DD2437 – Artificial Neural Networks and Deep Architectures (annda)

Lecture 2: **From perceptron learning rules to backpropagation – supervised learning**

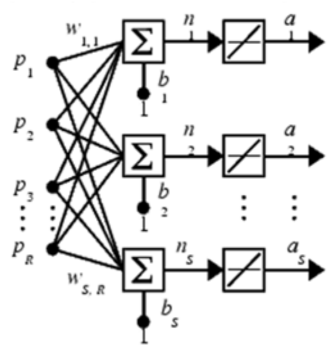
Pawel Herman

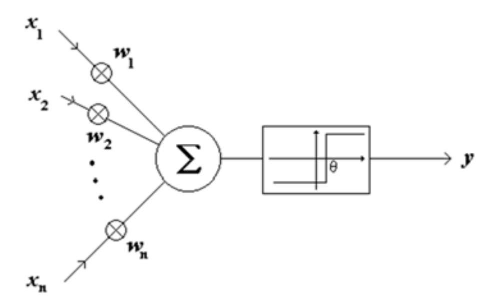
Computational Science and Technology (CST)

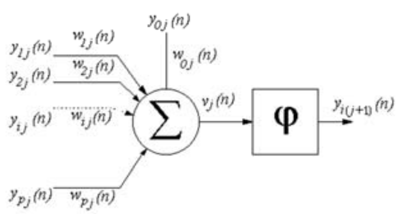
KTH Royal Institute of Technology

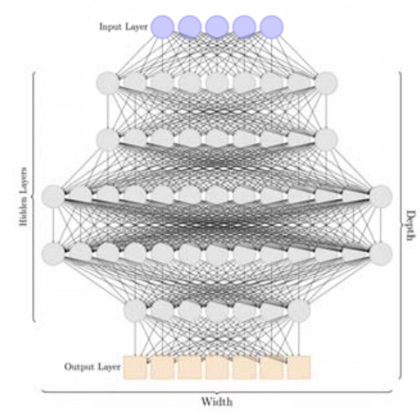
Lecture overview

- A quick recap
- Linear feed-forward networks
- Thresholded single-layer networks
- Perceptron learning, delta rule
- Multi-layer perceptron
- Backpropagation

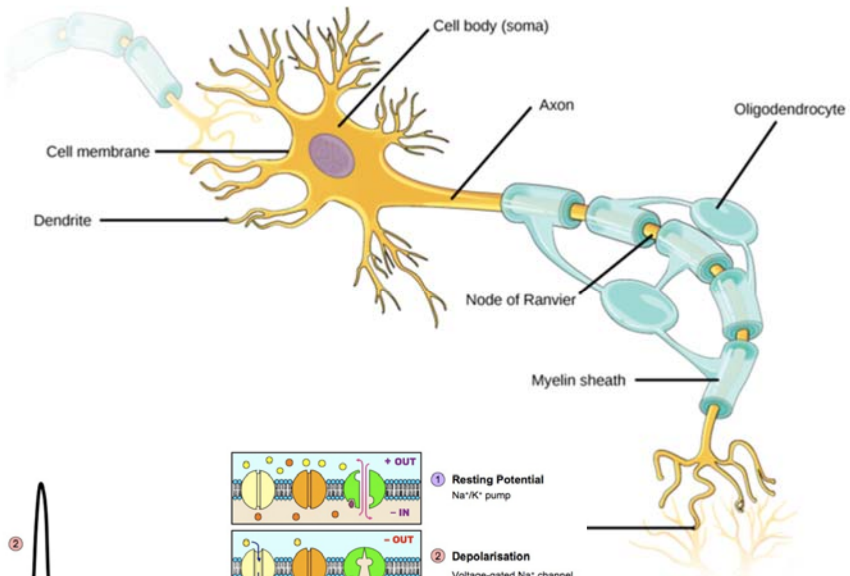


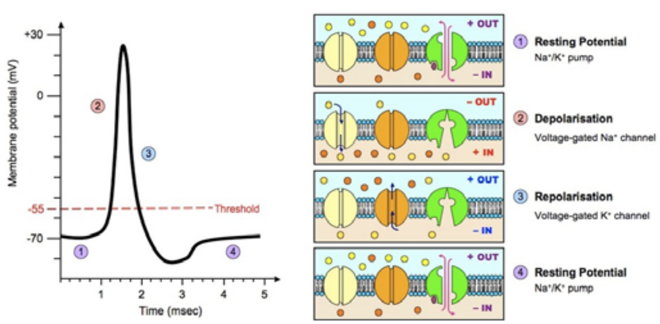


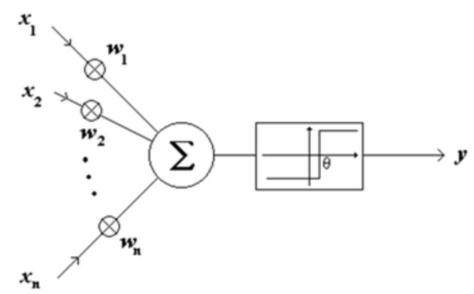




From biological inspirations to ANNs

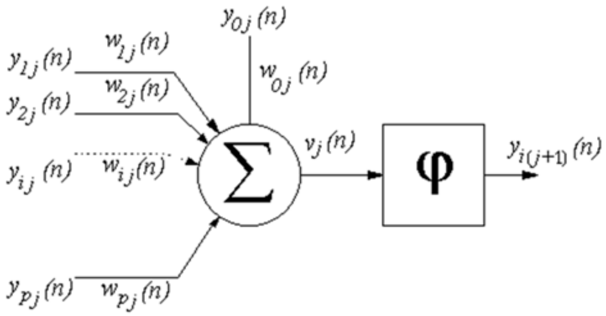






Fundamental aspects

nodes



Fundamental aspects

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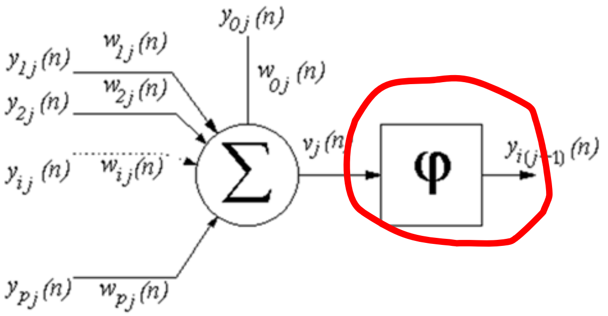
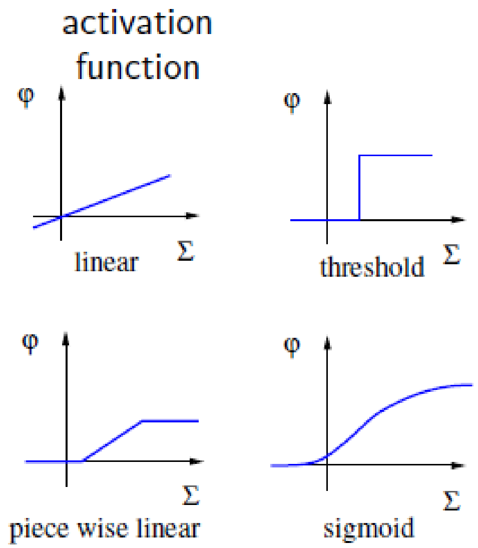


activation function



Fundamental aspects

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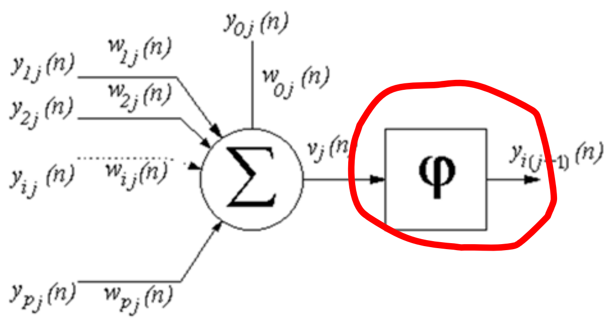
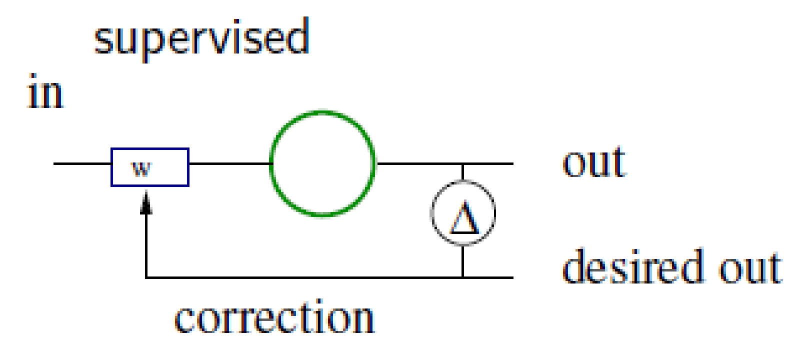


learning rule



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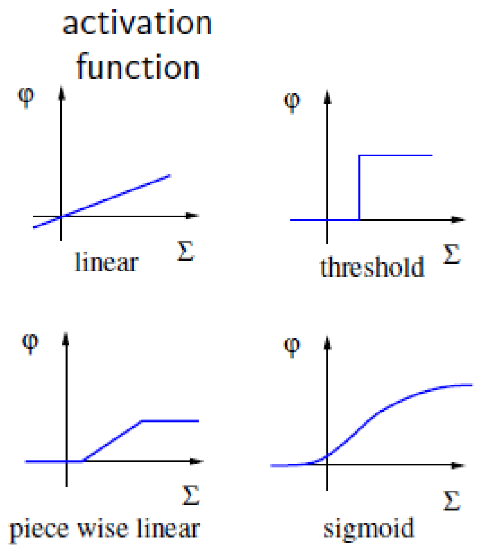

Fundamental aspects

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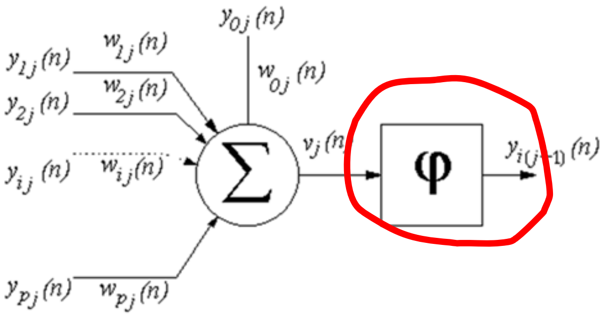

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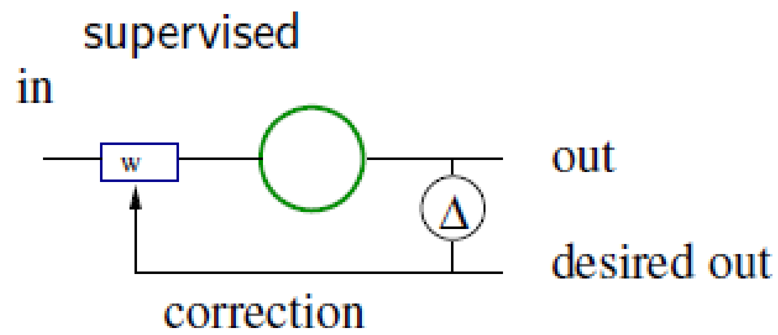
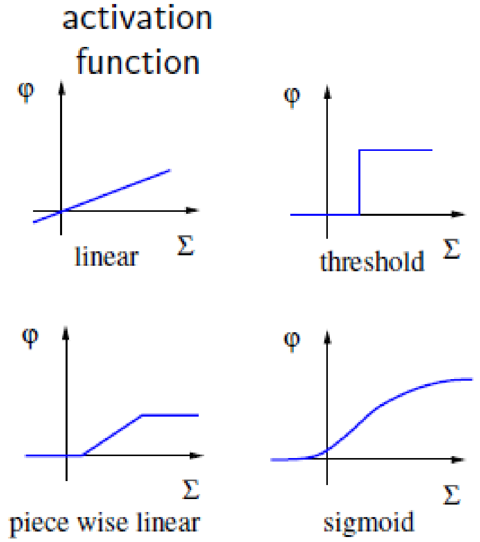
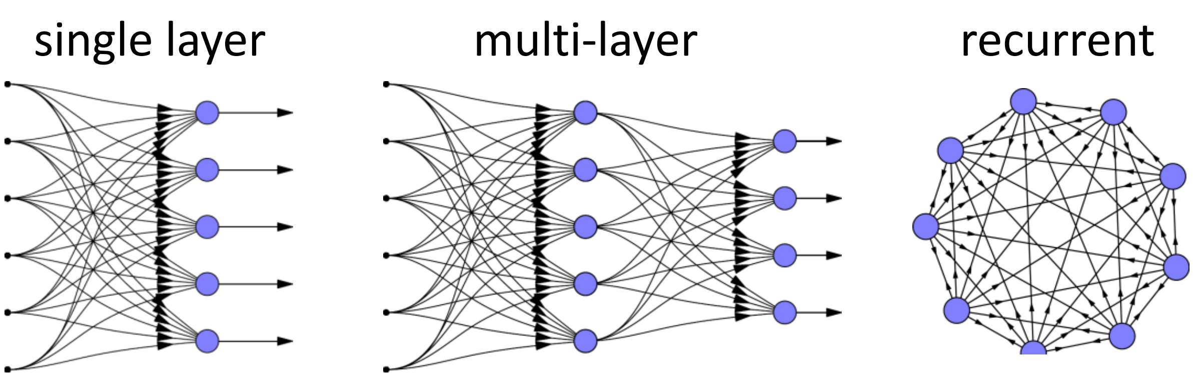


activation function

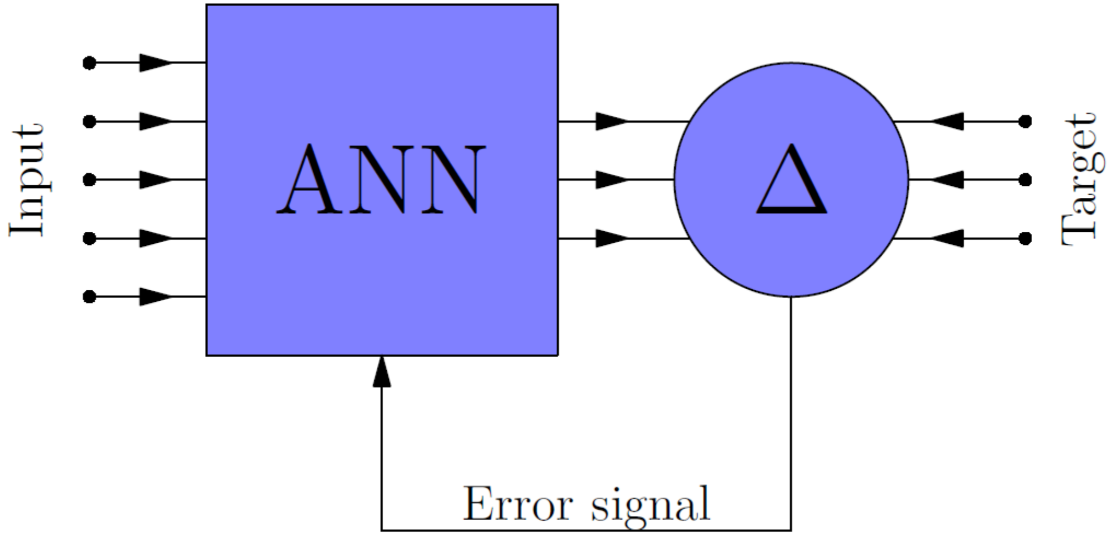


topologies, architectures



Learning principles

- **Error correction**
- Competitive learning
- Coincidence detection



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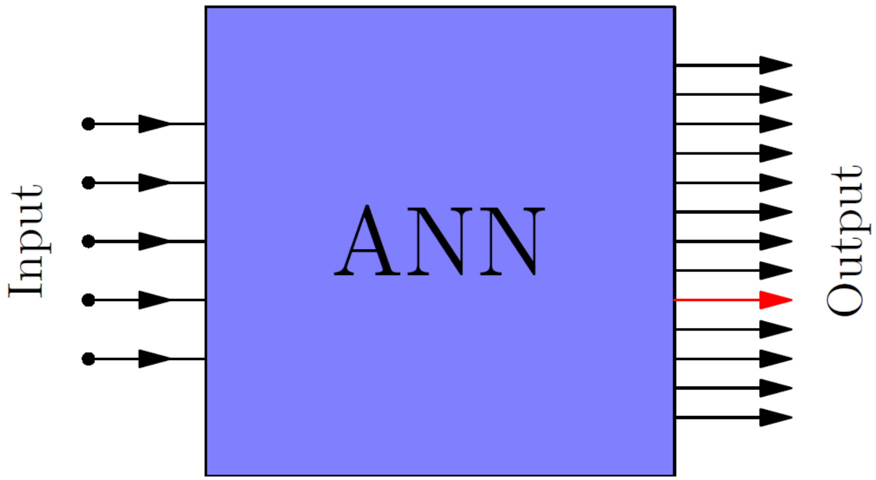

