X} - Z(h), \frac{g}{h} \rangle) = \langle V, f \rangle = \langle U, f \rangle$ .  $\varphi$  is injective because if  $\varphi(\langle U, f \rangle) = 0$ , then  $f = 0$  on  $U \cap X$ . Note  $X$  is open in  $\bar{X}$  because  $X$  is quasi-affine (indeed,  $\bar{X}$  is the unique affine variety containing  $X$ ), so  $U \cap X$  is open in  $U$ . But  $U$  is irreducible, so closure of  $U \cap X$  in  $U$  equals  $U$ , then continuity of  $f$  as a regular function on  $U$  implies  $f$  vanishes on the whole  $U$ . Therefore, we see that  $\varphi$  is isomorphism. So  $\dim \mathcal{O}_{Y, X} = \dim \mathcal{O}_{\bar{Y}, \bar{X}} = \dim \bar{X} - \dim \bar{Y} = \dim X - \dim Y$  where we have used Prop.1.10 in the last step. Finally, for the projective case, the arguments are similar as the affine case, except that now we consider  $S(X)_{(I(Y))}$ , the homogenized version of localization.

**Ex.3.14.**(a) By Lemma 5,  $\mathbb{P}^{n+1}$  minus a hyperplane are all isomorphic no matter what the specific of the hyperplane is. Thus we can assume the embedding of  $\mathbb{P}^n$  in  $\mathbb{P}^{n+1}$  is given by  $Z(x_0)$  where  $(x_0, \dots, x_{n+1})$  is the homogeneous coordinate on  $\mathbb{P}^{n+1}$ . Let  $P = (a_0 : \dots : a_{n+1})$  where  $a_0 \neq 0$  because  $P \notin \mathbb{P}^n$ . For any  $Q = (b_0 : \dots : b_{n+1}) \in \mathbb{P}^{n+1} - \{P\}$ , some calculation shows  $\varphi(Q) = (0 : a_1 b_0 - a_0 b_1 : \dots : a_{n+1} b_0 - a_0 b_{n+1})$ . Note  $\varphi$  is well defined, i.e. it does not depend on the specific representative of the equivalence class of homogeneous coordinates. Define  $\psi : k[x_0, \dots, x_{n+1}] \rightarrow k[x_0, \dots, x_{n+1}]$  by  $x_0 \mapsto 0, x_i \mapsto a_i x_0 - a_0 x_i$  for all  $1 \leq i \leq n+1$ . Note  $\psi$  sends homogeneous polynomial to homogeneous polynomial of the same degree.

We first show  $\varphi$  is continuous. It suffices to prove  $\varphi^{-1}(Z(f) \cap Z(x_0))$  is closed in  $\mathbb{P}^{n+1} - \{P\}$  for any homogeneous  $f \in k[x_0, \dots, x_{n+1}]$ . It is easy to verify  $\varphi^{-1}(Z(f) \cap Z(x_0)) = Z(\psi(f)) \cap (\mathbb{P}^{n+1} - \{P\})$  because  $\forall Q = (b_0 : \dots : b_{n+1}) \in \mathbb{P}^{n+1} - \{P\}$ ,  $\psi(f)(Q) = f(\varphi(Q))$ . Thus  $\varphi$  is continuous. Then take any  $V$  open in  $Z(x_0)$  and  $f : V \rightarrow k$  regular on  $V$ .  $f \circ \varphi : \varphi^{-1}(V) \rightarrow k$  is also regular, because  $\forall Q = (b_0 : \dots : b_{n+1}) \in \varphi^{-1}(V)$ , we can find  $U$  open in  $V$ ,  $U$  containing  $\varphi(Q)$ , such that  $f = \frac{g}{h}$  on  $U$  where  $g, h$  are homogeneous polynomials of same degree. Then on  $\varphi^{-1}(U) \ni Q$ ,  $f \circ \varphi = \frac{\psi(g)}{\psi(h)}$ . So  $\varphi$  is a morphism.

(b) Let  $(x, y, z, w)$  be homogeneous coordinates on  $\mathbb{P}^3$ , then in this problem the projection map  $\varphi : \mathbb{P}^3 - (0 : 0 : 1 : 0) \rightarrow Z(z)$  is  $\varphi(x : y : z : w) = (x : y : 0 : w)$ . Note  $(0 : 0 : 1 : 0) \notin Y$ , and  $\varphi(Y) = \varphi(\{(t^3 : t^2 u : t u^2 : u^3) | t, u \in k, t \neq 0 \text{ or } u \neq 0\}) = \{(t^3 : t^2 u : 0 : u^3) | t \neq 0 \text{ or } u \neq 0\}$ . We have  $\varphi(Y) = Z(z) \cap Z(x^2 w - y^3)$ . The " $\supseteq$ " is obvious. To see " $\supseteq$ ", take  $(x_0 : y_0 : z_0 : w_0) \in Z(z) \cap Z(x^2 w - y^3)$ . Then  $x_0^2 w_0 = y_0^3$ . If  $y_0 = 0$ , then either  $x_0 = 0$  or  $w_0 = 0$ , so either  $(x_0 : y_0 : z_0 : w_0) = (0 : 0 : 0 : 1)$  or  $(x_0 : y_0 : z_0 : w_0) = (1 : 0 : 0 : 0)$ . In both cases  $(x_0 : y_0 : z_0 : w_0) \in \varphi(Y)$ . So assume  $y_0 \neq 0$ , then  $x_0 \neq 0$  and  $w_0 \neq 0$ . Then  $\varphi(Y) \ni ((\frac{x_0}{y_0})^3 : (\frac{x_0}{y_0})^2 : 0 : 1) = (\frac{x_0}{w_0} : \frac{y_0}{w_0} : 0 : 1) = (x_0 : y_0 : 0 : w_0) = (x_0 : y_0 : w_0 : z_0)$ . Therefore,  $\varphi(Y) = Z(z) \cap Z(x^2 w - y^3)$ , and we see that the projection of  $Y$  from  $P$  is a cuspidal cubic curve in the plane  $Z(z)$ .

