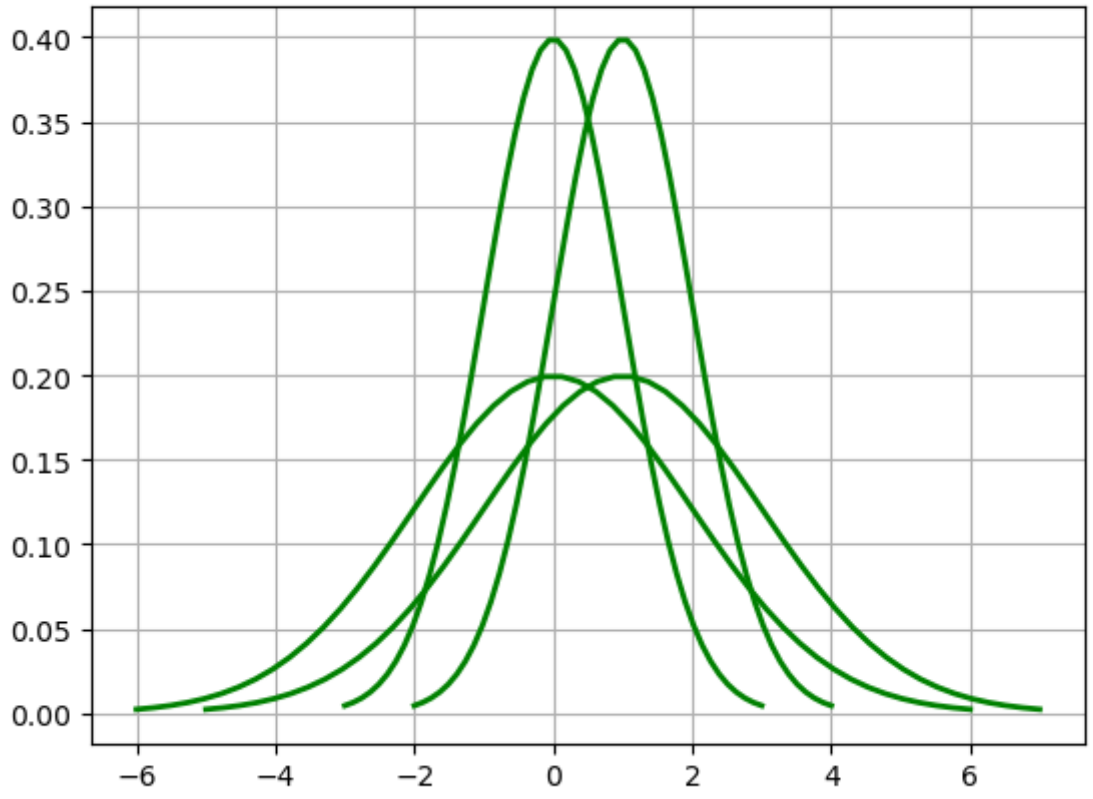
Assignment 5

2 NumPy Warm-up

2.1 (10 points) Choose a few different one-dimensional Gaussian functions (by choosing different mean and variance values), plot them.



我选择了四条高斯函数曲线，分别是均值为0和1、标准差为1和2

2.2 (10 points) Verify the above identity for each Gaussian function.

```
part2_2: the integrals of these 4 Gaussian functions are
0.9973443008306536
0.9973223077808391
0.9973443008306555
0.9973223077808402
```

经过求四条曲线从 μ - σ * 3 到 μ + σ * 3 的积分可以看出，图像面积为1

3 Numerics and Linear Algebra

3.1 (25 points) Use an LU solve (scipy.linalg.lu from SciPy package) to estimate the monomial coefficients c. Report the residual L2 norm for both linear systems when N = 8 and N = 16.

用vandermonde和fourier函数来构造两种矩阵：

```
def vandermonde(N):
    xis = np.arange(0, 1.001, 1 / (N - 1))
    v1 = np.vander(xis, increasing=True)
    return v1


def fourier(N, M=-1):
    if M == -1:
        M = N

    m = N - 1
    xis = np.arange(0, 1.01, 1 / m)

    ff = []
    for x in xis:
        for i in range(1, M + 1):
            if i <= M / 2:
                ff.append(np.sin(np.pi * i * x))
            else:
                ff.append(np.cos(np.pi * (i - M / 2) * x))

    ff = np.array([ff])
    ff = ff.reshape(N, M)
    return ff
```