Learning principles

- Error correction
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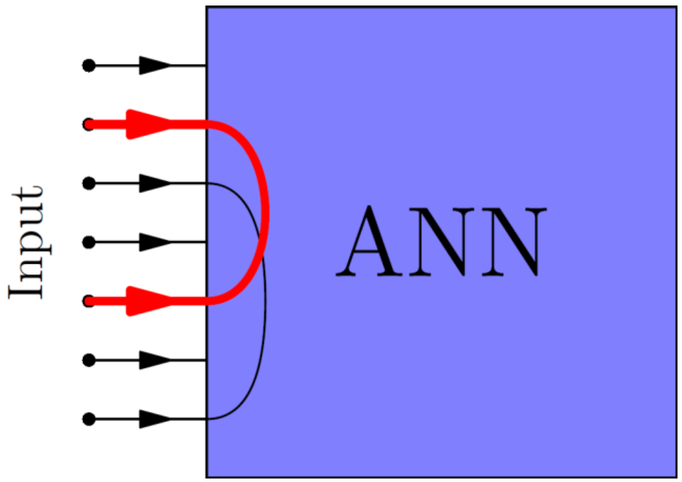


Learning principles

Learning approaches

- supervised
  - with a teacher that provides a correct answer
  - error correction paradigm

Learning principles

Learning approaches

- supervised
- unsupervised (input data only)
  - only input data is available
  - ability to organise information without any error signal to evaluate a potential solution – an explorative approach
  - detecting statistical regularities of the input data and forming internal representations that encode features of the input data

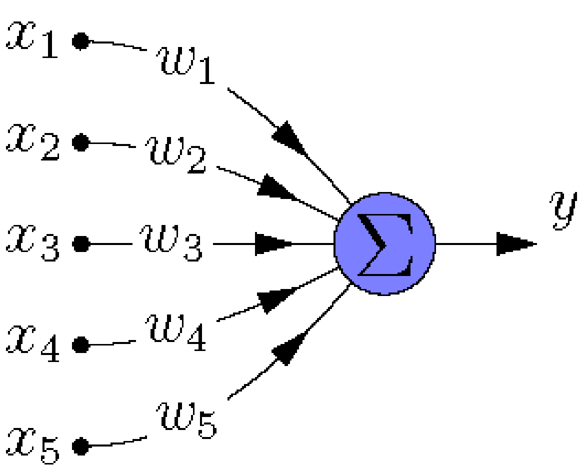
Learning principles

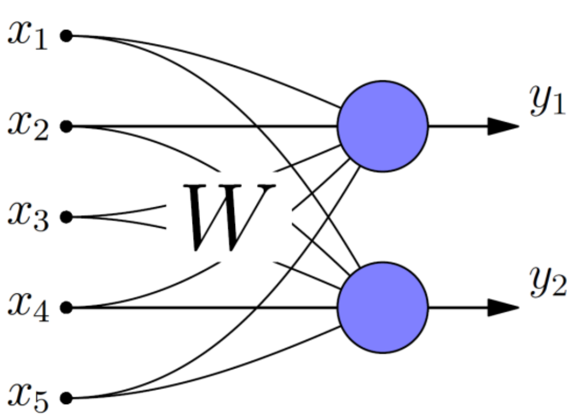
Learning approaches

- supervised
- unsupervised (input data only)
- reinforcement
  - simple scalar “reward” signal gives feedback on success

Linear networks

What can be computed?





$$y = \vec{w}^{\mathrm{T}} \cdot \vec{x}$$

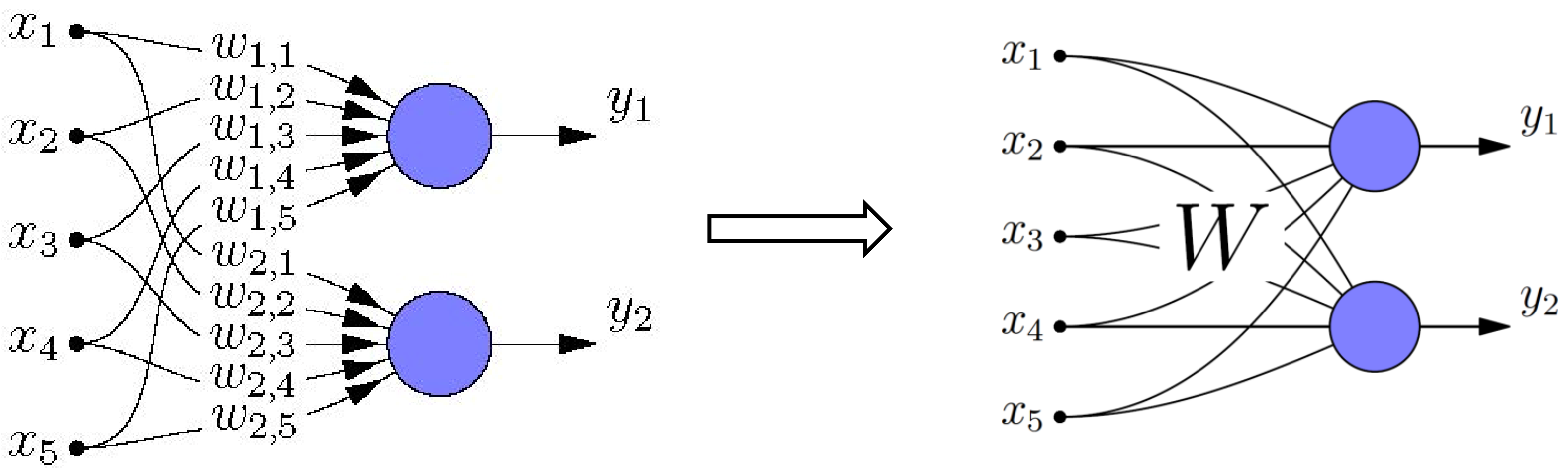
$$y = \mathrm{W} \cdot \vec{x}$$

$\vec{w}$ - weight vector

W - weight matrix

Linear networks

What can be computed?

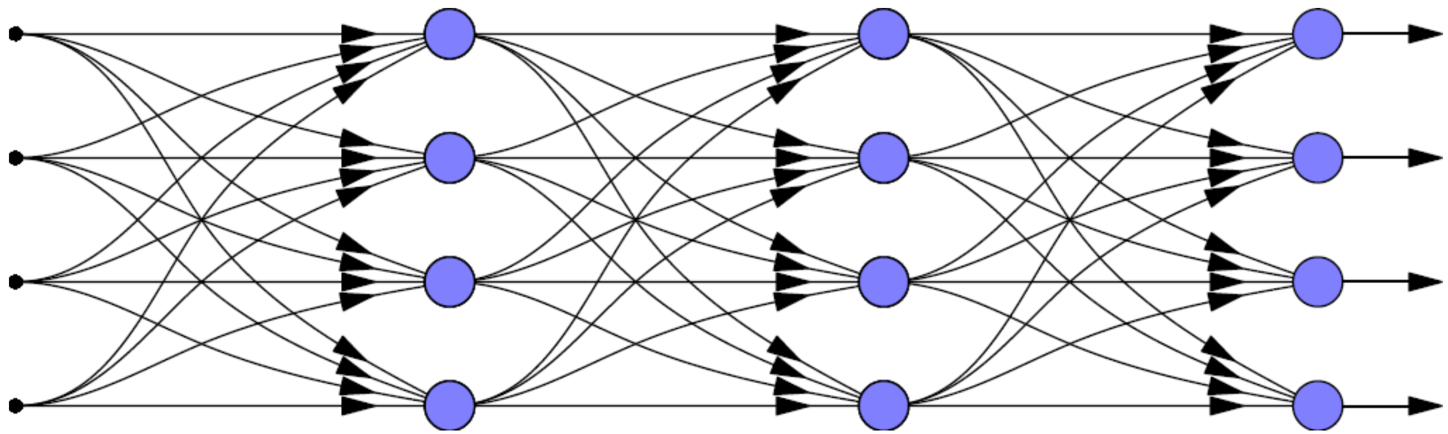


$$y = \mathrm{W} \cdot \vec{x}$$

W - weight matrix

Linear networks

What happens when we concatenate several linear networks?



Linear networks

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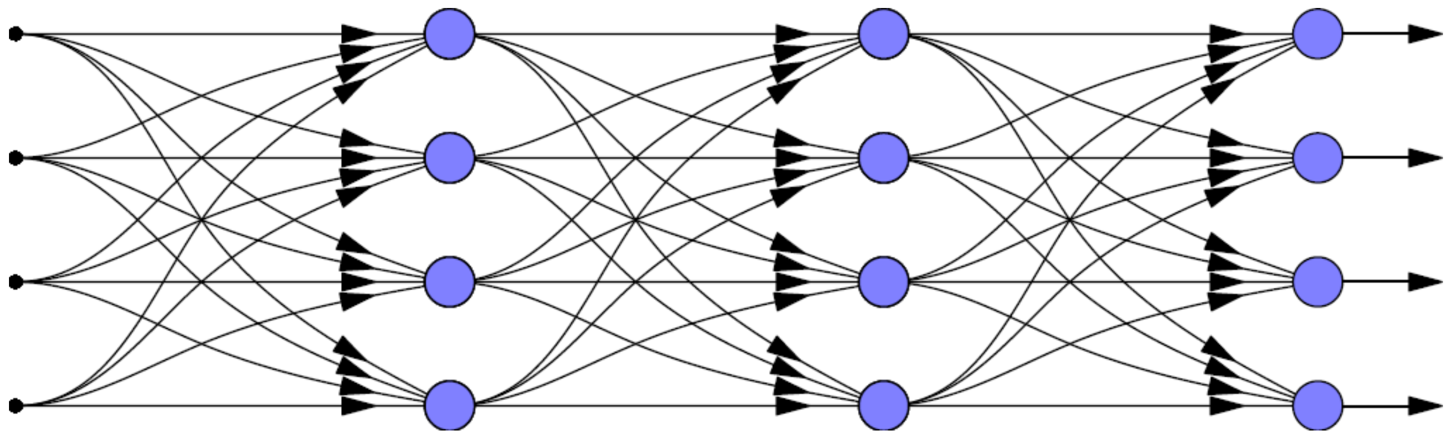

$$\vec{y} = \mathrm{W}_3\left(\mathrm{W}_2\left(\mathrm{W}_1\vec{x}\right)\right) = \left(\mathrm{W}_3\mathrm{W}_2\mathrm{W}_1\right)\vec{x}$$

Linear networks

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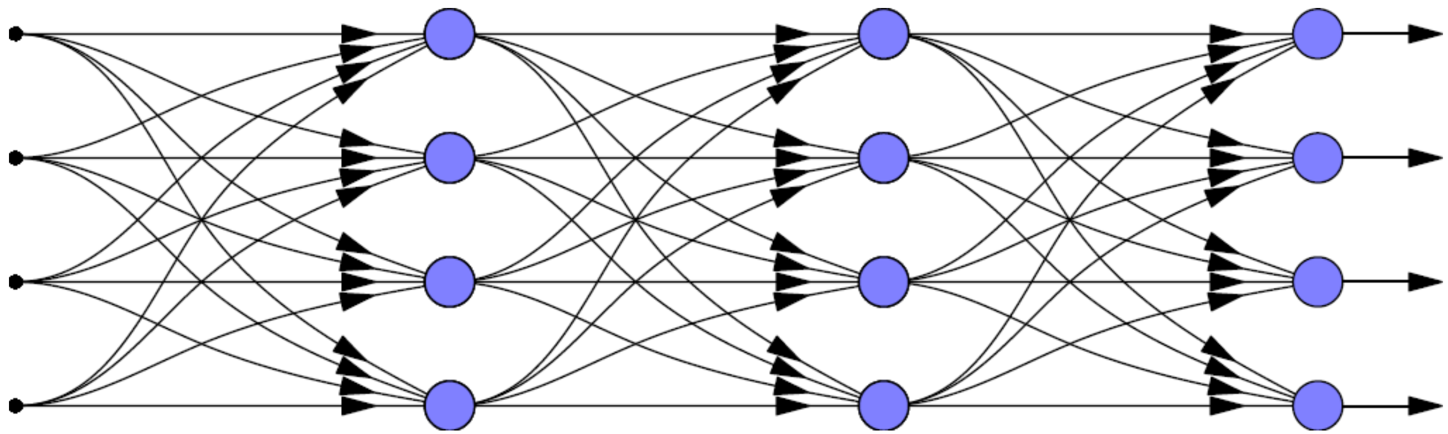

$$\vec{y} = W_3\left(W_2\left(W_1\vec{x}\right)\right) = \left(W_3W_2W_1\right)\vec{x}$$

$$\text{Let } \; W = W_3W_2W_1 \quad \Rightarrow \quad \vec{y} = W\vec{x}$$

Linear networks

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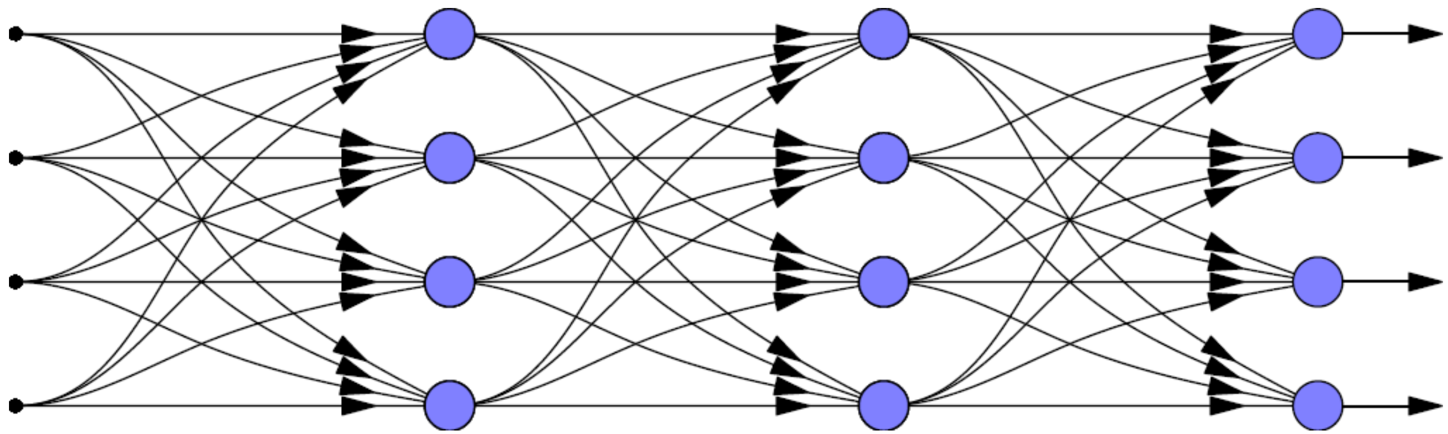

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$$\text{Let } W = W_3W_2W_1 \quad \Rightarrow \quad \vec{y} = W\vec{x}$$

It is still a linear mapping !

Storing mappings (memorising)

The program “resides” in weights

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But how do we find suitable weights?

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Learning corresponds to adapting weights, often *iteratively*, to achieve better performance

Storing mappings (memorising)

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Learning corresponds to adapting weights, often *iteratively*, to achieve better performance

$$w^{(new)} = w^{(old)} + \Delta w_{ij}$$

Storing mappings (memorising)

The program "resides" in weights

But how do we find suitable weights?

Learning corresponds to adapting weights, often *iteratively*, to achieve better performance

Hebb's learning hypothesis

Simultaneous activation of two neurons strengthens their synaptic inter-connection

Storing mappings (memorising)

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Hebb's learning hypothesis

Simultaneous activation of two neurons strengthens their synaptic inter-connection

Common interpretation:

$$\Delta w_{ij} = x_j y_i$$

Storing mappings (memorising)

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But how do we find suitable weights?