**3.16.** We will prove (a) and (b) together. Let  $\varphi : \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^{mn+m+n}$  be the Segre embedding defined by  $\varphi((a_0 : \dots : a_n), (b_0 : \dots : b_m)) = (a_0 b_0 : a_0 b_1 : \dots : a_0 b_m : \dots : a_n b_0 : \dots : a_n b_m)$ . Let  $\psi : k[\{z_{ij}\}] \rightarrow k[x_0, \dots, x_n, y_0, \dots, y_m]$  be the corresponding map mentioned in Ex.2.14. Then we have showed in Ex.2.14 that  $\ker \psi$  is a prime homogeneous ideal and  $\varphi$  is a bijection onto  $Z(\ker \psi)$ . We force  $\varphi$  to be a homeomorphism, then  $\mathbb{P}^n \times \mathbb{P}^m$  has the topology of a projective variety.

Now fix closed sets  $X \subseteq \mathbb{P}^n, Y \subseteq \mathbb{P}^m$ . We claim  $\varphi(X \times Y)$  is closed. In fact  $\varphi(X \times Y) = Z(\ker \bar{\psi})$  where  $\bar{\psi} : k[\{z_{ij}\}] \rightarrow k[x_0, \dots, x_n, y_0, \dots, y_m]/(I(X) + I(Y))$  is the map induced from  $\psi$ . Note  $I(X) + I(Y)$  is homogeneous ideal because it is generated by homogeneous elements, so  $\ker \bar{\psi}$  is a homogeneous ideal. We first prove  $\varphi(X \times Y) \subseteq Z(\ker \bar{\psi})$ . Take  $P \in X \times Y$  and any homogeneous  $f \in \ker \bar{\psi}$ , then  $f(\varphi(P)) = \psi(f)(P)$ , but  $\psi(f) \in I(X) + I(Y)$ , so  $f(\varphi(P)) = 0$ . Then we prove  $\varphi(X \times Y) \supseteq Z(\ker \bar{\psi})$ . We have  $\ker \bar{\psi} \supseteq \ker \psi$ , so  $Z(\ker \bar{\psi}) \subseteq Z(\ker \psi) = \text{im } \varphi$ , so for any  $Q \in Z(\ker \bar{\psi})$ , there exists  $P \in \mathbb{P}^n \times \mathbb{P}^m$  such that  $\varphi(P) = Q$ . Take any homogeneous  $f \in I(X)$ . Suppose  $P = ((a_0 : \dots : a_n), (b_0 : \dots : b_m))$ . Pick  $i, j$  such that  $a_i \neq 0, b_j \neq 0$ . Let  $\tilde{f}$  be the polynomial got from  $f$  by replacing each  $x_i$  with  $z_{ij}$  where  $j$  is our fixed choice. Then  $\psi(\tilde{f}) = y_j^{\deg f} f \in I(X) \subseteq I(X) + I(Y)$ , so  $\tilde{f}(Q) = 0$ . But  $\tilde{f}(Q) = \tilde{f}(\varphi(P)) = \psi(\tilde{f})(P) = b_j^{\deg f} f(a_0 : \dots : a_n)$  and  $b_j \neq 0$ , so  $f(a_0 : \dots : a_n) = 0$ . So  $(a_0 : \dots : a_n) \in Z(I(X)) = X$ . Similarly we can show  $(b_0 : \dots : b_m) \in Y$ . Thus  $P \in X \times Y$ . So we have shown  $\varphi(X \times Y) = Z(\ker \bar{\psi})$ .

In particular, when  $X$  and  $Y$  are projective varieties,  $I(X)$  and  $I(Y)$  are prime ideals, then  $I(X) + I(Y)$  is prime in  $k[x_0, \dots, x_n, y_0, \dots, y_m]$  by this link: <https://mathoverflow.net/questions/76772/>. Then  $\ker \bar{\psi}$  is a prime ideal, so  $\varphi(X \times Y)$  is projective variety. This proves part (b).

For part (a), when  $X$  and  $Y$  are quasi-projective varieties, we have  $X = C_1 \cap U_1$  and  $Y = C_2 \cap U_2$  where  $C_1, C_2$  are projective varieties and  $U_1, U_2$  are open sets in projective space. Then  $\varphi(X \times Y) = \varphi((C_1 \cap U_1) \times (C_2 \cap U_2)) = \varphi(C_1 \times C_2) - \varphi(C_1 \times (C_2 - U_2)) - \varphi((C_1 - U_1) \times C_2)$  where the last step is true because  $\varphi$  is bijection onto  $Z(\ker \psi)$ . In the last step, the latter two sets are closed subsets of  $\varphi(C_1 \times C_2)$ , and  $\varphi(C_1 \times C_2)$  is projective variety, so  $\varphi(X \times Y)$  is a quasi-projective variety.

(c)

## 4 Rational Maps

**Ex.4.1.** Let  $h$  be the function on  $U \cup V$  which is  $f$  on  $U$  and  $g$  on  $V$ . For any point  $P \in U \cup V$ , if  $P \in U$ , then  $f = \frac{f_1}{f_2}$  on some open set  $U' \subseteq U \subseteq X$ , and so is  $h$ . Similar argument follows if  $P \in V$ . So  $h$  is regular. Now if  $\langle U, f \rangle \in K(X)$ , let  $(U_i, f_i)_{i \in I}$  be the set of all pairs of open subsets and regular functions in the equivalence class of  $\langle U, f \rangle \in K(X)$ . Consider  $\langle \bigcup_{i \in I} U_i, f^* \rangle$  where  $f^*$  is defined using the  $f_i$ 's. Then  $f^*$  is a well-defined regular function, so  $\bigcup_{i \in I} U_i$  is the domain of definition of  $f$ .

