Hartshorne Chapter I Exercise Solution

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January 11, 2023

This is my ongoing attempt to solve exercises in the first chapter of Hartshorne. The solution to the first three sections is complete. I worked out most (more than 90%) exercises here by myself with no help from other people or sources. For the other exercises, I consulted online sources mostly for results from algebra, and tried my best to complete other parts of the exercises. I have given references to sources which have helped me whenever possible. I have also proved some lemmas which are useful and can be separated from particular exercises.

In this document, $\subset$ means strict set containment, and $\subseteq$ is any set containment. "WLOG" means "without loss of generality". "FSOC" means "for sake of contradiction". "iff" means "if and only if". $K(R)$ means field of fraction of the integral domain R (not to be confused with function field of a variety Y, $K(Y)$). Unless stated otherwise, $A = k[x_1, ..., x_n]$, $S = k[x_0, ..., x_n]$, and k is algebraically closed field of any characteristic. $\langle S \rangle$ denotes ideal generated by the set S. Homogeneous coordinate is represented by $(a_0 : ... : a_n)$. All references like "Theorem" and "Proposition" refer to Hartshorne's Algebraic Geometry.

If you have any question, advice, or correction, you can email me at jerry14790@outlook.com.

1 Affine Varieties

Ex.1.1. We see $y - x^2$ and $xy - 1$ are irreducible polynomials by viewing them as polynomials in y with coefficients in $k[x]$, then look at degree (in y) of possible factors.
(a) $A(Y) = k[x,y]/(y-x^2) \cong k[t]$ where the isomorphism is induced by $k[x,y] \to k[t]$ defined by $x \mapsto t, y \mapsto t^2$.
(b) $A(Z) = k[x,y]/(xy-1) \cong k[t,t^{-1}]$ where the isomorphism is induced by $k[x,y] \to k[t,t^{-1}]$ defined by $x \mapsto t, y \mapsto t^{-1}$. Because t, t^{-1} are units in $k[t,t^{-1}]$, any ring homomorphism $k[t,t^{-1}] \to k[t]$ must send t and t^{-1} to units, so the image of this ring homomorphism is k, so in particular $k[t] \ncong k[t,t^{-1}]$.
(c) Write $f = ax^2 + bxy + cy^2 + dx + ey + g$ irreducible. If $a = c = 0$, then $b \neq 0$. Since $A(W) = k[x,y]/(f)$ is invariant under scaling f by a unit, we can assume $b = 1$. Then $f = (x+e)(y+d) + (g - ed)$. Do a change of variable $k[x',y'] \to k[x,y]$ by $x' \mapsto x + e, y' \mapsto y + d$, then this isomorphism $k[x',y'] \cong k[x,y]$ induces isomorphism $\frac{k[x',y']}{(x'y'+(g-ed))} \cong A(W)$. Note $g - ed \neq 0$ because f is irreducible. Then we have $\frac{k[x',y']}{(x'y'+(g-ed))} \cong k[t,t^{-1}]$ by the argument in part (b).
If $a \neq 0$ or $c \neq 0$, WLOG assume $c \neq 0$, then we can assume $c = 1$. Temporarily working in fractional field of $k[x,y]$, we have $ax^2 + bxy + y^2 = x^2(a + b\frac{y}{x} + (\frac{y}{x})^2) = x^2(\frac{y}{x} - u_1)(\frac{y}{x} - u_2) = (y - u_1x)(y - u_2x)$ where u_1 and u_2 are two roots of the polynomial $z^2 + bz + a$. Our classification of $A(W)$ will depend on whether $u_1 = u_2$.
If $u_1 \neq u_2$, then we let $x' = y - u_1x, y' = y - u_2x$, then $k[x',y'] \cong k[x,y]$ because we can inverse the equations and write x, y in terms of x', y'. Under this isomorphism f becomes $x'y' + c_1x' + c_2y' + c_3 = (x'+c_2)(y'+c_1) + (c_3 - c_1c_2)$ for some $c_i \in k$, where $c_3 - c_1c_2 \neq 0$ because f is irreducible. This isomorphism induces $A(W) \cong \frac{k[x',y']}{(x'+c_2)(y'+c_1)+(c_3-c_1c_2)} \cong \frac{k[x'',y'']}{x''y''+(c_3-c_1c_2)} \cong k[t,t^{-1}]$ where we have done another change of variables in the second step.
If $u_1 = u_2$, then let u be $u = u_1 = u_2$, and $f = (y - ux)^2 + dx + ey + g$. Let $x' = y - ux, y' = -(dx + ey)$, then $f = x'^2 - y' + g$. We note that x and y can be expressed in linear homogeneous polynomial of x'

and y' because $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -u & 1 \\ -d & -e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} -u & 1 \\ -d & -e \end{pmatrix}$ is invertible, since if its determinant $eu + d = 0$, $f = (y - ux)^2 + e(y - ux) + g$ is reducible. Thus we have $k[x, y] \cong k[x', y']$, which induces $A(W) \cong \frac{k[x',y']}{(x'^2 - y' + g)} \cong k[x']$ where the last isomorphism is induced by $x' \mapsto x', y' \mapsto x'^2 + g$.

Thus we see $A(W)$ is isomorphic of $A(Y)$ or $A(Z)$, and the specific case conditions are described in the course of this proof.

Ex.1.2. We can verify $Y = Z(y - x^2, z - x^3)$. The ideal $(y - x^2, z - x^3)$ is prime, because it is kernel of the map $k[x, y, z] \to k[t]$ given by identity on k, $x \mapsto t$, $y \mapsto t^2$, $z \mapsto t^3$. Then Y is affine variety, and $\dim(Y) = \dim A(Y) = \dim k[t] = 1$.

Ex.1.3. By assumption $Y = Z(x^2 - yz, xz - x)$. We have $Y = Z(x, y) \cup Z(x, z) \cup Z(z - 1, x^2 - y)$. Since (x, y), (x, z), $(z - 1, x^2 - y)$ are prime ideals none of which contain another, $Z(x, y)$, $Z(x, z)$, $Z(z - 1, x^2 - y)$ are irreducible components of Y. Their ideals are (x, y), (x, z), $(z - 1, x^2 - y)$.

Ex.1.4. Let (x, y) be affine coordinates on $\mathbb{A}^2$. Then $Z(x - y)$ is closed in $\mathbb{A}^2$. FSOC, suppose $Z(x - y)$ is closed in $\mathbb{A}^1 \times \mathbb{A}^1$ with product topology. Then there is a basic open set $U \times V$ such that $U \times V \subseteq Z(x - y)^c$. Proper closed sets in $\mathbb{A}^1$ are finite sets, so $U = \mathbb{A}^1 - Y_1$, $V = \mathbb{A}^1 - Y_2$ where Y_1, Y_2 are finite sets in $\mathbb{A}^1$. But k is algebraically closed, so k is infinite, so there exists $a \in \mathbb{A}^1$ such that $a \in U \cap V$. Then $(a, a) \in U \times V \subseteq Z(x - y)^c$ implies $a \neq a$, contradiction. So the topology on $\mathbb{A}^2$ is different from the topology on $\mathbb{A}^1 \times \mathbb{A}^1$.

Ex.1.5. ($\Longrightarrow$) Suppose $B \cong \frac{k[x_1,...,x_n]}{I(Y)}$ as k-algebra, where $Y \subseteq \mathbb{A}^n$ is algebraic set. Then B is finitely generated by $x_1, ..., x_n$ as k-algebra. $I(Y)$ is radical ideal, because if $f^n \in I(Y)$, then $f \in I(Y)$. This implies B has no (nonzero) nilpotent elements.
($\Longleftarrow$) Let $\pi : k[x_1, ..., x_n] \to B$ be a projection. Then $\ker \pi$ is a radical ideal because B has no nonzero nilpotent elements. Let $Y = Z(\ker \pi)$, then $A(Y) = \frac{k[x_1,...,x_n]}{I(Z(\ker \pi))} = \frac{k[x_1,...,x_n]}{\ker \pi} = B$.

Ex.1.6. Let $Y \subseteq X$ be a nonempty open subset of irreducible topological space. If Y is not dense in X, then $X = \overline{Y} \cup Y^c$ is union of two proper closed subsets, contradiction. So Y is dense in X. FSOC, suppose Y is not irreducible, then $Y = C_1 \cup C_2$ where C_1, C_2 are proper closed in Y. Then $C_1 = C_1' \cap Y$, $C_2 = C_2' \cap Y$ where C_1' and C_2' are closed in X, and $C_1 \cup C_2$ is proper closed subset of X because X is irreducible. But then $X = (C_1 \cup C_2) \cup Y^c$ is union of two proper closed subset, contradiction. Also note Y is nonempty by assumption. So Y is irreducible with induced topology.
Next suppose $Y \subseteq X$ is irreducible with induced topology. Then $\overline{Y}$ is nonempty, and if $\overline{Y}$ is not irreducible, $\overline{Y} = Y_1 \cup Y_2$ where Y_1, Y_2 are proper closed in $\overline{Y}$. Then $Y = (Y_1 \cap Y) \cup (Y_2 \cap Y)$ is union of two closed subsets, so one of them must be Y. WLOG, say $Y_1 \cap Y = Y$. But note Y_1 is closed in X as well, so $\overline{Y} \subseteq Y_1$ by definition of closure. So $\overline{Y} = Y_1$, contradiction. So $\overline{Y}$ is irreducible.

Ex.1.7.(a) (i)$\Longrightarrow$(ii): If there is no minimal element, then we can construct a descending chain that never stabilizes.
(ii)$\Longrightarrow$(iii): Obvious.
(iii)$\Longrightarrow$(iv): Same argument as in (i)$\Longrightarrow$(ii).
(iv)$\Longrightarrow$(i): First we have (ii). Then apply (ii) to any descending chain of closed sets.

(b) If an open covering has no finite subcover, then we can construct an ascending chain of open sets which never stabilizes by choosing appropriate open sets (then taking union) from the open sets of the open covering.

(c) Let X be noetherian topological space, $Y \subseteq X$ be a subset with induced topology. Let $D_1 \supseteq D_2 \supseteq ...$ be a descending chain of closed sets in Y. Then $D_i = D_i' \cap Y$ where D_i' is closed in X. For $i \in \mathbb{Z}_+$, let $S_i = \cap_{k=1}^{i} D_k''$, then S_i is a descending chain of closed sets in X, so there exists n such that $S_i = S_n$ for all $i \geq n$. Then $D_i' \supseteq \cap_{k=1}^{n} D_k''$ for all $i \geq n$. Then for all $i \geq n$, $D_i' \cap Y \supseteq (\cap_{k=1}^{n} D_k'') \cap Y$, which implies $D_i \supseteq D_n$ for all $i \geq n$, so Y is noetherian.

(d) Let X be noetherian and Hausdorff. FSOC, suppose X is infinite. Then we can pick $x, y \in X$, $x \neq y$. Because X is Hausdorff, $\exists U, V$ open sets such that $x \in U$, $y \in V$, $U \cap V = \emptyset$. Then $X = U^c \cup V^c$. Because

X is infinite, U^c or V^c is infinite. WLOG, suppose U^c is infinite. By part (c), U^c with its induced topology is noetherian. It is also straightforward to verify that U^c with its induced topology is Hausdorff. Then we can find an infinite closed subset of U^c. Continue this process, we get a descending chain of closed subsets in X which does not stabilize. Contradiction. So X must be finite. Because in a Hausdorff space, each point is closed, and because finite union of closed sets is closed set, we conclude that X has discrete topology.

Ex.1.8. Suppose $H = Z(f)$ where f is irreducible and has positive degree. We have $A = k[x_1, ..., x_n] \supseteq I(Y) + (f) \supset I(Y)$, where the second inclusion is strict because $H \supsetneq Y$. Because ideals of A containing Y naturally correspond to ideals of $A(Y)$, it is quick to verify that $I(Y) + (f) \subseteq A$ correspond to $(\overline{f}) \subseteq A(Y)$ where $\overline{f}$ denotes equivalence class of f in $A(Y)$. $\overline{f} \neq 0$ because $I(Y) + (f) \neq I(Y)$. Because $A(Y)$ is integral domain, $\overline{f}$ is not a zero-divisor. If $\overline{f}$ is a unit, then $(\overline{f}) = A(Y)$, so $I(Y) + (f) = A$, so $Y \cap H = Z(I(Y) + (f)) = Z(A) = \emptyset$, then we are trivially done. So suppose $\overline{f}$ is not a unit in $A(Y)$. Let S_i be any irreducible component of $Y \cap H$. Then $I(S_i)/I(Y) \subseteq A(Y)$ is a minimal prime ideal containing $(\overline{f})$. By Krull's Haupidealsatz (Theorem 1.11A), $I(S_i)/I(Y)$ has height 1. By Theorem 1.8A, we have $\dim \frac{A(Y)}{I(S_i)/I(Y)} = \dim(A(Y)) - 1 = r - 1$. We also have $\frac{A(Y)}{I(S_i)/I(Y)} \cong A/I(S_i) = A(S_i)$, so $\dim S_i = \dim A(S_i) = r - 1$, and we are done.

Ex.1.9. Induction on r. The case $r = 1$ is easy to verify. For a general $r > 1$, suppose $\mathfrak{a} = (f_1, ..., f_r)$. Let $Y_1, ..., Y_m$ be irreducible components of $Z(f_1, ..., f_{r-1})$, then because $Z(\mathfrak{a}) = \cup_{i=1}^m (Y_m \cap Z(f_r))$, each irreducible component of $Z(\mathfrak{a})$ is irreducible component of some $Y_i \cap Z(f_r)$. Suppose $Z(f_r) = \cup_{j=1}^k H_j$ where H_j's are hypersurfaces. Then by a similar argument, each irreducible component of $Y_i \cap Z(f_r)$ is irreducible component of some $Y_i \cap H_j$. By Ex.1.8 and induction hypothesis, dimension of each irreducible component of $Y_i \cap H_j$ is at least $n - r$, so we are done.

Ex.1.10. (a) Given any ascending chain of irreducible closed distinct sets in Y, we can a get a corresponding ascending chain of irreducible closed distinct sets in X by taking closure.
(b) Intersection of any ascending chain of irreducible closed distinct sets $X_0 \subset X_1 \subset X_2 ...$ with an U_i where $U_i \cap X_0 \neq \emptyset$ gives an ascending chain of irreducible closed distinct subsets of U_i.
(c) Let $X = \{1, 2\}$ with closed sets: X, $\{1\}$, and $\emptyset$. Let $U = \{2\}$.
(d) If $Y \neq X$, take an ascending chain of closed irreducible distinct subsets of Y of maximum length. Append X to it, and we see contradiction to $\dim Y = \dim X$.
(e) Let $X = \mathbb{N}$ with closed sets $\{0, 1, ..., n\}$ where $n \in \mathbb{N}$.

Ex.1.11. It's straightforward to verify that $Y = Z(y^3 - x^4, z^3 - x^5, z^4 - y^5)$, so Y is closed in $\mathbb{A}^3$. Let $\varphi : k[x, y, z] \to k[t]$ be identity on k and $x \mapsto t^3$, $y \mapsto t^4$, $z \mapsto t^5$. Then $I(Y) = \ker \varphi$. Indeed, if $f \in I(Y)$, then $\varphi(f)(a) = f(a^3, a^4, a^5) = 0$ for all $a \in k$. k is infinite, so $\varphi(f) = 0$, so $f \in \ker \varphi$. Conversely if $f \in \ker \varphi$, then for all $(a^3, a^4, a^5) \in Y$, $f(a^3, a^4, a^5) = \varphi(f)(a) = 0$. so $f \in I(Y)$. Thus $I(Y) = \ker \varphi$ is a prime ideal. height $I(Y) = \dim k[x, y, z] - \dim \frac{k[x,y,z]}{I(Y)} = 3 - \dim k[t] = 2$. By my answer here: `https://math.stackexchange.com/questions/4365408`, $I(Y)$ cannot be generated by 2 elements, and in fact $I(Y) = (zx - y^2, yz - x^3, yx^2 - z^2)$.

Ex.1.12. Let $f(x) = x^2 + 1$, then $Z(f) = \emptyset$, which is by definition not irreducible.

2 Projective Varieties

Ex.2.1. If $\mathfrak{a} = S$ then the statement is trivial. Suppose $\mathfrak{a} \neq S$, then $0 \in Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$, where $Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$ means the zero set of $\mathfrak{a}$ in the affine space. Let $\psi : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ be the canonical projection map. Then it is easy to verify $\psi^{-1}(Z_{\mathbb{P}^n}(\mathfrak{a})) \cup \{0\} = Z_{\mathbb{A}^{n+1}}(\mathfrak{a})$. Then since $\deg f > 0$ and f is homogeneous, $f(0) = 0$, and then $f(Z_{\mathbb{P}^n}(\mathfrak{a})) = 0$ implies $f(Z_{\mathbb{A}^{n+1}}(\mathfrak{a})) = 0$, and we are done by Hilbert's Nullstellensatz for affine case (Theorem 1.3A).

Ex.2.2. (i⇒ii) Use 2.1. (ii⇒iii) Since $\{x_0, ..., x_n\} \subseteq \sqrt{\mathfrak{a}}$, $\exists k_0, ..., k_n$ such that $\forall i, x_i^{k_i} \in \mathfrak{a}$. Let $d = \sum_{i=0}^n k_i$. Then $\mathfrak{a} \supseteq S_d$ by pigeon hole principle. (iii⇒i) $\mathfrak{a} \supseteq \{x_0^d, x_1^d, ..., x_n^d\}$, so $Z(\mathfrak{a}) = \emptyset$.

Ex.2.3. These are similar to the affine case and easy to verify. Note we need Ex.2.1 for the "$\subseteq$" direction of (d).

Ex.2.4. (a) By 2.3(d), I sends algebraic sets to homogeneous radical ideals. If $\exists$ algebraic set Y such that $I(Y) = S_+$, then applying IZ to both sides we see $I(Y) = S$, contradiction. By 2.3(e), $ZI(Y) = Y$ when Y is algebraic set. For $\mathfrak{a}$ a homogeneous radical ideal not equal to S_+, If $Z(\mathfrak{a}) = \emptyset$, then by 2.2, $\mathfrak{a} = S$, then $IZ(\mathfrak{a}) = \mathfrak{a}$. If $Z(\mathfrak{a}) \neq \emptyset$, then $IZ(\mathfrak{a}) = \mathfrak{a}$ by 2.3(d).
(b) ($\Longrightarrow$) Pick $f, g \in S$ such that $fg \in I(Y)$. Let $\{f_0, ..., f_n\}$ and $\{g_0, ..., g_m\}$ be homogeneous parts of f and g respectively, then $Z(f_0, ..., f_n) \cup Z(g_0, ..., g_m) \supseteq Y$. Indeed, $\forall y \in Y$, because $I(Y)$ is homogeneous, $\forall i,(fg)_i \in I(Y)$, so $(fg)_i(y) = 0$ where $(fg)_i$ means homogeneous component of degree i in fg. But each $(fg)_i$ can also be written as a finite sum of products of components of f and g, so if $y \notin Z(g_0, ..., g_m)$, then pick the smallest i such that $g_i(y) \neq 0$, and consider $(fg)_i(y) = 0$,$(fg)_{i+1}(y) = 0$,..., we see that $y \in Z(f_0, ..., f_n)$.
Then because Y is irreducible, we can suppose $Z(f_0, ..., f_n) \supseteq Y$. Then each $f_i \in I(Y)$ so $f \in I(Y)$.
($\Longleftarrow$) $Y \neq \emptyset$ because otherwise $I(Y) = S$, not a prime ideal. Let C_1, C_2 be two closed subsets of Y such that $Y = C_1 \cup C_2$. Then by 2.3(c) $I(Y) = I(C_1) \cap I(C_2)$. Since $I(Y)$ is prime, WLOG we have $I(C_1) = I(Y)$. Apply Z to both sides and we are done.
(c) $I(\mathbb{P}^n) = (0)$ because any homogeneous polynomial vanishing on $\mathbb{P}^n$ also vanishes on $\mathbb{A}^{n+1}$, and $I(\mathbb{A}^{n+1}) = (0)$. Then use (b).

Ex.2.5. (a) It follows from S is noetherian, the inclusion-reversing function I in 2.4(a), and 2.3(e).
(b) It follows from part(a) and proposition 1.5.