Learning corresponds to adapting weights, often *iteratively*, to achieve better performance

Hebb's learning hypothesis

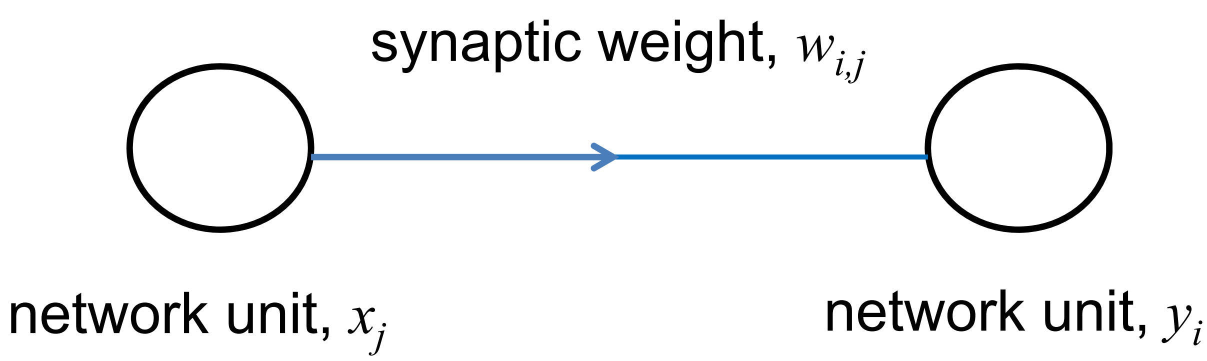
Simultaneous activation of two neurons strengthens their synaptic inter-connection

Common interpretation:

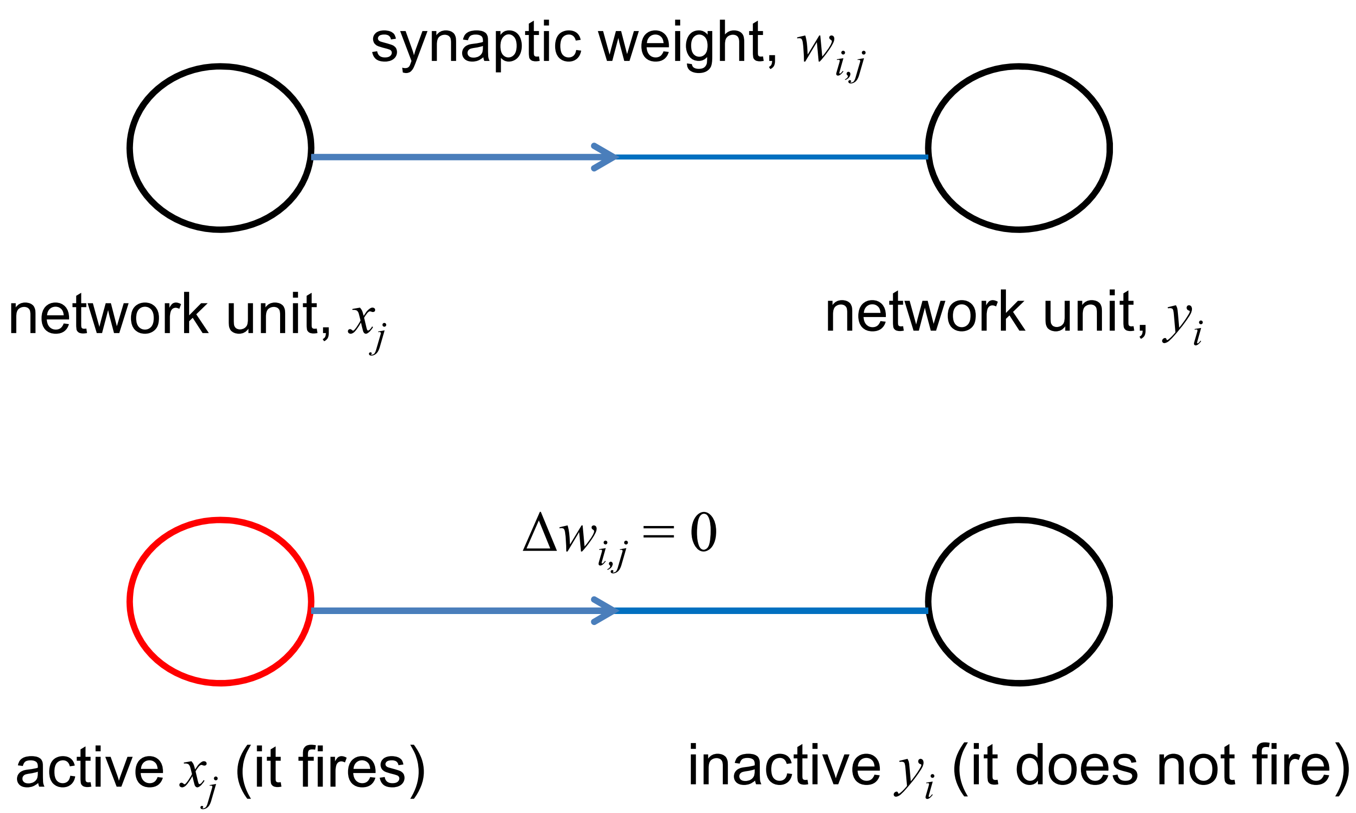
covariance rule

$$\Delta w_{ij} = x_j y_i \qquad \text{or } \ldots \; \Delta w_{ij} = (x_j - \bar{x})\,(y_i - \bar{y})$$

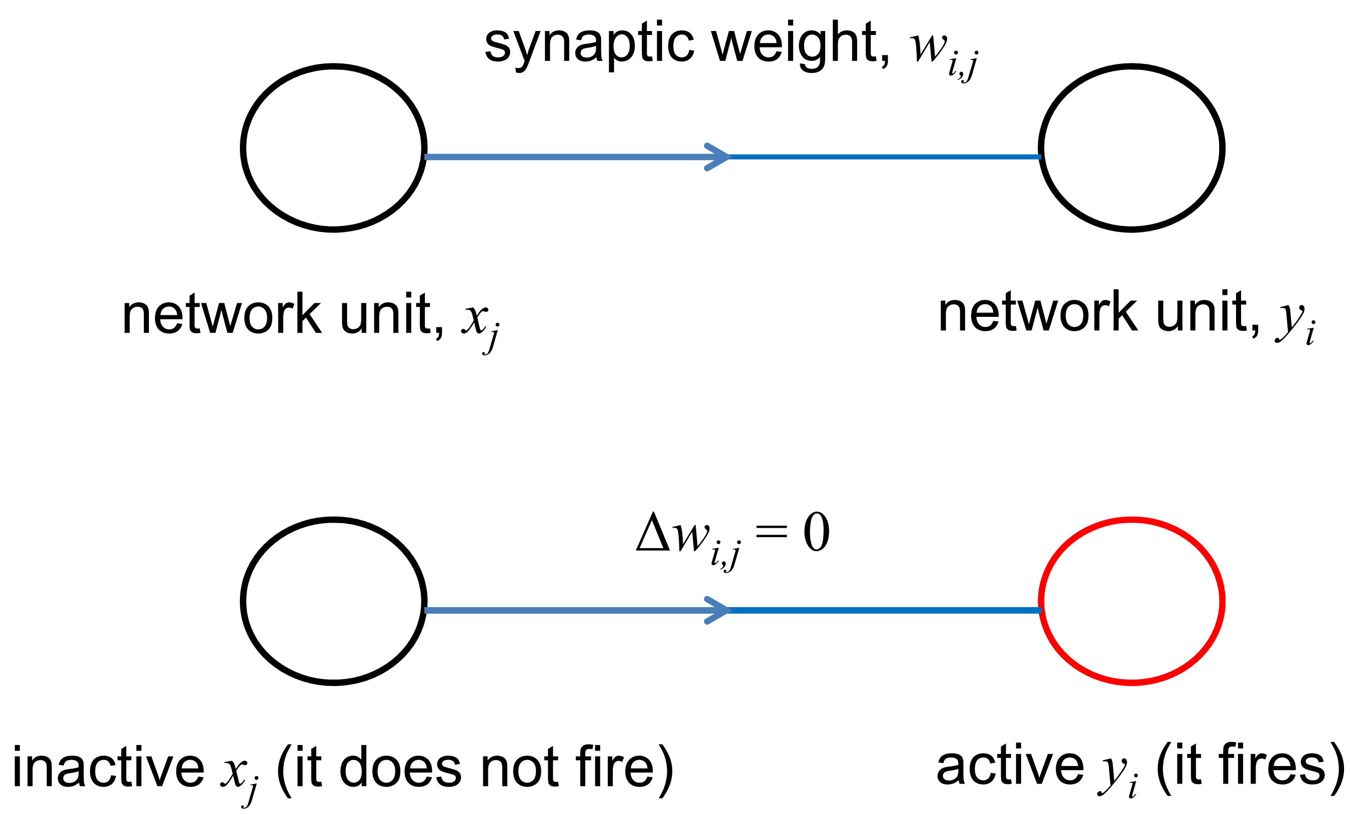
Hebbian learning rule



Hebbian learning rule



Hebbian learning rule



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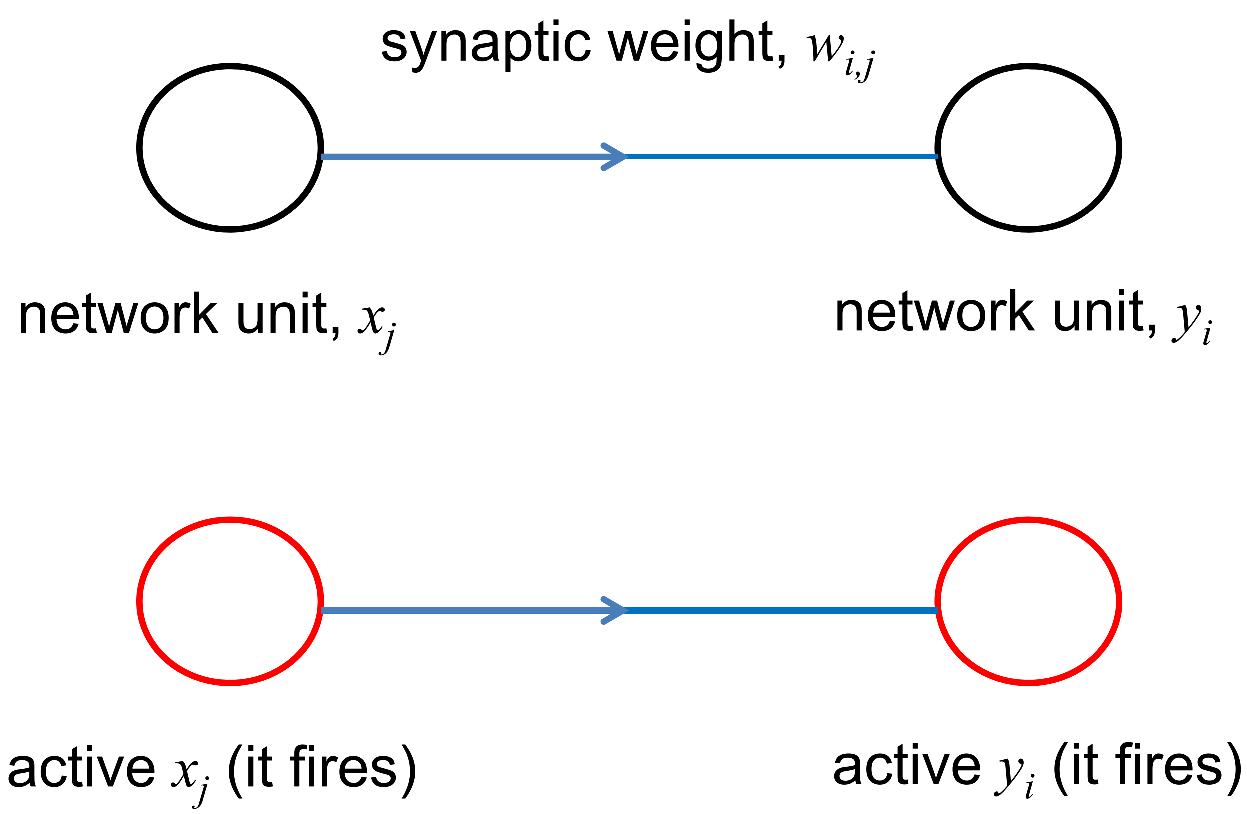
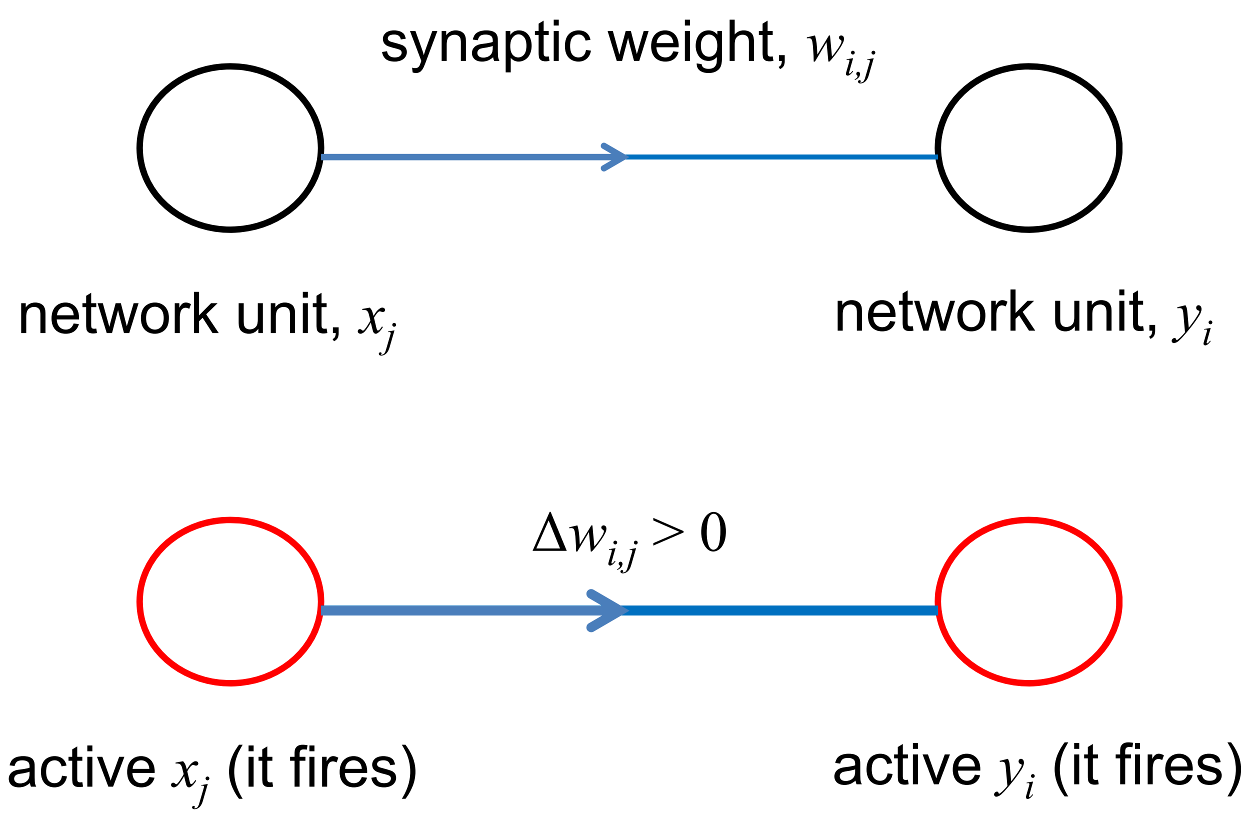


Hebbian learning rule



$$\Delta w_{i,j} = x_j \, y_i$$

"Fire together, wire together"

Storing mappings (memorising)

Storing a mapping using Hebb's rule

$$\vec{x}_1 \to \vec{y}_1 \qquad \vec{x}_2 \to \vec{y}_2 \qquad \vec{x}_3 \to \vec{y}_3 \quad \ldots \quad \vec{x}_n \to \vec{y}_n$$

Storing mappings (memorising)

Storing a mapping using Hebb's rule

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Hebb's rule $$\Delta w_{ij} = x_i y_i$$

Storing mappings (memorising)

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Hebb's rule

$$\Delta w_{ij} = x_i y_j$$

Result

$$\mathrm{W} = \sum_{p=1}^{n} \vec{y}_p \cdot \vec{x}_p^{\mathrm{T}}$$

(outer product of vector patterns)

Storing mappings (memorising)

Storing a mapping using Hebb's rule

$$\vec{x}_1 \rightarrow \vec{y}_1 \qquad \vec{x}_2 \rightarrow \vec{y}_2 \qquad \vec{x}_3 \rightarrow \vec{y}_3 \quad \ldots \quad \vec{x}_n \rightarrow \vec{y}_n$$

Hebb's rule $\quad \Delta w_{ij} = x_i y_j$

Result $\quad \mathrm{W} = \sum_{p=1}^{n} \vec{y}_p \cdot \vec{x}_p^{\mathrm{T}}$

Correlational memory!

Storing mappings (memorising)

Retrieving a memory trace

$$\mathrm{W} = \sum_{p=1}^{n} \vec{y}_p \cdot \vec{x}_p^{\mathrm{T}}$$

$$\vec{x}_k \rightarrow \; ?$$

Storing mappings (memorising)

Retrieving a memory trace

$$\mathrm{W} = \sum_{p=1}^{n} \vec{y}_p \cdot \vec{x}_p^{\mathrm{T}}$$

$$\vec{x}_k \rightarrow \;?$$

$$\vec{y}_{out} = \mathrm{W}\,\vec{x}_k = \sum_{p=1}^{n} (\vec{y}_p \vec{x}_p^{\mathrm{T}})\,\vec{x}_k = \sum_{p=1}^{n} \vec{y}_p (\vec{x}_p^{\mathrm{T}} \vec{x}_k)$$

Storing mappings (memorising)

Retrieving a memory trace

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$$= \vec{y}_k \underbrace{(\vec{x}_k^{\mathrm{T}} \vec{x}_k)}_{\alpha} + \sum_{p \neq k}^{n} \vec{y}_p (\vec{x}_p^{\mathrm{T}} \vec{x}_k)$$

Storing mappings (memorising)

Retrieving a memory trace

$$\mathrm{W} = \sum_{p=1}^{n} \vec{y}_p \cdot \vec{x}_p^{\mathrm{T}}$$

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Storing mappings (memorising)

Retrieving a memory trace

$$W = \sum_{p=1}^{n} \vec{y}_p \cdot \vec{x}_p^{\mathrm{T}}$$

$$\vec{x}_k \rightarrow \; ?$$

$$\vec{y}_{out} = W\vec{x}_k = \sum_{p=1}^{n} (\vec{y}_p \vec{x}_p^{\mathrm{T}})\vec{x}_k = \sum_{p=1}^{n} \vec{y}_p (\vec{x}_p^{\mathrm{T}} \vec{x}_k) =$$

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$$\approx 0$$

Storing mappings (memorising)

Retrieving a memory trace

$$\mathrm{W} = \sum_{p=1}^{n} \vec{y}_p \cdot \vec{x}_p^{\mathrm{T}}$$

$$\vec{x}_k \rightarrow \ ?$$

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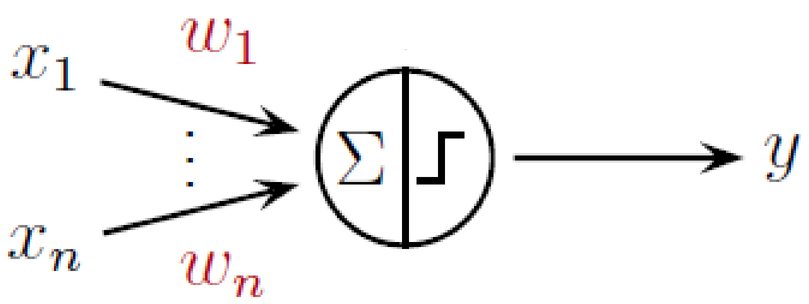
$$= \vec{y}_k (\vec{x}_k^{\mathrm{T}} \vec{x}_k) + \sum_{p \neq k}^{n} \vec{y}_p (\vec{x}_p^{\mathrm{T}} \vec{x}_k) \approx \alpha \vec{y}_k$$

Perfect memory only if the patterns $\vec{x}_p$ are orthogonal

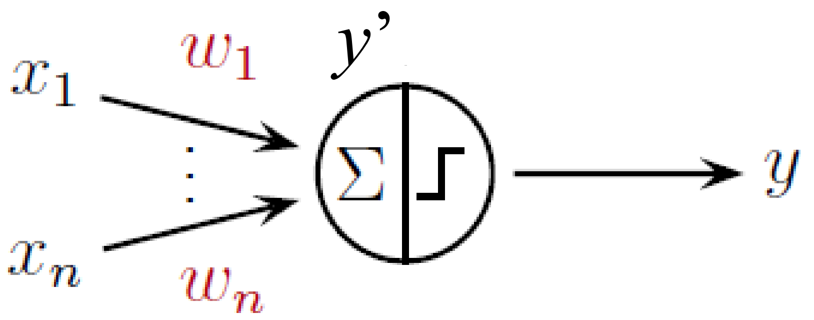
TLU – how it all started….