**Ex.4.2.** Let  $\varphi = \langle U, \varphi' \rangle$  be a rational map from  $X$  to  $Y$ . Let  $(U_i, \varphi_i)_{i \in I}$  be the set of all pairs of open subsets of  $X$  and morphisms in the equivalence class of  $\langle U, \varphi' \rangle$ . Consider  $\varphi^* : \bigcup_{i \in I} U_i \rightarrow Y$  defined by using the  $\varphi_i$ 's. For any  $V$  open in  $Y$ ,  $(\varphi^*)^{-1}(V) = \bigcup_{i \in I} \varphi_i^{-1}(V)$  is open in  $\bigcup_{i \in I} U_i$ , so  $\varphi^*$  is continuous. We can also easily show  $\varphi^*$  is a morphism, using the fact that all the  $\varphi_i$ 's are morphisms.

**Ex.4.3.**(a)  $f$  is defined on the open set  $Z(x_0)^c$ . We note that, as an element of  $K(\mathbb{P}^2)$ , domain of  $f$  cannot be further extended. Indeed, if  $\langle U, h \rangle = \langle Z(x_0)^c, f \rangle$  where  $U \cap Z(x_0) \neq \emptyset$ , then pick  $P \in U \cap Z(x_0)$ , then there is nonempty open subset  $V \subseteq \mathbb{P}^2$  such that  $h = \frac{h_1}{h_2} = \frac{x_1}{x_0}$  on  $V$  where  $h_1, h_2$  are homogeneous of same degree,  $h_2, x_0$  do not vanish on  $V$ ,  $h_2(P) \neq 0$ . This means the polynomial  $x_0 h_1 - x_1 h_2$  kills  $V$ , which is dense in  $\mathbb{P}^2$ , so  $x_0 h_1 - x_1 h_2 \in I(\mathbb{P}^2) = 0$ , so we have  $x_0 h_1 = x_1 h_2$  as polynomials. But  $x_0 \nmid h_2$  since  $h_2(P) \neq 0$ . Contradiction! Thus  $Z(x_0)^c$  is the maximal domain of definition of  $f$ .

(b) Note  $f$  is a continuous function from  $Z(x_0)^c$  to  $\mathbb{A}^1$  because regular function is continuous. It is straightforward to show  $f$  is a morphism from  $Z(x_0)^c$  to  $\mathbb{A}^1$  and  $f$  is surjective, so  $\langle Z(x_0)^c, f \rangle$  is a dominant rational map from  $\mathbb{P}^2$  to  $\mathbb{A}^1$ . The embedding of  $\mathbb{A}^1$  in  $\mathbb{P}^1$  is also obviously a dominant rational map, so composing these two maps we get dominant rational map  $\varphi : \mathbb{P}^2 \rightarrow \mathbb{P}^1$ . Explicitly,  $\varphi = \langle Z(x_0)^c, f \rangle$  where  $f(x_0 : x_1 : x_2) = (x_0 : x_1)$ . We note that the set of points where  $\varphi$  is defined is strictly larger than the set of points where  $f$  is defined! Indeed,  $\varphi = \langle (Z(x_0) \cap Z(x_1))^c, f \rangle$  where  $f$  has the same definition as before. Note  $\varphi$  cannot be extended to the whole  $\mathbb{P}^2$ , due to a similar argument as in part (a). Specifically, suppose it can be extended to  $(0 : 0 : 1)$ , then WLOG suppose  $\varphi(0 : 0 : 1) = (a_0 : a_1)$  where  $a_0 \neq 0$ , then pull back the regular function  $\frac{x_1}{x_0}$  on  $\mathbb{P}^1$  and we see locally around  $(0 : 0 : 1)$ ,  $(\frac{x_1}{x_0}) \circ \varphi = \frac{h_1}{h_2}$  where  $h_1, h_2$  are homogeneous polynomials of same degree. But we also have  $\varphi = \langle (Z(x_0) \cap Z(x_1))^c, f \rangle = \langle (Z(x_0)^c, f) \rangle$ , so pull back  $\frac{x_1}{x_0}$  using this description gives us  $(\frac{x_1}{x_0}) \circ \varphi = \frac{x_1}{x_0}$  on  $Z(x_0)^c$ . Then similarly as argued in part (a),  $x_0 h_1 = x_1 h_2$  as polynomials. This implies  $x_0 \mid h_2$ , but  $h_2(0 : 0 : 1) \neq 0$ , contradiction! Therefore the maximal domain of definition of  $\varphi$  is  $\mathbb{P}^2 - (0 : 0 : 1)$ .

**Ex.4.4.** We first prove the equivalence mentioned in the parenthesis: a variety  $Y$  is birationally equivalent to  $\mathbb{P}^n$  iff  $K(Y)$  is a purely transcendental extension of  $k$ . First suppose  $Y$  is birationally equivalent to  $\mathbb{P}^n$ , then by Corollary 4.5,  $K(Y) \cong K(\mathbb{P}^n)$ . By the result in section 3,  $K(\mathbb{P}^n) \cong k[x_0, \dots, x_n]_{((0))}$ . Because these isomorphisms are all over  $k$ , it suffices to prove  $k[x_0, \dots, x_n]_{((0))}$  is a purely transcendental extension of  $k$ . This is true because  $\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}$  is a pure transcendental basis. Conversely, suppose  $K(Y)$  is a pure transcendental extension of  $k$ . We can assume  $Y$  is affine variety, because if it is quasi-projective, we can use the affine charts in  $\mathbb{P}^n$  to map  $Y$  to an isomorphic quasi-affine variety in  $\mathbb{A}^n$ , then  $K(Y)$  is isomorphic to function field of closure of this quasi-affine variety. Then, using result of section 3, we know transcendence degree of  $K(Y)$  is finite, say  $K(Y) = k(a_1, \dots, a_m)$  where the  $a_i$ 's are algebraically independent. We also have  $K(\mathbb{P}^m) = k(b_1, \dots, b_m)$  where  $b_1, \dots, b_m$  are algebraically independent. This is explained in the forward direction. Thus we see  $K(Y) \cong K(\mathbb{P}^m)$ . By Corollary 4.5,  $Y$  is birationally equivalent to  $\mathbb{P}^m$ .

