# 18.335 Problem Set 2

Due March 17, 2023 at 11:59pm. You should submit your problem set **electronically** on the 18.335 Gradescope page. Submit **both** a *scan* of any handwritten solutions (I recommend an app like TinyScanner or similar to create a good-quality black-and-white "thresholded" scan) and **also** a *PDF printout* of the Julia notebook of your computer solutions. A **template Julia notebook is posted** in the 18.335 web site to help you get started.

## Problem 0: Pset Honor Code

Include the following statement in your solutions:

> *I will not look at 18.335 pset solutions from previous semesters. I may discuss problems with my classmates or others, but I will write up my solutions on my own. <your signature>*

## Problem 1: Stability and conditioning for linear systems

(a) (From Trefethen and Bau, Exercise 18.1.) Consider the example

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1.0001 \\ 1 & 1.0001 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 0.0001 \\ 4.0001 \end{pmatrix}.$$

(i) What are the matrices $A^+$ and $P$ in this example? Give exact answers.

(ii) Find the exact solutions $x$ and $y = Ax$ to the least squares problem $\mathrm{x} = \mathrm{argmin}_v \|Av - b\|_2$.

(iii) What are $\kappa(A)$, $\theta$, and $\eta$? Numerical answers computed with, e.g., Julia, are acceptable.

(iv) What numerical values do the four condition numbers of Theorem 18.1 take for this problem?

(v) Give examples of perturbations $\delta b$ and $\delta \mathrm{A}$ that approximately attain these four condition numbers.

(b) (From Trefethen and Bau, Exercise 21.6.) Suppose $A \in \mathbb{C}^{m\times m}$ is *strictly column diagonally dominant*, which means that for each column index $k$,

$$|a_{kk}| > \sum_{j\neq k} |a_{jk}|.$$

Show that if Gaussian elimination with partial pivoting is applied to $A$, no row interchanges take place.

(c) (From Trefethen and Bau, Exercise 23.2.) Using the proof of Theorem 16.2 as a guide, derive Theorem 23.3 from Theorems 23.2 and 17.1. In other words, show that solving symmetric positive definite (SPD) linear systems with a Cholesky factorization followed by forward and backward substitution is backward stable.

## Problem 2: Banded factorization of a finite-difference matrix

Consider the advection equation $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ with $(t, x) \in [0, T] \times [0, 1]$for some $T > 0$. Let $u(x, t)$ have initial condition $u(x, 0) = b(x)$ and boundary conditions $u(0, t) = u(1, t) = 0$. For numerical approximation with finite-differences at equispaced times $0 < t_1, t_2, \ldots, t_m < T$ and points $0 < x_1, x_2, \ldots, x_n < 1$, we can approximate

$$\frac{\partial u_{jk}}{\partial t} \approx \frac{u_{jk} - u_{j(k-1)}}{\Delta t},$$

where $u_{jk} = u(x_j, t_k)$, $\Delta t = t_k - t_{k-1}$, $\Delta x = x_j - x_{j-1}$, and

$$\frac{\partial u_{jk}}{\partial x} \approx \frac{-u_{(j+2)k} + 8u_{(j+1)k} - 8u_{(j-1)k} + u_{(j-2)k}}{12\Delta x}.$$

(a) Show that the vector $u_k = \big(\, u(t_k, x_1), \;\; u(t_k, x_2), \;\; \ldots, \;\; u(t_k, x_n) \,\big)^T$, representing an approximate solution on the grid at time $t_k$, can be obtained from $u_{k-1}$ by solving the $n \times n$ linear system $(I + \sigma D)u_k = u_{k-1}$, where $\sigma = \Delta t/\Delta x < 2/3$ and

$$D = \begin{pmatrix} 0 & 2/3 & -1/12 & & \\ -2/3 & 0 & 2/3 & \ddots & \\ 1/12 & -2/3 & 0 & \ddots & -1/12 \\ & \ddots & \ddots & \ddots & 2/3 \\ & & 1/12 & -2/3 & 0 \end{pmatrix}.$$

(b) Describe a banded elimination algorithm to factor the matrix $A = I + \sigma D$ into the product $A = LDU$: here, $L$ and $U^T$ are lower triangular with unit diagonal and have the same lower bandwidth as $A$, while $D$ is a diagonal matrix. How does the memory cost (number of nonzeros stored explicitly) and flop count for your algorithm scale as $n \to \infty$? Implement your algorithm in the accompanying Julia notebook.

(c) Can your algorithm fail without pivoting when $\sigma < 2/3$? How should one refine the approximation in space (decrease $\Delta x$) and in time (decrease $\Delta t$) to maintain stability without row exchanges? Explain why. (Hint: Problem 1(b) may be useful here.)

(d) Use $A = LDU$ to compute solutions $u_1, u_2, \dots, u_k$, with the problem parameters set in the accompanying Julia notebook.

## Problem 3: Regularized least-squares

Consider the regularized least-squares problem with regularization parameter $\lambda > 0$, given by

$$x_* = \text{argmin}_x \|Ax - b\|_2^2 + \lambda \|x\|_2^2.$$

(a) Show that the regularized least-squares problem is equivalent to a standard least-squares problem,

$$x_* = \text{argmin}_x \left\| \begin{pmatrix} A \\ \sqrt{\lambda} I \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\|_2^2.$$

(b) If the SVD of $A$ is $A = U\Sigma V^*$, show that the unique solution is

$$x_* = V(\Sigma^*\Sigma + \lambda I)^{-1}\Sigma^* U^* b.$$

(c) Under what conditions on $A$ does the regularized solution converge to the usual least-squares solution $\text{argmin}_x ||Ax - b||_2^2$ in the limit $\lambda \to 0$?

(d) Describe a structure-exploiting Givens-based QR solver for the equivalent standard least-squares problem in part (a). How does the flop count compare to the standard QR solver for $\text{argmin} ||Ax - b||_2^2$?

(e) (Not for credit.) Play with the example in the accompanying Julia notebook!