Threshold logic unit – McCulloch Pitts neuron (1942)



TLU – McCulloch Pitts

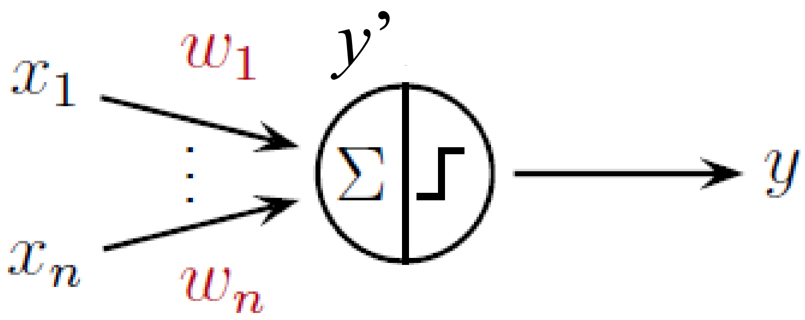
Threshold logic unit – McCulloch Pitts neuron (1942)



$$y' = w_1 x_1 + w_2 x_2 \qquad y = f_{step}(y')$$

TLU – McCulloch Pitts

Threshold logic unit – McCulloch Pitts neuron (1942)



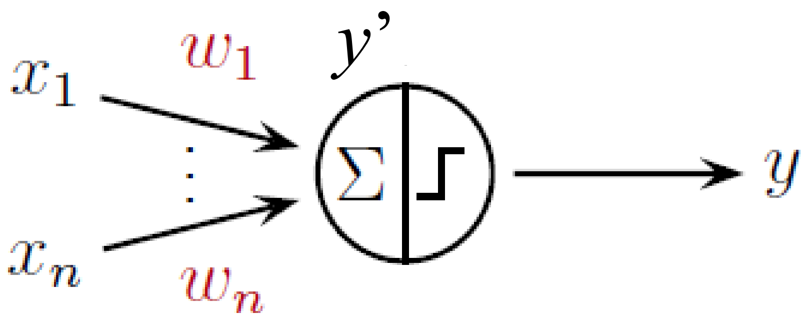
$$y' = w_1 x_1 + w_2 x_2 \qquad y = f_{step}(y')$$

If threshold is 0, then:

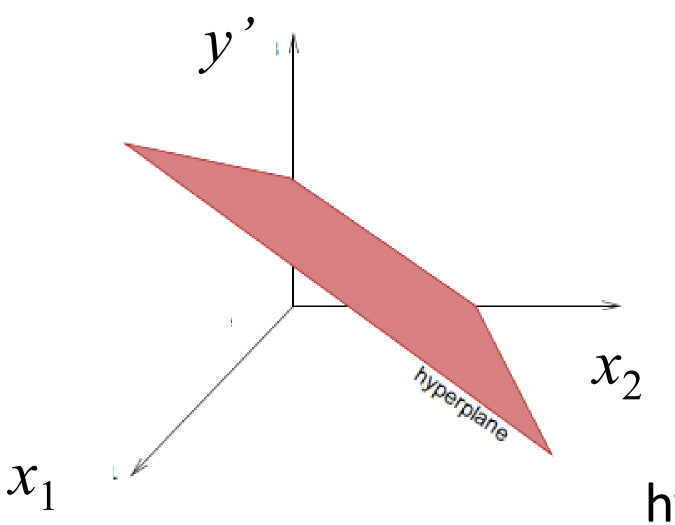
$$w_1 x_1 + w_2 x_2 > 0 \rightarrow y' > 0 \rightarrow y = 1$$

$$w_1 x_1 + w_2 x_2 <= 0 \rightarrow y' <= 0 \rightarrow y = 0$$

Geometrical interpretation

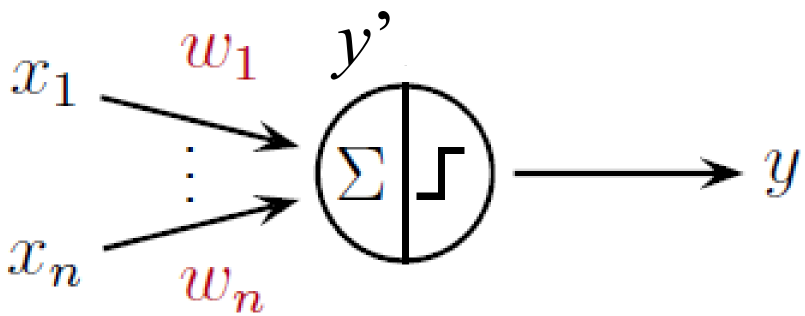


$$y' = w_1 x_1 + w_2 x_2$$



hyperplane

Geometrical interpretation



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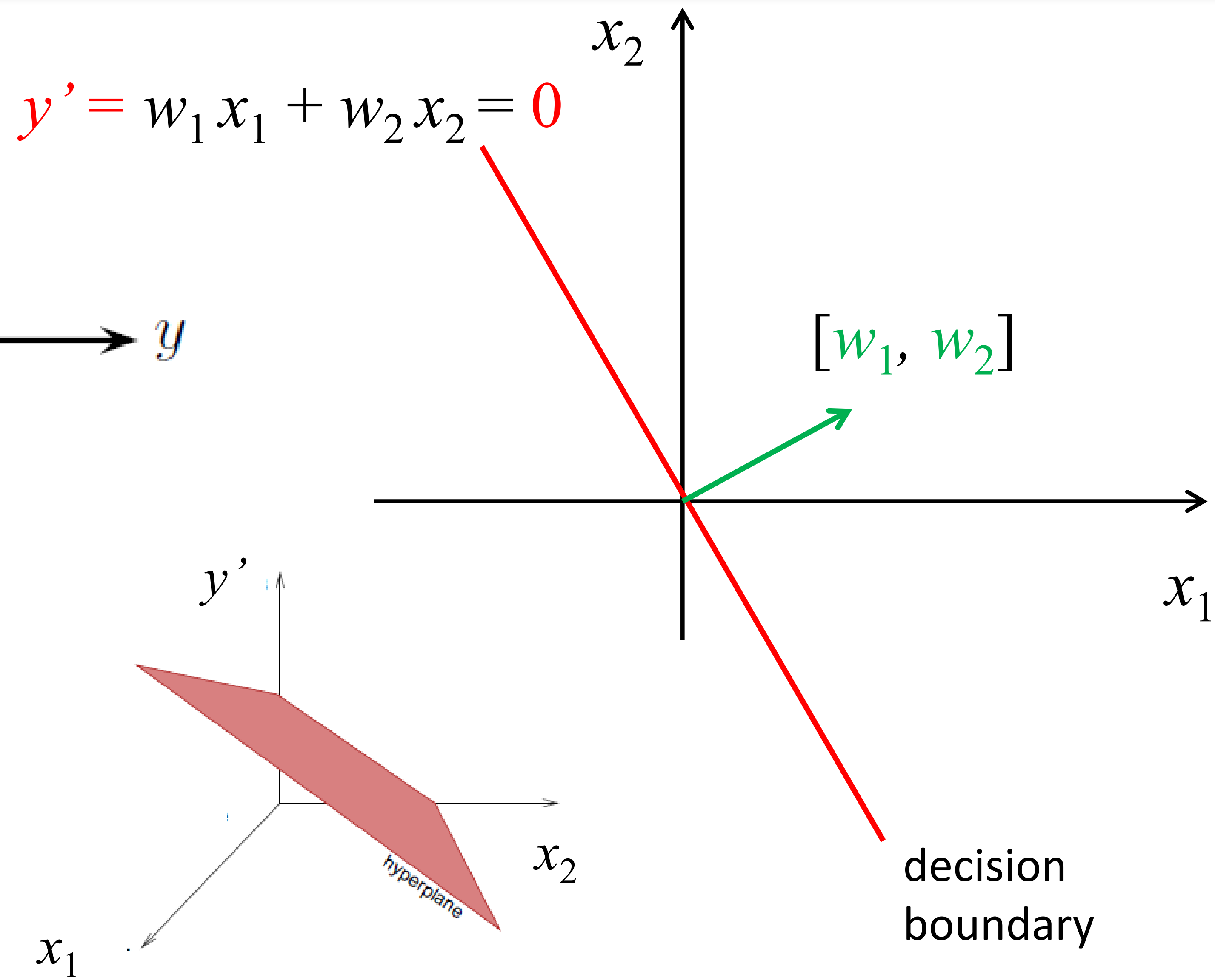
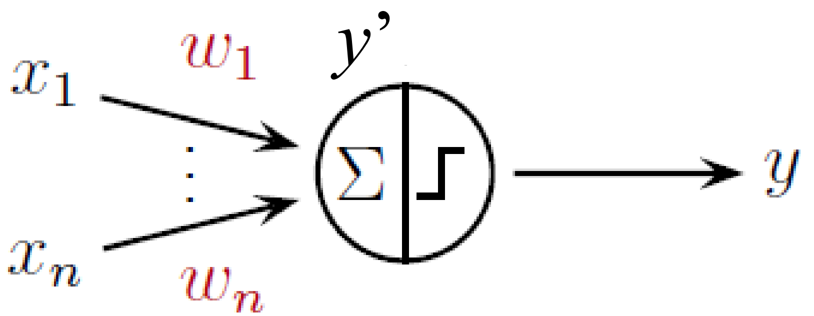
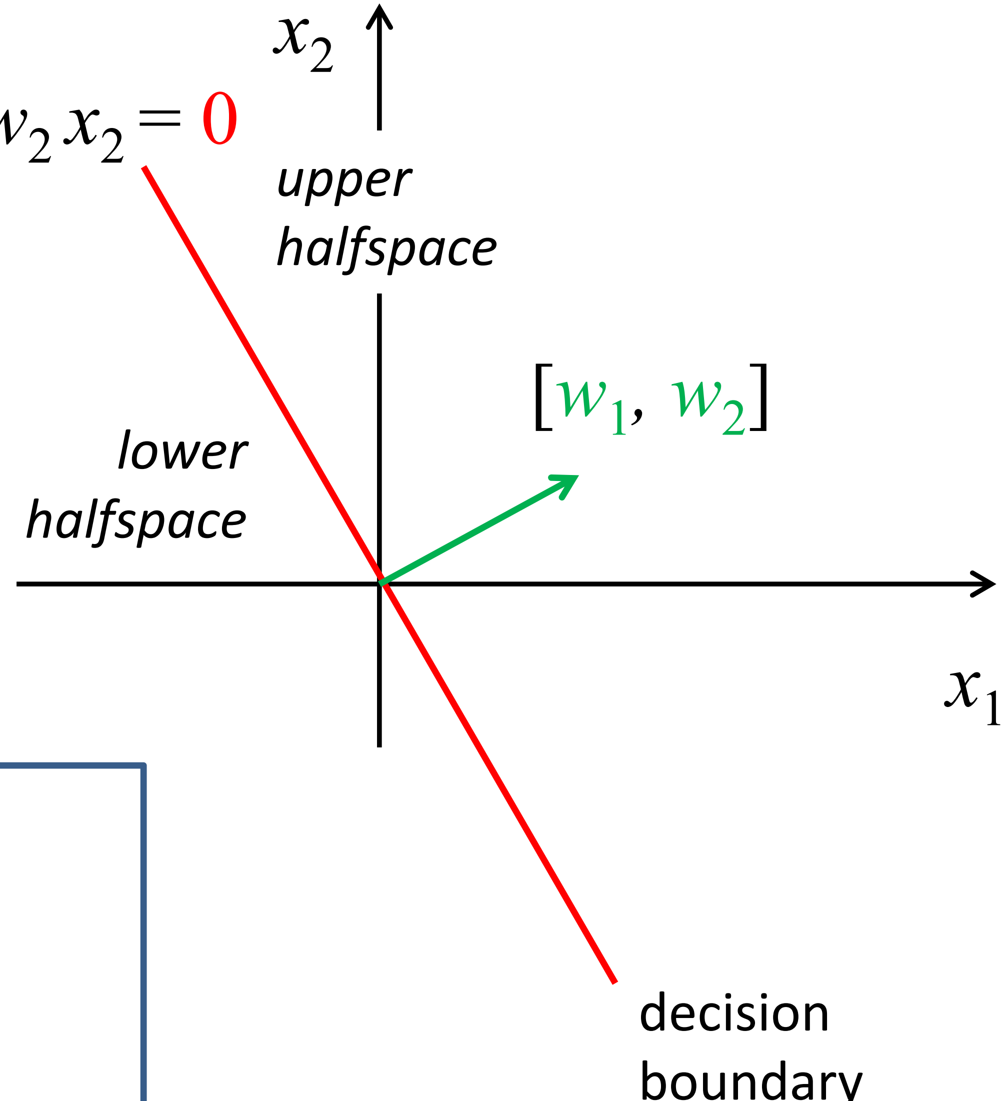


Threshold in TLU

$$y' = w_1 x_1 + w_2 x_2 = 0$$



$$y' = w_1 x_1 + w_2 x_2$$



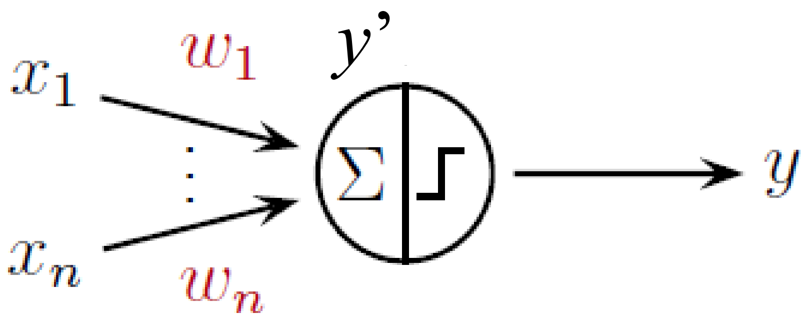
THRESHOLDING with $\theta = 0$:

$$y = f_{step}(y')$$

$$w_1 x_1 + w_2 x_2 > 0 \rightarrow y' > 0 \rightarrow y = 1$$

$$w_1 x_1 + w_2 x_2 <= 0 \rightarrow y' <= 0 \rightarrow y = 0$$

Threshold in TLU

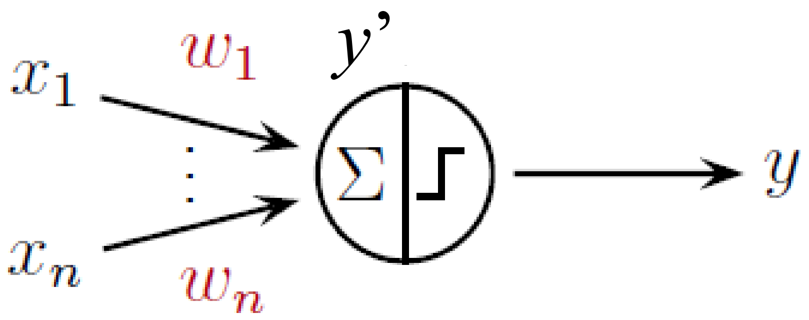


$$y' = w_1 x_1 + w_2 x_2$$

THRESHOLDING with $\theta \sim= 0$:

$$y' > \theta \rightarrow y = 1$$
$$y' <= \theta \rightarrow y = 0$$

Threshold in TLU



$$y' = w_1 x_1 + w_2 x_2$$

THRESHOLDING with $\theta \sim= 0$:

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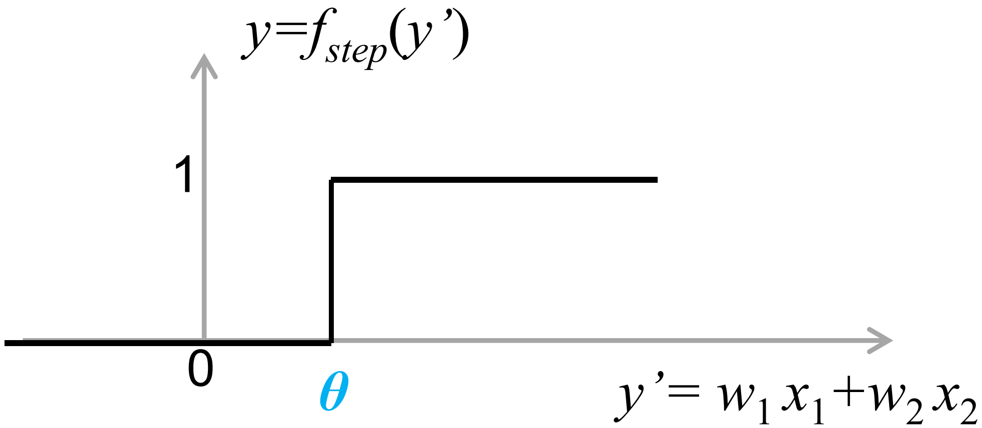
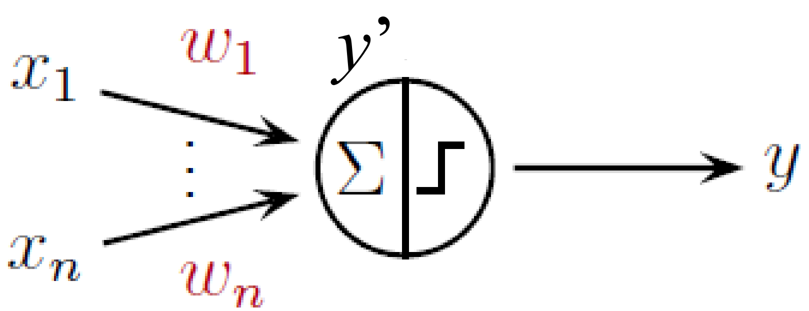
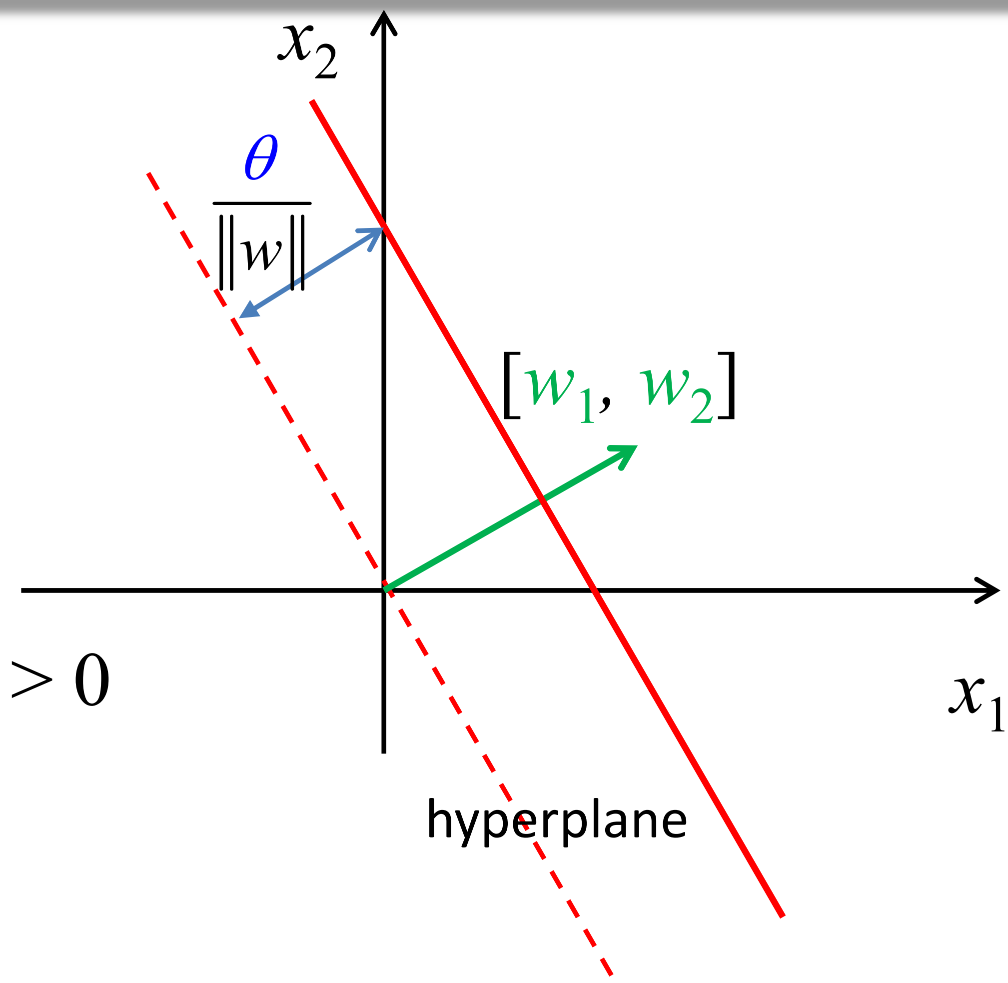
$$y' <= \theta \rightarrow y = 0$$



$$w_1 x_1 + w_2 x_2 > \theta \rightarrow y = 1$$

Threshold in TLU – geometrical interpretation

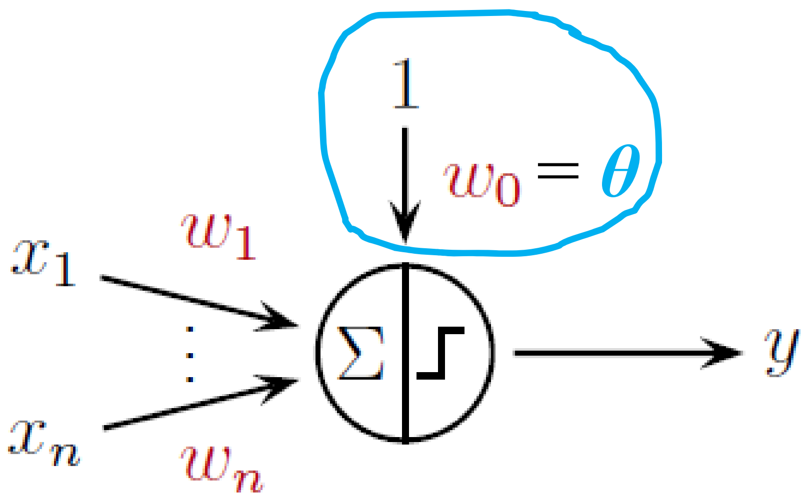


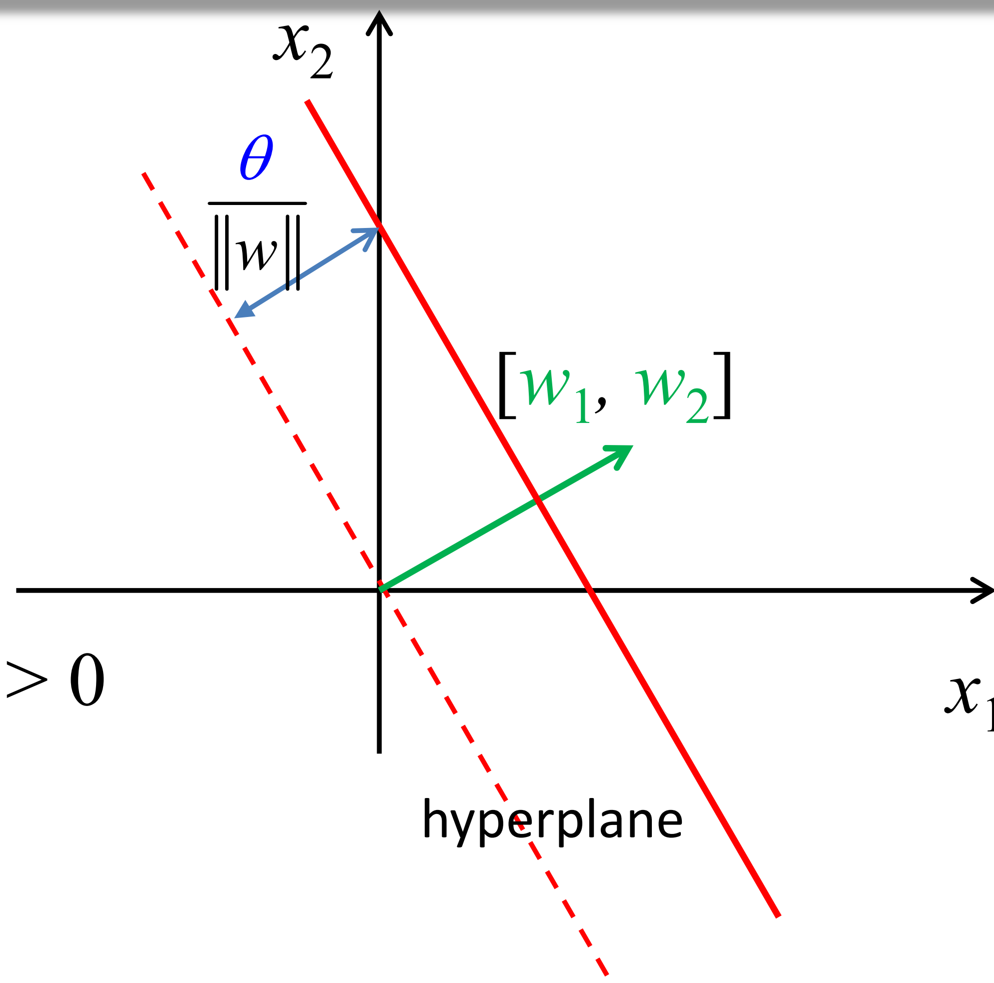


$$w_1 x_1 + w_2 x_2 > \theta \rightarrow w_1 x_1 + w_2 x_2 - \theta > 0$$

$$y' = w_1 x_1 + w_2 x_2 - \theta$$

Threshold in TLU – bias trick





$$w_1 x_1 + w_2 x_2 > \theta \rightarrow w_1 x_1 + w_2 x_2 - \theta > 0$$

$$y' = w_1 x_1 + w_2 x_2 - \theta$$

$$y' = w_1 x_1 + w_2 x_2 + w_0 1$$

where: $\quad bias \quad w_0 = -\theta$

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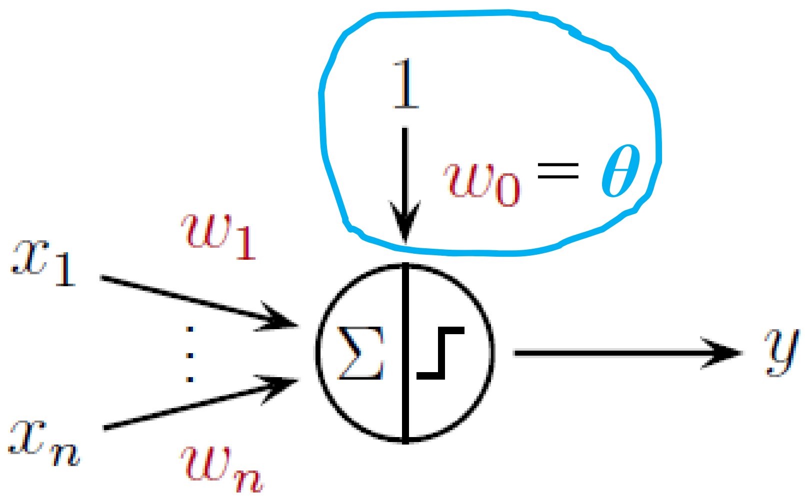

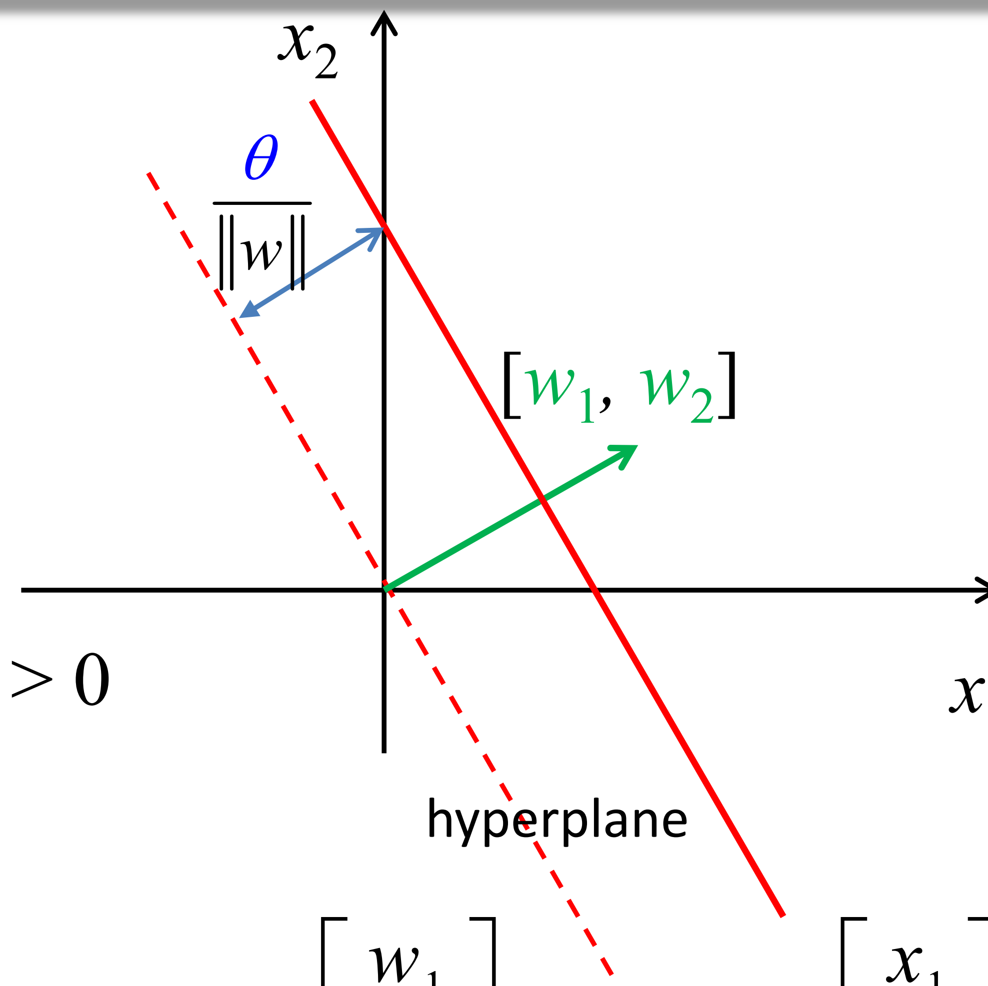

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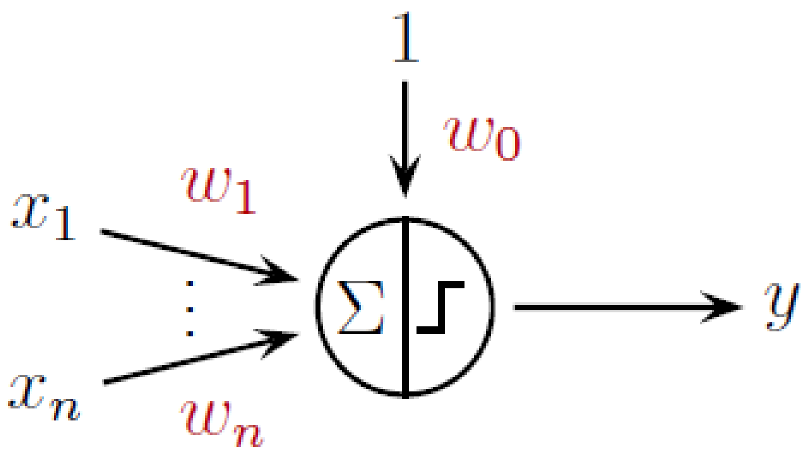
$$y' = w_1 x_1 + w_2 x_2 + w_0 1$$

where: $bias \quad w_0 = -\theta$

$$\vec{\boldsymbol{w}} = \begin{bmatrix} w_1 \\ w_2 \\ -\theta \end{bmatrix} \qquad \vec{\boldsymbol{x}} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

Linear separability with TLU – geometrical interpret.