用do_c函数来求范德蒙德和傅里叶矩阵下的c，以及L2范数：

```python
def do_c(N):
    m = N - 1
    xis = []  # 列向量x
    fis = []  # 列向量f
    for i in range(0, N):
        xi = i / m
        fi = f(xi)
        xis.append(xi)
        fis.append(fi)

    van = vandermonde(N)  # 范德蒙德矩阵
    ff = fourier(N)  # F矩阵
    van_p, van_l, van_u = lu(van)
    ff_p, ff_l, ff_u = lu(ff)
    van_c = np.linalg.solve(van_u, np.linalg.solve(van_l, fis))
    ff_c = np.linalg.solve(ff_u, np.linalg.solve(ff_l, fis))

    van_norm = la.norm(np.matmul(van, xis)-fis)
    ff_norm = la.norm(np.matmul(ff, xis)-fis)
    return van_c, ff_c, van_norm, ff_norm
```

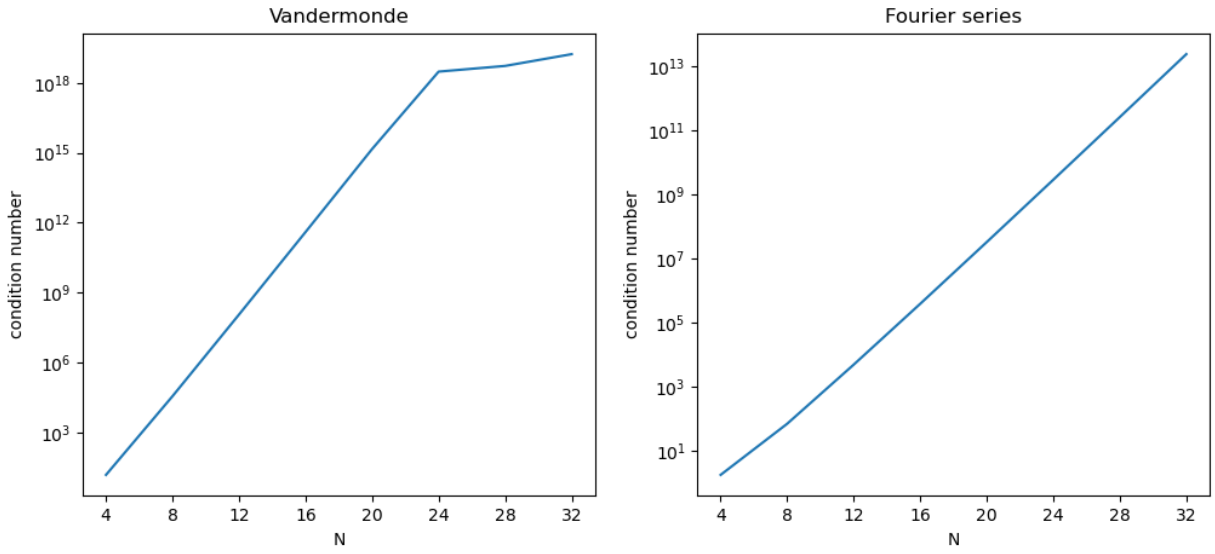
结果:

```
when N = 8
van_c : [ 1.00000000e+00  4.17062913e+01 -6.96422870e+02  4.07116491e+03
 -1.14004573e+04  1.65633688e+04 -1.20297269e+04  3.45034704e+03]
ff_c : [ 1.25946573  7.50388326  0.0125608  -2.13427132 -5.53669891  0.5715248
  5.5825409   0.13647935]
van_norm : 4.157635643795263
ff_norn : 4.103804100707729

when N = 16
van_c : [ 1.00000000e+00  5.39351587e+03 -2.58839425e+05  5.24587537e+06
 -6.04966965e+07  4.48554453e+08 -2.28138795e+09  8.26393577e+09
 -2.17785622e+10  4.21564954e+10 -5.98826802e+10  6.16573475e+10
 -4.47549142e+10  2.17055599e+10 -6.31014333e+09  8.31299146e+08]
ff_c : [-3.32664506e+01 -3.16180714e+04  1.65955455e+02  2.24249288e+04
 -1.18132691e+02 -4.73825802e+03  1.99294173e+01  1.59078874e+02
  2.02303755e+04 -1.08951683e+02 -3.11098089e+04  1.65951898e+02
  1.21026150e+04 -5.97095217e+01 -1.22316003e+03  3.46622709e+00]
van_norm : 8.857592171313605
ff_norn : 8.237097290612189
```

3.2 (10 points) Using the numpy.linalg.cond function in NumPy, plot N vs. cond(V) and N vs. cond(F) for N = 4, 6, 8, ...32. Write a couple of sentences explaining the reasons for the trends in these two plots.

