Solution to analysis in Home Assignment 2

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Analysis

In this report I will present my independent analysis of the questions related to home assignment 1. I have discussed the solution with Xinying Wang but I swear that the analysis written here are my own.

1 A first Kalman filter and its properties

(a) The state sequence and measurement sequence as illustrated in the Figure 1.1. We can find that the measurement sequence does not fit well with the state sequence, since the motion and measurement noises. But the measurement sequence is still follow the trend of the state sequence. Thus, we can say that the measurement behave according to the model.

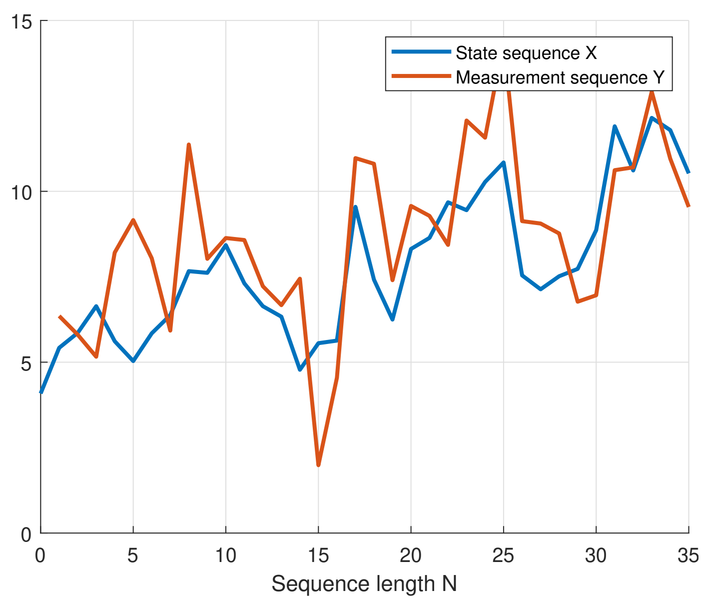

Figure 1.1: The state and measurement sequences.

(b) The Figure 1.2 shows the the sequence of estimates $\hat{x}_k$. As we can see from the figure, the measurement and state sequences are in the range of $\hat{x}_k \pm 3\sigma$. This indicates that the estimates provided by the Kalman filter are consistent with both the measurements and the predicted states. Based on this, we can conclude that the estimates output by the filter are reasonable.

The fact that the measurements and states are within the range of $\hat{x}_k \pm 3\sigma$ also demonstrates that the filter's estimates are appropriately capturing this uncertainty. So the error covariance represent the uncertainty in the estimates well.

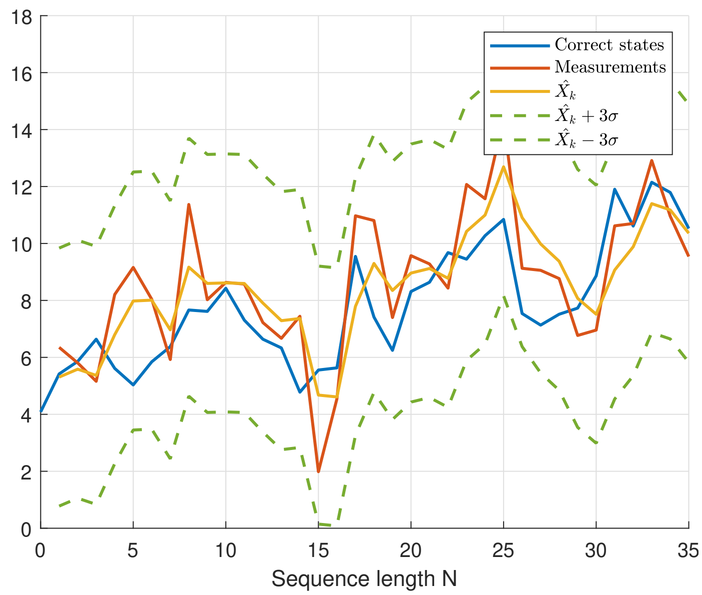

Figure 1.2: The sequence of estimates.

The Figure1.3 illustrates the the error density around zero-mean for time instances k = [1, 2, 4, 30]. The error density is represented by a Gaussian distribution with a mean of zero. As we observe, the error density becomes narrower and taller as the time instance k increases. It indicates that the uncertainty of the estimate decreases with the time instance k increases. As time progresses, the filter receives more measurements and integrates them into the state estimate, leading to a decrease in the uncertainty of the estimate. Consequently, the value of σ will decreases accordingly.

