18.335 Problem Set 2

Due March 17, 2023 at 11:59pm. You should submit your problem set **electronically** on the 18.335 Gradescope page. Submit **both** a *scan* of any handwritten solutions (I recommend an app like TinyScanner or similar to create a good-quality black-and-white "thresholded" scan) and **also** a *PDF printout* of the Julia notebook of your computer solutions. A **template Julia notebook is posted** in the 18.335 web site to help you get started.

Problem 0: Pset Honor Code

Include the following statement in your solutions:

> *I will not look at 18.335 pset solutions from previous semesters. I may discuss problems with my classmates or others, but I will write up my solutions on my own. <your signature>*

Problem 1: Stability and conditioning for linear systems

(a) (From Trefethen and Bau, Exercise 18.1.) Consider the example

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1.0001 \\ 1 & 1.0001 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 0.0001 \\ 4.0001 \end{pmatrix}.$$

(i) What are the matrices A^+ and P in this example? Give exact answers.

(ii) Find the exact solutions x and $y = Ax$ to the least squares problem $\mathrm{x} = \mathrm{argmin}_v \|Av - b\|_2$.

(iii) What are $\kappa(A)$, θ, and η? Numerical answers computed with, e.g., Julia, are acceptable.

(iv) What numerical values do the four condition numbers of Theorem 18.1 take for this problem?

(v) Give examples of perturbations δb and $\delta\mathrm{A}$ that approximately attain these four condition numbers.

(b) (From Trefethen and Bau, Exercise 21.6.) Suppose $A \in \mathbb{C}^{m\times m}$ is *strictly column diagonally dominant*, which means that for each column index k,

$$|a_{kk}| > \sum_{j\neq k} |a_{jk}|.$$

Show that if Gaussian elimination with partial pivoting is applied to A, no row interchanges take place.

(c) (From Trefethen and Bau, Exercise 23.2.) Using the proof of Theorem 16.2 as a guide, derive Theorem 23.3 from Theorems 23.2 and 17.1. In other words, show that solving symmetric positive definite (SPD) linear systems with a Cholesky factorization followed by forward and backward substitution is backward stable.

Problem 2: Banded factorization of a finite-difference matrix

Consider the advection equation $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ with $(t, x) \in [0, T] \times [0, 1]$for some $T > 0$. Let $u(x, t)$ have initial condition $u(x, 0) = b(x)$ and boundary conditions $u(0, t) = u(1, t) = 0$. For numerical approximation with finite-differences at equispaced times $0 < t_1, t_2, \ldots, t_m < T$ and points $0 < x_1, x_2, \ldots, x_n < 1$, we can approximate

$$\frac{\partial u_{jk}}{\partial t} \approx \frac{u_{jk} - u_{j(k-1)}}{\Delta t},$$

where $u_{jk} = u(x_j, t_k)$, $\Delta t = t_k - t_{k-1}$, $\Delta x = x_j - x_{j-1}$, and

$$\frac{\partial u_{jk}}{\partial x} \approx \frac{-u_{(j+2)k} + 8u_{(j+1)k} - 8u_{(j-1)k} + u_{(j-2)k}}{12\Delta x}.$$

(a) Show that the vector $u_k = \begin{pmatrix} u(t_k, x_1), & u(t_k, x_2), & \ldots, & u(t_k, x_n) \end{pmatrix}^T$, representing an approximate solution on the grid at time t_k, can be obtained from u_{k-1} by solving the $n \times n$ linear system $(I + \sigma D)u_k = u_{k-1}$, where $\sigma = \Delta t/\Delta x < 1$ and

$$D = \begin{pmatrix} 0 & 2/3 & 1/12 & & \\ -2/3 & 0 & 2/3 & \ddots & \\ -1/12 & -2/3 & 0 & \ddots & 1/12 \\ & \ddots & \ddots & \ddots & 2/3 \\ & & -1/12 & -2/3 & 0 \end{pmatrix}.$$

(b) Describe a banded elimination algorithm to factor the matrix $A = I + \sigma D$ into the product $A = LDU$: here, L and U^T are lower triangular with unit diagonal and have the same lower bandwidth as A, while D is a diagonal matrix. How does the memory cost (number of nonzeros stored explicitly) and flop count for your algorithm scale as $n \to \infty$? Implement your algorithm in the accompanying Julia notebook.

(c) Can your algorithm fail without pivoting when $\sigma \leq 1$? How should one refine the approximation in space (decrease Δx) and in time (decrease Δt) to maintain stability without row exchanges? Explain why. (Hint: Problem 1(b) may be useful here.)

(d) Use $A = LDU$ to compute solutions $u_1, u_2, \ldots, u_k$, with the problem parameters set in the accompanying Julia notebook.

Problem 3: Regularized least-squares

Consider the regularized least-squares problem with regularization parameter $\lambda > 0$, given by

$$x_* = \text{argmin}_x \|Ax - b\|_2^2 + \lambda \|x\|_2^2.$$

(a) Show that the regularized least-squares problem is equivalent to a standard least-squares problem,

$$x_* = \text{argmin}_x \left\| \begin{pmatrix} A \\ \sqrt{\lambda} I \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\|_2^2.$$

(b) If the SVD of A is $A = U\Sigma V^*$, show that the unique solution is

$$x_* = V(\Sigma^*\Sigma + \lambda I)^{-1}\Sigma^* U^* b.$$

(c) Under what conditions on A does the regularized solution converge to the usual least-squares solution $\text{argmin}_x ||Ax - b\|_2^2$ in the limit $\lambda \to 0$?

(d) (Not for credit.) Play with the example in the accompanying Julia notebook!