Ex.2.6. To work with dim $S(Y)$ we need to work with its prime ideals, but all the theory we know is about homogeneous ideals. So the first step is to convert to the more familiar affine case.
Using 1.10(b) $\exists U_i$ such that dim Y = dim $U_i \cap Y$. Let $Y_i = \varphi_i(Y \cap U_i)$, then by Proposition 1.7 dim Y = dim $A(Y_i)$. For convenience of notation, assume $i = 0$. The map $k[x_1, ..., x_n] \to S(Y)_{x_0}$ defined by $x_i \mapsto \frac{x_i}{x_0}$ and identity on k induces an embedding $A(Y_0) \hookrightarrow S(Y)_{x_0}$. This furthermore induces isomorphism

$$A(Y_0)[x_0, x_0^{-1}] \cong S(Y)_{x_0} \tag{1}$$

where $x_0 \mapsto \frac{x_0}{1}$.
For any integral domain R, $K(R[x, x^{-1}]) = K(R[x])$ by canonically embedding $K(R[x])$ into $K(R[x, x^{-1}])$. Thus, when R is both integral domain and finitely generated k-algebra and $\dim R$ is finite, using Proposition 1.8A(a) we have $\dim R + 1 = \dim R[x, x^{-1}]$.
Also, for any integral domain and finitely generated k-algebra R, $\forall x \in R \setminus \{0\}$, denote localization of R at x by R_x. Then $\dim R = \dim R_x$, by considering transcendental basis using Proposition 1.8A(a).
Applying the previous two paragraphs to equation (1) we conclude $\dim S(Y) = \dim(Y) + 1$.
Remark: $\forall i$ such that $Y_i \neq \emptyset$, equation (1) holds when 0 is replaced by i, so $\dim Y_i = \dim Y$.

Ex.2.7.(a) Use 2.6.
(b) Note that for any quasi-affine (quasi-projective) variety Y, there can be only one affine (projective) variety containing Y as an open subset, and it is $\bar{Y}$. By 1.10(b), $\exists U_i$ such that $\dim U_i \cap Y = \dim Y$. Then $\dim Y = \dim \varphi_i(U_i \cap Y) = \dim \overline{\varphi_i(U_i \cap Y)} = \dim \varphi_i(U_i \cap \bar{Y}) = \dim U_i \cap \bar{Y} = \dim \bar{Y}$, where we used Proposition 1.10 in the second step and used the remark at the end of 2.6 in the last step.

Ex.2.8.($\Longrightarrow$) $\dim S(Y) = n$ by 2.6. Then by Theorem 1.8A(b), height $I(Y) = 1$. By Proposition 1.12A, $I(Y)$ is principal, say $I(Y) = (f)$. Then $Y = Z(f)$. Because (f) is prime, and because in integral domain non zero prime element is irreducible, f is irreducible (thus f also has positive degree). f is homogeneous, otherwise (f) is not homogeneous.
($\Longleftarrow$) $\dim S(Y) = \dim k[x_0, ..., x_n]/(f) = \dim k[x_0, ..., x_n] - \text{height } (f) = (n+1)-1 = n$, where height $(f) = 1$ by Theorem 1.11A. Then by 2.6, $\dim Y = n - 1$.

Ex.2.9. (a) α and β will denote functions between A and S used in proof of Proposition (2.2). For $\subseteq$ direction, let $f \in S$ be a homogeneous polynomial killing $\bar{Y}$. Then $\forall y \in Y$, $\alpha(f)(y) = f(\varphi_0^{-1}(y)) = 0$, so $\alpha f \in I(Y)$. Then since $f = \beta\alpha(f) \cdot x_0^k$ for some $k \geq 0$, $f \in \langle\beta(I(Y))\rangle$. For $\supseteq$ direction, take $f \in A$ such that f kills Y. Then $\forall y' \in \varphi_0^{-1}(Y)$, $y' = \varphi_0^{-1}(y)$ for some $y \in Y$, and $\beta(f)(y') = \beta(f)(\varphi_0^{-1}(y)) = f(y) = 0$. So $Z(\beta(f))$ is a closed set containing $\varphi_0^{-1}(Y)$, so $\beta(f)$ kills $\overline{Y}$.
Remark: Exactly the same arguments show $I(\varphi_0^{-1}(Y)) = I(\overline{Y})$, and this exercise gives a good description of

the ideal generated by projective closure of an affine variety.

(b) By Ex1.2, $I(Y) = (x_2 - x_1^2, x_3 - x_1^3)$. It is tempting to use part (a) to conclude generators for $I(\overline{Y})$ using generators of $I(Y)$, but as the problem statement emphasizes, this is not the case. Instead, we try to first describe $\overline{Y}$. One obvious description of $\overline{Y}$ is $\overline{\{(1:t:t^2:t^3)|t \in k\}}$, which follows from parametric representation of the twisted cubic in $\mathbb{A}^3$. We have $\overline{Y} = Z(x_0x_3^2 - x_2^3, x_1^2 - x_0x_2, x_1^3 - x_0^2x_3)$. The $\subseteq$ direction is obvious. For $\supseteq$ direction, take any $(a_0 : a_1 : a_2 : a_3)$ which satisfies the equations. If $a_0 \neq 0$, $(a_0 : a_1 : a_2 : a_3) = (1 : \frac{a_1}{a_0} : \frac{a_2}{a_0} : \frac{a_3}{a_0})$ is in Y. Otherwise, $a_0 = a_1 = a_2 = 0$, and we are left with $(a_0 : a_1 : a_2 : a_3) = (0:0:0:a_3)$ which is not in Y but in $\overline{Y}$. Indeed, any homogeneous polynomial f which kills all points of Y must be zero polynomial after the substitution $x_0 \mapsto 1, x_1 \mapsto t, x_2 \mapsto t^2, x_3 \mapsto t^3$ where t is a variable. Because it is homogeneous, there cannot be a monomial only in x_3 in f (otherwise the polynomial after t-substitution is nontrivial). Thus all x_3 in f is in a product with other variables x_i, so $f(0:0:0:a_3) = 0$, so $(0:0:0:a_3) \in \overline{Y}$.
We also know that $\overline{Y}$ is irreducible, because Y in irreducible in the affine space, and closure of irreducible space is irreducible. Therefore $I(\overline{Y}) = \sqrt{(x_0x_3^2 - x_2^3, x_1^2 - x_0x_2, x_1^3 - x_0^2x_3)}$ is a prime ideal. I guess that $(x_0x_3^2 - x_2^3, x_1^2 - x_0x_2, x_1^3 - x_0^2x_3)$ is a prime ideal (which will allow me to conclude the problem), but I do not know how to prove that.

Ex.2.10.(a) $C(Y)$ is zero set of the same set of polynomials whose zero set in projective space is Y. It follows that Y and $C(Y)$ has the same ideal.
(b) $C(Y)$ is irreducible if and only if $I(C(Y))$ is prime ideal if and only if $I(Y)$ is prime ideal if and only if Y is irreducible.
(c) I don't know if the result holds for a general projective algebraic set, but if we assume Y is irreducible, then $\dim C(Y) = \dim A(C(Y)) = \dim S(Y) = \dim Y + 1$ where the second step is true by part (a) and the last step is true by Ex2.6.

Ex.2.11.(a) (i$\Longrightarrow$ii) Say $I(Y) = \langle S \rangle$ where the S is a set of linear polynomials. (We can assume S is finite because $k[x_0, ..., x_n]$ is noetherian, $\langle S \rangle$ is a finitely generated. Each of the generator can be written as a finite sum of products of some element from S with some element from $k[x_0, ..., k_n]$. Put all elements from S which generate all of these generators together, we get a finite set of generators of the ideal $\langle S \rangle$ where each generator is in S.) Then from Ex.2.3(e), $Y = \overline{Y} = ZI(Y) = Z(\langle S \rangle) = Z(S) = \cap_{f \in S} Z(f)$ is an intersection of hyperplanes.
(ii$\Longrightarrow$i)If $Y = \cap_{i \in I} Z(f_i)$ where f_i are linear polynomials, then $Y = Z(\cup_{i \in I} f_i) = Z(J)$ where J is the ideal generated by f_i for all $i \in I$. Then $I(Y) = IZ(J) = \sqrt{J}$. As explained in part (a), J is finitely generated by linear polynomials. Thus it suffices to prove that J is prime.
Thus we will prove the following:
Lemma 1: Any ideal $J \subseteq k[x_0, ..., x_n]$ finitely generated by linear polynomials is prime ideal.
Proof of Lemma 1: Note for each polynomial, the $(n+1)$-tuple of its coefficients can be viewed as an element of the k-vector space k^{n+1}. Let S be the set of these vectors (From now on by "vector" I will mean either the element in k^{n+1} or the corresponding polynomial whose coefficients form this vector, depending on the situation). Eliminate some vectors until they become linearly independent. Then the ideal generated by these vectors is the same as before. Let each vector form a row of a matrix M, and do Gaussian Elimination on the matrix. Call the new matrix M'. Then rows of M' generate the same ideal as rows of M do, because $(f, g) = (f + ag)$ for all $f, g \in k[x_0, ..., x_n]$ and $a \in k - \{0\}$. If M' has rank $n+1$, then the ideal generated by rows is $(x_0, ..., x_n)$, which is maximal ideal. Otherwise, by renaming the variables x_i if necessary, we can assume the matrix is $(I|A)$ where I is identity matrix of size less than $n+1$, and A is any matrix. Thus, the corresponding ideal J is generated by $m+1$ polynomials $(p_i)_{0 \le i \le m}$ where $m < n$ and coefficient of x_i in p_i is 1, and coefficient of x_j in p_i is 0 for j from 0 to m, $j \neq i$.
Next we use polynomial division to prove J is prime. Suppose $fg \in J$. Divide f by p_0 as polynomials in x_0, call the remainder r_0, which is a polynomial involving no x_0. then divide r_0 by p_1 as polynomials in x_1, call the remainder r_1, which is a polynomial involving no x_0, x_1. Continue this process, we get the final remainder r_m, which involves no $x_0, x_1, ..., x_m$. Do the same for g and call the final remainder r'_m. Then multiply f and g using this expansion. Because J is generated by all the p_i and $fg \in J$, we conclude $r'_m r_m \in J$. But r_m and r'_m are polynomials involving no $x_0, x_1, ..., x_m$, and J is generated by all the p_i which each has a coefficient equal to 1 at x_i. Thus $r'_m r_m = 0$ (In more detail, we conclude $r'_m r_m = 0$ by viewing $r'_m r_m$ as a

polynomial in x_i and consider its degree.) Then WLOG $r_m = 0$, then $f \in J$. $\square$

(b) It suffices to prove that for any Y a projective variety, if $I(Y)$ is generated by s linear polynomials, then $\dim Y \geq n - s$. We have $Y = ZI(Y) = \cap_{i=1}^{s} Z(f_i)$ where f_i are linear generators of $I(Y)$. By Ex.1.10(b), $\exists i$ such that $\dim Y = \dim Y_i$ where $Y_i = Y \cap U_i$. For sake of convenience of notation, assume $i = 0$. Then $\varphi_0(Y_0) \subseteq \mathbb{A}^n$ is an affine variety of dimension equal to $\dim Y$. We claim $\varphi_0(Y_0) = Z(\alpha(f_1), ..., \alpha(f_s))$ where α is defined in proof of Proposition 2.2. The proof is indeed just using definitions and is easy to check. Then by Ex1.9, $\dim \varphi_0(Y_0) \geq n - s$, therefore $\dim Y \geq n - s$.
Remark: I think this problem shows that intersection theory works easier in the affine case than in the projective case (maybe because we have extra details and more complicated structure to worry about in projective geometry than in affine geometry), thus it is a good idea to convert problems in projective space to problems in affine space.
Remark: Extension to any projective variety? Algebraic set?

(c) Before anything, we first prove one important observation for this problem, which is beautiful in itself:
Lemma 2: View $\mathbb{A}^n$ as a k-vector space, then any nonempty k-linear subspace V of $\mathbb{A}^n$ is affine variety, and $\dim V = \dim_k V$, where the first "dim" means dimension of topological space (with induced Zariski topology), and the second "dim" means dimension of vector space.
Proof of Lemma 2: Let $v_1, ..., v_m$ be a basis of V as k-vector space. Then $m \leq n$. We can also assume $m < n$, otherwise there is nothing to prove. Let M be a m-by-n matrix where i-th row is v_i. Let $\varphi : k^n \to k^m$ be the linear map represented by M (with respect to the standard basis). rank$M = m$, so φ is surjective. From linear algebra, $\dim \ker \varphi + \dim k^m = \dim k^n$, so $\dim \ker \varphi = n - m$. Let $w_1, ..., w_{n-m}$ be a basis of $\ker \varphi$. Let $f_1, ..., f_{n-m} \in k[x_1, ..., x_n]$ be linear polynomials where coefficient of f_i is the vector w_i. Then we claim

$$\mathrm{span}_k(v_1, ..., v_m) = Z(f_1, ..., f_{n-m}). \tag{2}$$

To prove (2), first note because f_i are linear polynomials, the RHS of (2) is a k-subspace of $\mathbb{A}^n$. The $\subseteq$ direction is true because each v_i is killed by all the f's as the coefficients of f's are in $\ker \varphi$. To prove $\supseteq$, note RHS is exactly kernel of the linear map represented by an $n - m$-by-n matrix whose rows are w_i, and this linear map is surjective because the matrix has full rank. Thus, dimension of this kernel (as k-vector space) is equal to $\dim k^n - \dim k^{n-m} = m$. Because both LHS and RHS of (2) as dimension m and LHS is a subspace of RHS, by linear algebra conclude the equality.
As proved in (b), $(f_1, ..., f_{n-m})$ is prime ideal, so $V = \mathrm{span}_k(v_1, ..., v_m) = Z(f_1, ..., f_{n-m})$ is an affine variety. Therefore $\{0\} \subset \mathrm{span}_k(v_1) \subset \mathrm{span}_k(v_1, v_2) \subset ... \subset \mathrm{span}_k(v_1, ..., v_m)$ is a chain of distinct irreducible closed subsets of V, so $\dim V \geq \dim_k V$. For the other inequality, note that using Theorem 1.8A(b) and height$(f_1, ..., f_{n-m}) \geq n - m$ (as a result of Lemma 1) we have $\dim V = \dim k[x_1, ..., x_n]/(f_1, ..., f_{n-m}) = n - \mathrm{height}(f_1, ..., f_{n-m}) \leq n - (n - m) = m = \dim_k V$. Above all, $\dim V = \dim_k V$. $\square$
Now we prove part(c). Assume Y and Z are linear varieties, $\dim Y = r$, $\dim Z = s$, and $r + s - n \geq 0$. By Ex.2.10, $C(Y)$ and $C(Z)$ are affine varieties, $\dim C(Y) = r + 1$ and $\dim C(Z) = s + 1$. Thus $\dim C(Y) + \dim C(Z) \geq \dim \mathbb{A}^{n+1} + 1$. Also by Ex.2.10, $C(Y)$ and $C(Z)$ are zeros sets of linear polynomials (the same polynomials as those which define Y and Z). Thus $C(Y)$ and $C(Z)$ are linear k-subspaces of $\mathbb{A}^{n+1}$. Apply Lemma 2, we have $\dim_k C(Y) + \dim_k C(Z) \geq \dim_k k^{n+1} + 1$. If $C(Y) \cap C(Z) = 0$, then $\dim_k C(Y) + \dim_k C(Z) = \dim_k C(Y) \oplus C(Z) \leq \dim_k k^{n+1} = n + 1$, so $C(Y) \cap C(Z) \supset \{0\}$. This implies $Y \cap Z \neq \emptyset$. Furthermore, if $Y \cap Z \neq \emptyset$, then first $Y \cap Z$ is a projective variety because it is the zero set of a finite set of linear polynomials (finiteness comes from explanation in part (a)) and the ideal generated by finitely many linear polynomials is prime ideal by Lemma 1. Finally, $\dim Y \cap Z = \dim C(Y \cap Z) - 1 = \dim C(Y) \cap C(Z) - 1 = \dim_k C(Y) \cap C(Z) - 1 = (\dim_k C(Y) + \dim_k C(Z) - \dim_k(C(Y) + C(Z))) - 1 \geq (r + 1) + (s + 1) - (n + 1) - 1 = r + s - n$, where we have used the fact from linear algebra that for U, W k-sub-vector spaces of V and U, W finite-dimensional, $\dim_k(U + W) = \dim_k U + \dim_k W - \dim_k U \cap W$.

Ex.2.12. As an aside, the example of "conic" in problem statement is equal to $Z(x_1^2 - x_0 x_2)$. $x_1^2 - x_0 x_2$ is irreducible, so the conic is a projective variety. Using Theorem 1.11A and Ex.2.6, dimension of the conic is 1.
(a) (Note the definition of θ is "reverse" to ρ_d.) $\mathfrak{a}$ is prime because it is kernel of a map to a integral domain. For any $f \in \mathfrak{a}$ and for any $k \geq 0$, the homogeneous component of f of degree k is sent to the homogeneous part of $\theta(f)$ of degree dk. Because $\theta(f) = 0$, the homogeneous component of any degree of $\theta(f)$ must be 0

(as a result of the graded ring structure, 0 has only one representation as a sum of homogeneous elements). In particular, each homogeneous component of f is sent to 0, so each homogeneous component of f is in $\mathfrak{a}$, so $\mathfrak{a}$ is homogeneous.

(b) For $\operatorname{im}\rho_d \subseteq Z(\mathfrak{a})$, take any $(a_0 : ... : a_n) \in \mathbb{P}^n$ and any homogeneous polynomial $f \in \mathfrak{a}$, then $f(\rho_d(a_0 : ... : a_n)) = \theta(f)(a_0 : ... : a_n) = 0$, so $\operatorname{im}\rho_d \subseteq Z(\mathfrak{a})$. For $\operatorname{im}\rho_d \supseteq Z(\mathfrak{a})$, take any $b = (b_0 : ... : b_N) \in Z(\mathfrak{a})$. $\forall 0 \leq i \leq n$, let s_i be an integer from 0 to N such that the s_i-th component of ρ_d is a_i^d. Then $\exists i$ such that $b_{s_i} \neq 0$, because if $b_{s_i} = 0$ for all i, then $\forall 0 \leq m \leq N$, b_m^d can be written as a product of powers of b_{s_i} by $b \in Z(\mathfrak{a})$ (we can imagine as if b is the image of $(a_0 : ... : a_n)$, then components of b satisfy equations they should satisfy when they are written in products of a_i), then $b_m = 0$, then all coordinates of b are 0, which is impossible. Now fix i where $b_{s_i} \neq 0$, that is, the s_i-th component of ρ_d is a_i^d. Now let $(r_j)_{0 \leq j \leq n}$ be a set of integers from 0 to N such that the r_i-th component of ρ_d is a_i^d (i.e. $r_i = s_i$), and for $j \neq i$, the r_j-th component of ρ_d is $a_i^{d-1}a_j$. Now we claim $\rho_d(b_{r_0} : b_{r_1} : ... : b_{r_n}) = b$. $\forall 0 \leq m \leq N$, suppose the m-th coordinate of $\rho_d(b_{r_0} : b_{r_1} : ... : b_{r_n})$ is $b_{r_0}^{c_0}...b_{r_n}^{c_n}$, we claim $b_{r_0}^{c_0}...b_{r_n}^{c_n} = b_{r_i}^{d-1}b_m$. To see this, let's count the "power" of a_j in $b_{r_0}^{c_0}...b_{r_n}^{c_n}$ and in $b_{r_i}^{d-1}b_m$ (This is valid, because $b \in Z(\mathfrak{a})$). When $j = i$, the power of a_j in $b_{r_0}^{c_0}...b_{r_n}^{c_n}$ is $c_i d + \sum_{j\neq i} c_j(d-1) = d(d-1) + c_i$ because $\sum c_i = d$. The power of a_j in $b_{r_i}^{d-1}b_m$ is $(d-1)d + c_i$, so the power of a_i is the same. For $j \neq i$, the power of a_j in $b_{r_0}^{c_0}...b_{r_n}^{c_n}$ is c_j, and the power of a_j in $b_{r_i}^{d-1}b_m$ is c_j, so again they are equal. So $b_{r_0}^{c_0}...b_{r_n}^{c_n} = b_{r_i}^{d-1}b_m$. So $\rho_d(b_{r_0} : b_{r_1} : ... : b_{r_n}) = (b_{r_i}^{d-1}b_0 : b_{r_i}^{d-1}b_1 : b_{r_i}^{d-1}b_2 : ... : b_{r_i}^{d-1}b_N) = (b_0 : b_1 : b_2 : ... : b_N) = b$ where the second to last step is true because $b_{r_i} = b_{s_i} \neq 0$..