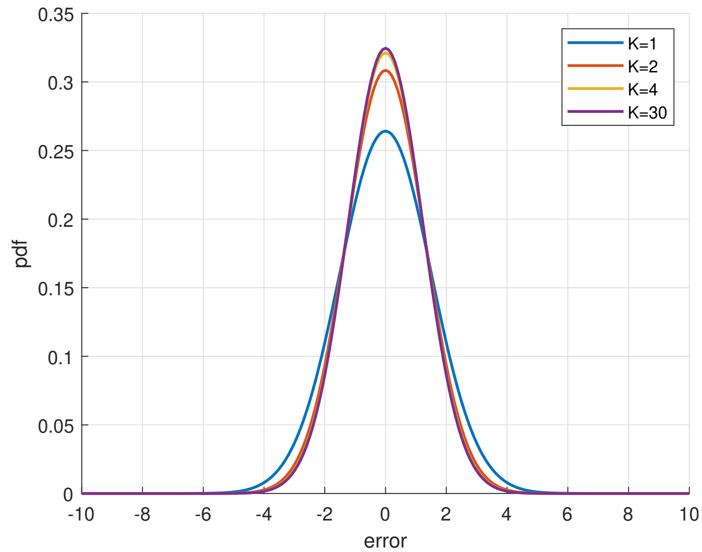

Figure 1.3: The error density around zero-mean for time instances $\mathrm{k} = [1, 2, 4, 30]$.

(c) From the Figure 1.4, we can find that the estimate output by the Kalman filter with the incorrect prior(the initial mean is instead 12) is initially far away from the correct states and correct estimate. As time progresses, the filter observes an increasing amount of measurement data. This accumulation of data causes estimates with incorrect priors to gradually converge towards the correct estimate, ultimately reaching the same value. With the incorporation of new measurements into the estimate, the filter's state estimate becomes progressively more accurate over time.

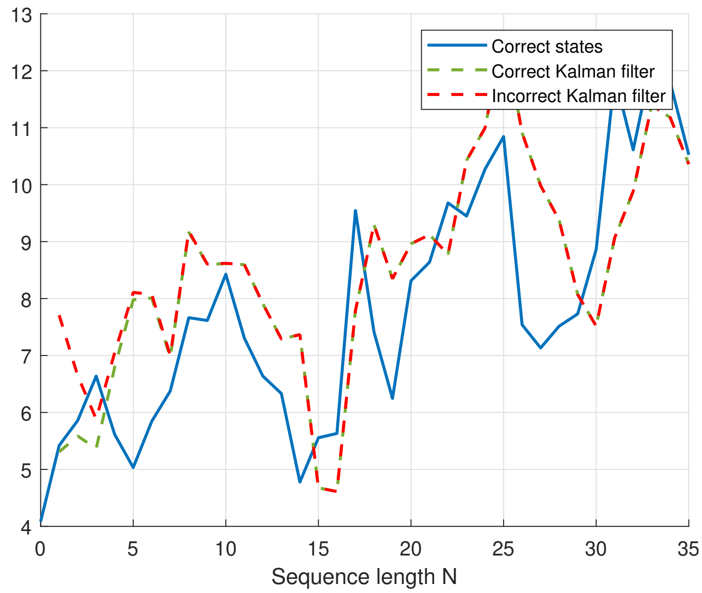

Figure 1.4: The estimate with correct prior and the incorrect prior.

(d) The Figure 1.5 shows the $p(x_{k-1}|y_{1:k-1}), p(x_k|y_{1:k-1}), y_k$ *and* $p(x_k|y_{1:k})$.

Prediction step: The $p(x_{k-1}|y_{1:k-1})$ represents the prior. The probability distribution $p(x_k|y_{1:k-1})$ represents the predicted state, exhibiting greater uncertainty compared to the prior distribution while sharing the same mean.

Update step: The $p(x_k|y_{1:k})$ represents the posterior. We observe that the mean of the posterior distribution lies between the prediction and the measurement y_k, with reduced uncertainty compared to the prediction.

The prediction step incorporates prior but maintains a certain level of uncertainty to account for potential errors in the system. During the update step, new measurements are utilized to enhance the precision of the estimates, diminish uncertainty, and align them more closely with the true state.

In conclusion, the behaviour of the prediction and update step of the Kalman filter is reasonable based on the information given by the Figure 1.5.