(a) A stronger conclusion follows from Ex.3.1(c). Below is another approach.

Let  $C$  be our conic. WLOG, Suppose  $U_0 \cap C \neq \emptyset$  where  $U_0 = Z(x_0)^c$ . Then  $K(C) \cong k(U_0 \cap C) \cong K(\varphi^{-1}(U_0 \cap C)) = K(Y)$  where  $\varphi$  is the canonical isomorphism between  $\mathbb{A}^2$  and  $U_0$ , and we have let  $Y := \varphi^{-1}(U_0 \cap C)$  be the affine variety. Furthermore,  $Y = Z(f)$  where  $f$  is the affinization (with respect to  $x_0$ ) of the homogeneous polynomial defining  $C$ .  $f$  is irreducible. (There are two ways to see this. First, if  $f$  is reducible, then  $Y$  becomes reducible, but we know  $Y$  is irreducible since it is variety. Or, alternatively, we can use the general algebraic fact that affinization of irreducible homogeneous polynomial is again irreducible.)  $f$  is quadratic. Otherwise, the polynomial defining  $C$  is reducible. Applying Ex.1.1, we know  $A(Y) \cong k[x]$  or  $A(Y) \cong k[x, x^{-1}] = k[x]_x$ , so its fractional field  $K(A(Y)) \cong k(x)$ . So  $K(C) \cong K(Y) \cong K(A(Y)) \cong k(x)$ , a pure transcendental extension of  $k$ . So  $C$  is birational to  $\mathbb{P}^1$ .

(b) Let  $Y := Z(y^2 - x^3)$ . Let  $Y' = Y - \{(0, 0)\}$ . Let  $U := \mathbb{P}^1 - ((1 : 0) \cup (0 : 1))$ . It suffices to prove  $Y' \cong U$ . Let  $\varphi : Y' \rightarrow U$  be  $\varphi(x, y) = (x : y)$ , let  $\psi : U \rightarrow Y'$  be  $\psi(x : y) = ((\frac{y}{x})^2, (\frac{y}{x})^3)$ . It is quick to see that these two maps are well defined and are inverses to each other.  $\varphi$  is obviously a morphism. Also, if we view  $\psi$  as taking value in  $Y$ , then by Lemma 3.6,  $\psi$  is a morphism to  $Y$ , and this implies  $\psi$  is a morphism to  $Y'$ .

(c) The projection  $\varphi$  is a morphism from  $\mathbb{P}^2 - (0 : 0 : 1)$  to  $Z(z)$  given by  $\varphi(x : y : z) = (x : y : 0)$  (by Ex.3.14). Denote by  $\varphi'$  the restriction of  $\varphi$  to  $Y - (0 : 0 : 1)$ . Note  $\text{im } \varphi' \subseteq Z(z) - Z(y^2 - x^2)$ . Let  $\psi : Z(z) - Z(y^2 - x^2) \rightarrow Y - (0 : 0 : 1)$  be defined by  $(x : y : 0) \mapsto (x(y^2 - x^2) : y(y^2 - x^2) : x^3)$ . It is straightforward to see that  $\varphi'$  and  $\psi$  are well-defined inverse functions.  $\varphi'$  is a morphism because it is induced from  $\varphi$  which is obviously a morphism.  $\psi$  is a morphism. Indeed, if we let  $\theta : k[x, y, z] \rightarrow k[x, y, z]$  be  $x \mapsto x(y^2 - x^2), y \mapsto y(y^2 - x^2), z \mapsto x^3$ , then for any homogeneous  $f \in k[x, y, z]$ ,  $\psi^{-1}(Z(f)) = Z(\theta(f)) \cap (Z(z) - Z(y^2 - x^2))$ , since  $f(\psi(x)) = \theta(f)(x)$ . So  $\psi$  is continuous. Pullback of regular functions on open subsets of  $Y - (0 : 0 : 1)$  by  $\psi$  are regular function on open subsets of  $Z(z) - Z(y^2 - x^2)$  because of definition of  $\psi$ . So  $\psi$  is a morphism. So  $Y - (0 : 0 : 1) \cong Z(z) - Z(y^2 - x^2)$ . So  $Y$  is birational to  $Z(z)$ .  $Z(z)$  is isomorphic to  $\mathbb{P}^1$ , so  $Y$  is birational to  $\mathbb{P}^1$ .