使用log坐标轴对不同N值的范德蒙德矩阵和傅里叶矩阵条件数进行折线图绘制：

```
def do3_2():
    axes = np.arange(4, 33, 4)
    print("part3_2:")
    print("axes :", axes)

    condV = []
    condF = []
    for x in axes:
        tempV = vandermonde(x-1)
        condV.append(la.cond(tempV))

        tempF = fourier(x)
        condF.append(la.cond(tempF))

    print("cond(V) :", condV)
    print("cond(F) :", condF)
    print("\n")
```



左图随着N增长先是指数爆炸，而后呈现出稳定状态，可能是因为x的次方较大后，轻微的扰动对结果影响被巨大的结果舍掉了，右图则不断震荡，条件数一直呈现指数爆炸趋势

3.3 (15 points)

N	isposdef(Av)	isposdef(Af)	cond(V)	cond(F)
4.0	1.0	1.0	98.86773850722767	1.7320508075688776
8.0	1.0	1.0	267816.7009075937	68.55137085190182
12.0	1.0	1.0	883478688.3299552	4830.611029146389
16.0	0.0	1.0	3121670884816.2505	385010.13413578254
20.0	0.0	1.0	1.12601329949494e+16	32732760.09396631
24.0	0.0	0.0	2.032325395715027e+18	2898147362.6024175
28.0	0.0	0.0	1.0756071857681015e+19	263888245558.10617
32.0	0.0	0.0	3.332826977094868e+18	24485885032569.48

```
the largest value of N where Av is positive definite : 12
883478688.3299552
the largest value of N where Af is positive definite : 20
32732760.09396631
```

- 正定矩阵的定义是，对所有非零向量x，有 $x^T Ax > 0$
- 这些条件数之间没有关联

3.4 (10 points)

```python
def do3_4():
    Av = vandermonde(8)
    Af = fourier(8)

    Lv = np.linalg.cholesky(Av.T @ Av)
    Lf = np.linalg.cholesky(Af.T @ Af)

    # 构造一维向量，作为xi
    a = np.arange(0, 1.001, 1 / 7)

    # 得到对应的函数值，作为f向量
    b = [((1 + x ** 2) ** -1) for x in a]

    yv = np.linalg.solve(Lv, Av.T @ b)
    cv = np.linalg.solve(Lv.T, yv)

    yf = np.linalg.solve(Lf, Af.T @ b)
    cf = np.linalg.solve(Lf.T, yf)
```

使用Cholesky分解对 N = 8 时的方程求解：

```
vector c of vandermonde is : [ 1.00000000e+00 -1.92981042e-03 -9.64341028e-01 -2.55940760e-01
  1.93414873e+00 -1.83753052e+00  7.29013454e-01 -1.03420064e-01]
verify using np.allclose(V c, b) : True
residual of vandermonde : 3.608904809161406e-22
vector c of fourier series is : [ 1.35488014 -0.04116264 -0.35430384  0.0015691   0.26057558  0.83058785
 -0.01057558 -0.08058785]
verify using np.allclose(V c, b) : True
residual of fourier series : 1.8858706015439813e-30
```

LU的L2残差要比Cholesky分解小几个数量级

4 Least Squares Problems and QR

4.1 (15 points) Solve the least square system with QR decomposition(numpy.linalg.qr ()) when M = 16, N = 4, 8