(c) First we show ρ_d is injective. Suppose $\rho_d(a_0 : ... : a_n) = \rho_d(b_0 : ... : b_n)$. WLOG, suppose $a_0 \neq 0$, then $b_0 \neq 0$. Then $\exists \lambda \in k^\times$ such that $a_0^d = \lambda b_0^d$, and $\forall 1 \leq i \leq n$, $a_0^{d-1}a_i = \lambda b_0^{d-1}b_i$. Then $\forall 1 \leq i \leq n$, if $a_i = 0$ then $b_i = 0$, and $a_i = \frac{a_0}{b_0}b_i$. Otherwise $a_i \neq 0$, then $b_i \neq 0$, then $\frac{a_i}{b_i} = \lambda(\frac{b_0}{a_0})^{d-1} = (\frac{a_0}{b_0})^d(\frac{b_0}{a_0})^{d-1} = \frac{a_0}{b_0}$, so $\forall 0 \leq i \leq n$, $a_i = \frac{a_0}{b_0}b_i$, so ρ_d is injective.
Next we prove ρ_d takes closed set to closed set. It suffices to take any homogeneous $f \in k[x_0, ..., x_n]$ and prove $\rho_d(Z(f))$ is closed in $Z(\mathfrak{a})$. Note $f^d \in \operatorname{im}\theta$. We can take $\tilde{f} \in k[y_0, ..., y_N]$ such that $\theta(\tilde{f}) = f^d$ and $\tilde{f}$ is homogeneous, and we claim $\rho_d(Z(f)) = Z(\mathfrak{a}) \cap Z(\tilde{f})$. For $\subseteq$, $\forall p \in \mathbb{P}^n$ we have $\tilde{f}(\rho_d(p)) = \theta(\tilde{f})(p) = f^d(p)$, so when $p \in Z(f)$, $\tilde{f}(\rho_d(p)) = 0$. For $\supseteq$, take any $q \in Z(\mathfrak{a}) \cap Z(\tilde{f})$, then because $Z(\mathfrak{a}) = \operatorname{im}\rho_d$, $\exists p \in \mathbb{P}^n$ such that $\rho_d(p) = q$ (and such p is unique because ρ_d is injective). Then $f^d(p) = \theta(\tilde{f})(p) = \tilde{f}(\rho_d(p)) = \tilde{f}(q) = 0$, so $f(p) = 0$. Thus ρ_d takes closed set to closed set.
Next we prove preimage of closed subsets of $Z(\mathfrak{a})$ under ρ_d is closed in $\mathbb{P}^n$. It suffices to take any homogeneous $f \in k[y_0, ..., y_N]$ and prove $\rho_d^{-1}(Z(f) \cap Z(\mathfrak{a}))$ is closed in $\mathbb{P}^n$. We claim $\rho_d^{-1}(Z(f) \cap Z(\mathfrak{a})) = Z(\theta(f))$. Both directions follow from $\theta(f)(p) = f(\rho_d(p))$. Therefore, ρ_d is a homeomorphism from $\mathbb{P}^n$ to $Z(\mathfrak{a})$.

(d) Denote the twisted cubic curve by C. From Ex.2.9(c) we know that $C = Z(y_0y_3^2 - y_2^3, y_1^2 - y_0y_2, y_1^3 - y_0^2y_3)$. Another description for C is $C = \overline{\{(1 : t : t^2 : t^3 | t \in k)\}}$. On the other hand, we also have two descriptions for the 3-uple embedding of $\mathbb{P}^1$ to $\mathbb{P}^3$: $Z(\mathfrak{a})$ and $\operatorname{im}\rho_d = \{(a_0^3 : a_0^2a_1 : a_0a_1^2 : a_1^3) | a_0 \neq 0 \text{ or } a_1 \neq 0\}$. Note $(y_0y_3^2 - y_2^3, y_1^2 - y_0y_2, y_1^3 - y_0^2y_3) \subseteq \mathfrak{a}$, so $C = Z(y_0y_3^2 - y_2^3, y_1^2 - y_0y_2, y_1^3 - y_0^2y_3) \supseteq Z(\mathfrak{a})$. Also, $\{(1 : t : t^2 : t^3 | t \in k)\} \subseteq \{(a_0^3 : a_0^2a_1 : a_0a_1^2 : a_1^3) | a_0 \neq 0 \text{ or } a_1 \neq 0\}$ by setting $a_0 = 1$ and $a_1 = t$. Because the RHS is closed, taking closure on both sides we see $C \subseteq \operatorname{im}\rho_d$. Therefore, the twisted cubic curve is the same as the 3-uple embedding of $\mathbb{P}^1$ to $\mathbb{P}^3$.

Ex.2.13. By Ex.2.12, $\rho_d : \mathbb{P}^2 \to Y$ is a homeomorphism, so $\rho_d^{-1}(Z)$ is projective variety in $\mathbb{P}^2$ with dimension 1. By Ex.2.8, $\rho_d^{-1}(Z) = Z(f)$ for some irreducible, homogeneous $f \in k[x_0, x_1, x_2]$. Let $d \geq 1$ be the smallest integer such that there exists homogeneous $g \in k[y_0, ..., y_5]$, $\theta(g) = f^d$ (d is at most 2). Then g is irreducible, because if $g = g_1g_2$ for some $g_1, g_2 \in k[y_0, ..., y_5]$, then g_1 and g_2 must be homogeneous because g is homogeneous. Then $\theta(g) = f^d = \theta(g_1)\theta(g_2)$. f is irreducible, so $\theta(g_1) = f^i$ for some $0 \leq i \leq d$ up to multiplication by unit. But by our choice of d, i can only be 0 or d. WLOG, assume $i = d$. Then g_1 and g must be homogeneous of the same degree, then g_2 must have degree 0, so g_2 is a unit, so g is irreducible. Then we can show $\rho_d(Z(f)) = Y \cap Z(g)$ using the same proof as in Ex.2.12(c). Then $Z = Y \cap Z(g)$ where $Z(g)$ is a hypersurface by Ex.2.8.

Ex.2.14. Let $\varphi : k[\{z_{ij}\}] \to k[x_0, ..., x_r, y_0, ..., y_s]$ be the ring homomorphism sending z_{ij} to x_iy_j. Then $\ker\varphi$ is a homogeneous prime ideal for the same reason as explained in Ex.2.12(a), so $Z(\ker\varphi)$, if nonempty, is a projective variety. Next we show $\operatorname{im}\psi = Z(\ker\varphi)$. $\operatorname{im}\psi \subseteq Z(\ker\varphi)$ because for all homogeneous $f \in \ker\varphi$ and $p \in \mathbb{P}^r \times \mathbb{P}^s$, $f(\psi(p)) = \varphi(f)(p) = 0(p) = 0$. To see $\operatorname{im}\psi \supseteq Z(\ker\varphi)$, suppose $(c_{ab}) \in Z(\ker\varphi)$, then there exists $0 \le i \le r$ and $0 \le j \le s$ such that $c_{ij} \neq 0$ by definition of projective space. Then $\forall 0 \le a \le r, 0 \le b \le s$, the ab-th coordinate of $\psi((c_{0j} : c_{1j} : ... : c_{rj}), (c_{i0} : c_{i1} : ... : c_{is}))$ is $c_{aj}c_{ib} = c_{ij}c_{ab}$ because $(c_{ij}) \in \ker\varphi$. Thus $\psi((c_{0j} : c_{1j} : ... : c_{rj}), (c_{i0} : c_{i1} : ... : c_{is})) = (c_{ij}c_{00} : ... : c_{ij}c_{ab} : ... : c_{ij}c_{rs}) = (c_{ab})$ where we cancel c_{ij} because $c_{ij} \neq 0$. Thus $\operatorname{im}\psi = Z(\ker\varphi)$ is a projective variety.

Ex.2.15.(a) Let the homogeneous coordinates of $\mathbb{P}^3$ be (x, z, w, y). The Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ is $S = \{(a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1) | \exists i, j, a_ib_j \neq 0\}$, so obviously $S \subseteq Q$. To see $S \supseteq Q$, take any $(x : z : w : y) \in Q$. WLOG, suppose $x \neq 0$. Then $\psi((x : w), (x : z)) = (x^2 : xz : xw : wz) = (x^2 : xz : xw : xy) = (x : z : w : y)$, so $(x : z : w : y) \in S$, so $S \supseteq Q$. Thus $S = Q$.
(b) Fix $t = (t_0 : t_1) \in \mathbb{P}^1$. Let $L_t = Z(xt_0 - zt_1, wt_0 - yt_1)$ and $M_t = Z(xt_0 - wt_1, zt_0 - yt_1)$. Note L_t and M_t are well defined, i.e. they do not depend on representative of t. It's easy to verify L_t and M_t are subsets of Q. By Lemma 1 we proved in Ex.2.11, the ideals defining L_t and M_t are prime ideals, and it's easy to verify L_t and M_t are nonempty, so they are linear varieties. They have dimension 1 by Ex.2.10(c) and Lemma 2 we proved in Ex.2.11. When $L_t \neq L_u$, $t \neq u$, then $L_t \cap L_u = \emptyset$ because the ratio of two coordinates of t is different from the ratio of two coordinates of u (the case for one coordinate being 0 is also easy to check). Similarly, when $M_t \neq M_u$, $M_t \cap M_u = \emptyset$. For all t, u, $L_t \cap M_u =$ one point, because for any $(x : z : w : y) \in L_t \cap M_u$, the ratios of its coordinates are determined, thus there can be at most one solution. On the other hand, $(u_1t_1 : t_0u_1 : u_0t_1 : u_0t_0)$ is one solution.
(c) Consider $Y := Z(xy - zw, x - y) = Z(x^2 - zw, x - y)$. We first prove Y is projective variety. It suffices to prove $(x^2 - zw, x - y)$ is prime ideal. By polynomial division, for each $f \in k[x, z, w, y]$, there exists unique polynomials $h_1 \in k[x, z, w, y], h_2 \in k[x, z, w], h_3, h_4 \in k[z, w]$ such that $f = (x-y)h_1 + (x^2 - zw)h_2 + xh_3 + h_4$. Suppose $fg \in (x^2 - zw, x - y)$. If $g = (x-y)h'_1 + (x^2 - zw)h'_2 + xh'_3 + h'_4$ where h'_i satisfies the same condition as h_i, then $fg = (x - y)h''_1 + (x^2 - zw)h''_2 + x(h_3h'_4 + h_4h'_3) + (h_4h'_4 + zwh_3h'_3)$ where $h''_1 \in k[x, z, w, y], h''_2 \in k[x, z, w]$. Now uniqueness of such expression and $fg \in (x - y, x^2 - zw)$ imply $h_3h'_4 + h_4h'_3 = 0$ and $h_4h'_4 + zwh_3h'_3 = 0$. If $h'_4 = 0$ or $h_4 = 0$, then manipulating the previous two equations a little bit tells us $f \in (x^2 - zw, x - y)$ or $g \in (x^2 - zw, x - y)$. Otherwise, $h_3 = \frac{-h_4h'_3}{h'_4}$, and substituting into the latter equation we get $h_4^{'2} = zwh_3^{'2}$. Viewed as polynomials in one variable (either z or w), the LHS has even degree, while the RHS has odd degree, so the only possibility is that both sides are 0. Thus $g \in (x^2 - zw, x - y)$, so $(x^2 - zw, x - y)$ is prime ideal. $Y = Z(x^2 - zw, x - y)$ is nonempty because $(1 : 1 : 1 : 1)$ is in it, thus we conclude Y is a projective variety.
Using Ex.2.10 and its notation, $C(Y)$ is an affine variety given by $C(Y) = Z(xy - zw, x - y)$. Because this is intersection of two hypersurfaces neither of which contains the other, by Ex.1.9 we have $\dim C(Y) = 2$. Then by Ex.2.10(c), $\dim Y = 1$, so Y is a curve in Q. Y is not any line of form L_t, because all points on L_t have fixed ratio between x and z, which is not the case in Y. Similarly Y is not any line of form M_t.
Now we prove that ψ is not a homeomorphism from $\mathbb{P}^1 \times \mathbb{P}^1$ to Q. First note that $\mathbb{P}^1$ has cofinite topology (obviously finite sets are closed. Conversely, if X is closed in $\mathbb{P}^1$, then we can assume $X = Z(f)$ where $f \in k[x_0, x_1]$ is homogeneous. If X is infinite, Then there are infinitely many $x_1 \in k$ such that $f(1, x_1) = 0$, this implies f is zero polynomial, then $Z(f) = \mathbb{P}^1$). If $\psi^{-1}(C(Y))$ is closed, then its complement is open in $\mathbb{P}^1 \times \mathbb{P}^1$ with product topology. By definition of product topology, complement of $\psi^{-1}(C(Y))$ is a union of basic open sets of form $U \times V$ where U, V are open on $\mathbb{P}^1$. Pick any such U, V. Because U and V are complements of finite sets and k is infinite, $\exists t \in k$ such that $(1 : t) \in U, (t : 1) \in V$. Then $\psi((1 : t), (t : 1)) = (t : 1 : t^2 : t) \in C(Y)$, so $((1 : t), (t : 1)) \in \psi^{-1}(C(Y))$. But $((1 : t), (t : 1)) \in U \times V \subseteq (\mathbb{P}^1 \times \mathbb{P}^1) - \psi^{-1}(C(Y))$, contradiction. Thus ψ is not a homeomorphism from $\mathbb{P}^1 \times \mathbb{P}^1$ to Q.

Ex.2.16.(a) Let $C = Z(wz^2 - y^3, x^2 - wy, x^3 - w^2z)$, then C is a twisted cubic curve by Ex.2.9. Then we have

$$Z(wz^2 - y^3, x^2 - wy, x^3 - w^2z) \cup Z(w, x) = Z(x^2 - yw, xy - zw). \tag{3}$$

The equality is easy to verify. By Lemma 1, (w, x) is a prime ideal, so $Z(w, x)$ is a variety. It is a linear variety of dimension 1 because $C(Z(w, x))$ (cone over $Z(w, x)$) has dimension 2 by Lemma 2, then by Ex. 2.10(c), $\dim(Z(w, x)) = 1$. So $Z(w, x)$ is a line.
(b) If $x^2 - yz = 0$ and $y = 0$ then $x = 0$, so $C \cap L = \{(0 : 0 : a) | a \in k^\times\}$ is a point P. It's easy to verify $I(C) = (x^2 - yz), I(L) = (y), I(P) = (x, y)$. Then $I(P) \supsetneq I(C) + I(L)$ because $x \in I(P)$ but $x \notin I(C) + I(L)$.

Ex.2.17.(a) By Ex.2.10, $C(Y) = Z(\mathfrak{a})$ where $\mathfrak{a}$ is viewed not as homogeneous ideal, but a general ideal in

$k[x_0,..,x_n]$. By Ex.1.9, $\dim(C(Y)) \geq n+1-q$, so by Ex.2.10(c) $\dim Y \geq n-q$.
(b) Let's assume that in the definition of strict complete intersection, the ideal of Y is generated by $n-r$ homogeneous polynomials. (This is not explicitly stated in Hartshorne, but is intuitional and consistent with Wikipedia.) Let $f_1,...,f_{n-r}$ be homogeneous generators of $I(Y)$, then $Y = ZI(Y) = Z(f_1,...,f_{n-r}) = \cap_{i=1}^{n-r} Z(f_i)$ is intersection of $n-r$ hypersurfaces (Here I am being loose and allowing hypersurface to be given by any single homogeneous polynomial, not necessarily irreducible).
(c) By Ex2.12, Y is $Z(\mathfrak{a})$ where a is kernel of $\theta: k[y_0,...,y_3] \to k[x_0,x_1]$ given by $y_0 \mapsto x_0^3, y_1 \mapsto x_0^2 x_1, y_2 \mapsto x_0 x_1^2, y_3 \mapsto x_1^3$. Since $\mathfrak{a}$ is prime, $I(Y) = I(Z(\mathfrak{a})) = \mathfrak{a}$. $\mathfrak{a}$ cannot be generated by two elements for a similar reason as in Ex.1.11. Specifically, we look at terms of $f \in \mathfrak{a}$ which will be sent to $x_0^4 x_1^2, x_0^3 x_1^3, x_0^2 x_1^4$. $\theta(f) = 0$ implies coefficient of y_1^2 in f is the negative of coefficient of $y_0 y_2$, coefficient of $y_0 y_3$ in f is the negative of coefficient of $y_1 y_2$, coefficient of y_2^2 in f is the negative of coefficient of $y_1 y_3$. Define a k-linear map $\psi: \mathfrak{a} \to k^3$ sending a polynomial to its coefficients of $y_1^2 - y_0 y_2$, coefficients of $y_0 y_3 - y_1 y_2$, coefficients of $y_2^2 - y_1 y_3$. This map is surjective because $y_1^2 - y_0 y_2, y_0 y_3 - y_1 y_2, y_2^2 - y_1 y_3 \in \mathfrak{a}$. Now if $\mathfrak{a}$ is generated by two elements, this would imply im ψ is k-subspace of k^3 spanned by coefficients of $y_1^2 - y_0 y_2, y_0 y_3 - y_1 y_2, y_2^2 - y_1 y_3$ of the two generators. Then $\dim_k \operatorname{im} \psi \leq 2$, contradiction (more detailed explanation is in my solution to Ex.1.11). So $\mathfrak{a}$ cannot be generated by two elements.

3 Morphisms

We first prove some results which will be used later possibly without mentioning.
Lemma 3: Let X and Y be any varieties and $\varphi: X \to Y$ be a morphism. Let $X' \subseteq X$ and $Y' \subseteq Y$ be subsets which are also varieties and $\varphi(X') \subseteq (Y')$. Then $\varphi|_{X'}: X' \to Y'$ is a morphism of varieties.
Proof: $\varphi|_{X'}$ is continuous because restriction of continuous function to a subset of its domain is continuous (when all relevant spaces have induced topology). To see $\varphi|_{X'}$ is a morphism, take any open subset $U \subseteq Y'$ and regular function $f: U \to k$. We want to prove $f \circ \varphi|_{X'}: \varphi|_{X'}^{-1}(U) \to k$ is regular. Take any $p \in \varphi|_{X'}^{-1}(U)$. Then $\varphi(p) \in U$, so there exists an open subset of U containing $\varphi(p)$ such that $f = g/h$ where g, h are polynomials on this open subset. Because U has topology induced from Y, we can write this open set as $V \cap U$ where V is open in Y. The function g/h is regular on $V - Z(h)$, so $(g/h) \circ \varphi: \varphi^{-1}(V - Z(h)) \to k$ is a regular function. Note $V - Z(h) \supseteq V \cap U$ and $(g/h) \circ \varphi$ agrees with $f \circ \varphi|_{X'}$ on $\varphi|_{X'}^{-1}(V \cap U)$. Since $p \in \varphi|_{X'}^{-1}(V \cap U)$, $f \circ \varphi|_{X'}$ can be written as a rational function in some open set (of $\varphi|_{X'}^{-1}(U)$) near p by first using $(g/h) \circ \varphi$ to find such open set in $\varphi^{-1}(V - Z(h))$ and then restrict it to $\varphi|_{X'}^{-1}(U)$. $\square$

Lemma 4: Any automorphism of the underlying k-vector space of $\mathbb{P}^n$ induces an automorphism of $\mathbb{P}^n$.
Proof: Let $\varphi: k^{n+1} \to k^{n+1}$ be an isomorphism of k-vector space. It is obvious that φ induces a bijective function $\tilde{\varphi}: \mathbb{P}^n \to \mathbb{P}^n$. Let $\varphi^*: k[x_0,...,x_n] \to k[x_0,...,x_n]$ be k-algebra homomorphism defined by $\varphi^*(x_i) = \mathbf{x}^t (M^{-1})^t \mathbf{e}_i$ where $\mathbf{x} = (x_0,...,x_n)$, M is the matrix of φ w.r.t. the standard basis, $\mathbf{e}_i$ is the standard unit vector. For any closed set $Z(f) \subseteq \mathbb{P}^n$, $\tilde{\varphi}(Z(f)) = Z(\varphi^*(f))$, because for any $P = (a_0 : ... : a_n) \in Z(f)$, let $\mathbf{a} = (a_0,...,a_n)^t$, then $\varphi^*(f)(\tilde{\varphi}(P)) = f(\varphi^*(x_0),...,\varphi^*(x_n))(\tilde{\varphi}(P))$ and $\varphi^*(x_i)(\tilde{\varphi}(P)) = (M\mathbf{a})^t (M^{-1})^t \mathbf{e}_i = \mathbf{a}^t M^t (M^{-1})^t \mathbf{e}_i = a_i$, so $\varphi^*(f)(\tilde{\varphi}(P)) = f(P) = 0$. Conversely if $Q = (b_0 : ... : b_n) \in Z(\varphi^*(f))$, let $\mathbf{b} = (b_0,...,b_n)^t$, then $0 = \varphi^*(f)(Q) = f(\varphi^*(x_0),...,\varphi^*(x_n))(Q)$ and $\varphi^*(x_i)(b_0,...,b_n) = \mathbf{b}^t (M^{-1})^t \mathbf{e}_i = (M^{-1}\mathbf{b})^t \mathbf{e}_i = \varphi^{-1}(\mathbf{b})_i$, so $0 = f(\varphi^*(x_0),...,\varphi^*(x_n))(Q) = f(\varphi^{-1}(b)) = f(\tilde{\varphi}^{-1}(Q))$. Therefore, $\tilde{\varphi}(Z(f)) = Z(\varphi^*(f))$. Thus we see $\tilde{\varphi}$ is homeomorphism. $\tilde{\varphi}$ is an isomorphism because the definition of $\tilde{\varphi}$ uses polynomial expression at each coordinate.