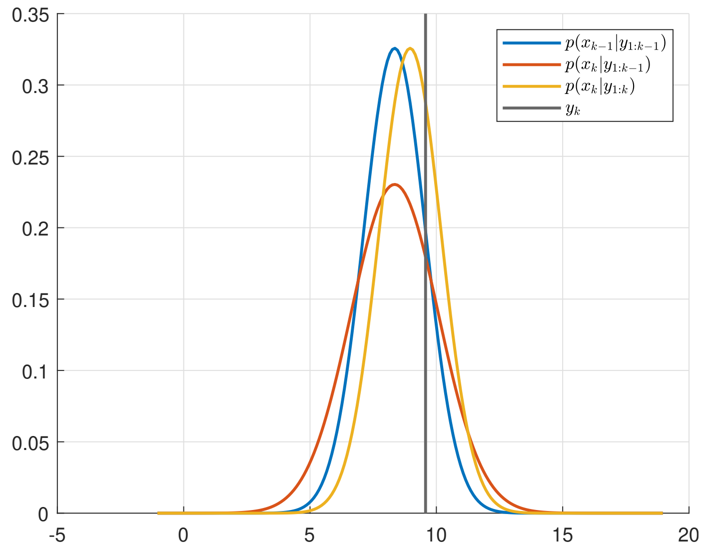

Figure 1.5: Plot of $p(x_{k-1}|y_{1:k-1}), p(x_k|y_{1:k-1}), y_k$ *and* $p(x_k|y_{1:k})$ *for* $k = 20$.

(e) Figure 1.6 illustrates the histogram of the error$x_k - \hat{x}_{k|k}$ and the pdf $\mathcal{N}(x; 0, P_{N|N})$. From the figure, we can find that the histogram of the error$x_k - \hat{x}_{k|k}$ fits the pdf. This indicates that the Kalman filter's estimate $\hat{x}_{k|k}$ is effectively capturing the true state x_k with an error conform to Gaussian distribution. It implies that the Kalman filter efficiently diminishes measurement noise, thereby yielding reliable state estimates. Furthermore, this fact demonstrates that the covariance matrix $P_{N|N}$ effectively captures the uncertainty in the estimates.

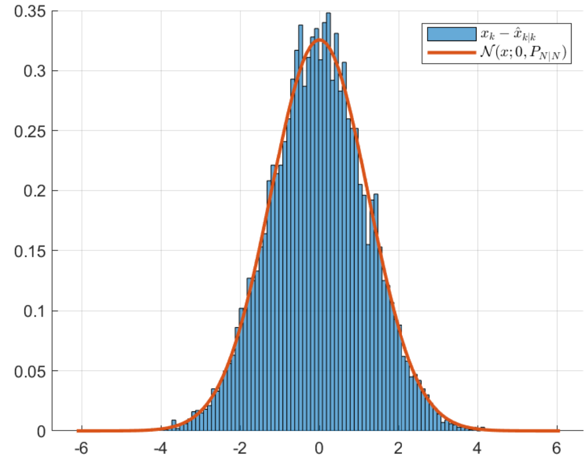

Figure 1.6: The histogram of the error $x_k - \hat{x}_{k|k}$ *and the pdf* $\mathcal{N}(x; 0, P_{N|N})$.

The innovation $v_k = y_k - H_k \hat{x}_{k|k-1}$ should satisfy the equation below:

$$p(v_k | y_{1:k-1} = \mathcal{N}(v_k; 0, S_k)$$

$$Cov(v_k, v_{k-l}) = \begin{cases} Cov\{v_k\} & if \quad l = 0 \\ 0 & otherwise \end{cases}$$

And we can use the auto correlation function to check the correlation of the innovation process.

As we can see in the Figure 1.7, the $S_k^{(-1/2)} v_k$ is inside the region of $\pm\sigma$. Thus the innovation process of the Kalman filter fulfill the consistency. From the Figure 1.8, we can find that for $l > 0, \rho(l) \approx 0$. This fact confirms the correlation of the innovation process.

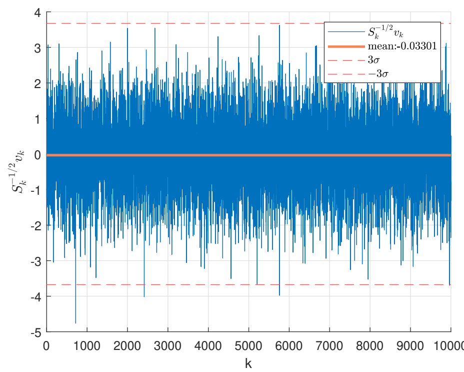

Figure 1.7: The consistency.