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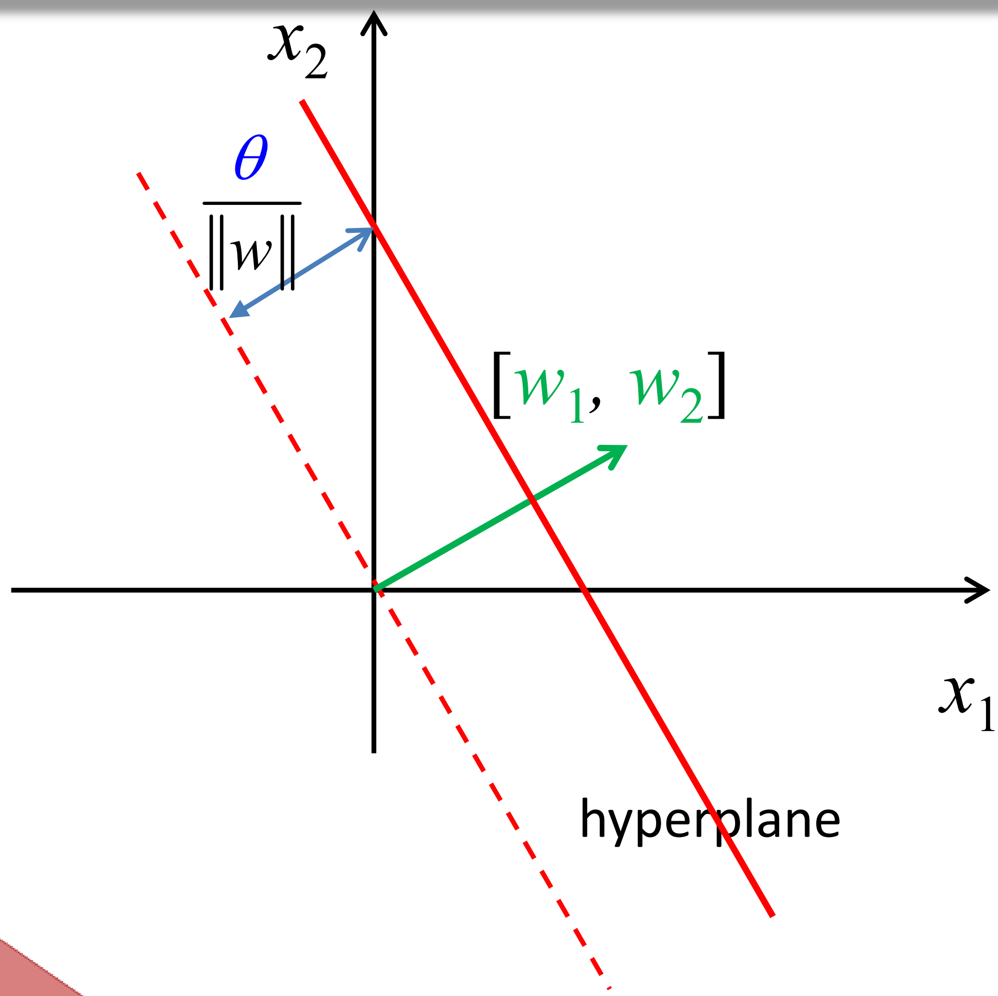

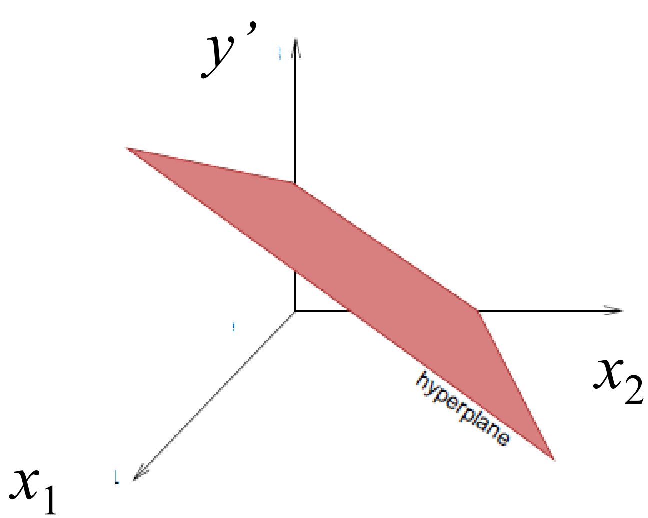

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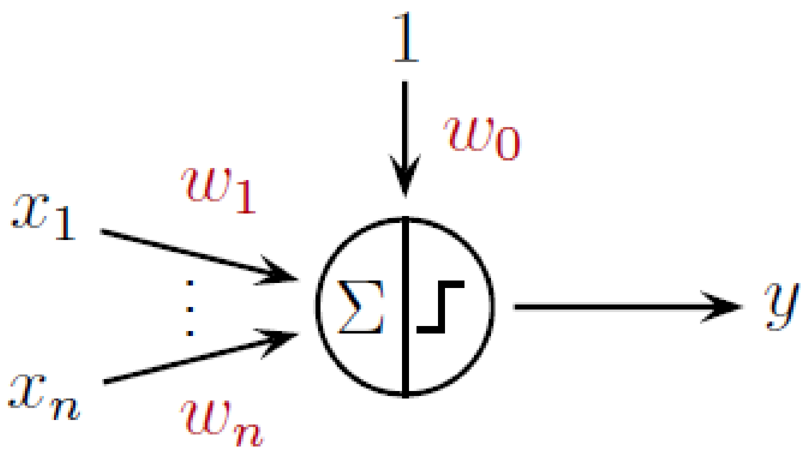

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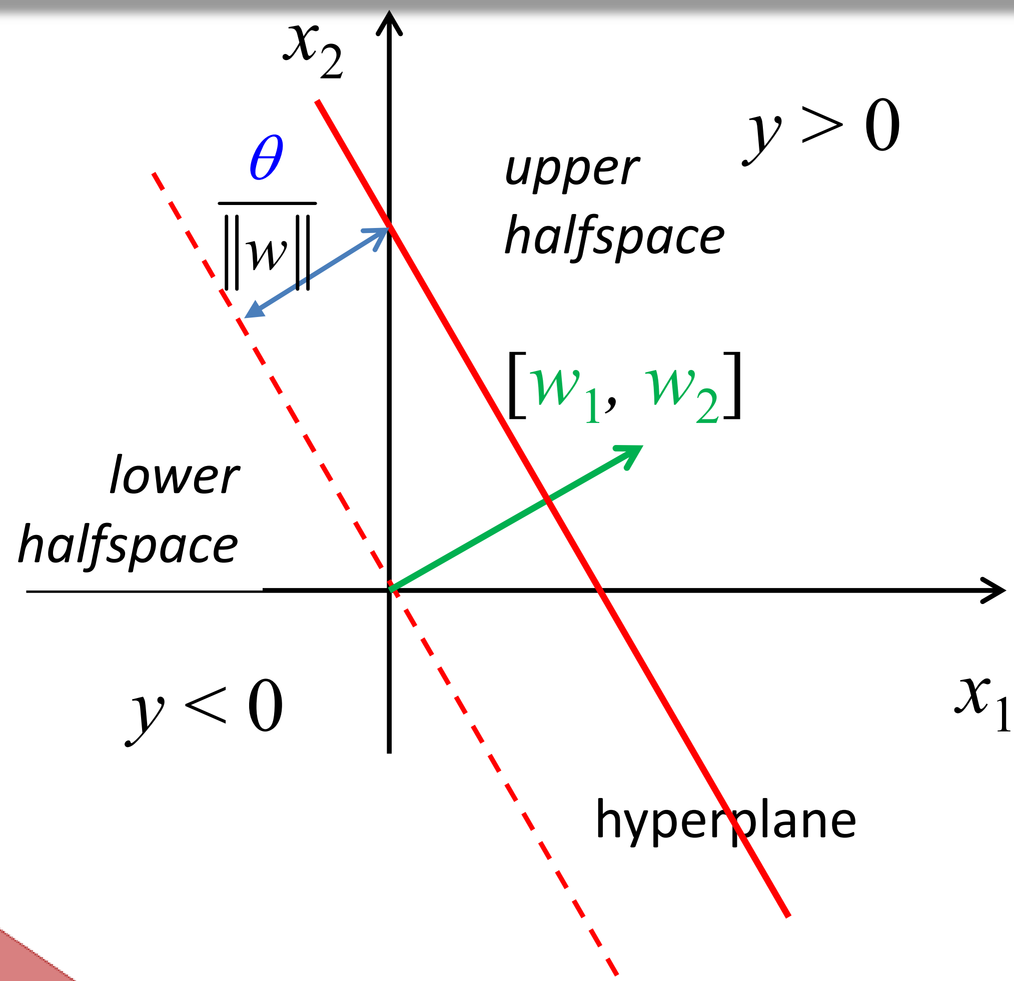
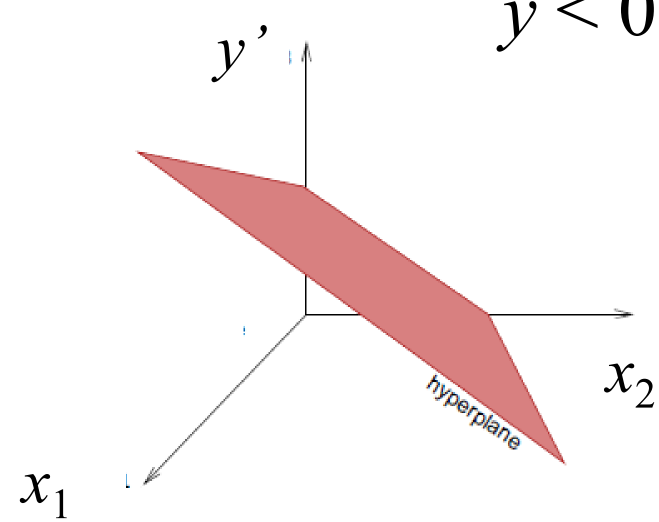
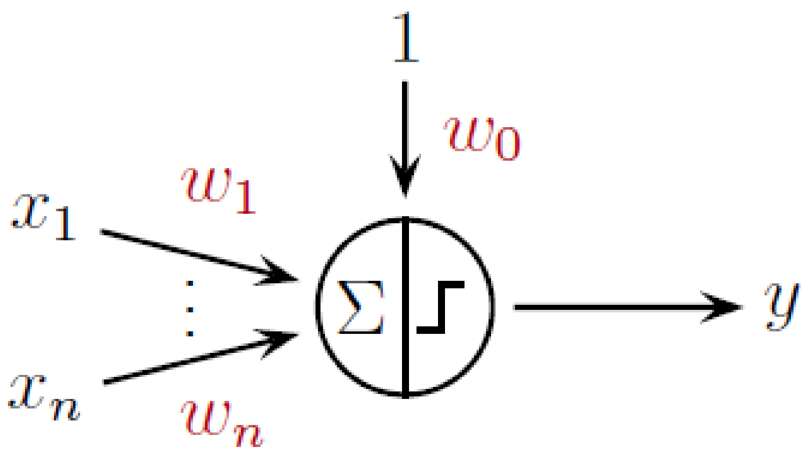
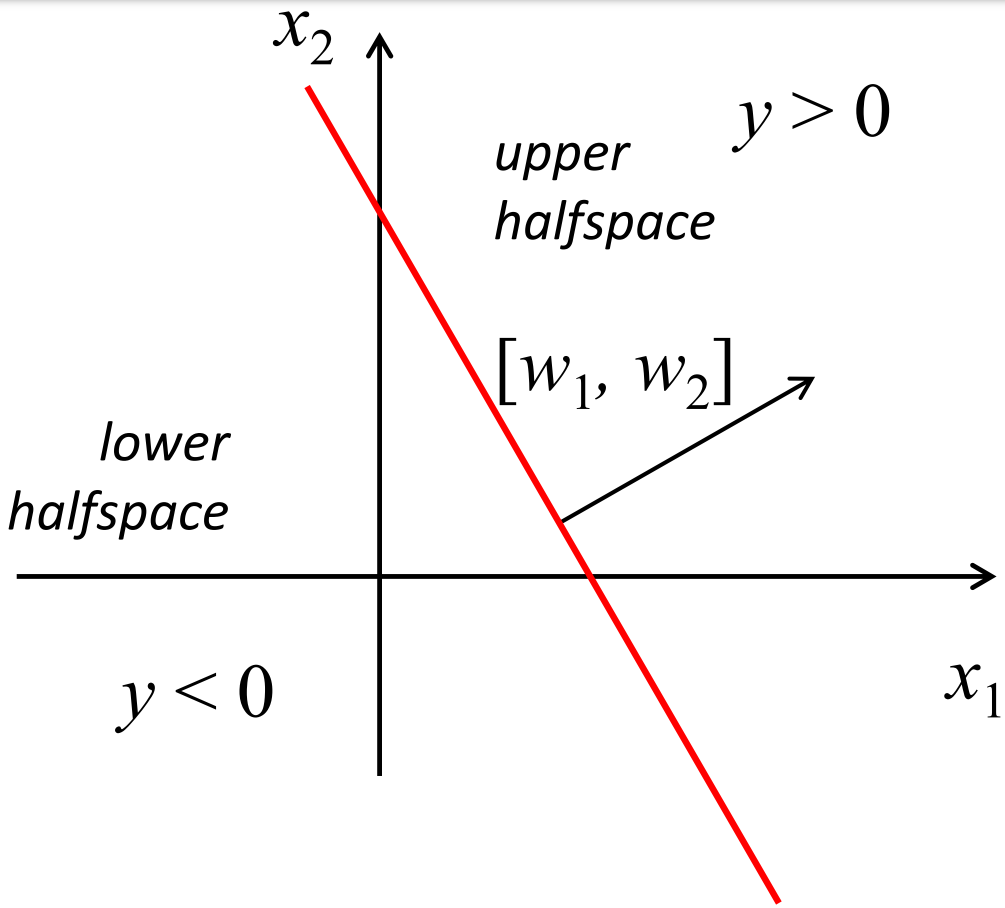


