C^∞ Manifold: Grassmannian $G(k, n)$

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Dec 3rd, 2021

Outline

Topological Space $G(k, n)$

- Intuition of $G(k, n)$

k-dimensional subspaces of $\mathbb{R}^n$, which can be described by k linearly independent vectors in $\mathbb{R}^n$.

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- $F(k, n)/\sim$

Different matrices in $F(k, n)$ may represent the same subspace, up to a change of basis. So for $A, B \in F(k, n)$, let $A \sim B$ if $\exists g \in GL(k, \mathbb{R})$ such that $A = Bg$. g is the change of basis matrix.

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- Definition of $G(k, n)$

Let $G(k, n) = F(k, n)/\sim$, with the quotient topology.

Smooth (C^∞) Manifold

Let's recall definition of smooth manifold:

- Topological Space which is
  - Hausdorff
  - Second Countable
- Existence of C^∞ atlas

We will prove that $G(k, n)$ (topologized as before) is smooth manifold.

$G(k, n)$ Second Countable

- $\sim$ is open equivalence relation

$\forall U$ open in $F(k, n)$, $\pi^{-1}(\pi(U)) = \bigcup_{g \in GL(k,\mathbb{R})} Ug$. Right multiplication by g is homeomorphism on $F(k, n)$ since coordinate-wise, it is just multipication and addition. So each Ug is open, so the union is open.

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- Countable basis in $F(k, n)$ gives countable basis in $G(k, n)$

$G(k, n)$ Hausdorff

Theorem

Let S be a topological space, let $\sim$ an open equivalence relation on S, then $S/\sim$ Hausdorff if and only if the graph R of the equivalence relation $\sim$ is closed in $S \times S$. $(R := \{(x, y) \in S \times S | x \sim y\})$

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Let's show graph R of $\sim$ for $F(k,n)$ is closed:
$R = \{(a,b) \in F(n,k) \times F(n,k) | \exists g \in GL(k,\mathbb{R}), a = bg\}$
$= (\mathrm{F(n,k)} \times F(n,k)) \cap \{A \in M(n,2k) | rk\ A \leq k\}$.
$\{A \in M(n,2k) | rk\ A \leq k\}$ is closed in $\mathbb{R}^{n\times 2k}$ since $rk\ A \leq k$ if and only if all $(k+1)\times(k+1)$ minors of A vanish.

Open Covering of $G(k, n)$

First let's find an open covering of $F(k, n)$.
Let $I = \{1 \leq i_1 < i_2 < ... < i_k \leq n\}$ be a multi-index. For $A \in F(k, n)$, let A_I denote the submatrix of A consisting of $i_1 th, ..., i_k th$ rows of A. Let $V_I = \{A \in F(k, n) | det(A_I) \neq 0\}$, then $\{V_I\}$ forms an open cover of $F(k, n)$. Let $U_I = \pi(V_I)$, then $\{U_I\}$ is an open cover of $G(k, n)$.

C^∞ atlas on $G(k, n)$

First define $\tilde{\phi}_I : V_I \to \mathbb{R}^{(n-k)\times k}$ by $\tilde{\phi}_I(A) = (AA_I^{-1})_{I'}$ where I' is complement of I. Obviously $\tilde{\phi}_I$ is surjective. $\forall A, B \in V_I$, if $A = Bg$ for some $g \in GL(k, \mathbb{R})$, $\tilde{\phi}_I(A) = (AA_I^{-1})_{I'} = ((Bg)(Bg)_I^{-1})_{I'} = ((Bg)(B_I g)^{-1})_{I'} = (Bgg^{-1}B_I^{-1})_{I'} = (BB_I^{-1})_{I'} = \tilde{\phi}_I(B)$, so $\tilde{\phi}_I$ induces $\phi_I : U_I \to \mathbb{R}^{(n-k)\times k}$, which is injective: $(AA_I^{-1})_{I'} = (BB_I^{-1})_{I'}$ implies $A \sim B$.

We get an atlas $\{(\phi_I, U_I)\}$, and conclude that $G(k, n)$ is a $k(n-k)$ dimensional C^∞ manifold.

Thank you!