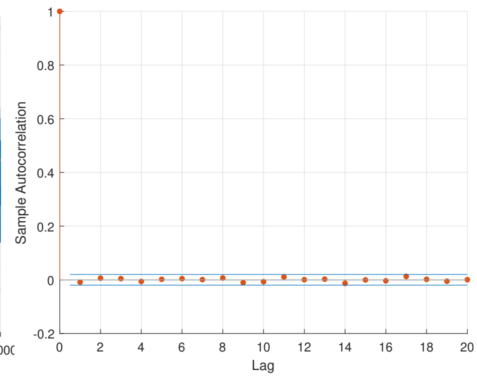

Figure 1.8: Sample auto correlation function.

2 Tuning a Kalman filter

(a) Since the supplier proposes the following sensor model.

$$y_k^v = C(v_k + r_k^v)$$

Then we can get:

$$\mathbb{E}[y_k^v] = \mathbb{E}[Cv_k] + \mathbb{E}[Cr_k^v]$$

$$C = \frac{\mathbb{E}[y_k^v] - \mathbb{E}[Cr_k^v]}{v_k}$$

$$Var[y_k^v] = C^2 Var[r_k^v]$$

Using the measurements in **SensorMeasurements.mat**, we can get that for $v_k = 10m/s$, $C_{10} = 1.1100$ and for $v_k = 20m/s$, $C_{20} = 1.1059$. By calculating the mean value of C_{10} and C_{20}, we can determine the scaling constant $C = 1.107954$.

By $Var[y_k^v] = C^2 Var[r_k^v]$, we can get the $Var[r_k^v] = 2.461195$ with similar approach for the scaling constant C.

(b) Comparing to the speed measurement, the position measurement will lack half of the data. Our method to handle this type of data is to remove the couple of the speed measurement and the position measurement where it lacks the data from the measurement data sequence. Then using the remaining data to reform a new measurement data sequence. Thus our Kalman filter can process the new measurement data sequence. The Matlab code for this method shown as bellow.

```
Y_2b = Generate_y_seq();
n_2b = size(Y_2b,2);
temple = [];
for i = 1:n_2b
    if ~isnan(Y_2b(1,i))
        temple = [temple,Y_2b(:,i)];
    end
end
Y_2b = temple;
```

(c) For the constant velocity model:

$$X = \begin{bmatrix} p \\ v \end{bmatrix}, \quad A_{k-1} = \begin{bmatrix} I_n & TI_n \\ 0_n & I_n \end{bmatrix}$$

$$H_{cv} = \begin{bmatrix} I_n & 0_n \\ 0_n & C \end{bmatrix}$$

$$Q_{cv} = \begin{bmatrix} 0_n & 0_n \\ 0_n & Q \end{bmatrix}$$

$$R_{cv} = \begin{bmatrix} I_n & 0_n \\ 0_n & C^2R \end{bmatrix}$$

For the constant acceleration velocity model:

$$X = \begin{bmatrix} p \\ v \\ a \end{bmatrix}, \quad A_{k-1} = \begin{bmatrix} I_n & TI_n & T^2I_n \\ 0_n & I_n & TI_n \\ 0_n & 0_n & I_n \end{bmatrix}$$

$$H_{ca} = \begin{bmatrix} I_n & 0_n & 0_n \\ 0_n & C & 0_n \end{bmatrix}$$

$$Q_{ca} = \begin{bmatrix} 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n \\ 0_n & 0_n & Q \end{bmatrix}$$

$$R_{ca} = \begin{bmatrix} I_n & 0_n \\ 0_n & C^2R \end{bmatrix}$$

We design our prior:

$$x_{cv0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad P_{cv0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_{ca0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad P_{ca0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

such that train is initially stationary. As mentioned earlier, the parameter that significantly influences the results of the models is Q. The larger the value of Q, the more weight we place on the measurements. Conversely, the smaller the value of Q, the more emphasis we place on the motion model.

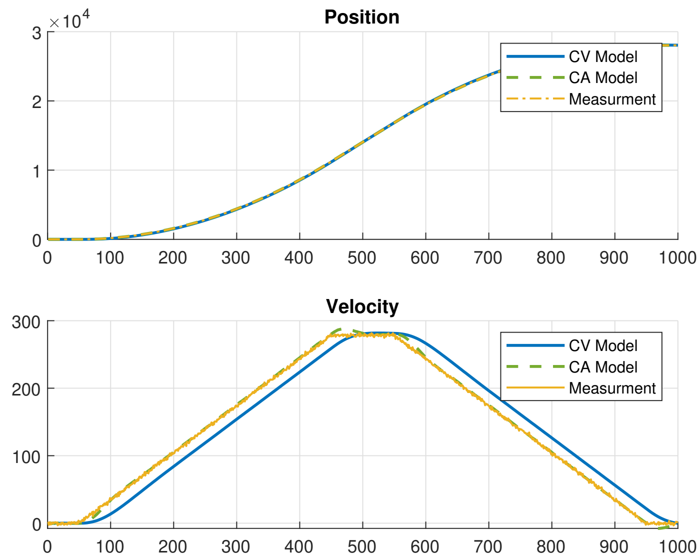

Figure 2.1: Position and velocity with Q = 0.0001.

