Sum of 4 Squares

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Outline

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Statement of Problem

- We are interested in the number of ways to write an integer n as sum of four squares, $r_n := |\{(x, y, z, w) \in \mathbb{Z}^4 \mid n = x^2 + y^2 + z^2 + w^2\}|$ for each integer n.
- Lagrange proved the existence of such representation in 1770.

Statement of the Problem in Complex Fourier Series

- Let $q := e^{2\pi iz}$ where $z \in H := \{z \in \mathbb{C} | Im\, z > 0\}$, let $\Theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi izn^2} = \sum_{n\in\mathbb{Z}} q^{n2}$ (Note the series converges uniformly on compact subsets of H).
- To calculate r_n is equivalent to calculating coefficients of the Series $\Theta^4(z) = \sum_{n\in\mathbb{N}} r_n q^n$.

Group action of $SL_2(\mathbb{Z})$ and its congruence subgroups on H

- $\Gamma := SL_2(\mathbb{Z}) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid ad - bc = 1\}$
- $H := \{z \in \mathbb{C} \mid Im(z) > 0\}$
- Some subgroups of Γ
  - principle subgroup of level N, $\Gamma(N) := \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N\}$
  - $\Gamma_1(N) := \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N\}$ (a congruence subgroup of level N)
  - $\Gamma_0(N) := \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N\}$ (a congruence subgroup of level N)
- We will be interested in the subgroup $\Gamma_0(4)$
- Group action of $SL_2(\mathbb{Z})$ (and its subgroups) on H:
  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$ (Fractional Linear Transformation)

Modular/Cusp form for congruence subgroup of level N

Let $q_N = e^{2\pi iz}$. $f(z)$ be holomorphic on H, and let $\Gamma' \subset \Gamma$ be a congruence subgroup of level N. We call f a modular form of weight k for Γ' $(M_k(\Gamma'))$if $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'$, $f(\gamma z) = f(z)(cz+d)^k$, and if $\forall \gamma_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in \Gamma$, the q_N-expansion of $f(\gamma_0 z)(c_0 z + d_0)^{-k}$ has 0 coefficients in negative powers (This is equivalent to saying that this function goes to a finite number as $z \to i\infty$). Furthermore, we say f is a cusp form of weight k $(S_k(\Gamma'))$if the q_N-expansion has 0 coefficients in non-positive powers (This is equivalent to saying that this function goes to 0 number as $z \to i\infty$)

$M_2(\Gamma_0(4))$

- It can be shown that $\Theta^4 \in M_2(\Gamma_0(4))$, and $F := \sum_{\text{n odd}} \sigma_1(n)q^n \in S_2(\Gamma_0(4))$. It can also be shown that $dim_{\mathbb{C}}(M_2(\Gamma_0(4))) = 2$, thus Θ^4, F is a basis for $M_2(\Gamma_0(4))$.
- By defining Hecke operators T_n on $M_2(\Gamma_0(4))$, we can show that for n odd, T_n is just multiplication by $\sigma_1(n)$, and Θ^4 is eigenvector of T_n with eigenvalue $\frac{1}{8}r_n$. Thus $r_n = 8\sigma_1(n)$.

The Result

- $r_n = \begin{cases} 8\sigma_1(n), \text{n odd} \\ 24\sigma_1(n_0), n = 2^k n_0, 2 \nmid n_0, \text{ n even} \end{cases}$