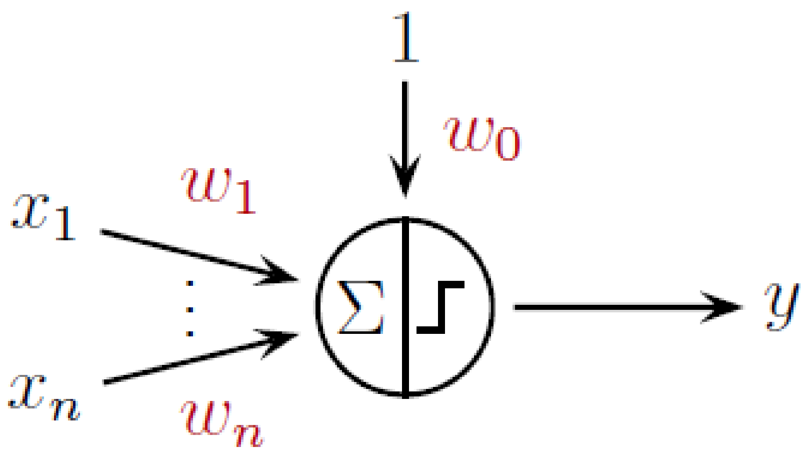


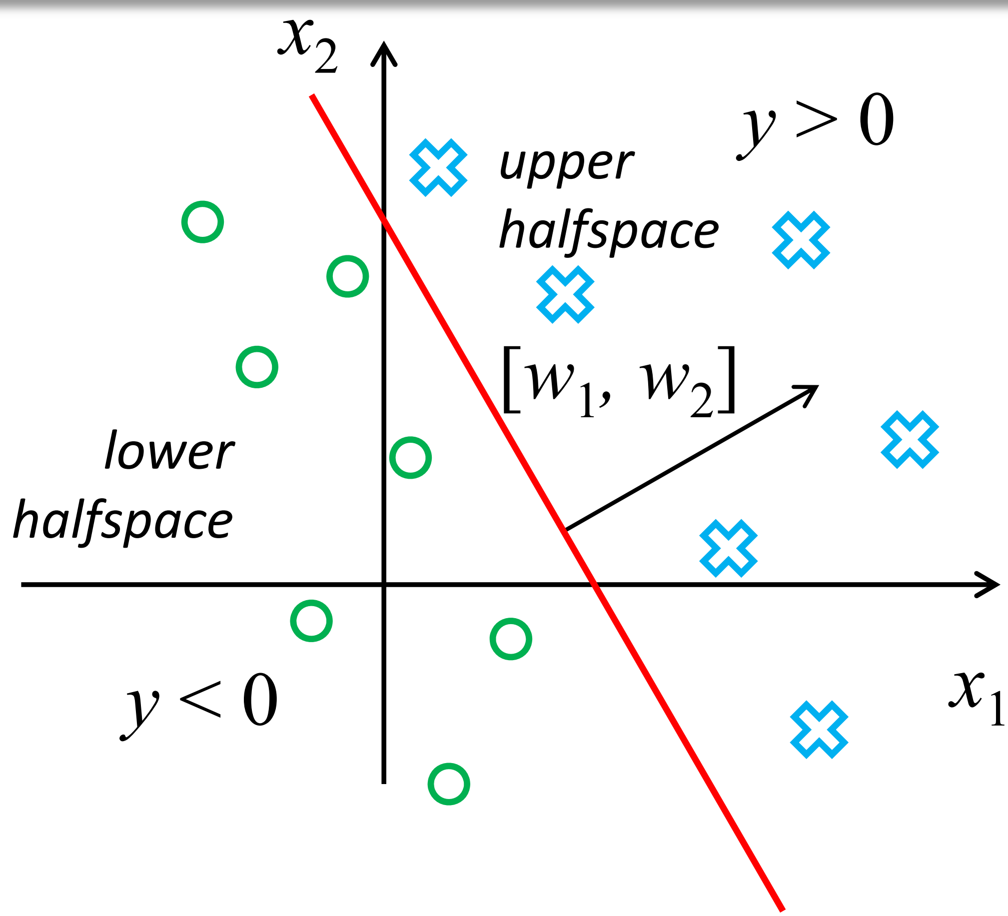
Binary classification with perceptron



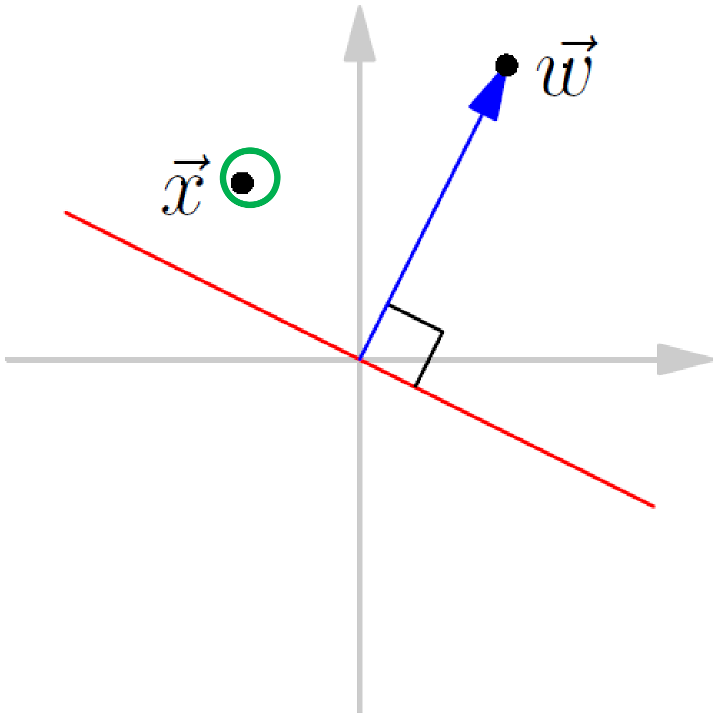


Binary classification with perceptron





Space of weights and inputs - perceptron



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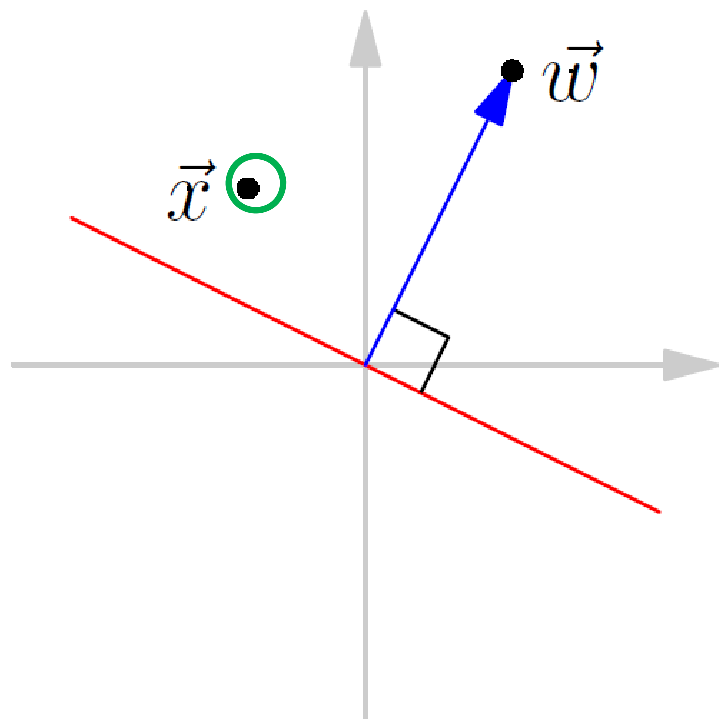
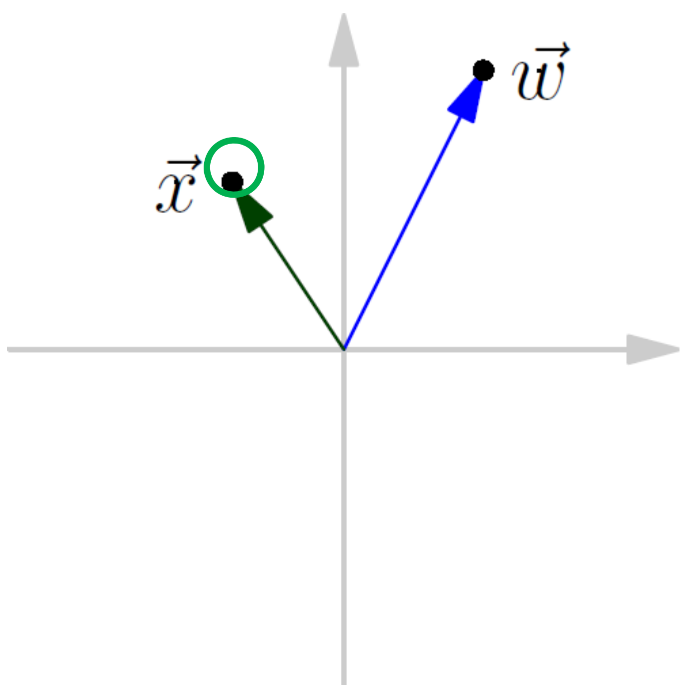
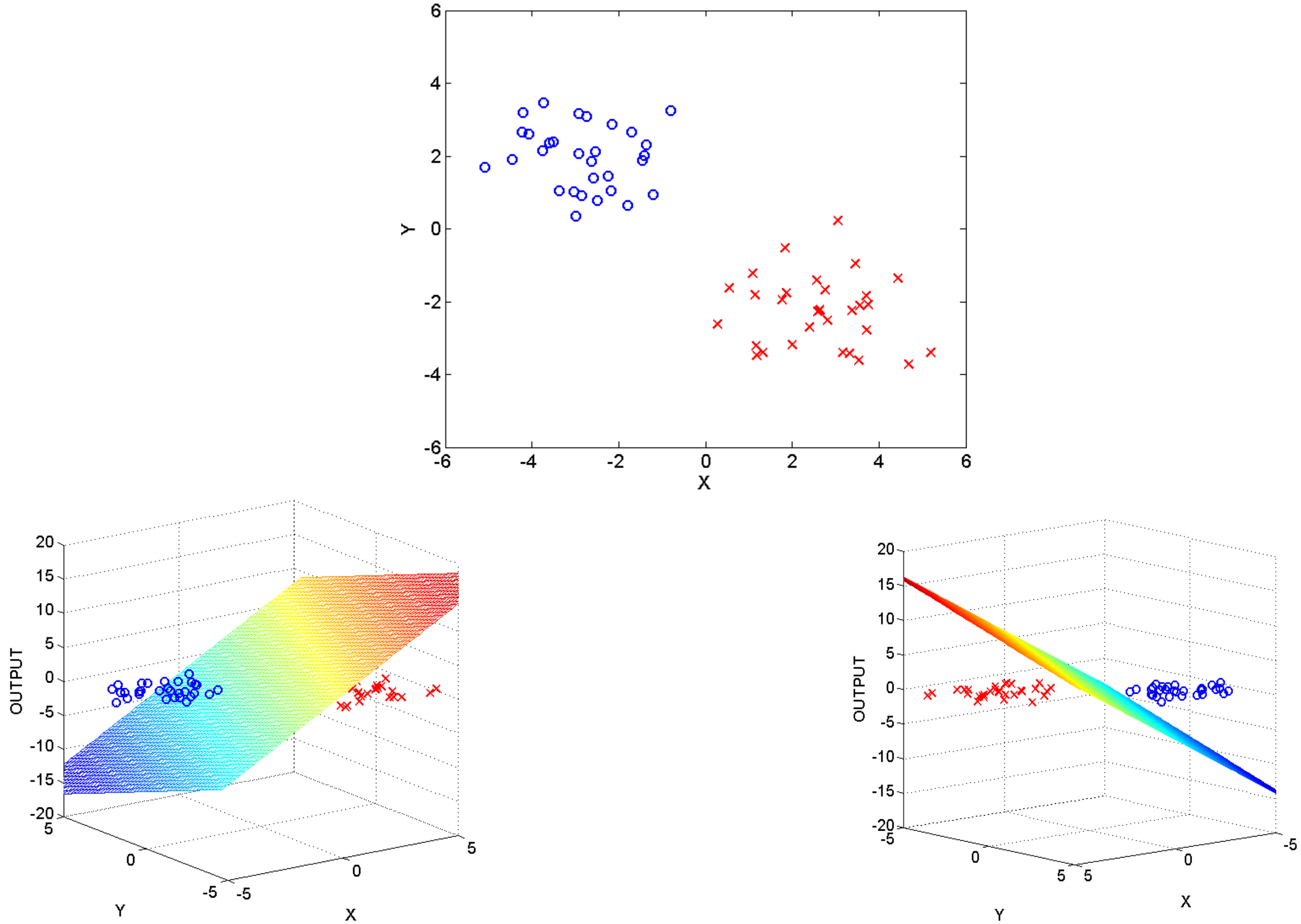




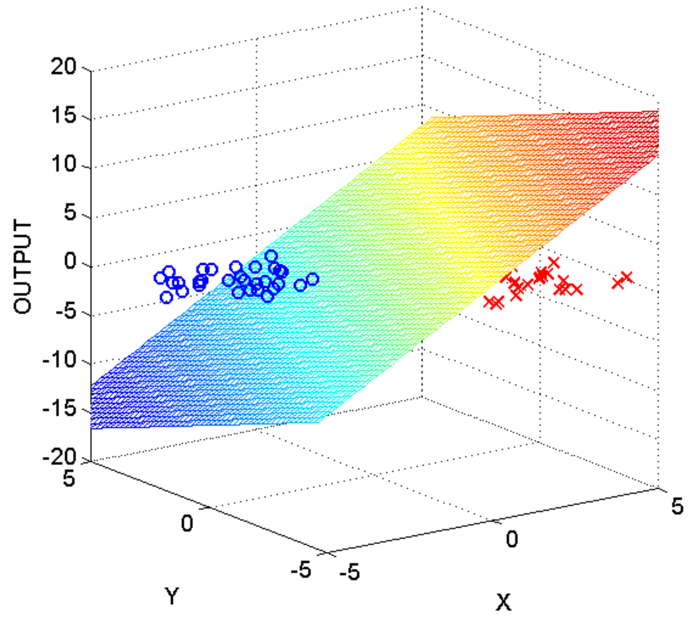
Classification with perceptron – how does it work?

2D input space and 3D network's linear output

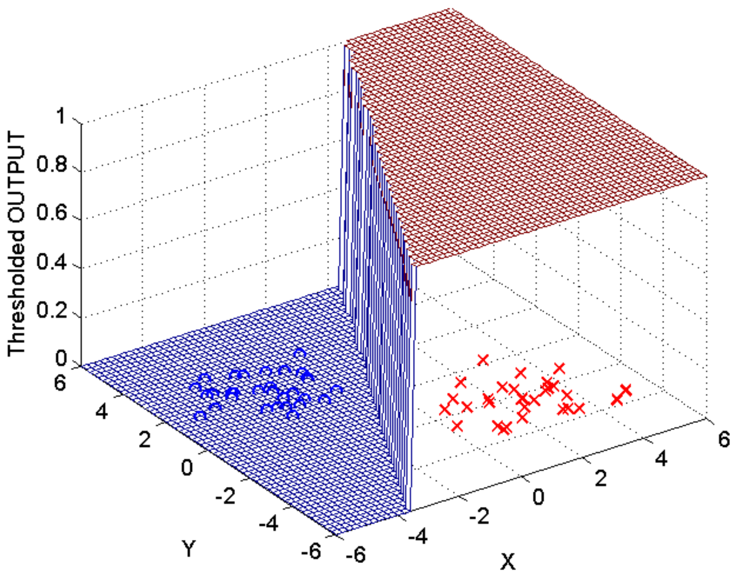


Classification with perceptron

Linear output and
perceptron’s thresholded output



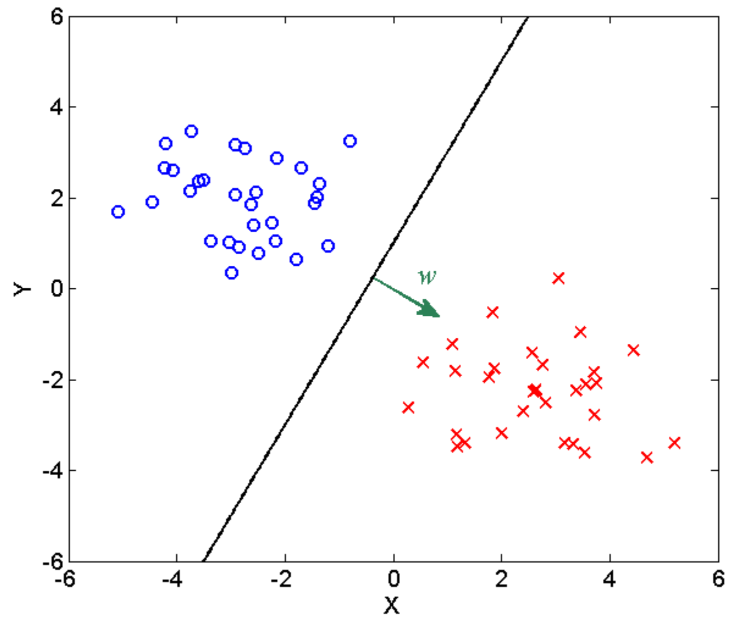
Separating hyperplane – network’s linear output

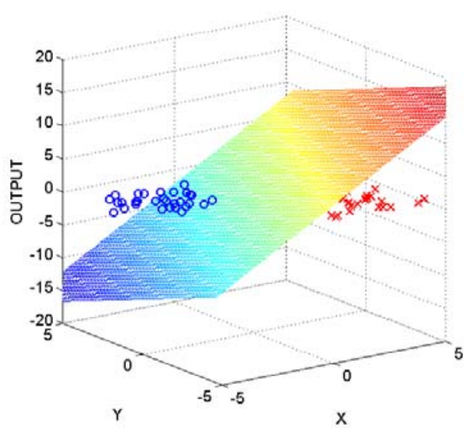


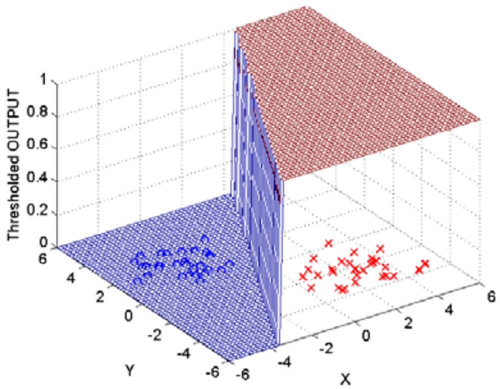
Output surface – percpetron’s thresholded output

Classification with perceptron

Decision boundary in the input space







Perceptron learning for classification

Perceptron learning for thresholded single-layer networks

Basic principle: weights are modified if and only if a pattern is erroneously classified:

When the network *output* = 0 but it should be 1 (*target* = 1)

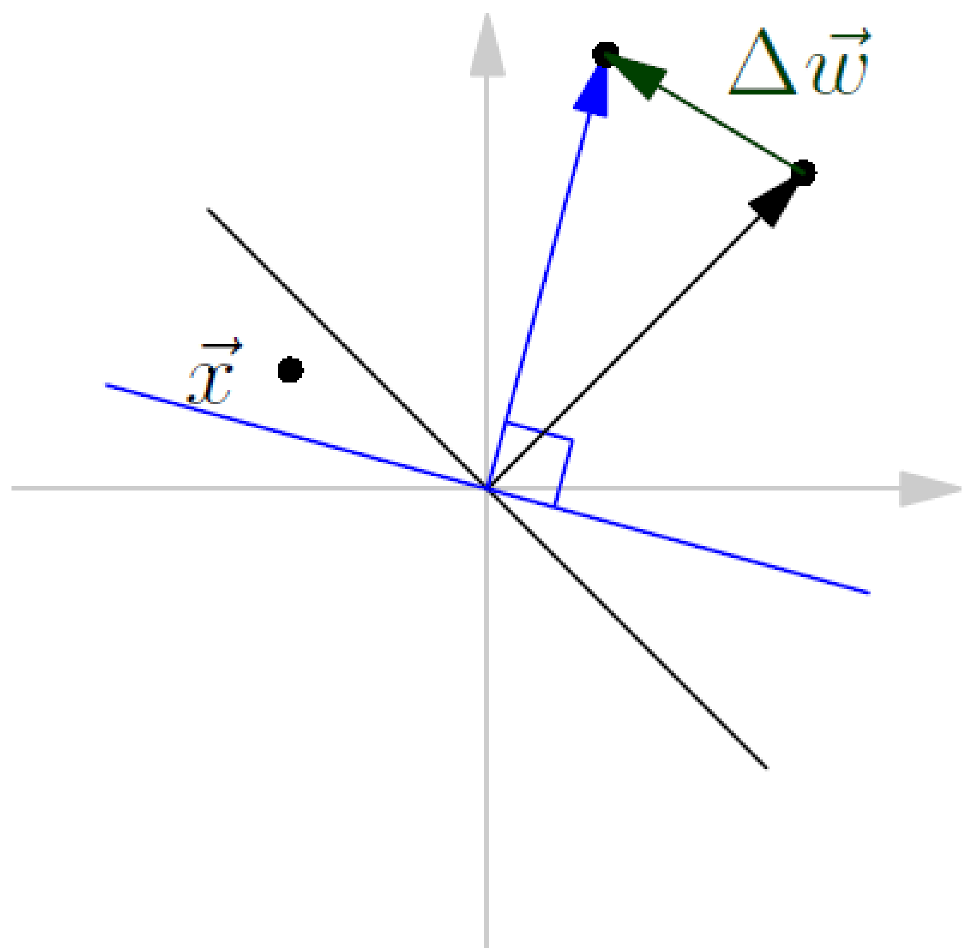
$$\Delta\vec{w} = \eta\vec{x}$$

When the network *output* = 1 but it should be 0 (*target* = 0)