Lemma 5: Let X by any variety and $Y \in \mathbb{P}^m$ be quasi-projective variety. A set function $\varphi: X \to Y$ is a morphism if and only if $\frac{y_i}{y_j} \circ \varphi$ is regular function on open subsets of X for all possible i, j.
Remark: This result is analogous to Lemma 3.6 in the book.
Proof: One direction is obvious. For the other direction, first we prove φ is continuous. Take any homogeneous $f \in k[y_0,...,y_m]$, let $f_i = \frac{f}{y_i^{\deg f}}$, let $U_i = \varphi^{-1}(Y - Z(y_i))$, let $C_i = (f_i \circ \varphi)^{-1}(0)$, then $\varphi^{-1}(Z(f)) = \cup_{i=0}^m C_i$. Note that by assumption, U_i is an open subset of X and $f_i \circ \varphi$ is regular function on U_i (because regular functions on a variety form a ring). Since regular function is continuous, each C_i is closed in U_i. Note for $j \neq i$, $C_j \cap U_i \subseteq C_i$, because if $x \in C_j \cap U_i$, write $\varphi(x) = (a_0 : ... : a_m)$, then

$0 = f_j \circ \varphi(x) = \frac{f}{y_j^{\deg f}}(a_0 : ... : a_m) = \frac{f(a_0:...:a_m)}{a_j^{\deg f}}$. Since $a_j \neq 0$ and $a_i \neq 0$, we have $\frac{f(a_0:...:a_m)}{a_i^{\deg f}} = 0$, so $x \in C_i$. Thus $\varphi^{-1}(Z(f)) \cap U_i = \cup_{j=0}^m (C_j \cap U_i) = C_i$ is closed in U_i. Since (U_i) is an open covering of X, we see $\varphi^{-1}(Z(f))$ is closed in X, so φ is continuous.

Next we show φ is a morphism. Pick open subset $U \subseteq Y$, regular function $f : U \to k$, consider $f \circ \varphi : \varphi^{-1}(U) \to k$. Pick any $P \in \varphi^{-1}(U)$, then there exists open subset $V \subseteq U$ such that $\varphi(P) \in V \subseteq U$ and $f|_V = \frac{g}{h}$ where g, h are homogeneous of the same degree and h is non-vanishing on V. Assume the i-th coordinate of $\varphi(P)$ is nonzero. Then we can find open subset V' such that $\varphi(P) \in V' \subseteq V \subseteq U$ and $f|_{V'} = \frac{g/y_i^{\deg g}}{h/y_i^{\deg h}}$. Then $f|_{V'} \circ \varphi : \varphi^{-1}(V') \to k$ is regular by assumption, so $f \circ \varphi : \varphi^{-1}(U) \to k$ is regular, so φ is a morphism.

Ex.3.1.(a) By a similar argument as the one I give for Ex.3.6, $\mathcal{O}(\mathbb{A}^1 - \{0\}) \cong k[t, t^{-1}]$ as k-algebra. We also know $A(\mathbb{A}^1) = k[t]$. Let W be any conic. By Ex.1.1, $A(W) \cong k[t]$ or $A(W) \cong k[t, t^{-1}]$. By Proposition 3.5, $W \cong \mathbb{A}^1$ or $W \cong \mathbb{A}^1 - \{0\}$.

(b) Let $U \subset \mathbb{A}^1$ be a proper open subset. FSOC, suppose $\mathbb{A}^1 \cong U$. Then $k[t] \cong \mathcal{O}(U)$ as k-algebra. By a similar argument as the one I give for Ex.3.6, we get $\mathcal{O}(U) = \{\frac{f}{g} | f, g \in k[t], Z(g) \subseteq \mathbb{A}^1 - U\}$, a sub-$k$-algebra of $k(t)$. Let $\varphi : \mathcal{O}(U) \to k[t]$ be a k-algebra isomorphism. Pick any $P \notin U$, then $t - P$ is a unit in $\mathcal{O}(U)$, so $\varphi(t - P)$ is a unit in $k[t]$, so $\varphi(t) - P = \varphi(t - P) \in k$, so $\varphi(t) \in k$, so $\operatorname{im} \varphi \subseteq k$, which contradicts with φ being surjective. Thus $\mathbb{A}^1$ is not isomorphic to any proper open subset.

(c) By Ex.3.4, the 2-uple embedding $\rho_2 : \mathbb{P}^1 \to \mathbb{P}^2$ is isomorphism. Since $\operatorname{im} \rho_2 \subseteq Z(xz - y^2)$ and both sides are projective curves, we have $\operatorname{im} \rho_2 = Z(xz - y^2)$. So it remains to prove that any projective curve in $\mathbb{P}^2$ given by $C = Z(ax^2 + by^2 + cz^2 + dxy + exz + fyz)$ is isomorphic to $Z(xz - y^2)$. By Lemma 4 and its proof, we can do linear change of coordinates on $\mathbb{P}^n$ as long as the corresponding matrix is invertible. The rest is just manipulation of algebra similar to what we did in Ex.1.1(c). We note $ace \neq 0$, so $ax^2 + cz^2 + exz$ is a product of 2 linear factors.

In the first case, suppose the two linear factors are distinct, write them as (new) x and z, we get $C = Z(xz + ay^2 + bxy + czy)$ for some (new) coefficients a, b, c. Since $xz + ay^2 + bxy + czy = x(z + by) + y(ay + cz)$ and since $z + by$ and $ay + cz$ should be relatively prime (otherwise the curve is reducible), we let $y := z + by, z := ay + cz$, then we get $C = Z(xy + ayz + bz^2)$ for some new coefficients a and b. Then because $xy + ayz = y(x + az)$ we let $x := x + az$ and get $C = Z(xy + bz^2)$. Note $b \neq 0$ because C is irreducible. Another change of coordinates gives $C = Z(xz - y^2)$.

In the second case, suppose the two linear factors are not distinct. WLOG suppose the linear factor has nonzero coefficient of x, then let (new) x be the linear factor we get $C = Z(x^2 + ay^2 + bxy + cyz)$ for some new coefficients a, b, c. Now $x^2 + ay^2 + bxy$ is a product of 2 linear factors. If these two are distinct factors, then a change of coordinates gives $C = Z(xy + axz + byz)$ for new nonzero coefficients a and b. Let $y := y + az$ we get $C = Z(xy + ayz + bz^2)$. Then let $x := x + az$ we get $C = Z(xy + bz^2)$. Another easy change of coordinates gives $C = Z(xz - y^2)$. If $x^2 + ay^2 + bxy$ is a product of 2 same factors, such factor must have nonzero coefficient in x, so we let our new x be this factor and get $C = Z(x^2 + cyz)$. Another easy change of coordinates gives $C = Z(xz - y^2)$.

Above all, all conics in $\mathbb{P}^2$ are isomorphic to $\mathbb{P}^1$.

Remark: This result is cleaner than the affine counterpart (part (a)).

(d) We first prove that any two curves in $\mathbb{P}^2$ have nonempty intersection. Let C_1, C_2 be two curves in $\mathbb{P}^2$, let $C(C_1)$, $C(C_2)$ be cones over these curves in $\mathbb{A}^3$ (Ex.2.10). Then $\dim C(C_1) = \dim C(C_2) = 2$, and $C(C_1) \cap C(C_2) \neq \emptyset$. By Ex.1.8, either $C(C_1) = C(C_2)$ or $C(C_1) \cap C(C_2)$ is a finite union of at least 1 affine curves. Since $C(C_1 \cap C_2) = C(C_1) \cap C(C_2)$, $C_1 \cap C_2 \neq \emptyset$. (Then since $\mathbb{P}^n$ is noetherian, it is obvious that $C_1 \cap C_2 = \{P_1, ..., P_n\}$ for some points P_i.)

But $\mathbb{A}^2$ can have infinitely many curves with pairwise empty intersection. For example $C_t = Z(t), t \in k$ is such a family of curves. So $\mathbb{A}^2$ is not homeomorphic to $\mathbb{P}^2$.

(e) Let X be an affine variety and Y be a projective variety. Then $X \cong Y$ implies $\mathcal{O}(X) \cong \mathcal{O}(Y)$ (under the natural map). Using Theorem 3.2 and 3.4, $A(X) \cong k$. Because $A(X) = k[x_1, ..., x_n]/I(X)$, $I(X)$ is maximal ideal. By Hilbert's Nullstenllensatz, $I(X) = I(P)$ for some $P \in \mathbb{A}^n$. Then $X = ZI(X) = P$.

Ex.3.2.(a) Let $C = Z(y^2 - x^3)$, then obviously $\operatorname{im}\varphi \subseteq C$. Let x_1, x_2 denote coordinate functions on $\mathbb{A}^2$, then $x_1 \circ \varphi(t) = t^2$, $x_2 \circ \varphi(t) = t^3$ are regular functions on $\mathbb{A}^1$, so by Lemma 3.6, φ is a morphism. Define $\psi : C \to \mathbb{A}^1$ by $(x, y) \mapsto \frac{y}{x}$ when $x \neq 0$ and $(x, y) \mapsto 0$ when $x = 0$. $\psi \circ \varphi = \mathrm{id}_{\mathbb{A}^1}$ is easy to verify. For any $(x, y) \in C$, if $x = 0$, then $y^2 = x^3 = 0$ so $y = 0$, so $\varphi \circ \psi(x, y) = (0, 0) = (x, y)$. If $x \neq 0$, then $\varphi \circ \psi(x, y) = (\frac{y^2}{x^2}, \frac{y^3}{x^3}) = (\frac{x^3}{x^2}, \frac{y^3}{y^2}) = (x, y)$, so we see $\varphi \circ \psi = \mathrm{id}_C$. ψ is continuous because any closed subset of $\mathbb{A}^1$ is finite, so its inverse image under ψ is obviously closed in C. Now we have shown φ is a bijective bicontinuous morphism onto C, but φ is not isomorphism, because if it is isomorphism, then by Corollary 3.7, $A(\mathbb{A}^1) \cong A(C)$, but $A(\mathbb{A}^1) = 0$ and $A(C) = k[x, y]/(y^2 - x^3) \neq 0$, contradiction.
(b) If $\varphi(a) = \varphi(b)$, then $a^p = b^p$, then $(a - b)^p = a^p - b^p = 0$ because k has characteristic p. So $a - b = 0$ and φ is injective. $\forall a \in k$, $x^p - a \in k[x]$ splits in $k[x]$ because k is algebraically closed. So φ is surjective. φ is bicontinuous because $\mathbb{A}^1$ has cofinite topology. Suppose φ is isomorphism. Let $f : \mathbb{A}^1 \to k$ be $f(t) = t$. Then $f \circ \varphi^{-1} : \mathbb{A}^1 \to k$ should be regular function. By Theorem 3.2(a), regular function on an affine variety has a polynomial expression. Thus $\exists g \in k[x]$ such that when we view g as a function, $\forall t \in k, g(t) = f \circ \varphi^{-1}(t) = \sqrt[p]{t}$ where $\sqrt[p]{t}$ denotes the unique element of k whose p-th power is t. Then $(g(x))^p(t) = t$ for all $t \in k$, so as a polynomial, $(g(x))^p = x$. But this is impossible because degree of $(g(x))^p$ should be a multiple of p.

Ex.3.3.(a) φ_P^* is defined by pullback using φ, By definition of a morphism, the pullback of regular function is still regular function. That this map is well-defined and is k-algebra homomorphism is straightforward to see.

(b) ($\Longrightarrow$) An isomorphism is automatically a homeomorphism. For all $P \in X$, we have $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$ and $(\varphi^{-1})^*_{\varphi(P)} : \mathcal{O}_{P,X} \to \mathcal{O}_{\varphi(P),Y}$. It is straightforward to see that these two maps are inverses of each other, so φ_P^* is an isomorphism.
($\Longleftarrow$) It suffices to prove φ^{-1} is morphism. Take any nonempty open set $U \subseteq X$ and $f : U \to k$ a regular function. We want to prove $f \circ \varphi^{-1} : \varphi(U) \to k$ is regular. Pick any $Q \in \varphi(U)$, let $P = \varphi^{-1}(Q)$, Since $\varphi_P^* : \mathcal{O}_{Q,Y} \to \mathcal{O}_{P,X}$ is surjective, there exists $[(V, g)] \in \mathcal{O}_{Q,Y}$ such that $\varphi_P^*([(V, g)]) = [(U, f)]$. Thus $g \circ \varphi = f$ on $\varphi^{-1}(V) \cap U$. Then $g = f \circ \varphi^{-1}$ on $V \cap \varphi(U)$. So $f \circ \varphi^{-1}$ is regular on $V \cap \varphi(U)$. Since Q is arbitrary, we conclude that $f \circ \varphi^{-1}$ is regular on $\varphi(U)$. So φ^{-1} is morphism, and φ is isomorphism.

(c) First we prove that for any nonempty open subset $V \subseteq X$, $\varphi(V)$ is dense in Y. Let $D \subseteq Y$ be a closed set containing $\varphi(V)$, then $\varphi^{-1}(D) \supseteq \varphi^{-1}(\varphi(V)) \supseteq V$. X is irreducible, and any nonempty open subset of irreducible space is dense, so V is dense in X. $\varphi^{-1}(D)$ is closed in X, so $\varphi^{-1}(D) = X$. Then $D \supseteq \varphi(\varphi^{-1}(D)) = \varphi(X)$ which is dense in Y by assumption. Thus $D = Y$ and $\varphi(V)$ is dense in Y.

Back to our problem, suppose $\varphi_P^*([(U, f)]) = 0$ for some $[(U, f)] \in \mathcal{O}_{\varphi(P),Y}$. Then $f \circ \varphi = 0$ on $\varphi^{-1}(U)$. Since f is continuous on U, the zero set of f contains the closure of $\varphi(\varphi^{-1}(U))$ in U. Since $\varphi(\varphi^{-1}(U))$ is dense in Y, it is also dense in U, so $f \equiv 0$ on U. so φ_P^* is injective.

Ex.3.4. We have shown in Ex.2.12 that the d-uple embedding $\rho_d : \mathbb{P}^n \to Z(\mathfrak{a}) \subseteq \mathbb{P}^N$ is a homeomorphism where $\mathfrak{a}$ is kernel of θ defined in that exercise. ρ_d is a morphism because if $U \subset Z(\mathfrak{a})$ is any open set and $\varphi : U \to k$ is regular function, then locally on some open $V \subseteq U$, $\varphi = \frac{f}{g}$ where $f, g \in k[y_0, ..., y_N]$ are homogeneous of same degree. Then $\varphi \circ \rho_d : \rho_d^{-1}(U) \to k$ restricted to $\rho_d^{-1}(V)$ is equal to $\frac{\theta(f)}{\theta(g)}$, so we see ρ_d is morphism.
Conversely, take any open $U \subseteq \mathbb{P}^n$, $\varphi : U \to k$ regular function, and take any $p \in \rho_d(U)$. First take open set V such that $\rho_d^{-1}(p) \in V \subseteq U$ and $\varphi = \frac{f}{g}$ on V where $f, g \in k[x_0, ..., x_n]$ are homogeneous of same degree. Then $p \in \rho_d(V)$. We know $p = \rho_d(a_0 : ... : a_n)$ for some $(a_0 : ... : a_n) \in \mathbb{P}^n$. WLOG, suppose $a_0 \neq 0$, and let i be the integer such that the i-th coordinate of ρ_d is a_0^d. Then $p \in \rho_d(V) - Z(x_i)$. When we restrict ρ_d^{-1} to $Z(\mathfrak{a}) - Z(x_i)$, we have $\rho_d^{-1}(c_0 : ... : c_N) = (c_i : c_{i_1} : c_{i_2} : ... : c_{i_n})$ where each i_k is an integer such that the i_k-th coordinate of ρ_d is $a_0^{d-1}a_k$. Therefore, the restriction of $\varphi \circ \rho_d^{-1}$ to $\rho_d(V) - Z(y_i)$ is a rational function. Because p is arbitrary, $\varphi \circ \rho_d^{-1} : \rho_d(U) \to k$ is a regular function, so ρ_d^{-1} is a morphism. Above all, ρ_d is isomorphism between $\mathbb{P}^n$ and $Z(\mathfrak{a})$.

Ex.3.5. First we prove a useful result.

Lemma 6: $\mathbb{P}^n - H \cong \mathbb{A}^n$ for any n and hyperplane H.
Proof: Suppose $H = Z(f)$ where f is a linear polynomial in $k[x_0, ..., x_n]$. WLOG, suppose the coefficient of x_0 in f is not zero. Define $\varphi : \mathbb{P}^n - H \to \mathbb{A}^n$ by $(a_0 : ... : a_n) \mapsto (\frac{a_1}{f(a)}, ..., \frac{a_n}{f(a)})$ where $a = (a_0 : ... : a_n)$. Note this is well defined. Define $\psi : \mathbb{A}^n \to \mathbb{P}^n - H$ by $(b_1, ..., b_n) \mapsto (b_0 : b_1 : ... : b_n)$ where b_0 is the unique element in k such that $f(b_0, ..., b_n) = 1$. It's straightforward to verify that φ and ψ are inverse maps. φ is a morphism by Lemma 3.6. ψ is continuous because for any closed set $Z(g) \cap (\mathbb{P}^n - H) \subseteq \mathbb{P}^n - H$, we can view f as a variable, replace the variable x_0 in g by $(f - c_1x_1 - ... - c_nx_n)/c_0$ where c_n is coefficient of x_n in f, then let $\tilde{g}$ be the polynomial got from g by replacing f with 1. Then $\varphi(Z(g) \cap (\mathbb{P}^n - H)) = Z(\tilde{g})$. ψ is then a morphism because each coordinate of ψ has polynomial expression. $\square$
Now we prove Ex.3.5. By Ex.2.8, $H = Z(f)$ for some $f \in k[x_0, ..., x_n]$ homogeneous, irreducible, and of positive degree. Suppose $d = \deg f > 0$. Let $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$ be the d-uple embedding, then $\mathbb{P}^n - H \cong \rho_d(\mathbb{P}^n - H)$ by Ex.3.4. Because of our choice of d, using notation of Ex.2.12, there exists linear polynomial $\tilde{f} \in k[y_0, ..., y_N]$ such that $\theta(\tilde{f}) = f$. We have $\rho_d(H) = Z(\tilde{f}) \cap \rho_d(\mathbb{P}^n)$ because $\forall p \in \mathbb{P}^n$, $\tilde{f}(\rho_d(p)) = f(p)$. Thus $\rho_d(\mathbb{P}^n - H) = \rho_d(\mathbb{P}^n) - Z(\tilde{f})$ is an irreducible closed subset of $\mathbb{P}^N - Z(\tilde{f})$. Then by Lemma 6 we see that $\rho_d(\mathbb{P}^n - H)$ is isomorphic to an affine variety, so $\mathbb{P}^n - H$ is isomorphic to an affine variety.

Ex.3.6. By the argument I give in `https://math.stackexchange.com/questions/373315`, the inclusion $i : X \to \mathbb{A}^2$ induces an k-algebra isomorphism $i_* : A(\mathbb{A}^2) \to O(X)$. If X is affine, then by Corollary 3.8, i is an isomorphism, which is obviously not. So X is not affine.

Ex.3.7.(a) This is a special case of part (b). Or, we gave another argument in Ex.3.1(d).
(b) If $Y \cap H = \emptyset$, then $Y \subseteq \mathbb{P}^n - H$. By Ex.3.5, $\mathbb{P}^n - H$ is isomorphic to an affine variety. Because Y is an irreducible closed set $\mathbb{P}^n - H$, Y is also isomorphic to an affine variety. By Ex.3.1(e), Y is a single point, then $\dim Y = 0$, contradiction.

Ex.3.8. Let $Y = \mathbb{P}^n - (H_i \cap H_j)$. Because $\overline{Y} = \mathbb{P}^n$, any regular function on Y has form $\frac{f}{g}$ where f, g are homogeneous polynomials of the same degree and g nonzero on Y. The reason is the same as the one I give here: `https://math.stackexchange.com/questions/373315`. But if $\deg g > 0$, then $n - 1 = \dim Z(g) > \dim H_i \cap H_j = n - 2$, then $Z(g) \nsubseteq H_i \cap H_j$, then g is not nonzero on Y, contradiction. So g has to be constant, and so is f.

Remark: We can replace $H_i \cap H_j$ by any closed set with dimension $\leq n - 2$, and the conclusion still holds.