使用QR分解对方称进行求解：

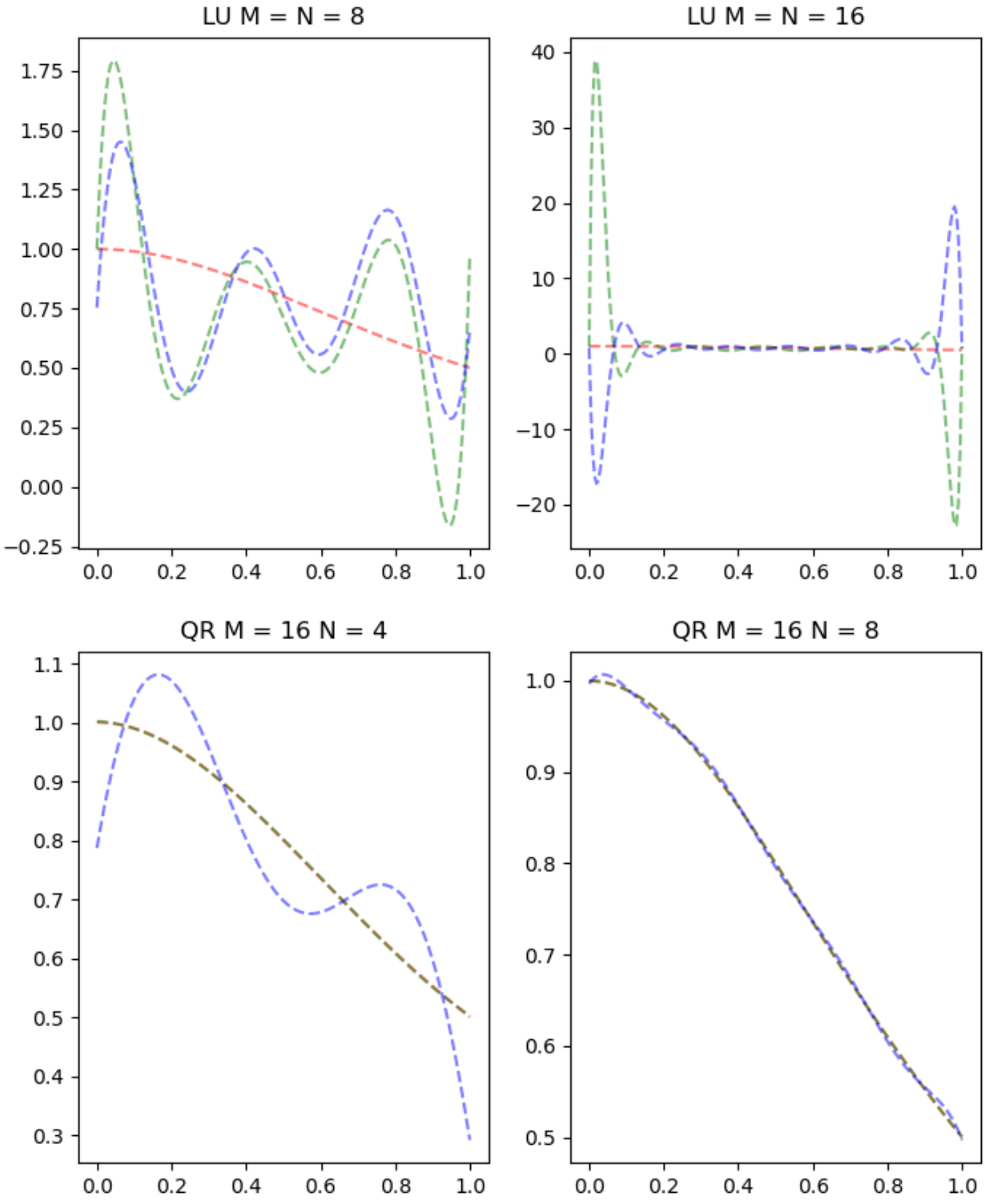
```
def qr_vandermonde(N):
    a = np.arange(0, 1.001, 1 / 15)
    # 得到对应的函数值，作为f向量
    b = [((1 + x ** 2) ** -1) for x in a]

    v = np.vander(a, N, increasing=True)

    q, r = np.linalg.qr(v.T @ v)

    y = np.linalg.solve(q, v.T @ b)

    c = np.linalg.solve(r, y)

    print(c)

    return c


def qr_fourier(M):
    a = np.arange(0, 1.001, 1 / 15)

    # 得到对应的函数值，作为f向量
    b = [((1 + x ** 2) ** -1) for x in a]

    f = fourier(16, M)

    q, r = np.linalg.qr(f.T @ f)

    c = np.linalg.inv(r) @ q.T @ f.T @ b
    # c = np.linalg.linalg.inv(r).dot(q.T).dot(b)

    print(c)

    return c
```

结果：

```
vandermonde 16 X 4 :
[ 1.00166564 -0.02698999 -1.01909544  0.54698731]
vandermonde 16 X 8 :
[ 1.00000188e+00 -1.07143526e-03 -9.75766181e-01 -2.00107712e-01
  1.80128197e+00 -1.67195242e+00  6.25188470e-01 -7.75731067e-02]
fourier series 16 X 4 :
[ 1.23673804 -0.02483006  0.24763933  0.53933343]
fourier series 16 X 8 :
[ 1.35569612 -0.04267561 -0.36313314  0.00197172  0.26167473  0.83497201
 -0.01168661 -0.08810734]
[ 1.00166564 -0.02698999 -1.01909544  0.54698731]
[ 1.23673804 -0.02483006  0.24763933  0.53933343]
[ 1.00000188e+00 -1.07143526e-03 -9.75766181e-01 -2.00107712e-01
  1.80128197e+00 -1.67195242e+00  6.25188470e-01 -7.75731067e-02]
[ 1.35569612 -0.04267561 -0.36313314  0.00197172  0.26167473  0.83497201
 -0.01168661 -0.08810734]
```

4.2 (10 points) Plot the gV , gF when M = 16, N = 4, 8, compare them with the analytical function f(x) and the interpolation function obtained in Question 3.1.

图中三条线为红、绿、蓝，分别表示标准，范德蒙德，傅里叶



可以看出除了M=16 N=4时的傅里叶曲线，其他曲线的拟合效果(在0，1范围内)都很好