**Ex.4.7.** (The conclusion of this exercise is so beautiful!) We can suppose both  $X$  and  $Y$  are affine varieties, because any variety has a basis of open affine subsets (Proposition 4.3). Let  $(y_1, \dots, y_n)$  be affine coordinates on  $Y$  and let  $x_1, \dots, x_m$  be affine coordinates on  $X$ .  $\theta : \mathcal{O}_{Q,Y} \rightarrow \mathcal{O}_{P,X}$  be the  $k$ -algebra isomorphism. Since  $A(Y) = \mathcal{O}(Y) \subseteq \mathcal{O}_{Q,Y}$ , we can restrict  $\theta$  to  $A(Y)$  to get  $k$ -algebra homomorphism  $A(Y) \rightarrow \mathcal{O}_{P,X}$ .  $A(Y)$  is finitely generated by  $y_1, \dots, y_n$ , and we can find open  $U \subseteq X$  such that  $P \in U$  and each  $\theta(y_i)$  is regular on  $U$ . This gives us  $k$ -algebra homomorphism  $\theta' : A(Y) \rightarrow \mathcal{O}(U)$ . By Proposition 3.5, this induces a morphism  $\varphi : U \rightarrow Y$  given by  $\varphi(x) = (\theta'(y_1)(x), \dots, \theta'(y_n)(x))$ . We observe that  $\varphi(P) = Q$ . Indeed, write  $Q = (Q_1, \dots, Q_n)$ , then  $\theta'(y_i - Q_i)(P) = \theta'(y_i)(P) - Q_i$  because  $\theta'$  is  $k$ -algebra homomorphism. On the other hand,  $y_i - Q_i \in \mathcal{O}_{Q,Y}$  is in the maximal ideal of  $\mathcal{O}_{Q,Y}$ , so  $\theta(y_i - Q_i)$  is in the maximal ideal of  $\mathcal{O}_{P,X}$ , so  $\theta'(y_i - Q_i)(P) = 0$ . Thus  $\theta'(y_i)(P) = Q_i$ , and  $\varphi(P) = Q$ .

In the same way we can find open  $V \subseteq Y$  and morphism  $\psi : V \rightarrow X$  such that  $Q \in V$  and  $\psi(Q) = P$ . Then it suffices to show  $\psi \circ \varphi$  and  $\varphi \circ \psi$  are identity on the open sets where they have definition (Then  $\varphi^{-1}(V) \cap U \cong \psi^{-1}(U) \cap V$ ). Take  $x \in \varphi^{-1}(V) \cap U$ , write  $x = (x_1, \dots, x_m)$ , then we want to show  $(\psi \circ \varphi(x))_i = x_i$ . We have

$$\begin{aligned} (\psi \circ \varphi(x))_i &= \psi(\theta(y_1)(x), \dots, \theta(y_n)(x))_i \\ &= \theta^{-1}(x_i)(\theta(y_1)(x), \dots, \theta(y_n)(x)) \\ &= \frac{f}{g}(\theta(y_1)(x), \dots, \theta(y_n)(x)) \\ &= \theta\left(\frac{f}{g}\right)(x) \\ &= \theta(\theta^{-1}(x_i))(x) \\ &= x_i(x) \\ &= x_i. \end{aligned}$$

where we have used the local description of  $\theta^{-1}(x_i) = \frac{f}{g}$  at the point  $(\theta(y_1)(x), \dots, \theta(y_n)(x))$  in the third line. So we are done.

**Ex.4.9.** (I could not figure it out by myself. I am basically following the answer here: [math.stackexchange.com/questions/2042583](https://math.stackexchange.com/questions/2042583). According to this link, we need a stronger version of the primitive element the-

orem, which I will assume here. Also, we will drop the requirement  $P \notin X$ , because we want to conclude birational equivalence anyway.)

WLOG, assume  $X \cap U_0 \neq \emptyset$ , then  $K(X)$  is generated over  $k$  by  $\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}$ . Using the affine chart  $U_0$ , Ex.2.7(b), Prop.1.10, and Theorem 3.2(d), we conclude that  $K(Y)$  has transcendence degree  $r$  over  $k$ . By Theorem 4.7A and 4.8A, we get finite separable extension  $K(X)/k(\frac{x_1}{x_0}, \dots, \frac{x_r}{x_0})$  after possibly a change of coordinates. By the stronger version of the primitive element theorem,  $K(X)$  is generated over  $k(\frac{x_1}{x_0}, \dots, \frac{x_r}{x_0})$  by  $\alpha$  where  $\alpha = \sum_{i=r+1}^n a_i \frac{x_i}{x_0}$ ,  $a_i \in k$ .

If all the  $a_i$  are 0, let  $\pi : \mathbb{P}^n \dashrightarrow Z(x_n)$  be  $(a_0 : \dots : a_n) \mapsto (a_0 : \dots : a_{n-1} : 0)$ . Let  $X' = \pi(X)$ , then  $\pi$  induces  $k$ -algebra homomorphism  $\pi^* : K(X') \rightarrow K(X)$  which takes  $\frac{x_i}{x_0}$  to  $\frac{x_i}{x_0}$  for all  $1 \leq i \leq r$ . Then  $\pi^*$  is surjective, so  $\pi$  is birational morphism between  $X$  and  $X'$ .

If some of the  $a_i$  is nonzero, we can assume  $\alpha = \frac{x_{r+1}}{x_0}$  after possibly another change of coordinates. Because  $r+1 \leq n-1$ , the same definition of  $\pi$  induces isomorphism of function fields  $K(X') \cong K(X)$ , so  $\pi$  is birational morphism between  $X$  and  $X'$ . So we are done.

Remark: I have assumed  $X'$  is a variety in this solution, which I think is true from intuition but I don't know how to prove. Assuming this fact, the effect of the construction of  $\pi$  is to construct a variety birational to  $X$  with codimension 1 less than  $X$ . By repeating such constructions, we explicitly get a birational equivalence between any variety of dimension  $r$  and a hypersurface in  $\mathbb{P}^{r+1}$ , recovering Proposition 4.9.