Figure 2.1 shows the position and velocity of the constant velocity model and constant acceleration model with Q = 0.0001.

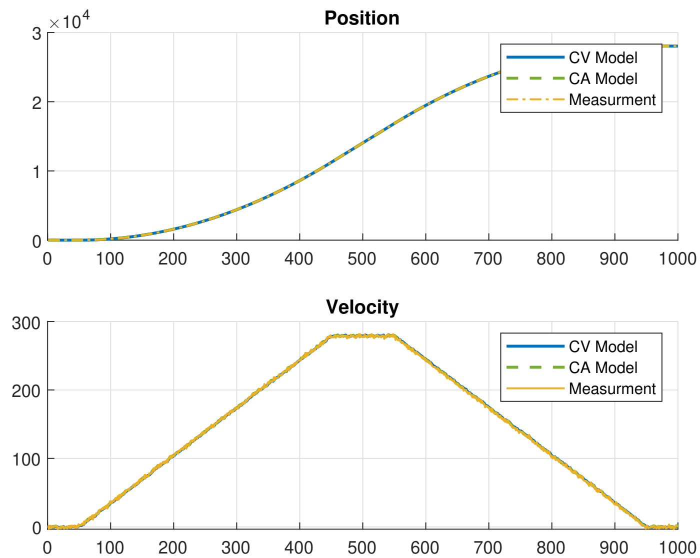

Figure 2.2: Position and velocity with Q = 1.

Figure 2.2 shows the position and velocity of the constant velocity model and constant acceleration model with Q = 1.

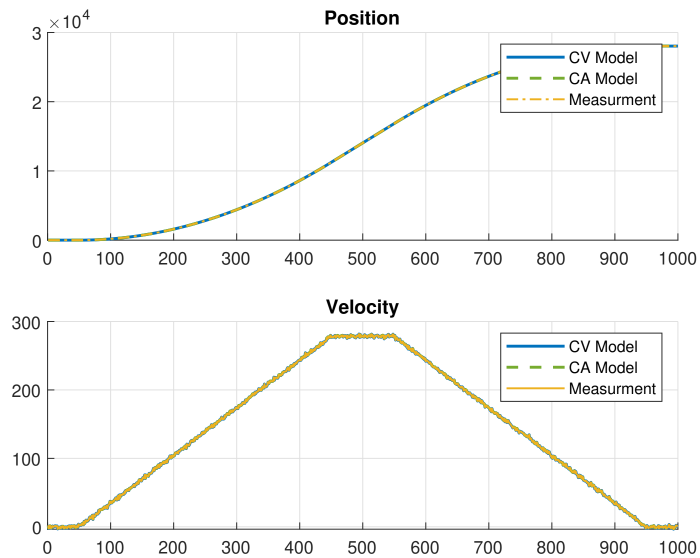

Figure 2.3: Position and velocity with Q = 10000.

Figure 2.3 shows the position and velocity of the constant velocity model and constant acceleration model with Q = 10000.

According to the plot shown above, we can know that: if Q is small, the kalman filter tend to trust on the motion model. The CA model will fit well with the measurement while the CV model fits bad. If Q is large, the filter place more weight on the measurements. The results fits well with the measurement, but the filter will ignore the motion model.

Comparing the result, we can find that the parameter Q = 1 is work best for each of the models.

(d) I think the constant acceleration model fits with the measurement better than the constant velocity model. From the figures above, we can find that whether Q is extreme large or small the CA model all fits well with the measurement.

Constant Velocity Model:
Advantages:
1. The constant velocity model is straightforward and simple. It is easy to understand.
2. It is suitable for describing some basic scenarios where the acceleration of the object can be assumed to be zero.
3. The estimates of the position and velocity are smoother since it is easier

to predict the position of the object at any given time.

Disadvantages:
1. The scope of application is limited. It can not be applied to some more realistic scenarios.
2. Since the model assumes that the velocity is constant, it can not capture the sudden changes in velocity and orientation.

Constant Velocity Model:
Advantages:
1. Can be applied to a wider range of more realistic scenarios.
2. It can capture the sudden changes in velocity and orientation.

Disadvantages:
1. The model is more complex.
2. It requires more data to describe the motion of the object accurately.