$$\Delta\vec{w} = -\eta\vec{x}$$

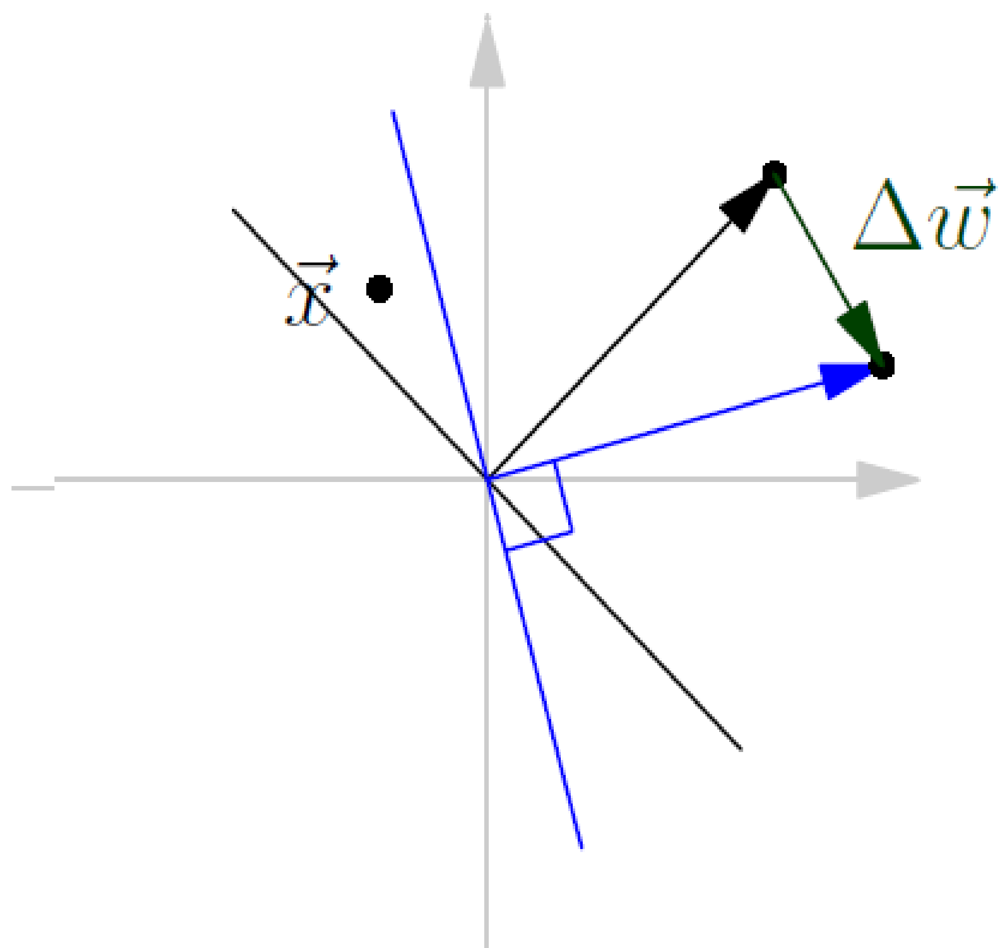
Perceptron learning – geometrical interpretation

When the result is 0 but should be 1: $\Delta\vec{w} = \eta\Delta\vec{x}$



Perceptron learning – geometrical interpretation

When the result is 1 but should be 0: $\Delta\vec{w} = -\eta\Delta\vec{x}$



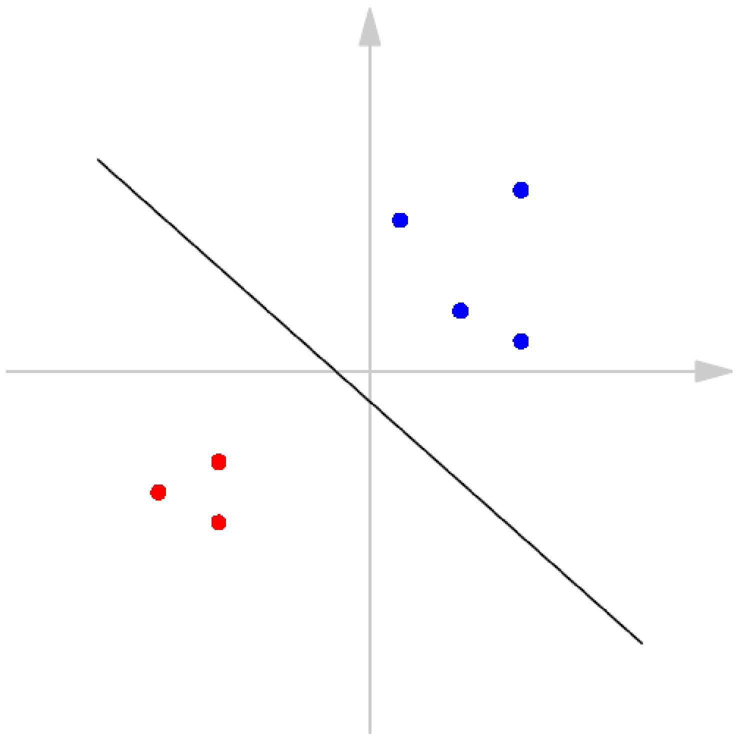
Perceptron learning – convergence theorem

Convergence theorem

If a solution exists for a finite training dataset then perceptron learning always converges after a finite number of sets (independent of step size/learning rate, η)

Perceptron learning

Problem: learning terminates prematurely.



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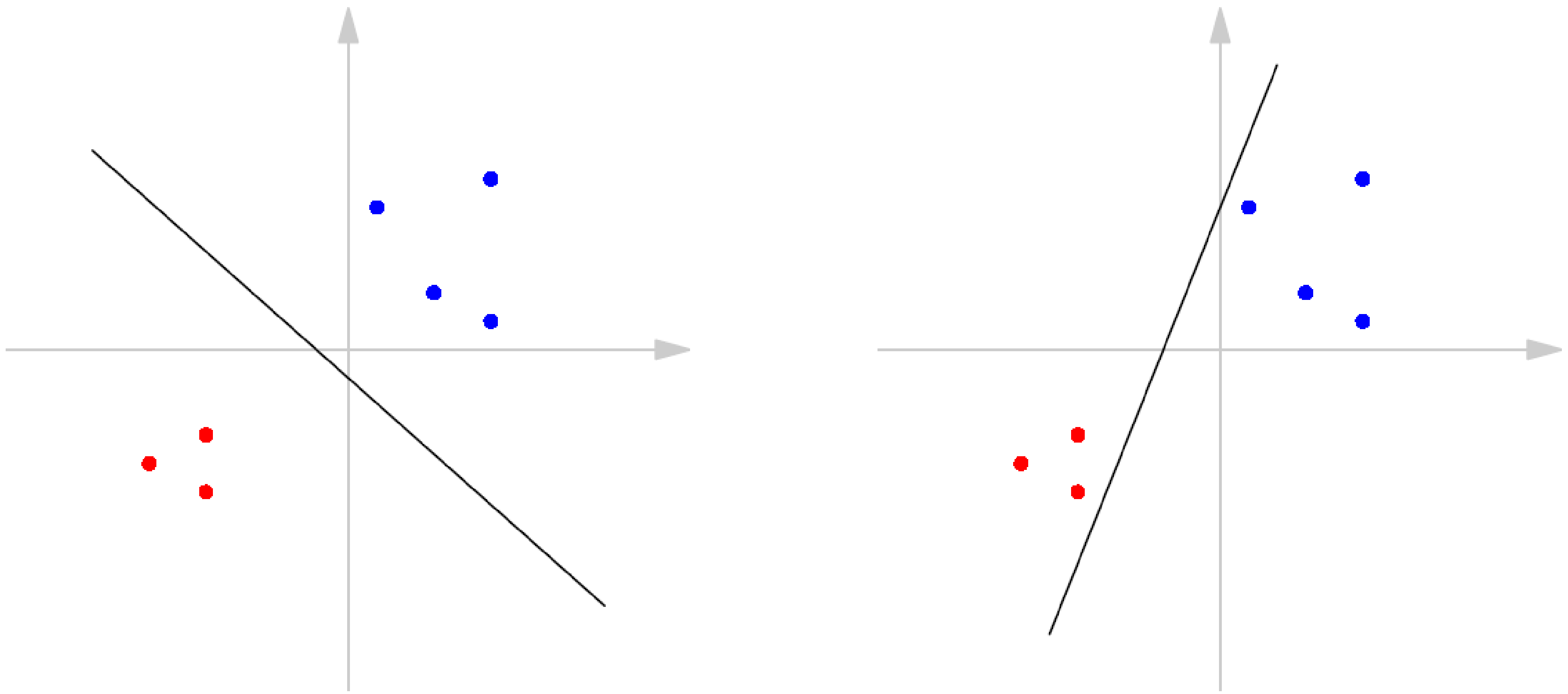

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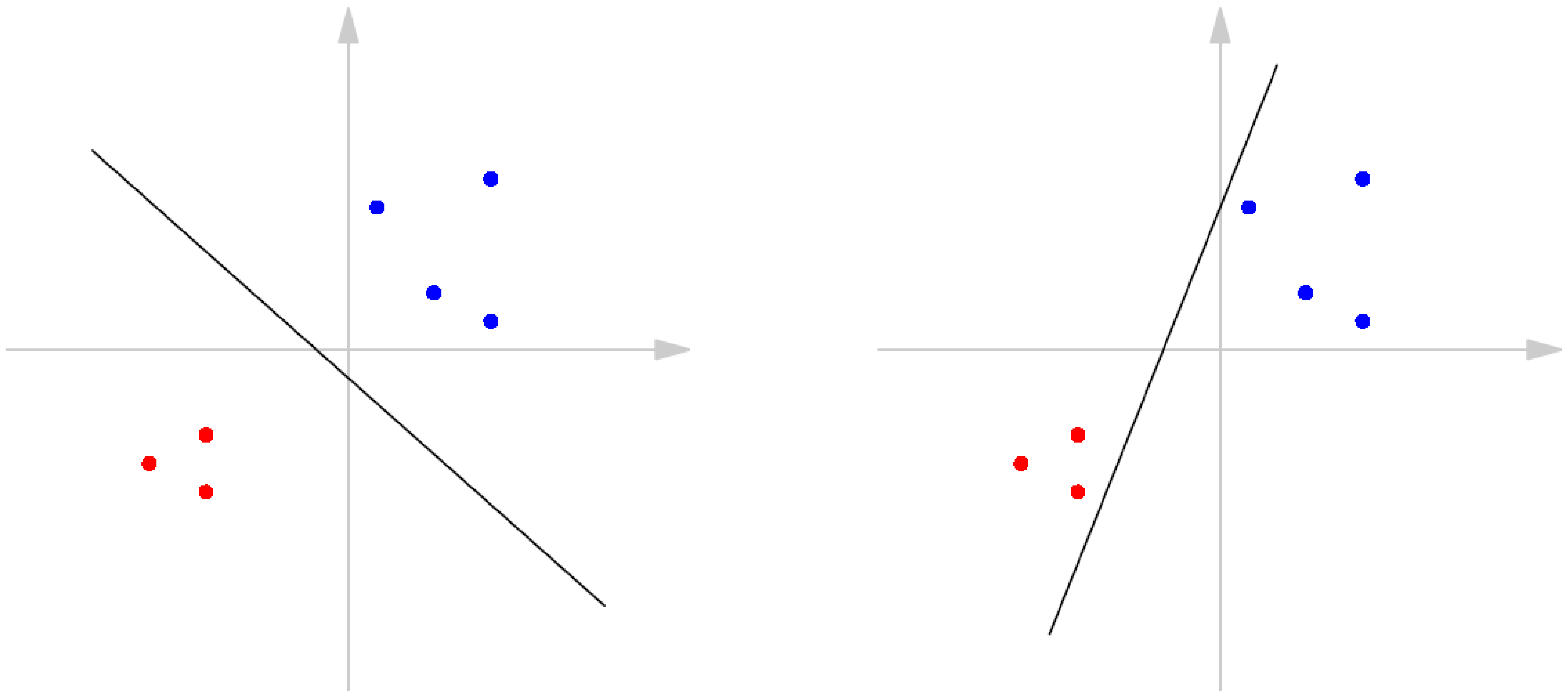

Negative consequences are likely when patterns are only *approximately similar* to those used for training

Delta rule

Delta rule (Widrow-Hoff rule, ADALINE)

Delta rule

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1. Symmetric target values: {-1, 1}
2. Error is measured before thresholding

$$e = t - \vec{w}^{\mathrm{T}} \vec{x}$$

3. Find weights that minimise the error cost function

$$\varepsilon = \frac{e^2}{2}$$

Delta rule

The task is to minimise the cost function $\varepsilon = \dfrac{e^2}{2}$

Simple algorithm: **steepest descent**

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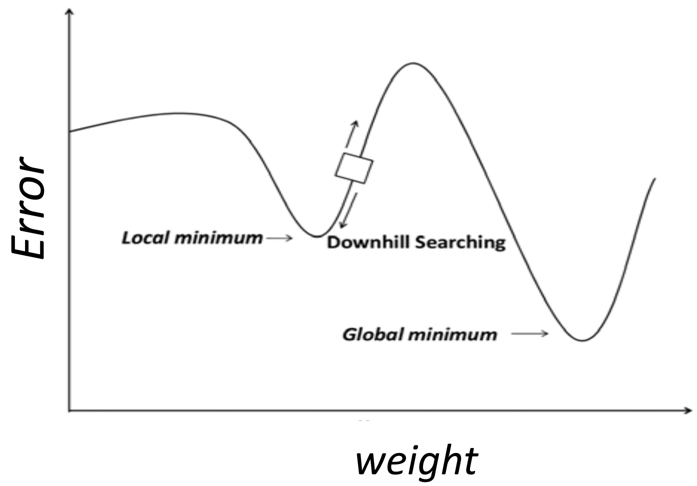

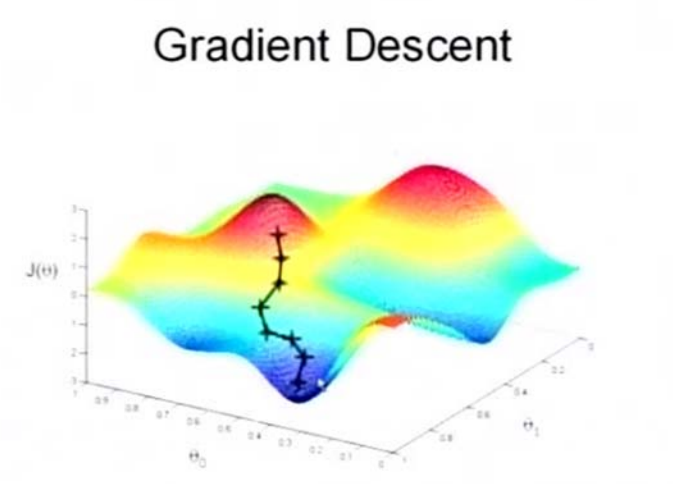

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$$\frac{\partial \varepsilon}{\partial \vec{w}} = e\frac{\partial e}{\partial \vec{w}} = e\frac{\partial (t - \vec{w}^{\mathrm{T}}\vec{x})}{\partial \vec{w}} = -e\vec{x}$$

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Training of thresholded single-layer networks

Perceptron learning:

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Separability with TLU / perceptron

Can all sets of patterns be separated?

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Classical counter-example is Exclusive OR (XOR)

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow 0 \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow 1 \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow 1 \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow 0$$

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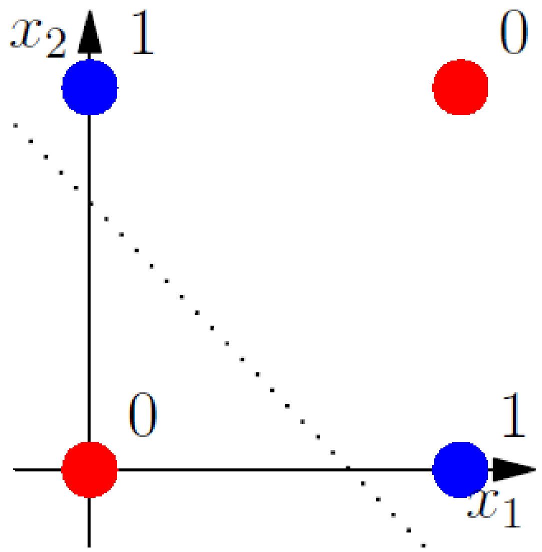

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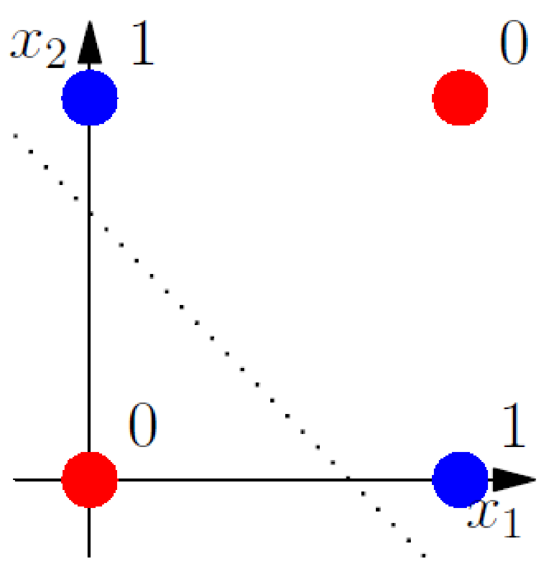

Not linearly separable!