**Ex.4.10.** (Blow-up) Let  $x, y$  be affine coordinates on  $\mathbb{A}^2$ , let  $t, u$  be projective coordinates on  $\mathbb{P}^1$ . Let  $X \subseteq \mathbb{A}^2 \times \mathbb{P}^1$  be the blow up of  $\mathbb{A}^2$  at  $(0, 0)$ , and let  $\varphi : X \rightarrow \mathbb{A}^2$  be projection. Then  $X = Z(xu - yt)$ ,  $\varphi^{-1}(Y) = Z(xu - yt, y^2 - x^3)$ . Because  $Y - (0, 0)$  contains no point on the  $y$ -axis, then  $\varphi^{-1}(Y - (0, 0))$  is in the affine chart  $t \neq 0$ , so we can assume  $t \neq 0$ , then we can assume  $t = 1$  and let  $u$  be an affine coordinate, and we get two equations  $y = xu, y^2 = x^3$ . Then  $x^2u^2 = x^3$ . If  $x = 0$ , then  $y = 0$  and  $u$  is arbitrary, so we get the exceptional line  $E$ . If  $x \neq 0$ ,  $u^2 = x$ , so  $\tilde{Y} = Z(u^2 - x, y - xu) = Z(u^2 - x, u^3 - y) \subseteq \mathbb{A}^3$ . (Note  $\tilde{Y}$  is twisted cubic curve.) To get  $\tilde{Y} \cap E$ , set  $x = y = 0$ , then  $u^2 = 0$ , so  $u = 0$ , so  $\tilde{Y} \cap E = (0, 0, 0)$  is a single point. Let  $\psi : \tilde{Y} \rightarrow \mathbb{A}^1$  be  $(x, y, u) \mapsto u$  then  $\psi$  is obviously an isomorphism between  $\tilde{Y}$  and  $\mathbb{A}^1$ .

## 5 Nonsingular Varieties

**Ex.5.1.** It is straightforward to verify that all four polynomials are irreducible. Denote the polynomial in each problem by  $f$ , then according to the definition of singular points on affine variety, we need to have  $f_x, f_y = 0$ .

(a) We want  $2x - 4x^3 = -4y^3 = x^2 - x^4 - y^4 = 0$ . Since  $\text{char } k \neq 2, 4 \neq 0$ , then  $y = 0$ . If  $x \neq 0$  then  $1 - 2x^2 = 1 - x^2 = 0$ , impossible. So  $x = 0$ . The graph corresponds to "tacnode".

(b) We want  $y - 6x^5 = x - 6y^5 = xy - x^6 - y^6 = 0$ . Then  $y = 6x^5$ , so  $x = 6^6x^{25}$ .  $5x^6 = 6^6x^{30}$ . If  $x \neq 0$  then  $1 = 5$ , impossible. So  $x = y = 0$ . This corresponds to "node".

(c) We want  $3x^2 - 4x^3 = -2y - 4y^3 = x^3 - y^2 - x^4 - y^4 = 0$ . If  $y \neq 0$ , then  $y^2 = -\frac{1}{2}$ , so  $x^3 - x^4 = -\frac{1}{4}$ . We also have  $3x^3 - 4x^4 = 0$ , so we see  $x^3 = -1$ ,  $x^4 = -\frac{3}{4}$ , so  $x = \frac{3}{4}$ , then  $x^3 = \frac{27}{64} = -1$ , so  $\frac{91}{64} = 0$ , so  $\text{char } k = 7$  or  $13$ . If  $\text{char } k \neq 7$  and  $\text{char } k \neq 13$ , we have  $y = 0$ , then if  $x \neq 0$ ,  $x = 1$ , so  $3 - 4 = 0$ , impossible, so  $x = y = 0$ .

Therefore, when  $\text{char } k = 7$  or  $13$ , singular points are  $\{(0, 0), (\frac{3}{4}, y_1), (\frac{3}{4}, y_2)\}$  where  $y_1, y_2$  are roots of  $y^2 + \frac{1}{2} = 0$ . Otherwise, the origin is the only singular point. The graph corresponds to "cusp".

(d) We want  $2xy + y^2 - 4x^3 = 2xy + x^2 - 4y^3 = x^2y + xy^2 - x^4 - y^4 = 0$ . Multiply the first equation by  $x$  and subtract from the third equation we get  $x^2y - 3x^4 + y^4 = 0$ . So  $x^2y = 3x^4 - y^4 = x^4 + y^4 - xy^2$ , so  $2x^4 - 2y^4 = -xy^2$ . Multiply the second equation by  $y$  and subtract from the third equation we get  $xy^2 - 3y^4 + x^4 = 0$ . So  $xy^2 = 3y^4 - x^4 = x^4 + y^4 - x^2y$ , so  $2y^4 - 2x^4 = -x^2y$ . Thus  $x^2y = xy^2$ , then  $xy(x - y) = 0$ . If  $x = 0$ , then  $y = 0$ . If  $y = 0$ , then  $x = 0$ . If  $x = y$ , then  $x = y = 0$ . Therefore the origin is the only singularity point. The graph corresponds to "triple point".

**Ex.5.2.** It is straightforward to verify that all three polynomials are irreducible. Denote the polynomial



in each problem by  $f$ , then according to the definition of singular points on affine variety, we need to have  $f_x, f_y, f_z = 0$ .

(a) We want  $y^2 = 2xy = -2z = xy^2 - z^2 = 0$ . Then  $y = z = 0$ ,  $x$  is arbitrary. This corresponds to "pinch point".

(b) We want  $2x = 2y = -2z = x^2 + y^2 - z^2 = 0$ . Then  $x = y = z = 0$ . This corresponds to "conical double point".

(c) We want  $y + 3x^2 = x + 3y^2 = xy + x^3 + y^3 = 0$ . If  $\text{char } k = 3$ , then  $x = y = 0$ ,  $z$  is arbitrary. Assume  $\text{char } k \neq 3$ . Then  $y = -3x^2$ , so  $x(1 + 27x^3) = x^3(2 + 27x^3) = 0$ . So either  $x = y = 0$  and  $z$  is arbitrary, or  $x \neq 0$ . But if  $x \neq 0$ , then  $-1 = 2$ , impossible. So singular points are  $x = y = 0$ ,  $z$  arbitrary, This corresponds to "double line".

**Ex.5.3.** (a) Suppose  $Y = Z(f)$  and  $\mu_P(Y) = 1$ . Write  $P = (P_1, P_2)$ . Let  $g = f(x + P_1, y + P_2)$ , then  $g_0 = 0$ ,  $g_1 \neq 0$ . This implies  $P \in Y$  and  $\text{rank}(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y})_{(0,0)} = 1$ , so  $\text{rank}(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})_{(P_1, P_2)} = 1$ , so  $P$  is a nonsingular point of  $Y$ . The reverse direction follows by reversing the above arguments.