Ex.3.9. It is easy to use the definition of $\rho_2 : \mathbb{P}^1 \to \mathbb{P}^2$ to show that $Y = Z(y_0y_2 - y_1^2)$. $(y_0y_2 - y_1^2)$ is prime ideal because $y_0y_2 - y_1^2$ is irreducible, so $I(Y) = (y_0y_2 - y_1^2)$ and $S(Y) = k[y_0, y_1, y_2]/(y_0y_2 - y_1^2)$. In $S(Y)$, $\bar{y_0}$ (the class of y_0) is nonzero, nonunit, and irreducible. The proof is to use existence and uniqueness of division by $y_0y_2 - y_1^2$ where we view polynomials as in y_1 with coefficients in $k[y_0, y_2]$. The process is tedious so I will omit it here. Similarly $\bar{y_1}$ and $\bar{y_2}$ are irreducible in $S(Y)$, and we can prove these $\bar{y_i}$ are distinct irreducibles up to multiplication by units. Then since $\bar{y_0}\bar{y_2} = \bar{y_1}^2$ in $S(Y)$, $S(Y)$ is not UFD. But $S(X) = k[x_0, x_1]$ is UFD, so $S(X) \ncong S(Y)$.

Ex.3.10. By Lemma 3.

Ex.3.11. First assume X is affine variety. Closed subvarieties of X containing P are just the affine varieties contained in X containing P. Since $\mathcal{O}_{P,X} \cong A(X)_{\mathfrak{m}_P}$, prime ideals of $\mathcal{O}_{P,X}$ correspond to prime ideals of $A(X)$ contained in $\mathfrak{m}_P$, which correspond to prime ideals of $A = k[x_1, ..., x_n]$ between $I(X)$ and $\mathfrak{m}_P$, which correspond exactly to affine varieties between P and X.
Next, assume X is quasi-affine, then $\mathcal{O}_{P,X} \cong \mathcal{O}_{P,\overline{X}}$, and closed subvarieties of X containing P correspond to closed subvarieties of $\overline{X}$ containing P (One map is given by taking closure in $\overline{X}$, the reverse map is given by restriction to X. Arguments in general topology show these are indeed inverse maps).
Last, assume X is quasi-projective. Suppose $P \in U_i$ where $U_i = Z(x_i)^c$. Let $\varphi_i : U_i \to \mathbb{A}^n$ be the canonical isomorphism, then $\mathcal{O}_{P,X} \cong \mathcal{O}_{\varphi_i(P),\varphi_i(U_i \cap X)}$, closed subvarieties of $\varphi_i(U_i \cap X)$ containing $\varphi_i(P)$ correspond to closed subvarieties of $U_i \cap X$ containing P (because φ_i is isomorphism), which correspond to closed subvarieties of X containing P (under the same maps defined at the end of previous paragraph).

Ex.3.12. If X is quasi-affine, then we have $\mathcal{O}_{P,X} \cong \mathcal{O}_{P,\overline{X}}$ and $\dim X = \dim \overline{X}$, so we reduce to the affine case, which is true by Theorem 3.2(c). If X is quasi-projective, then there exists U_i such that $P \in U_i$

where $U_i = Z(x_i)^c$. Let $\varphi_i : U_i \to \mathbb{A}^n$ be the canonical isomorphism then $\mathcal{O}_{P,X} \cong \mathcal{O}_{\varphi(P),\varphi(X\cap U_i)}$ and $\dim X = \dim \varphi(X \cap U_i)$ (by Ex.2.7, two quasi-projective varieties have the same dimension if they have the same closure, so $\dim X = \dim X \cap U_i$), so we have reduced to the quasi-affine case, which is true by the first sentence.

Ex.3.13. The relation defined in problem is an equivalence relation because any finite number of nonempty open subsets of a variety have nonempty intersection and regular function is continuous. The addition in $\mathcal{O}_{Y,X}$ is defined by $\langle U, f\rangle + \langle V, g\rangle = \langle U\cap V, f+g\rangle$, and multiplication is defined by $\langle U, f\rangle \cdot \langle V, g\rangle = \langle U\cap V, fg\rangle$, and it's easy to verify that these are well defined operations, and that $\mathcal{O}_{Y,X}$ is an integral domain. We claim that $\mathfrak{m} := \{\langle U, f\rangle | f(a) = 0\ \forall a \in U\cap Y\}$ is the unique maximal ideal. First note that $\mathfrak{m}$ is well-defined as a set because if $\langle U, f\rangle = \langle V, g\rangle$ where $f(a) = 0$ for all $a \in U \cap Y$, then because Y is a variety, $(U\cap Y)\cap(V\cap Y) \neq \emptyset$, so g is zero on a nonempty open subset of $V\cap Y$, so by continuity of regular functions, we see that g vanishes on $V \cap Y$. It is straightforward to see that $\mathfrak{m}$ is an ideal. To see $\mathfrak{m}$ is the unique maximal ideal, it suffices to prove that any $\langle U, f\rangle \notin \mathfrak{m}$ is a unit. Pick $a \in U \cap Y$ such that $f(a) \neq 0$. Because f is regular, there exists open set $V \subseteq X$ such that $a \in V$ and $f = \frac{g}{h}$ on V where g, h are polynomials, h nowhere zero on V. Then $\langle U, f\rangle \cdot \langle V - Z(g), \frac{h}{g}\rangle = \langle X, 1\rangle$, the multiplicative identity in $\mathcal{O}_{Y,X}$, so $\mathfrak{m}$ is the unique maximal ideal, and $\mathcal{O}_{Y,X}$ is a local ring.
Define $\varphi : \mathcal{O}_{Y,X} \to K(Y)$ by $\varphi(\langle U, f\rangle) = \langle U \cap Y, f|_{U\cap Y}\rangle$. It's easy to see that this is a well-defined ring homomorphism. φ is surjective because $\forall\langle U, f\rangle \in K(Y)$, $f = \frac{g}{h}$ on some open set $V \subseteq U$ where g, h are polynomials and h is nowhere zero on V. Then $\varphi(\langle X - Z(h), \frac{g}{h}\rangle) = \langle Y - Z(h), \frac{g}{h}\rangle = \langle V, f\rangle = \langle U, f\rangle$. Obviously $\ker\varphi = \mathfrak{m}$. So we have $\frac{\mathcal{O}_{Y,X}}{\mathfrak{m}} \cong K(Y)$.
Finally we show $\dim \mathcal{O}_{Y,X} = \dim X - \dim Y$. First assume X and Y are affine varieties. Then there is a natural map $\psi : A(X)_{I(Y)} \to \mathcal{O}_{Y,X}$ given by $\frac{[f]}{[g]} \mapsto \langle X - Z(g), \frac{f}{g}\rangle$. Note by definition of localization, $g \notin I(Y)$, so $(X - Z(g)) \cap Y \neq \emptyset$. Obviously this is a well-defined ring homomorphism. ψ is surjective because $\forall\langle U, h\rangle \in \mathcal{O}_{Y,X}$, $h = \frac{f}{g}$ on some open subset of X which has nonempty intersection with Y. In particular $g \notin I(Y)$, so $\psi(\frac{[f]}{[g]}) = \langle U, h\rangle$. It's easy to see that ψ is also injective. Thus ψ is an isomorphism and $\dim \mathcal{O}_{Y,X} = \dim A(X)_{I(Y)} = \text{height}\, I(Y)/I(X) = \dim A(X) - \dim \frac{A(X)}{I(Y)/I(X)} = \dim X - \dim A(Y) = \dim X - \dim Y$ where the second step is true by the general fact that dimension of localization of integral domain at prime ideal is equal to height of the ideal.
Next assume X, Y are any quasi-affine varieties. Note we have natural map $\varphi : \mathcal{O}_{\overline{Y},\overline{X}} \to \mathcal{O}_{Y,X}$ given by $\langle U, f\rangle \mapsto \langle U \cap X, f|_{U\cap X}\rangle$. This is well defined because if V is any open set in $\mathbb{A}^n$ and $V \cap \overline{X} \neq \emptyset$, then $V \cap X \neq \emptyset$. It's easy to see φ is ring homomorphism. φ is surjective because $\forall\langle U, f\rangle \in \mathcal{O}_{Y,X}$, there exists some V open in X such that $V \cap Y \neq \emptyset$ and $f = \frac{g}{h}$ on V where g, h are polynomials, h nonzero on V. Then $\varphi(\langle\overline{X} - Z(h), \frac{g}{h}\rangle) = \langle V, f\rangle = \langle U, f\rangle$. φ is injective because if $\varphi(\langle U, f\rangle) = 0$, then $f = 0$ on $U \cap X$. Note X is open in $\overline{X}$ because X is quasi-affine (indeed, $\overline{X}$ is the unique affine variety containing X), so $U \cap X$ is open in U. But U is irreducible, so closure of $U \cap X$ in U equals U, then continuity of f as a regular function on U implies f vanishes on the whole U. Therefore, we see that φ is isomorphism. So $\dim \mathcal{O}_{Y,X} = \dim \mathcal{O}_{\overline{Y},\overline{X}} = \dim \overline{X} - \dim \overline{Y} = \dim X - \dim Y$ where we have used Prop.1.10 in the last step.
Finally, for the projective case, the arguments are are similar as the affine case, except that now we consider $S(X)_{(I(Y))}$, the homogenized version of localization.

Ex.3.14.(a) By Lemma 5, $\mathbb{P}^{n+1}$ minus a hyperplane are all isomorphic no matter what the specific of the hyperplane is. Thus we can assume the embedding of $\mathbb{P}^n$ in $\mathbb{P}^{n+1}$ is given by $Z(x_0)$ where $(x_0, ..., x_{n+1})$ is the homogeneous coordinate on $\mathbb{P}^{n+1}$. Let $P = (a_0 : ... : a_{n+1})$ where $a_0 \neq 0$ because $P \notin \mathbb{P}^n$. For any $Q = (b_0 : ... : b_{n+1}) \in \mathbb{P}^{n+1} - \{P\}$, some calculation shows $\varphi(Q) = (0 : a_1 b_0 - a_0 b_1 : ... : a_{n+1} b_0 - a_0 b_{n+1})$. Note φ is well defined, i.e. it does not depend on the specific representative of the equivalence class of homogeneous coordinates. Define $\psi : k[x_0, ..., x_{n+1}] \to k[x_0, ..., x_{n+1}]$ by $x_0 \mapsto 0, x_i \mapsto a_i x_0 - a_0 x_i$ for all $1 \leq i \leq n+1$. Note ψ sends homogeneous polynomial to homogeneous polynomial of the same degree.
We first show φ is continuous. It suffices to prove $\varphi^{-1}(Z(f) \cap Z(x_0))$ is closed in $\mathbb{P}^{n+1} - \{P\}$ for any homogeneous $f \in k[x_0, ..., x_{n+1}]$. It is easy to verify $\varphi^{-1}(Z(f) \cap Z(x_0)) = Z(\psi(f)) \cap (\mathbb{P}^{n+1} - \{P\})$ because $\forall Q = (b_0 : ... : b_{n+1}) \in \mathbb{P}^{n+1} - \{P\}$, $\psi(f)(Q) = f(\varphi(Q))$. Thus φ is continuous. Then take any V open in $Z(x_0)$ and $f : V \to k$ regular on V. $f \circ \varphi : \varphi^{-1}(V) \to k$ is also regular, because $\forall Q = (b_0 : ... : b_{n+1}) \in \varphi^{-1}(V)$, we can find U open in V, U containing $\varphi(Q)$, such that $f = \frac{g}{h}$ on U where g, h are homogeneous polynomials

of same degree. Then on $\varphi^{-1}(U) \ni Q$, $f \circ \varphi = \frac{\psi(g)}{\psi(h)}$. So φ is a morphism.
(b) Let (x, y, z, w) be homogeneous coordinates on $\mathbb{P}^3$, then in this problem the projection map $\varphi : \mathbb{P}^3 - (0 : 0 : 1 : 0) \to Z(z)$ is $\varphi(x : y : z : w) = (x : y : 0 : w)$. Note $(0 : 0 : 1 : 0) \notin Y$, and $\varphi(Y) = \varphi(\{(t^3 : t^2u : tu^2 : u^3)|t, u \in k, t \neq 0 \,\text{or}\, u \neq 0\}) = \{(t^3 : t^2u : 0 : u^3)|t \neq 0 \,\text{or}\, u \neq 0\}$. We have $\varphi(Y) = Z(z) \cap Z(x^2w - y^3)$. The "$\subseteq$" is obvious. To see $\supseteq$, take $(x_0 : y_0 : z_0 : w_0) \in Z(z) \cap Z(x^2w - y^3)$. Then $x_0^2w_0 = y_0^3$. If $y_0 = 0$, then either $x_0 = 0$ or $w_0 = 0$, so either $(x_0 : y_0 : z_0 : w_0) = (0 : 0 : 0 : 1)$ or $(x_0 : y_0 : z_0 : w_0) = (1 : 0 : 0 : 0)$. In both cases $(x_0 : y_0 : z_0 : w_0) \in \varphi(Y)$. So assume $y_0 \neq 0$, then $x_0 \neq 0$ and $w_0 \neq 0$. Then $\varphi(Y) \ni ((\frac{x_0}{y_0})^3 : (\frac{x_0}{y_0})^2 : 0 : 1) = (\frac{x_0}{w_0} : \frac{y_0}{w_0} : 0 : 1) = (x_0 : y_0 : 0 : w_0) = (x_0 : y_0 : w_0 : z_0)$. Therefore, $\varphi(Y) = Z(z) \cap Z(x^2w - y^3)$, and we see that the projection of Y from P is a cuspidal cubic curve in the plane $Z(z)$.

Ex.3.15. (a) Suppose $X = Z(S_1)$ and $Y = Z(S_2)$ where $S_1 \subset k[x_1, ..., x_n]$, $S_2 \subset k[y_1, ...y_m]$ are finite sets. Write the coordinates on $\mathbb{A}^{n+m}$ as $x_1, ..., x_n, y_1, ..., y_m$, we have $X \times Y = Z(S_1 \cup S_2)$, so $X \times Y$ is closed in $\mathbb{A}^{n+m}$. To see $X \times Y$ is irreducible with induced topology, use the same notations in the hint, suppose $Z_1 = Z(f_1, ..., f_k)$, and fix $x = (a_1, ..., a_n) \in X$, then $Y_{x,1} := \{y \in Y|(x, y) \in Z_1\} = Z(f_1(a_1, ..., a_n, y_1, ..., y_m), ..., f_k(a_1, ..., a_n, y_1, ..., y_m))$ is closed in Y. Similarly, $Y_{x,2} := \{y \in Y|(x, y) \in Z_2\}$ is closed in Y. $Y = Y_1 \cup Y_2$ because $X \times Y = Z_1 \cup Z_2$. Y is irreducible, so either $Y_{x,1} = Y$ or $Y_{x,2} = Y$, so either $x \times Y \subseteq Z_1$ or $x \times Y \subseteq Z_2$, so $X = X_1 \cup X_2$.

Next, for a fixed $y \in Y$, let $X_{y,i} := \{x \in X|(x, y) \in Z_i\}$ for $i = 1, 2$. Because of previous arguments, $X_{y,i}$ is closed in X. Note we have $X_i = \bigcap_{y \in Y} X_{y,i}$ is closed in X. X is irreducible, so $X_1 = X$ or $X_2 = X$, so $Z_1 = X \times Y$ or $Z_2 = X \times Y$. So $X \times Y$ is irreducible, and is affine variety (under induced topology from $\mathbb{A}^{n+m}$).

(b) We first establish $A(X \times Y) \cong A(X) \otimes_k A(Y)$ as k-vector spaces. By the universal property of tensor product, it suffices to prove that, there exists k-bilinear map $\iota : A(X) \times A(Y) \to A(X \times Y)$, such that for any k-vector space V, any k-bilinear map $\varphi : A(X) \times A(Y) \to V$, there exists unique k-linear map $\tilde{\psi} : A(X \times Y) \to V$ such that $\tilde{\psi} \circ \iota = \varphi$.

Define ι by $\iota([f], [g]) = [fg]$. It is easy to check ι is well-defined and is k-bilinear. To define $\tilde{\psi}$, we first define k-linear map $\psi : k[x_1, ..., x_n, y_1, ..., y_m] \to V$ by $\psi(x_i^a y_j^b) = \varphi([x_i^a], [y_j^b])$ and extend ψ by k-linearity.

Suppose $I(X) = (f_1, ..., f_k)$ and $I(Y) = (g_1, ..., g_l)$. By `https://mathoverflow.net/questions/76772/`, $I(X + Y) = (f_1, ..., f_k, g_1, ..., g_l)$. By k-linearity of $\varphi(\_, [1]) : A(X) \to V$ and $\varphi([1], \_) : A(Y) \to V$, we see $\ker \psi \supseteq (f_1, ..., f_k, g_1, ..., g_l)$, so ψ induces k-linear map $\tilde{\psi} : A(X \times Y) \to V$. $\tilde{\psi} \circ \iota = \varphi$ by k-bilinearity of φ. $\tilde{\psi}$ is obviously unique. Above all, $A(X \times Y) \cong A(X) \otimes_k A(Y)$ as k-vector spaces.

The algebra structure on $A(X) \otimes_k A(Y)$ is defined by linearly extending $([f_1] \otimes [g_1]) \cdot ([f_2] \otimes [g_2]) = ([f_1f_2] \otimes [g_1g_2])$. Since the multiplication on $A(X \times Y)$ satisfies the same law, i.e, $[f_1g_1] \cdot [f_2g_2] = [f_1f_2 \cdot g_1g_2]$ and is linear with respect to both multipliers, we see $A(X \times Y) \cong A(X) \otimes_k A(Y)$ as k-algebras.

(c) (i) is obvious. Call the morphism $X \times Y \to X$ by π_1, call the morphism $X \times Y \to Y$ by π_2, call the morphism $Z \to X$ by φ_1, call the morphism $Z \to Y$ by φ_2. Define a function $\psi : Z \to X \times Y$ by $\psi(z) = (\varphi_1(z), \varphi_2(z))$. For each $i = 1, ..., n$, $x_i \circ \psi = x_i \circ \varphi_1$ is regular; for each $j = 1, ..., m$, $y_j \circ \psi = y_j \circ \varphi_2$ is regular. By Lemma 3.6, ψ is morphism. ψ is the obviously unique map making the diagram commute. Thus $X \times Y$ is categorical product.

(d) To prove $\dim X \times Y = \dim X + \dim Y$, we prove $\dim X \times Y \geq \dim X + \dim Y$ using geometry, and prove $\dim X \times Y \leq \dim X + \dim Y$ using algebra.

Specifically, we proved in part (a) that (set-theoretic) product of affine varieties is still affine variety is the larger space, so using definition of dimension of topological space we can take two longest chains of irreducible closed subsets of X and Y, then we consider their products in a suitable order to get $\dim X \times Y \geq \dim X + \dim Y$.

For the other direction, note we stated in part (b) that $I(X \times Y) = I(X) + I(Y)$. Pick longest chains of prime ideals of $I(X)$ and $I(Y)$, concatenate these two chains, we see height $I(X \times Y) \geq$ height $I(X) +$ height $I(Y)$. Because for any integral domain and finitely-generated k-algebra A and prime ideal $\mathfrak{p} \subset A$, $\dim A/\mathfrak{p} =$

$\dim A - \text{height}\,\mathfrak{p}$ (Theorem 1.8A), we see $\dim X \times Y \le n+m-\text{height}\, I(X) - \text{height}\, I(Y) = \dim X + \dim Y$.

Therefore, $\dim X \times Y = \dim X + \dim Y$.

Ex.3.16. We will prove (a) and (b) together. Let $\varphi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{mn+m+n}$ be the Segre embedding defined by $\varphi((a_0 : ... : a_n), (b_0 : ... : b_m)) = (a_0b_0 : a_0b_1 : ... : a_0b_m : ... : a_nb_0 : ... : a_nb_m)$. Let $\psi : k[\{z_{ij}\}] \to k[x_0, ..., x_n, y_0, ..., y_m]$ be the corresponding map mentioned in Ex.2.14. Then we have showed in Ex.2.14 that $\ker\psi$ is a prime homogeneous ideal and φ is a bijection onto $Z(\ker\psi)$. We give $\mathbb{P}^n \times \mathbb{P}^m$ the structure of projective variety by identifying it with $Z(\ker\psi)$ via φ.