(b) For the tacnode,  $\mu_0(Y) = 2$ . For the node,  $\mu_0(Y) = 2$ . For the cusp, if  $\text{char } k \neq 7$  and  $\text{char } k \neq 13$ , then 0 is the only singularity and  $\mu_0(Y) = 2$ . Otherwise there are three singular points described in Ex5.1(c), all with multiplicity 2. For the triple point,  $\mu_0(Y) = 3$ .

**Ex.5.5.** When  $p \nmid d$ ,  $x^d + y^d + z^d$  gives such curve. Indeed, we view this curve via the affine chart  $x \neq 0$ , then the equation of the curve in  $\mathbb{A}^2$  is  $1 + y^d + z^d$ . If both partial derivatives are 0, then  $y = z = 0$ , but no point on the curve satisfies  $y = z = 0$ , so  $Z(1 + y^d + z^d)$  is a nonsingular affine curve. Using the other affine charts and the intrinsic description of singularity, we see that  $Z(x^d + y^d + z^d) \subset \mathbb{P}^2$  is nonsingular.

When  $p \mid d$ ,  $zx^{d-1} + y^d + yz^{d-1}$  gives such curve. Affinize using the affine chart  $x \neq 0$ , the curve becomes  $Z(z + y^d + yz^{d-1}) \subset \mathbb{A}^2$ . Setting the partial derivative w.r.t.  $y$  to 0, we get  $z = 0$ . Then setting the partial derivative w.r.t  $z$  to 0 we get  $1 + (d-1)yz^{d-2} = 0$ . When  $d > 2$  this is impossible, and when  $d = 2$ , we get  $y = -1$ , but  $(y, z) = (-1, 0)$  is not on the curve. Similar arguments show that, under other affine charts, the curve is nonsingular. By the intrinsic description of singularity, we see that  $Z(zx^{d-1} + y^d + yz^{d-1}) \subset \mathbb{P}^2$  is nonsingular.

**Ex.5.8.** Let  $M = (\partial f_i / \partial x_j(a_0, \dots, a_n))_{i,j}$ .  $\text{rank } M$  is independent of the homogeneous coordinates of  $P$  because entries of  $i$ -th row of  $(\partial f_i / \partial x_j)_{i,j}$  are homogeneous polynomials of degree  $\deg f_i - 1$ . Thus, using another homogeneous coordinate will multiply the original row of  $M$  by a nonzero scalar, which does not change the (row) rank of the whole matrix. We also note that  $\text{rank } M$  is also independent of the choice of generators  $(f_i)$ . The reason is similar as the one for the affine case.

WLOG, suppose  $P \in U_0$ . Let  $\varphi_0 : \mathbb{A}^n \rightarrow U_0$  be the canonical isomorphism. Then  $Y_0 := \varphi_0^{-1}(Y \cap U_0)$  is an affine variety of dimension  $r$ . Let  $M_0 = (\partial((f_i)_a) / \partial x_j(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}))_{i,j}$  where  $1 \leq j \leq n$  and  $(f_i)_a$  is the affinization of  $f_i$  w.r.t  $x_0$ . Note  $M$  has  $n+1$  columns while  $M_0$  has  $n$  columns. Note we have  $I(Y_0) = ((f_1)_a, \dots, (f_t)_a)$ . Also note that for  $j > 0$ ,  $\partial((f_i)_a) / \partial x_j(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}) = (\partial f_i / \partial x_j)_a(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}) = \frac{1}{a_0^{\deg f_i - 1}} (\partial f_i / \partial x_j)(a_0, \dots, a_n)$ .

Finally note that by Euler's lemma (which is easy to prove) we have  $\sum_{j=0}^n x_j (\partial f_i / \partial x_j) = \deg f_i \cdot f_i$ . Evaluating this equation at  $(a_0, \dots, a_n)$  we get  $a_0 (\partial f_i / \partial x_0)(a_0, \dots, a_n) = - \sum_{j=1}^n a_j \cdot (\partial f_i / \partial x_j)(a_0, \dots, a_n)$ , so we see the first column of  $M$  is a linear combination of the rest  $n$  columns of  $M$ , so  $\text{rank } M$  is equal to  $\text{rank}$  of the last  $n$  columns of  $M$ , which is equal to  $\text{rank}$  of  $M_0$ .

Therefore,  $P$  is nonsingular on  $Y$  iff  $\varphi_0^{-1}(P)$  is nonsingular on  $Y_0$  iff  $\text{rank } M_0 = n - r$  iff  $\text{rank } M = n - r$ .

**Ex.5.9.** FSO, suppose  $f$  is reducible. Let  $f_1, f_2$  be two irreducible factors of  $f$ . Suppose  $f = f_1 f_2 g$  where  $g \in k[x, y, z]$  homogeneous. By Ex.3.7,  $Z(f_1) \cap Z(f_2) \neq \emptyset$ , so we can pick  $P \in Z(f_1) \cap Z(f_2)$ .  $(\partial f / \partial x)(P) = (\partial f_1 / \partial x)(P) \cdot (f_2 g)(P) + f_1(P) \cdot (\partial f_2 g / \partial x)(P) = 0$ . Similarly, all other partial derivatives of  $f$  at  $P$  is 0, contradiction with assumption, so  $f$  must be irreducible.

**Ex.5.10.** (a)  $\dim T_P(X) = \dim \mathfrak{m} / \mathfrak{m}^2 \geq \dim \mathcal{O}_P = \dim X$ , where the second inequality is true by proposition 5.2A. Then equality holds if and only if  $P$  is nonsingular.