Now fix closed sets $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$. We claim $\varphi(X \times Y)$ is closed. In fact $\varphi(X \times Y) = Z(\ker\overline{\psi})$ where $\overline{\psi} : k[\{z_{ij}\}] \to k[x_0, ..., x_n, y_0, ..., y_m]/(I(X) + I(Y))$ is the map induced from ψ. Note $I(X) + I(Y)$ is homogeneous ideal because it is generated by homogeneous elements, so $\ker\overline{\psi}$ is a homogeneous ideal. We first prove $\varphi(X \times Y) \subseteq Z(\ker\overline{\psi})$. Take $P \in X \times Y$ and any homogeneous $f \in \ker\overline{\psi}$, then $f(\varphi(P)) = \psi(f)(P)$, but $\psi(f) \in I(X) + I(Y)$, so $f(\varphi(P)) = 0$. Then we prove $\varphi(X \times Y) \supseteq Z(\ker\overline{\psi})$. We have $\ker\overline{\psi} \supseteq \ker\psi$, so $Z(\ker\overline{\psi}) \subseteq Z(\ker\psi) = \operatorname{im}\varphi$, so for any $Q \in Z(\ker\overline{\psi})$, there exists $P \in \mathbb{P}^n \times \mathbb{P}^m$ such that $\varphi(P) = Q$. Take any homogeneous $f \in I(X)$. Suppose $P = ((a_0 : ... : a_n), (b_0 : ... : b_m))$. Pick i, j such that $a_i \ne 0$, $b_j \ne 0$. Let $\tilde{f}$ be the polynomial got from f by replacing each x_i with z_{ij} where j is our fixed choice. Then $\psi(\tilde{f}) = y_j^{\deg f} f \in I(X) \subseteq I(X) + I(Y)$, so $\tilde{f}(Q) = 0$. But $\tilde{f}(Q) = \tilde{f}(\varphi(P)) = \psi(\tilde{f})(P) = b_j^{\deg f} f(a_0 : ... : a_n)$ and $b_j \ne 0$, so $f(a_0 : ... : a_n) = 0$. So $(a_0 : ... : a_n) \in Z(I(X)) = X$. Similarly we can show $(b_0 : ... : b_m) \in Y$. Thus $P \in X \times Y$. So we have shown $\varphi(X \times Y) = Z(\ker\overline{\psi})$.

In particular, when X and Y are projective varieties, $I(X)$ and $I(Y)$ are prime ideals, then $I(X) + I(Y)$ is prime in $k[x_0, ..., x_n, y_0, ..., y_m]$ by this link: `https://mathoverflow.net/questions/76772/`. Then $\ker\overline{\psi}$ is a prime ideal, so $\varphi(X \times Y)$ is projective variety. This proves part (b).

For part (a), when X and Y are quasi-projective varieties, we have $X = C_1 \cap U_1$ and $Y = C_2 \cap U_2$ where C_1, C_2 are projective varieties and U_1, U_2 are open sets in projective space. Then $\varphi(X \times Y) = \varphi((C_1 \cap U_1) \times (C_2 \cap U_2)) = \varphi(C_1 \times C_2) - \varphi(C_1 \times (C_2 - U_2)) - \varphi((C_1 - U_1) \times C_2)$ where the last step is true because φ is bijection onto $Z(\ker\psi)$. In the last step, the latter two sets are closed subsets of $\varphi(C_1 \times C_2)$, and $\varphi(C_1 \times C_2)$ is projective variety, so $\varphi(X \times Y)$ is a quasi-projective variety.

(c) First, let us study in more detail the variety structure of $X \times Y$. Let $f \in k[(z_{ij})]$ be homogeneous, then $\varphi^{-1}(Z(f) \cap \varphi(X \times Y)) = Z(\psi(f)) \cap (X \times Y)$, so closed subsets of $X \times Y$ include zero sets of polynomials homogeneous in both $x_0, ..., x_n$ and $y_0, ..., y_m$ and having the same homogeneous degree in $x_0, ..., x_n$ and $y_0, ..., y_m$. Furthermore, f can have different homogeneous degree in $x_0, ..., x_n$ and $y_0, ..., y_m$ because zero set of such f (which is obviously well defined) is intersection, over i, of zero sets of polynomials equal to f multiplied by certain powers of x_i (or y_i). Next we study possible forms of regular functions on $X \times Y$. Let $U \subseteq X \times Y$ be open, (re)define $f : U \to k$ to be a function which locally is equal to quotient of two polynomials in $x_0, ..., x_n$ and $y_0, ..., y_m$, so that both polynomials are homogeneous in $x_0, ..., x_n$ and $y_0, ..., y_m$, and both polynomials have the same degree. Consider $f \circ \varphi^{-1} : \varphi(U) \to k$. Choose $\varphi(P) \in \varphi(U)$ and suppose $\varphi(P)_{ab} \ne 0$, let $\varphi(U') = \varphi(U) - Z(z_{ab})$, then $\varphi^{-1}|_{\varphi(U')}(c_{ij}) = ((c_{0j}, ..., c_{nj}), (c_{i0}, ..., c_{im}))$. Then it is obvious that $f \circ \varphi^{-1}|_{\varphi(U')}$ is regular on $\varphi(U')$. P is arbitrarily chosen, so $f \circ \varphi^{-1}$ is regular on $\varphi(U)$, so f is regular on $X \times Y$.

Now that we know a bit about topology and regular functions of $X \times Y$ (we happily note that these results coincide with our natural expectation), we proceed to prove $X \times Y$ is categorical product. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be projections. We prove π_1 is morphism. Let $f \in k[x_0, ..., x_n]$ be homogeneous, then $\pi_1^{-1}(Z(f)) = Z(f)$ is closed in $X \times Y$. Let $U \subseteq X$ be open, $f : U \to k$ be regular function, pick $P \in \pi_1^{-1}(U)$, then there exists open subset $V \subseteq U$ such that $\pi_1(P) \in V$ and $f = \frac{g}{h}$ on V where g, h are homogeneous of same degree. Then $(f \circ \pi_1)|_{\pi_1^{-1}(V)}$ is equal to quotient of 2 polynomials in $x_0, ..., x_n$ of same degree, so $f \circ \pi_1$ is regular on $\pi_1^{-1}(U)$, so π_1 is a morphism. Similarly, π_2 is a morphism.

For any variety Z and morphism $\varphi_1 : Z \to X$, $\varphi_2 : Z \to Y$, define $\varphi^* : Z \to X \times Y$ by $\varphi^*(z) = (\varphi_1(z), \varphi_2(z))$. We use Lemma 5 to prove φ^* is a morphism. Choose a regular function $\frac{z_{ij}}{z_{ab}}$ on $\varphi(X \times Y) - Z(z_{ab})$. Then $\frac{z_{ij}}{z_{ab}} \circ \varphi \circ \varphi^*$ is defined on $Z - (\varphi_1^{-1}(Z(x_a)) \cup \varphi_2^{-1}(Z(y_b)))$, which is open in Z. Furthermore, $\frac{z_{ij}}{z_{ab}} \circ \varphi \circ \varphi^* = (\frac{x_i}{x_a} \circ \varphi_1) \cdot (\frac{y_j}{y_b} \circ \varphi_2)$ which is regular on its domain, so by Lemma 5, φ^* is a morphism. The uniqueness of φ^*

is trivial.

Above all, for quasi-projective varieties X and Y, identification of the set-theoretic product $X \times Y$ with its image under the Segre embedding is the categorical product of X and Y.

Lemma 7: Let A be an integrally closed domain and $S \subset A$ be a multiplicative subset not containing 0. Then $S^{-1}A$ is integrally closed domain.
Proof: It is easy to test that $S^{-1}A$ is an integral domain. Let K be fractional field of A. Note K is also fractional field of $S^{-1}A$. Let $a \in K$ be in the integral closure of $S^{-1}A$ in K, then there exists monic polynomial $f(x) = x^n + c_{n-1}x^{n-1} + ... + c_1 x + c_0 \in S^{-1}A[x]$ such that $f(a) = 0$. Let c be the product of all denominators of c_i. Since $c^n f(a) = 0$, ca is in the integral closure of A in K. But A is integrally closed, so $ca \in A$. So $a = ca \cdot c^{-1} \in S^{-1}A$. So $S^{-1}A$ is integrally closed.

Lemma 8: Let R be a UFD, then R is integrally closed.
Proof: Indeed, suppose there is $a \in K(R)$ and $f(x) = x^m + c_{m-1}x^{m-1} + ... + c_1 x + c_0 \in R[x]$ such that $f(a) = 0$. FSOC, suppose $a = \frac{a_1}{a_2}$ where $\gcd(a_1, a_2) = 1$, and a_2 is not unit. $a_2^m f(a) = 0$, so $-a_1^m = c_{m-1}a_2 a_1^{m-1} + ... + c_1 a_2^{m-1} a_1 + c_0 a_2^m$. a_2 divides the RHS, but a_2 does not divide the LHS, contradiction. Thus $a \in R$, and R is integrally closed.

Lemma 9: Let A be an integral domain. Then $A = \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$ where $\mathfrak{m}$ runs through all maximal ideals of A and all $A_{\mathfrak{m}}$ are embedded in $K(A)$.
Remark: This result appeared at the end of proof of Theorem 3.2 in the book.
Proof: "$\subseteq$" is trivial. For the other direction, pick any $\frac{a}{a'} \in \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$. Define $I := \{x \in A | \, xa \in (a')\}$. This is obviously an ideal of A. If $I \neq A$, then $I \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset A$ (by Zorn's Lemma). Because $\frac{a}{a'} = \frac{a''}{s}$ for some $\frac{a''}{s} \in A_{\mathfrak{m}}$, $as = a'a'' \in (a')$, so $s \in I \subseteq \mathfrak{m}$. But $s \notin \mathfrak{m}$ by our choice of $\frac{a''}{s}$. Contradiction. So $I = A$. In particular, $1 \in I$, so $a \in (a')$ and $\frac{a}{a'} \in A$.

Lemma 10: By Proposition 3.5 in the book, for any variety X and any affine variety Y, we have natural bijection $\alpha : \mathrm{Hom}(X, Y) \cong \mathrm{Hom}(A(Y), \mathcal{O}(X))$. I will prove furthermore that α induces a bijection between dense morphisms and injective k-algebra homomorphisms.
Proof: Suppose φ is dense. Suppose $\bar{f} \in A(Y)$ satisfies $\alpha(\varphi)(\bar{f}) = 0$. Indeed, $\alpha(\varphi)(\bar{f}) = f \circ \varphi = 0$, so $Z(f) \supseteq \overline{\mathrm{im}\, \varphi} = Y$, so $\bar{f} = 0$, so $\alpha(\varphi)$ is injective. Conversely, suppose $\varphi \in \mathrm{Hom}(A(Y), \mathcal{O}(X))$ is injective. Suppose $Y \subseteq \mathbb{A}^n$ with coordinates $y_1, ..., y_n$. To prove $\alpha^{-1}(\varphi)$ is dense, it suffices to take any $f \in k[y_1, ..., y_n]$ such that $Z(f) \supseteq \mathrm{im}\, \alpha^{-1}(\varphi)$ and prove $Z(f) \supseteq Y$. Take any $x \in X$, then $0 = f(\alpha^{-1}(\varphi)(x)) = f(\varphi(\bar{y_1})(x), ..., \varphi(\bar{y_n})(x)) = \varphi(\bar{f})(x)$, so $\varphi(\bar{f}) = 0$. φ is injective, so $\bar{f} = 0$, so $f \in I(Y)$. So we are done.

Lemma 11: Let A be integral domain and $\bar{A}$ be integral closure of A in $K(A)$. We prove a universal property of $\bar{A}$: The inclusion of A in $\bar{A}$ is initial in the category where objects are inclusions of A in integrally closed domains and morphisms are ring homomorphisms that make commutative diagrams.
Proof: Denote the inclusion of A in $\bar{A}$ by i_0. Let $i_1 : A \to B$ be any object in this category. By the universal property of localization, we have embedding $i_2 : K(A) \to K(B)$ (Note that to invoke this universal property we need $A \to B$ to be injective). Because B is integrally closed and each element of $\bar{A}$ is killed by a monic polynomial in A, we have $i_2(\bar{A}) \subseteq B$. This proves existence. For uniqueness, suppose $i : \bar{A} \to B$ is any ring homomorphism such that $i \circ i_0 = i_1$. i must be injective, otherwise pick nonzero $a \in \ker i$, and let $f \in A[x]$ be a monic polynomial with nonzero constant term such that $f(a) = 0$. Then $0 = i(f(a)) = f(0)$, contradiction. So i must be injective. Denote the inclusion of $\bar{A}$ in $K(A)$ by i_3 and inclusion of B in $K(B)$ by i_4. Because $K(A)$ is also fraction field of $\bar{A}$, by universal property of localization we have an embedding $i_5 : K(A) \to K(B)$ such that $i_5 \circ i_3 = i_4 \circ i$. By universal property of localization (of the pair $i_3 \circ i_0 : A \to K(A)$), i_5 is the unique map such that $i_5 \circ (i_3 \circ i_0) = i_4 \circ i_1$. Since i is induced from i_5, we conclude that i is unique, so we are done.

Ex.3.17. (a) Let C be a conic. By Ex.3.1(c), we have $C \cong \mathbb{P}^1$. Using $\mathbb{A}^1 \cong U_i$ where $U_i = Z(x_i)^c$, we see $\mathcal{O}_{P,C} \cong \mathcal{O}_{Q,\mathbb{A}^1} = k[x]_{\mathfrak{m}_Q}$ for some $Q \in \mathbb{A}^1$. By Lemma 7 and Lemma 8, $\mathcal{O}_{P,C}$ is integrally closed.

(b) Pick any $P \in Q_1$. WLOG, suppose $P \notin Z(x)$. Define $Y_1 := Z(y - zw) \subset \mathbb{A}^3$, then $\mathcal{O}_{P,Q_1} \cong \mathcal{O}_{P',Y_1}$ for some $P' \in Y_1$. $A(Y_1) = k[y, z, w]/(y - zw) \cong k[z, w]$, so $\mathcal{O}_{P',Y_1} = A(Y_1)_{\mathfrak{m}_{P'}} = k[z, w]_{\mathfrak{m}}$ where $\mathfrak{m}$ is some maximal ideal of $k[z, w]$. By Lemma 7 and Lemma 8, $\mathcal{O}_{P,Q_1}$ is integrally closed.

Next, pick $Q \in Q_2$. If $Q \notin Z(x)$, let $Y = Z(y - z^2) \subset \mathbb{A}^3$ then $\mathcal{O}_{Q,Q_2} \cong \mathcal{O}_{Q',Y}$ for some $Q' \in Y$. $A(Y) = k[y,z,w]/(y-z^2) = k[z,w]$, so $\mathcal{O}_{Q,Q_2} = k[z,w]_{\mathfrak{m}}$ where $\mathfrak{m}$ is some maximal ideal of $k[z,w]$. By Lemma 7 and Lemma 8, $\mathcal{O}_{Q,Q_2}$ is integrally closed. The argument is the same if $Q \notin Z(y)$. If $Q \in Z(x,y)$, then $Q = (0:0:0:a)$ for some $a \in k - \{0\}$. Let $Y = Z(xy - z^2) \subset \mathbb{A}^3$. Using the affine chart $Z(w)^c$, $\mathcal{O}_{Q,Q_2} \cong \mathcal{O}_{Q',Y} = A(Y)_{\mathfrak{m}_{Q'}}$ for some $Q' \in Y$. $A(Y)$ is integrally closed. Indeed, $A(Y) = k[x,y,z]/(xy - z^2)$. Let $\varphi : k[x,y,z] \to k[t,s]$ (t, s are variables) be $x \mapsto t^2$, $y \mapsto s^2$, $z \mapsto ts$. Then φ induces $A(Y) \cong R \subset k[t,s]$ where R is the subring of polynomials with vanishing odd-degree terms. If $\frac{f}{g} \in K(R)$ is killed by some monic polynomial in $R[x]$, then since $k[t,s]$ is integrally closed, $\frac{f}{g} \in k[t,s]$. So $f = gh$ for some $h \in k[t,s]$. We must have $h \in R$, otherwise gh contains some non-vanishing odd-degree term. Thus $\frac{f}{g} \in R$, so R is integrally closed, and so is $A(Y)$. By Lemma 7, $\mathcal{O}_{Q,Q_2} \cong A(Y)_{\mathfrak{m}_{Q'}}$ is integrally closed, so we are done.

(c) Let Y be the curve in question. We first note that $\mathcal{O}_{(0,0),Y}$ is a noetherian local domain of dimension 1. FSOC, suppose $\mathcal{O}_{(0,0),Y}$ is integrally closed, then by Theorem 6.2A, $\mathcal{O}_{(0,0),Y}$ is regular local ring, which means $(0,0)$ is a nonsingular point of Y, which obviously is not. So $\mathcal{O}_{(0,0),Y}$ is not integrally closed.

(d) The forward direction follows from Lemma 9. The reverse direction follows from Lemma 7.

(e) Let B be integral closure of $A(Y)$ in $K(A(Y))$. By Theorem 3.9A, B is a finitely-generated k-algebra. B is also integral domain, so $B \cong k[x_1, ..., x_n]/\mathfrak{p}$ for some n and prime ideal $\mathfrak{p}$. Let $\widetilde{Y} := Z(p) \subset \mathbb{A}^n$. $A(\widetilde{Y}) = B$ which is integrally closed, so $\widetilde{Y}$ is normal by part (d). By proposition 3.5, the inclusion $A(Y) \hookrightarrow A(\widetilde{Y})$ induces morphism $\pi : \widetilde{Y} \to Y$ (which is dense by Lemma 10). Denote the inclusion $A(Y) \hookrightarrow A(\widetilde{Y})$ by π^*.

Now we prove the universal property. Take any $\varphi : Z \to Y$ where φ is dense and Z is normal. This induces $\varphi^* : A(Y) \to \mathcal{O}(Z)$ which is injective by Lemma 10. $\mathcal{O}(Z)$ is integrally closed because $\mathcal{O}(Z) = \bigcap_{P \in Z} \mathcal{O}_{P,Z}$ (Considered in $K(Z)$. Also $K(\mathcal{O}(Z))$ may not be equal to $K(Z)$, but we can always embed $K(\mathcal{O}(Z))$ in $K(Z)$.). Then we are done by Lemma 11 and the natural bijection in Proposition 3.5.

Lemma 12: Let A be a integral domain graded over $\mathbb{Z}$. Let B be the subring of A consisted of elements of degree 0. If A is integrally closed, then B is integrally closed.
Proof: Suppose $\frac{a_1}{a_2} \in K(B)$ is killed by a monic polynomial $f \in B[x]$. Under the natural embedding $B \hookrightarrow A$ and $K(B) \hookrightarrow K(A)$, $\frac{a_1}{a_2} \in K(A)$ is killed by $f \in A[x]$. A is integrally closed, so $\frac{a_1}{a_2} = \frac{a}{1}$ for some $a \in A$. So $a_1 = a_2 a$. Consider the degree on both sides, we get $a \in B$. So $\frac{a_1}{a_2} \in B$. So B is integrally closed.

Ex.3.18.(a) Fix $P \in Y$. Let $\mathfrak{m}_P$ be the prime homogeneous ideal $I(P)/I(Y) \in S(Y)$. Let $T \subset S(Y)$ be the homogeneous elements not in $\mathfrak{m}_P$. Then T is a multiplicative subset, and $T^{-1}(S(Y))$ is integrally closed by assumption and Lemma 7. By Theorem 3.4, $\mathcal{O}_{P,Y} \cong S(Y)_{(\mathfrak{m}_p)}$ where $S(Y)_{(\mathfrak{m}_p)}$ is the subring of elements of degree 0 in $T^{-1}(S(Y))$. By Lemma 12, $\mathcal{O}_{P,Y}$ is integrally closed. So Y is normal.

(b) Let $\rho : \mathbb{P}^1 \to Y$ be $\rho(t:u) = (t^4 : t^3u : tu^3 : u^4)$. We first prove ρ is injective. Suppose $\rho(t:u) = \rho(a:b)$. If $t^4 = 0$, then $a^4 = 0$, so $(t:u) = (a:b)$. If $u^4 = 0$, then $b^4 = 0$, so $(t:u) = (a:b)$. If $t^4 \neq 0$ and $u^4 \neq 0$, then $t, u, a, b \neq 0$. Then $\frac{t^4}{a^4} = \frac{t^3u}{a^3b}$, so $\frac{t}{u} = \frac{a}{b}$, so $\frac{t}{a} = \frac{u}{b}$, so $(t:u) = (a:b)$ and ρ is injective. ρ is obviously surjective.

Let $\theta : k[x,y,z,w] \to k[t,u]$ be $x \mapsto t^4$, $y \mapsto t^3u$, $z \mapsto tu^3$, $w \mapsto u^4$. We have $Y = Z(\ker\theta)$. Indeed, "$\subseteq$" is obviously true. For "$\supseteq$", take $P = (x_0 : y_0 : z_0 : w_0) \in Z(\ker\theta)$. If $x_0 = 0$, then since $y^4 - xw, z^4 - xw^3 \in \ker\theta$, $y_0 = z_0 = 0$. So $P = \rho(0:1) \in Y$. If $x_0 \neq 0$, then since $y^3 - x^2z, y^4 - x^3w \in \ker\theta$, we have $\rho(x_0 : y_0) = (x_0^4 : x_0^3y_0 : x_0y_0^3 : y_0^4) = (x_0^4 : x_0^3y_0 : x_0^3z_0 : x_0^3w_0) = (x_0 : y_0 : z_0 : w_0)$. So $Y = Z(\ker\theta)$. $\ker\theta$ is a prime homogeneous ideal, so Y is a projective variety.