(b) Denote the maximal ideal of  $\mathcal{O}_{P,X}$  by  $\mathfrak{m}_1$ . Denote the maximal ideal of  $\mathcal{O}_{\varphi(P),Y}$  by  $\mathfrak{m}_2$ .  $\varphi$  induces a

$k$ -algebra homomorphism  $\varphi^* : \mathcal{O}_{\varphi(P),Y} \rightarrow \mathcal{O}_{P,X}$ . Because pullback of a regular function vanishing on  $\varphi(P)$  vanishes on  $P$ ,  $\varphi^*(\mathfrak{m}_2) \subseteq \mathfrak{m}_1$ . Because  $\varphi^*$  is ring homomorphism,  $\varphi^*(\mathfrak{m}_2^2) \subseteq \mathfrak{m}_1^2$ . Thus  $\varphi^*$  induces  $k$ -linear map:  $\frac{\mathfrak{m}_2}{\mathfrak{m}_2^2} \rightarrow \frac{\mathfrak{m}_1}{\mathfrak{m}_1^2}$ . Its dual map is the natural induced  $k$ -linear map  $T_P(\varphi) : T_P(X) \rightarrow T_{\varphi(P)}(Y)$ .

(c) Let  $X = Z(x - y^2)$ ,  $Y = Z(y)$ . Let 0 denote the origin. Then  $\mathcal{O}_{0,Y} = \mathcal{O}_{0,\mathbb{A}^1} = k[x]_{(x)}$  where the second step is true by Theorem 3.2(c). We know localization of Dedekind domain at nonzero prime ideal is discrete valuation ring, and here the generator of maximal ideal of  $\mathcal{O}_{0,Y}$  is  $x$ . Then  $\varphi^*(x) = x \circ \varphi = x = y^2 \in \mathfrak{m}_1^2$ , so  $\varphi^*$  induces the zero map:  $\frac{\mathfrak{m}_2}{\mathfrak{m}_2^2} \rightarrow \frac{\mathfrak{m}_1}{\mathfrak{m}_1^2}$ . So its dual map  $T_0(\varphi)$  is also zero map.

**Ex.5.14(a).** It suffices to prove for  $P = Q = (0,0)$ . Assume  $Y = Z(f)$  and  $Z = Z(g)$ , then by assumption  $\frac{k[[x,y]]}{(f)} \cong \frac{k[[x,y]]}{(g)}$ . By some algebra we get  $f$  and  $g$  should have the same order, so  $\mu_P(Y) = \mu_Q(Z)$ .

(b)

**Ex.5.15.** (a) Given a point  $P = (p_0 : \dots : p_N) \in \mathbb{P}^N$ , let  $f_P$  be the homogeneous polynomial of degree  $d$  with coefficients  $(p_0, \dots, p_N)$ . Then  $Z(f_P)$  is well defined i.e. it does not depend on the choice of representative of coordinate of  $P$ . This is the algebraic set we want. Conversely, given an algebraic set  $Y = Z(g)$  where  $g$  has degree  $d$ , we note that the equation defining  $Y$  is unique up to unit when  $g$  has distinct factors. Indeed,  $Y = Z(g) = Z(f)$  where  $\deg f = d$ , then  $I(Y) = \sqrt{(g)} = \sqrt{(f)}$ , so  $g|f^n$  for some  $n \geq 1$ . Since  $g$  has distinct factors,  $g|f$ . Since  $\deg g = \deg f$ , we have  $g = uf$  where  $u$  is a unit. Thus we get a unique point in  $\mathbb{P}^N$  from an algebraic set in  $\mathbb{P}^2$  which can be defined using a degree  $d$  polynomial with distinct factors. Such correspondence is obviously one-to-one when we restrict to points in  $\mathbb{P}^N$  which correspond to degree  $d$  polynomials with distinct factors.

We note that this correspondence cannot be extended to all algebraic sets which can be defined using a degree  $d$  polynomial. For example let  $d = 3$ , then  $x^2y$  and  $xy^2$  define the same algebraic set, but they correspond to distinct points in  $\mathbb{P}^N$ .

(b) Let  $C$  be a nonsingular curve of degree  $d$ , then  $C = Z(f)$  for some irreducible, homogeneous polynomial  $f$  of degree  $d$ . Let  $P_f \in \mathbb{P}^N$  be the point formed by coefficients of  $f$ . Because  $f$  has distinct factors, by part (a) we only need to show that  $P_f$  forms a nonempty open subset of  $\mathbb{P}^N$  when  $f$  ranges over all irreducible polynomials of degree  $d$  that define nonsingular curves. By Ex.5.5, such set is nonempty.

Given a homogeneous  $f \in k[x, y, z]$  of degree  $d$ , we have the following equivalence:

$$f \text{ irreducible and } Z(f) \text{ nonsingular} \iff Z(f, \partial f / \partial x, \partial f / \partial y, \partial f / \partial z) = \emptyset \iff (a_{ij}) \notin Z(g_1, \dots, g_t)$$

where  $a_{ij}$  are coefficients of  $f, \partial f / \partial x, \partial f / \partial y, \partial f / \partial z$ , and  $g_1, \dots, g_t$  are polynomials with integer coefficients in  $(a_{ij})$  and homogeneous in coefficients of each of  $f, \partial f / \partial x, \partial f / \partial y, \partial f / \partial z$  separately. The first equivalence follows from Ex.5.8 and Ex.5.9, and the second from Theorem 5.7A.

Theorem 5.7A outputs the  $g_i$  which have more than  $N$  indeterminates, but coefficients of  $\partial f / \partial x, \partial f / \partial y, \partial f / \partial z$  can be expressed using coefficients of  $f$ , so we can write  $g_i$  as integer-coefficient polynomials with exactly  $N$  indeterminates, thus the complement of their zero set is open in  $\mathbb{P}^N$ , so we are done.