For any homogeneous $f \in k[x,y,z,w]$, $\rho^{-1}(Z(f)) = Z(\theta(f))$, so ρ is continuous. Pullback of regular functions on Y via ρ is again regular function because ρ is defined using homogeneous polynomials of the same degree, so ρ is a morphism. By analysis of the second paragraph, $\rho^{-1}(x_0 : y_0 : z_0 : w_0) = (x_0 : y_0)$ when $x_0 \neq 0$ and , $\rho^{-1}(x_0 : y_0 : z_0 : w_0) = (z_0 : w_0)$ when $w_0 \neq 0$, so ρ^{-1} is a morphism. So ρ is an isomorphism. We showed in Ex.3.17(a) that $\mathbb{P}^1$ is normal, so Y is normal. On the other hand, $S(Y) \cong \operatorname{im}\theta$. $t^2u^2 = \frac{(t^3u)^2}{t^4} \in K(\operatorname{im}\theta)$, $t^2u^2 \notin \operatorname{im}\theta$, but $x^2 - t^4u^4 \in \operatorname{im}\theta[x]$ kills t^2u^2, so $\operatorname{im}\theta$ is not integrally closed. So Y is not projectively normal.

(c) Already shown in (b).

Ex.3.19.(a) φ induces k-algebra isomorphism $\varphi^* : \mathcal{O}_{\mathbb{A}^n} \to \mathcal{O}_{\mathbb{A}^n}$. Since $\mathcal{O}_{\mathbb{A}^n} \cong A(\mathbb{A}^n) = k[x_1, ..., x_n]$, we have isomorphism $\varphi^* : k[x_1, ..., x_n] \to k[x_1, ..., x_n]$ defined by $x_i \mapsto f_i$. Let $g_i = (\varphi^*)^{-1}(x_i)$. Let M and N be n-by-n matrices defined by $M_{ij} = \frac{\partial g_i}{\partial x_j}(f_1, ..., f_n)$, $N_{ij} = \frac{\partial f_i}{\partial x_j}$. Then $(MN)ij = \sum_{k=1}^n \frac{\partial g_i}{\partial x_k}(f_1, ..., f_n)\frac{\partial f_k}{\partial x_j} = \frac{\partial (g_i(f_1,...,f_n))}{\partial x_j} = \frac{\partial}{\partial x_j}((\varphi^* \circ (\varphi^*)^{-1})(x_i)) = \frac{\partial x_i}{\partial x_j} = \delta_{ij}$, so $MN = I$. So $\det N$ is a unit, so $\det N$ is a nonzero constant polynomial.

Ex.3.20.(a) By Proposition 4.3, we can assume Y is affine variety. I consulted `https://math.stackexchange.com/questions/1791250` to get the algebraic result here: `https://stacks.math.columbia.edu/tag/031T`, where the second point implies $\mathcal{O}_{P,Y} = A(Y)_{\mathfrak{m}_P} = \bigcap_{\text{height }\mathfrak{p}=1}(A(Y)_{\mathfrak{m}_P})_{\mathfrak{p}} = \bigcap_{\mathfrak{p}\subseteq\mathfrak{m}_P,\text{height }\mathfrak{p}=1} A(Y)_{\mathfrak{p}}$. Each prime ideal $\mathfrak{p}$ contained in $\mathfrak{m}_P$ with height 1 in $A(Y)$ correspond to an affine variety containing P, contained in Y, with dimension $= \dim Y - 1 \geq 1$. In particular, there exists $Q \in Z(\mathfrak{p}) - P$. Suppose $f = \frac{g}{h}$ around Q. Then $h(Q) \neq 0$, so $\frac{g}{h} \in A(Y)_{\mathfrak{p}}$. So $f \in \mathcal{O}_{P,Y}$, so f can be extended to P.
Remark: I spent a number of hours on this problem trying to work with local expression of f and find a monic polynomial in $\mathcal{O}_{P,Y}$ killing $f \in K(Y)$. But the problem becomes so easy after knowing the commutative algebra result, which translates the normal condition in a suitable way. I think it is a demonstration of the power (and necessity) of algebra.

(b) Let $Y = \mathbb{A}^1$, $P = \{0\}$, $f : Y - \{p\} \to k$ be $a \mapsto a^{-1}$. If f can be extended to the whole Y, then $f \in \mathcal{O}(Y) = A(Y) = k[x]$ and $f(a) = \frac{1}{a}$ for all $a \in k^*$, so $xf(x) - 1$ is zero polynomial. Then $x \in k[x]$ is a unit, which is false. So f cannot be extended to the whole Y.

Lemma 13: Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be quasi-affine varieties. If $\varphi : X \to Y$ is a function such that for all $P \in X$, there exists open set $U \subseteq X$ such that $P \in U$ and $\varphi|_U(a_1, ..., a_n) = (\frac{f_1}{g_1}, ..., \frac{f_m}{g_m})$ where the f_i and g_i are polynomials, then φ is a morphism.
Proof: Take any $f \in k[y_1, ..., y_m]$ and $P \in X$. Choose U in the same way as described in the lemma. Then $\varphi^{-1}(Z(f)) \cap U = (f \circ \varphi|_U)^{-1}(0)$ is closed in U because $f \circ \varphi|_U$ is regular on U and regular function is continuous. Because a subset of a topological space X is closed if and only if it is closed in a family of open sets which form a covering of X, $\varphi^{-1}(Z(f))$ is closed in X, so φ is continuous. Pullback of regular functions on open subsets of Y via φ is obviously again regular function. So φ is morphism.
Remark: It is not hard to show that a function between any two varieties is a morphism if it is locally equal to a rational function at each coordinate. Of course more care is needed when the domain is in projective space. This result shows that functions between varieties that look "nice" locally are indeed morphisms.

Ex.3.21.(a) By Lemma 13, μ is a morphism. $\mathbb{A}^1$ is obviously a group under μ whose identity is 0 (additive identity in k) and inverse map is defined by $y \mapsto -y$ (the additive inverse of y in k). The map $y \mapsto -y$ is a morphism by Lemma 13. So $\mathbf{G}_a$ is a group variety.

(b) Let $Y = \mathbb{A}^1 - \{(0)\}$. We give $Y \times Y$ the structure of quasi-affine variety by viewing it as a subset of $\mathbb{A}^2$. By Lemma 13, μ is a morphism. Y is obviously a group under μ whose identity is 1 (multiplicative identity in k) and inverse map is defined by $y \mapsto y^{-1}$ (the multiplicative inverse of y in k^*). The map $y \mapsto y^{-1}$ is a morphism by Lemma 13. So $\mathbf{G}_m$ is a group variety.

(c) Given $f, g \in \mathrm{Hom}(X, G)$, define group operation on $\mathrm{Hom}(X, G)$ by $(f \cdot g)(x) = f(x) \cdot g(x)$. By Ex.3.15 and Ex.3.16, $G \times G$ with canonical projection to its two factors is categorical product, so the function $X \to G \times G$ defined by $x \mapsto (f(x), g(x))$ is a morphism. Composing this morphism with μ, we see $x \mapsto f(x) \cdot g(x)$ is a morphism. Let identity on $\mathrm{Hom}(X, G)$ be the map sending every element to the identity of G. This is obviously a morphism. Let the inverse of $f \in \mathrm{Hom}(X, G)$ be $x \mapsto f(x)^{-1}$. This is a morphism because it is the composition of two morphisms. Under these prescriptions, it is obvious that $\mathrm{Hom}(X, G)$ becomes a group.

(d) Any $f \in \mathrm{Hom}(X, \mathbf{G}_a)$ is also a regular function on X. Indeed $\mathrm{id} : \mathbf{G}_a \to k$ is a regular function, so the pullback $\mathrm{id} \circ f = f$ is also a regular function. Conversely, given $f \in \mathcal{O}(X)$, f is a morphism from X to $\mathbf{G}_a$ because regular function is continuous and f is locally equal to quotient of two polynomials. Thus $\mathrm{Hom}(X, \mathbf{G}_a) \cong \mathcal{O}(X)$.

(e) If $f \in \mathcal{O}(X)$ is not equal to 0 anywhere, then the function $X \to k$ defined by $x \mapsto f(x)^{-1}$ is obviously a regular function. Conversely, if $f \in \mathcal{O}(X)^*$, f is not equal to 0 anywhere. So $\mathcal{O}(X)^*$ is exactly those regular functions on X which are not equal to 0 anywhere.

Any $f \in \mathcal{O}(X)^*$ is a morphism from X to $\mathbf{G}_m$, because regular function is continuous and f is locally equal to quotient of two polynomials. Conversely, any $f \in \mathrm{Hom}(X, \mathbf{G}_m)$ is also a regular function on X because the pullback of the inclusion $\iota : \mathbb{A}^1 - \{(0)\} \to k$ via f is equal to f. f is nowhere equal to 0, so $f \in \mathcal{O}(X)^*$. Thus $\mathrm{Hom}(X, \mathbf{G}_m) \cong \mathcal{O}(X)^*$.

4 Rational Maps

Ex.4.1. Let h be the function on $U \cup V$ which is f on U and g on V. For any point $P \in U \cup V$, if $P \in U$, then $f = \frac{f_1}{f_2}$ on some open set $U' \subseteq U \subseteq X$, and so is h. Similar argument follows if $P \in V$. So h is regular. Now if $\langle U, f \rangle \in K(X)$, let $(U_i, f_i)_{i \in I}$ be the set of all pairs of open subsets and regular functions in the equivalence class of $\langle U, f \rangle \in K(X)$. Consider $\langle \bigcup_{i \in I} U_i, f^* \rangle$ where f^* is defined using the f_i's. Then f^* is a well-defined regular function, so $\bigcup_{i \in I} U_i$ is the domain of definition of f.

Ex.4.2. Let $\varphi = \langle U, \varphi' \rangle$ be a rational map from X to Y. Let $(U_i, \varphi_i)_{i \in I}$ be the set of all pairs of open subsets of X and morphisms in the equivalence class of $\langle U, \varphi' \rangle$. Consider $\varphi^* : \bigcup_{i \in I} U_i \to Y$ defined by using the φ_i's. For any V open in Y, $(\varphi^*)^{-1}(V) = \bigcup_{i \in I} \varphi_i^{-1}(V)$ is open in $\bigcup_{i \in I} U_i$, so φ^* is continuous. We can also easily show φ^* is a morphism, using the fact that all the φ_i's are morphisms.

Ex.4.3.(a) f is defined on the open set $Z(x_0)^c$. We note that, as an element of $K(\mathbb{P}^2)$, domain of f cannot be further extended. Indeed, if $\langle U, h \rangle = \langle Z(x_0)^c, f \rangle$ where $U \cap Z(x_0) \neq \emptyset$, then pick $P \in U \cap Z(x_0)$, then there is nonempty open subset $V \subseteq \mathbb{P}^2$ such that $h = \frac{h_1}{h_2} = \frac{x_1}{x_0}$ on V where h_1, h_2 are homogeneous of same degree, h_2, x_0 do not vanish on V, $h_2(P) \neq 0$. This means the polynomial $x_0 h_1 - x_1 h_2$ kills V, which is dense in $\mathbb{P}^2$, so $x_0 h_1 - x_1 h_2 \in I(\mathbb{P}^2) = 0$, so we have $x_0 h_1 = x_1 h_2$ as polynomials. But $x_0 \nmid h_2$ since $h_2(P) \neq 0$. Contradiction! Thus $Z(x_0)^c$ is the maximal domain of definition of f.

(b) Note f is a continuous function from $Z(x_0)^c$ to $\mathbb{A}^1$ because regular function is continuous. It is straightforward to show f is a morphism from $Z(x_0)^c$ to $\mathbb{A}^1$ and f is surjective, so $\langle Z(x_0)^c, f \rangle$ is a dominant rational map from $\mathbb{P}^2$ to $\mathbb{A}^1$. The embedding of $\mathbb{A}^1$ in $\mathbb{P}^1$ is also obviously a dominant rational map, so composing these two maps we get dominant rational map $\varphi : \mathbb{P}^2 \to \mathbb{P}^1$. Explicitly, $\varphi = \langle Z(x_0)^c, f \rangle$ where $f(x_0 : x_1 : x_2) = (x_0 : x_1)$. We note that the set of points where φ is defined is strictly larger than the set of points where f is defined! Indeed, $\varphi = \langle (Z(x_0) \cap Z(x_1))^c, f \rangle$ where f has the same definition as before. Note φ cannot be extended to the whole $\mathbb{P}^2$, due to a similar argument as in part (a). Specifically, suppose it can be extended to $(0 : 0 : 1)$, then WLOG suppose $\varphi(0 : 0 : 1) = (a_0 : a_1)$ where $a_0 \neq 0$, then pull back the regular function $\frac{x_1}{x_0}$ on $\mathbb{P}^1$ and we see locally around $(0 : 0 : 1)$, $(\frac{x_1}{x_0}) \circ \varphi = \frac{h_1}{h_2}$ where h_1, h_2 are homogeneous polynomials of same degree. But we also have $\varphi = \langle (Z(x_0) \cap Z(x_1))^c, f \rangle = \langle (Z(x_0)^c, f \rangle$, so pull back $\frac{x_1}{x_0}$ using this description gives us $(\frac{x_1}{x_0}) \circ \varphi = \frac{x_1}{x_0}$ on $Z(x_0)^c$. Then similarly as argued in part (a), $x_0 h_1 = x_1 h_2$ as polynomials. This implies $x_0 | h_2$, but $h_2(0 : 0 : 1) \neq 0$, contradiction! Therefore the maximal domain of definition of φ is $\mathbb{P}^2 - (0 : 0 : 1)$.

Ex.4.4. We first prove the equivalence mentioned in the parenthesis: a variety Y is birationally equivalent to $\mathbb{P}^n$ iff $K(Y)$ is a purely transcendental extension of k. First suppose Y is birationally equivalent to $\mathbb{P}^n$, then by Corollary 4.5, $K(Y) \cong K(\mathbb{P}^n)$. By the result in section 3, $K(\mathbb{P}^n) \cong k[x_0, ..., x_n]_{((0))}$. Because these isomorphisms are all over k, it suffices to prove $k[x_0, ..., x_n]_{((0))}$ is a purely transcendental extension of k. This is true because $\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}$ is a pure transcendental basis. Conversely, suppose $K(Y)$ is a pure transcendental extension of k. We can assume Y is affine variety, because if it is quasi-projective, we can use the affine charts in $\mathbb{P}^n$ to map Y to an isomorphic quasi-affine variety in $\mathbb{A}^n$, then $K(Y)$ is isomorphic to function field of closure of this quasi-affine variety. Then, using result of section 3, we know transcendence degree of $K(Y)$ is finite, say $K(Y) = k(a_1, ..., a_m)$ where the a_i's are algebraically independent. We also have $K(\mathbb{P}^m) = k(b_1, ..., b_m)$ where $b_1, ..., b_m$ are algebraically independent. This is explained in the forward direction. Thus we see $K(Y) \cong K(\mathbb{P}^m)$. By Corollary 4.5, Y is birationally equivalent to $\mathbb{P}^m$.

(a) A stronger conclusion follows from Ex.3.1(c). Below is another approach.

Let C be our conic. WLOG, Suppose $U_0 \cap C \neq \emptyset$ where $U_0 = Z(x_0)^c$. Then $K(C) \cong k(U_0 \cap C) \cong K(\varphi^{-1}(U_0 \cap C)) = K(Y)$ where φ is the canonical isomorphism between $\mathbb{A}^2$ and U_0, and we have let $Y := \varphi^{-1}(U_0 \cap C)$ be the affine variety. Furthermore, $Y = Z(f)$ where f is the affinization (with respect to x_0) of the homogeneous polynomial defining C. f is irreducible. (There are two ways to see this. First, if f is reducible, then Y becomes reducible, but we know Y is irreducible since it is variety. Or, alternatively, we can use the general algebraic fact that affinization of irreducible homogeneous polynomial is again irreducible.) f is quadratic. Otherwise, the polynomial defining C is reducible. Applying Ex.1.1, we know $A(Y) \cong k[x]$ or $A(Y) \cong k[x, x^{-1}] = k[x]_x$, so its fractional field $K(A(Y)) \cong k(x)$. So $K(C) \cong K(Y) \cong K(A(Y)) \cong k(x)$, a pure transcendental extension of k. So C is birational to $\mathbb{P}^1$.

(b) Let $Y := Z(y^2 - x^3)$. Let $Y' = Y - \{(0,0)\}$. Let $U := \mathbb{P}^1 - ((1:0) \cup (0:1))$. It suffices to prove $Y' \cong U$. Let $\varphi : Y' \to U$ be $\varphi(x,y) = (x:y)$, let $\psi : U \to Y'$ be $\psi(x:y) = ((\frac{y}{x})^2, (\frac{y}{x})^3)$. It is quick to see that these two maps are well defined and are inverses to each other. φ is obviously a morphism. Also, if we view ψ as taking value in Y, then by Lemma 3.6, ψ is a morphism to Y, and this implies ψ is a morphism to Y'.

(c) The projection φ is a morphism from $\mathbb{P}^2 - (0:0:1)$ to $Z(z)$ given by $\varphi(x:y:z) = (x:y:0)$ (by Ex.3.14). Denote by φ' the restriction of φ to $Y - (0:0:1)$. Note $\operatorname{im}\varphi' \subseteq Z(z) - Z(y^2 - x^2)$. Let $\psi : Z(z) - Z(y^2 - x^2) \to Y - (0:0:1)$ be defined by $(x:y:0) \mapsto (x(y^2 - x^2) : y(y^2 - x^2) : x^3)$. It is straightforward to see that φ' and ψ are well-defined inverse functions. φ' is a morphism because it is induced from φ which is obviously a morphism. ψ is a morphism. Indeed, if we let $\theta : k[x,y,z] \to k[x,y,z]$ be $x \mapsto x(y^2 - x^2), y \mapsto y(y^2 - x^2), z \mapsto x^3$, then for any homogeneous $f \in k[x,y,z]$, $\psi^{-1}(Z(f)) = Z(\theta(f)) \cap (Z(z) - Z(y^2 - x^2))$, since $f(\psi(x)) = \theta(f)(x)$. So ψ is continuous. Pullback of regular functions on open subsets of $Y - (0:0:1)$ by ψ are regular function on open subsets of $Z(z) - Z(y^2 - x^2)$ because of definition of ψ. So ψ is a morphism. So $Y - (0:0:1) \cong Z(z) - Z(y^2 - x^2)$. So Y is birational to $Z(z)$. $Z(z)$ is isomorphic to $\mathbb{P}^1$, so Y is birational to $\mathbb{P}^1$.

Ex.4.7. (The conclusion of this exercise is so beautiful!) We can suppose both X and Y are affine varieties, because any variety has a basis of open affine subsets (Proposition 4.3). Let $(y_1, ..., y_n)$ be affine coordinates on Y and let $x_1, ..., x_m$ be affine coordinates on X. $\theta : \mathcal{O}_{Q,Y} \to \mathcal{O}_{P,X}$ be the k-algebra isomorphism. Since $A(Y) = \mathcal{O}(Y) \subseteq \mathcal{O}_{Q,Y}$, we can restrict θ to $A(Y)$ to get k-algebra homomorphism $A(Y) \to \mathcal{O}_{P,X}$. $A(Y)$ is finitely generated by $y_1, ..., y_n$, and we can find open $U \subseteq X$ such that $P \in U$ and each $\theta(y_i)$ is regular on U. This gives us k-algebra homomorphism $\theta' : A(Y) \to \mathcal{O}(U)$. By Proposition 3.5, this induces a morphism $\varphi : U \to Y$ given by $\varphi(x) = (\theta'(y_1)(x), ..., \theta'(y_n)(x))$. We observe that $\varphi(P) = Q$. Indeed, write $Q = (Q_1, ..., Q_n)$, then $\theta'(y_i - Q_i)(P) = \theta'(y_i)(P) - Q_i$ because θ' is k-algebra homomorphism. On the other hand, $y_i - Q_i \in \mathcal{O}_{Q,Y}$ is in the maximal ideal of $\mathcal{O}_{Q,Y}$, so $\theta(y_i - Q_i)$ is in the maximal ideal of $\mathcal{O}_{P,X}$, so $\theta'(y_i - Q_i)(P) = 0$. Thus $\theta'(y_i)(P) = Q_i$, and $\varphi(P) = Q$.
In the same way we can find open $V \subseteq Y$ and morphism $\psi : V \to X$ such that $Q \in V$ and $\psi(Q) = P$. Then it suffices to show $\psi \circ \varphi$ and $\varphi \circ \psi$ are identity on the open sets where they have definition (Then $\varphi^{-1}(V) \cap U \cong \psi^{-1}(U) \cap V$. Take $x \in \varphi^{-1}(V) \cap U$, write $x = (x_1, ..., x_m)$, then we want to show $(\psi \circ \varphi(x))_i = x_i$. We have

$$\begin{aligned}
(\psi \circ \varphi(x))_i &= \psi(\theta(y_1)(x), ..., \theta(y_n)(x))_i \\
&= \theta^{-1}(x_i)(\theta(y_1)(x), ..., \theta(y_n)(x)) \\
&= \frac{f}{g}(\theta(y_1)(x), ..., \theta(y_n)(x)) \\
&= \theta(\frac{f}{g})(x) \\
&= \theta(\theta^{-1}(x_i))(x) \\
&= x_i(x) \\
&= x_i.
\end{aligned}$$

where we have used the local description of $\theta^{-1}(x_i) = \frac{f}{g}$ at the point $(\theta(y_1)(x), ..., \theta(y_n)(x))$ in the third line. So we are done.

Ex.4.8.(a) Any algebraically closed field is infinite, so k is infinite. Assuming axiom of choice, finite product of some copies of k is bijective with k, so $\mathbb{A}^n$ has the same cardinality as k. Since $\mathbb{P}^n = \bigcup_{i=0}^n \mathbb{A}^n$ as sets (where $\mathbb{A}^0$ is a point), and since (under axiom of choice) an infinite set has the same cardinality as the disjoint union of finitely many copies of itself, we see $\mathbb{P}^n \cong k$ as sets. Any hypersurface Y in $\mathbb{A}^n, n \geq 2$ has the same cardinality as k because it is embedded in $\mathbb{A}^n \cong k$ and there exists injective function $k \to Y$ sending a to $(a, y_2, ..., y_n)$ where the y_i depends on a. Such y_i exists because Y is defined by a single polynomial equation with at least 2 variable and k is algebraically closed. Thus any hypersurface in $\mathbb{P}^n, n \geq 2$ also has the same cardinality as k. Now, for any variety X, if $\dim X = 1$, then by Proposition 4.9 it is birational to a hypersurface $Y \subset \mathbb{P}^2$, so X and Y have isomorphic nonempty open subsets. Because proper closed subset of a dimension 1 variety is finite, we see $X \cong Y$ as sets. If $\dim X = n \geq 2$, again it is birational to a hypersurface $Y \subset \mathbb{P}^{n+1}$ and X and Y have isomorphic nonempty open subsets. Because proper closed subset of a variety has dimension strictly less than the variety, by induction on dimension we see $X \cong Y$ as sets.

(b) Any set isomorphism of two curves is automatically a homeomorphism because curves have cofinite topology.

Ex.4.9. (I could not figure it out by myself. I am basically following the answer here: `math.stackexchange.com/questions/2042583`. According to this link, we need a stronger version of the primitive element theorem, which I will assume here. Also, we will drop the requirement $P \notin X$, because we want to conclude birational equivalence anyway.)
WLOG, assume $X \cap U_0 \neq \emptyset$, then $K(X)$ is generated over k by $\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}$. Using the affine chart U_0, Ex.2.7(b), Prop.1.10, and Theorem 3.2(d), we conclude that $K(Y)$ has transcendence degree r over k. By Theorem 4.7A and 4.8A, we get finite separable extension $K(X)/k(\frac{x_1}{x_0}, ..., \frac{x_r}{x_0})$ after possibly a change of coordinates. By the stronger version of the primitive element theorem, $K(X)$ is generated over $k(\frac{x_1}{x_0}, ..., \frac{x_r}{x_0})$ by α where $\alpha = \sum_{i=r+1}^n a_i \frac{x_i}{x_0}$, $a_i \in k$.
If all the a_i are 0, let $\pi : \mathbb{P}^n \dashrightarrow Z(x_n)$ be $(a_0 : ... : a_n) \mapsto (a_0 : ... : a_{n-1} : 0)$. Let $X' = \pi(X)$, then π induces k-algebra homomorphism $\pi^* : K(X') \to K(X)$ which takes $\frac{x_i}{x_0}$ to $\frac{x_i}{x_0}$ for all $1 \leq i \leq r$. Then π^* is surjective, so π is birational morphism between X and X'.
If some of the a_i is nonzero, we can assume $\alpha = \frac{x_{r+1}}{x_0}$ after possibly another change of coordinates. Because $r + 1 \leq n - 1$, the same definition of π induces isomorphism of function fields $K(X') \cong K(X)$, so π is birational morphism between X and X'. So we are done.

Remark: I have assumed X' is a variety in this solution, which I think is true from intuition but I don't know how to prove. Assuming this fact, the effect of the construction of π is to construct a variety birational to X with codimension 1 less than X. By repeating such constructions, we explicitly get a birational equivalence between any variety of dimension r and a hypersurface in $\mathbb{P}^{r+1}$, recovering Proposition 4.9.

Ex.4.10. (Blow-up) Let x, y be affine coordinates on $\mathbb{A}^2$, let t, u be projective coordinates on $\mathbb{P}^1$. Let $X \subseteq \mathbb{A}^2 \times \mathbb{P}^1$ be the blow up of $\mathbb{A}^2$ at $(0, 0)$, and let $\varphi : X \to \mathbb{A}^2$ be projection. Then $X = Z(xu - yt)$, $\varphi^{-1}(Y) = Z(xu - yt, y^2 - x^3)$. Because $Y - (0, 0)$ contains no point on the y-axis, then $\varphi^{-1}(Y - (0, 0))$ is in the affine chart $t \neq 0$, so we can assume $t \neq 0$, then we can assume $t = 1$ and let u be an affine coordinate, and we get two equations $y = xu, y^2 = x^3$. Then $x^2u^2 = x^3$. If $x = 0$, then $y = 0$ and u is arbitrary, so we get the exceptional line E. If $x \neq 0$, $u^2 = x$, so $\tilde{Y} = Z(u^2 - x, y - xu) = Z(u^2 - x, u^3 - y) \subseteq \mathbb{A}^3$. (Note $\tilde{Y}$ is twisted cubic curve.) To get $\tilde{Y} \cap E$, set $x = y = 0$, then $u^2 = 0$, so $u = 0$, so $\tilde{Y} \cap E = (0, 0, 0)$ is a single point. Let $\psi : \tilde{Y} \to \mathbb{A}^1$ be $(x, y, u) \mapsto u$ then ψ is obviously an isomorphism between $\tilde{Y}$ and $\mathbb{A}^1$.

5 Nonsingular Varieties

Ex.5.1. It is straightforward to verify that all four polynomials are irreducible. Denote the polynomial in each problem by f, then according to the definition of singular points on affine variety, we need to have $f_x, f_y = 0$.

(a) We want $2x - 4x^3 = -4y^3 = x^2 - x^4 - y^4 = 0$. Since $\operatorname{char} k \neq 2$, $4 \neq 0$, then $y = 0$. If $x \neq 0$ then $1 - 2x^2 = 1 - x^2 = 0$, impossible. So $x = 0$. The graph corresponds to "tacnode".

(b) We want $y - 6x^5 = x - 6y^5 = xy - x^6 - y^6 = 0$. Then $y = 6x^5$, so $x = 6^6x^{25}$. $5x^6 = 6^6x^{30}$. If $x \neq 0$ then

$1 = 5$, impossible. So $x = y = 0$. This corresponds to "node".

(c) We want $3x^2 - 4x^3 = -2y - 4y^3 = x^3 - y^2 - x^4 - y^4 = 0$. If $y \neq 0$, then $y^2 = -\frac{1}{2}$, so $x^3 - x^4 = -\frac{1}{4}$. We also have $3x^3 - 4x^4 = 0$, so we see $x^3 = -1$, $x^4 = -\frac{3}{4}$, so $x = \frac{3}{4}$, then $x^3 = \frac{27}{64} = -1$, so $\frac{91}{64} = 0$, so char $k = 7$ or 13. If char $k \neq 7$ and char k $\neq 13$, we have $y = 0$, then if $x \neq 0$, $x = 1$, so $3 - 4 = 0$, impossible, so $x = y = 0$.

Therefore, when char $k = 7$ or 13, singular points are $\{(0,0), (\frac{3}{4}, y_1), (\frac{3}{4}, y_2)\}$ where y_1, y_2 are roots of $y^2 + \frac{1}{2} = 0$. Otherwise, the origin is the only singular point. The graph corresponds to "cusp".

(d) We want $2xy + y^2 - 4x^3 = 2xy + x^2 - 4y^3 = x^2y + xy^2 - x^4 - y^4 = 0$. Multiply the first equation by x and subtract from the third equation we get $x^2y - 3x^4 + y^4 = 0$. So $x^2y = 3x^4 - y^4 = x^4 + y^4 - xy^2$, so $2x^4 - 2y^4 = -xy^2$. Multiply the second equation by y and subtract from the third equation we get $xy^2 - 3y^4 + x^4 = 0$. So $xy^2 = 3y^4 - x^4 = x^4 + y^4 - x^2y$, so $2y^4 - 2x^4 = -x^2y$. Thus $x^2y = xy^2$, then $xy(x - y) = 0$. If $x = 0$, then $y = 0$. If $y = 0$, then $x = 0$. If $x = y$, then $x = y = 0$. Therefore the origin is the only singularity point. The graph corresponds to "triple point".

Ex.5.2. It is straightforward to verify that all three polynomials are irreducible. Denote the polynomial in each problem by f, then according to the definition of singular points on affine variety, we need to have $f_x, f_y, f_z = 0$.

(a) We want $y^2 = 2xy = -2z = xy^2 - z^2 = 0$. Then $y = z = 0$, x is arbitrary. This corresponds to "pinch point".

(b) We want $2x = 2y = -2z = x^2 + y^2 - z^2 = 0$. Then $x = y = z = 0$. This corresponds to "conical double point".

(c) We want $y + 3x^2 = x + 3y^2 = xy + x^3 + y^3 = 0$. If char $k = 3$, then $x = y = 0$, z is arbitrary. Assume char $k \neq 3$. Then $y = -3x^2$, so $x(1 + 27x^3) = x^3(2 + 27x^3) = 0$. So either $x = y = 0$ and z is arbitrary, or $x \neq 0$. But if $x \neq 0$, then $-1 = 2$, impossible. So singular points are $x = y = 0$, z arbitrary, This corresponds to "double line".

Ex.5.3. (a) Suppose $Y = Z(f)$ and $\mu_P(Y) = 1$. Write $P = (P_1, P_2)$. Let $g = f(x + P_1, y + P_2)$, then $g_0 = 0$, $g_1 \neq 0$. This implies $P \in Y$ and rank $(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y})_{(0,0)} = 1$, so rank $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})_{(P_1,P_2)} = 1$, so P is a nonsingular point of Y. The reverse direction follows by reversing the above arguments.
(b) For the tacnode, $\mu_0(Y) = 2$. For the node, $\mu_0(Y) = 2$. For the cusp, if char $k \neq 7$ and char k $\neq 13$, then 0 is the only singularity and $\mu_0(Y) = 2$. Otherwise there are three singular points described in Ex5.1(c), all with multiplicity 2. For the triple point, $\mu_0(Y) = 3$.

Ex.5.5. When $p \nmid d$, $x^d + y^d + z^d$ gives such curve. Indeed, we view this curve via the affine chart $x \neq 0$, then the equation of the curve in $\mathbb{A}^2$ is $1 + y^d + z^d$. If both partial derivatives are 0, then $y = z = 0$, but no point on the curve satisfies $y = z = 0$, so $Z(1 + y^d + z^d)$ is a nonsingular affine curve. Using the other affine charts and the intrinsic description of singularity, we see that $Z(x^d + y^d + z^d) \subset \mathbb{P}^2$ is nonsingular.

When $p|d$, $zx^{d-1} + y^d + yz^{d-1}$ gives such curve. Affinize using the affine chart $x \neq 0$, the curve becomes $Z(z + y^d + yz^{d-1}) \subset \mathbb{A}^2$. Setting the partial derivative w.r.t. y to 0, we get $z = 0$. Then setting the partial derivative w.r.t z to 0 we get $1 + (d-1)yz^{d-2} = 0$. When $d > 2$ this is impossible, and when $d = 2$, we get $y = -1$, but $(y, z) = (-1, 0)$ is not on the curve. Similar arguments show that, under other affine charts, the curve is nonsingular. By the intrinsic description of singularity, we see that $Z(zx^{d-1} + y^d + yz^{d-1}) \subset \mathbb{P}^2$ is nonsingular.

Ex.5.8. Let $M = (\partial f_i / \partial x_j(a_0, ..., a_n))_{i,j}$. rank M is independent of the homogeneous coordinates of P because entries of i-th row of $(\partial f_i / \partial x_j)_{i,j}$ are homogeneous polynomials of degree $\deg f_i - 1$. Thus, using another homogeneous coordinate will multiply the original row of M by a nonzero scalar, which does not change the (row) rank of the whole matrix. We also note that rank M is also independent of the choice of generators (f_i). The reason is similar as the one for the affine case.

WLOG, suppose $P \in U_0$. Let $\varphi_0 : \mathbb{A}^n \to U_0$ be the canonical isomorphism. Then $Y_0 := \varphi_0^{-1}(Y \cap U_0)$ is an affine variety of dimension r. Let $M_0 = (\partial((f_i)_a)/\partial x_j(\frac{a_1}{a_0}, ..., \frac{a_n}{a_0}))_{i,j}$ where $1 \leq j \leq n$ and $(f_i)_a$ is the affinization of f_i w.r.t x_0. Note M has $n+1$ columns while M_0 has n columns. Note we have $I(Y_0) = ((f_1)_a, ..., (f_t)_a)$. Also note that for $j > 0$, $\partial((f_i)_a)/\partial x_j(\frac{a_1}{a_0}, ..., \frac{a_n}{a_0}) = (\partial f_i/\partial x_j)_a(\frac{a_1}{a_0}, ..., \frac{a_n}{a_0}) = \frac{1}{a_0^{\deg f_i - 1}}(\partial f_i/\partial x_j)(a_0, ..., a_n)$. Finally note that by Euler's lemma (which is easy to prove) we have $\sum_{j=0}^n x_j(\partial f_i/\partial x_j) = \deg f_i \cdot f_i$. Evaluating this equation at $(a_0, ..., a_n)$ we get $a_0(\partial f_i/\partial x_0)(a_0, ..., a_n) = -\sum_{j=1}^n a_j \cdot (\partial f_i/\partial x_j)(a_0, ..., a_n)$, so we see the first column of M is a linear combination of the rest n columns of M, so rank M is equal to rank of the last n columns of M, which is equal to rank of M_0.

Therefore, P is nonsingular on Y iff $\varphi_0^{-1}(P)$ is nonsingular on Y_0 iff rank $M_0 = n - r$ iff rank $M = n - r$.

Ex.5.9. FSOC, suppose f is reducible. Let f_1, f_2 be two irreducible factors of f. Suppose $f = f_1 f_2 g$ where $g \in k[x, y, z]$ homogeneous. By Ex.3.7, $Z(f_1) \cap Z(f_2) \neq \emptyset$, so we can pick $P \in Z(f_1) \cap Z(f_2)$. $(\partial f/\partial x)(P) = (\partial f_1/\partial x)(P) \cdot (f_2 g)(P) + f_1(P) \cdot (\partial f_2 g/\partial x)(P) = 0$. Similarly, all other partial derivatives of f at P is 0, contradiction with assumption, so f must be irreducible.

Ex.5.10. (a) $\dim T_p(X) = \dim \mathfrak{m}/\mathfrak{m}^2 \geq \dim \mathcal{O}_P = \dim X$, where the second inequality is true by proposition 5.2A. Then equality holds if and only if p is nonsingular.

(b) Denote the maximal ideal of $\mathcal{O}_{P,X}$ by $\mathfrak{m}_1$. Denote the maximal ideal of $\mathcal{O}_{\varphi(P),Y}$ by $\mathfrak{m}_2$. φ induces a k-algebra homomorphism $\varphi^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$. Because pullback of a regular function vanishing on $\varphi(P)$ vanishes on P, $\varphi^*(\mathfrak{m}_2) \subseteq \mathfrak{m}_1$. Because φ^* is ring homomorphism, $\varphi^*(\mathfrak{m}_2^2) \subseteq \mathfrak{m}_1^2$. Thus φ^* induces k-linear map: $\frac{\mathfrak{m}_2}{\mathfrak{m}_2^2} \to \frac{\mathfrak{m}_1}{\mathfrak{m}_1^2}$. Its dual map is the natural induced k-linear map $T_P(\varphi) : T_P(X) \to T_{\varphi(P)}(Y)$.

(c) Let $X = Z(x - y^2)$, $Y = Z(y)$. Let 0 denote the origin. Then $\mathcal{O}_{0,Y} = \mathcal{O}_{0,\mathbb{A}^1} = k[x]_{(x)}$ where the second step is true by Theorem 3.2(c). We know localization of Dedekind domain at nonzero prime ideal is discrete valuation ring, and here the generator of maximal ideal of $\mathcal{O}_{0,Y}$ is x. Then $\varphi^*(x) = x \circ \varphi = x = y^2 \in \mathfrak{m}_1^2$, so φ^* induces the zero map: $\frac{\mathfrak{m}_2}{\mathfrak{m}_2^2} \to \frac{\mathfrak{m}_1}{\mathfrak{m}_1^2}$. So its dual map $T_0(\varphi)$ is also zero map.

Ex.5.14(a). It suffices to prove for $P = Q = (0, 0)$. Assume $Y = Z(f)$ and $Z = Z(g)$, then by assumption $\frac{k[[x,y]]}{(f)} \cong \frac{k[[x,y]]}{(g)}$. By some algebra we get f and g should have the same order, so $\mu_P(Y) = \mu_Q(Z)$.

(b)

Ex.5.15. (a) Given a point $P = (p_0 : ... : p_N) \in \mathbb{P}^N$, let f_P be the homogeneous polynomial of degree d with coefficients $(p_0, ..., p_N)$. Then $Z(f_P)$ is well defined i.e. it does dot depend on the choice of representative of coordinate of P. This is the algebraic set we want. Conversely, given an algebraic set $Y = Z(g)$ where g has degree d, we note that the equation defining Y is unique up to unit when g has distinct factors. Indeed, $Y = Z(g) = Z(f)$ where $\deg f = d$, then $I(Y) = \sqrt{(g)} = \sqrt{(f)}$, so $g|f^n$ for some $n \geq 1$. Since g has distinct factors, $g|f$. Since $\deg g = \deg f$, we have $g = uf$ where u is a unit. Thus we get a unique point in $\mathbb{P}^N$ from an algebraic set in $\mathbb{P}^2$ which can be defined using a degree d polynomial with distinct factors. Such correspondence is obviously one-to-one when we restrict to points in $\mathbb{P}^N$ which correspond to degree d polynomials with distinct factors.

We note that this correspondence cannot be extended to all algebraic sets which can be defined using a degree d polynomial. For example let $d = 3$, then x^2y and xy^2 define the same algebraic set, but they correspond to distinct points in $\mathbb{P}^N$.

(b) Let C be a nonsingular curve of degree d, then $C = Z(f)$ for some irreducible, homogeneous polynomial f of degree d. Let $P_f \in \mathbb{P}^N$ be the point formed by coefficients of f. Because f has distinct factors, by part (a) we only need to show that P_f forms a nonempty open subset of $\mathbb{P}^N$ when f ranges over all irreducible polynomials of degree d that define nonsingular curves. By Ex.5.5, such set is nonempty.

Given a homogeneous $f \in k[x, y, z]$ of degree d, we have the following equivalence:

$$f \text{ irreducible and } Z(f) \text{ nonsingular} \iff Z(f, \partial f/\partial x, \partial f/\partial y, \partial f/\partial z) = \emptyset \iff (a_{ij}) \notin Z(g_1, ..., g_t)$$

where a_{ij} are coefficients of $f, \partial f/\partial x, \partial f/\partial y, \partial f/\partial z$, and $g_1, ..., g_t$ are polynomials with integer coefficients in (a_{ij}) and homogeneous in coefficients of each of $f, \partial f/\partial x, \partial f/\partial y, \partial f/\partial z$ separately. The first equivalence follows from Ex.5.8 and Ex.5.9, and the second from Theorem 5.7A.

Theorem 5.7A outputs the g_i which have more than N indeterminates, but coefficients of $\partial f/\partial x, \partial f/\partial y, \partial f/\partial z$ can be expressed using coefficients of f, so we can write g_i as integer-coefficient polynomials with exactly N indeterminates, thus the complement of their zero set is open in $\mathbb{P}^N$, so we are done.