1 Useful Facts

Exponentiation rules:

1. $b^{n+m} = b^n\, b^m$.
2. $(b^n)^m = b^{nm}$.

The logarithm base b is defined as

$$\log_b(n) = x, \text{ where } b^x = n.$$

Some logarithm rules:

1. $b^{\log_b n} = n$.
2. $\log_b(b^n) = n$.
3. $\log_b(cn) = \log_b(c) + \log_b(n)$.
4. $\log_b(n^c) = c \log_b(n)$
5. $\log_b(a) = \frac{\log_x(a)}{\log_x(b)}$.

2 Big Oh Notation

Definition 1. Suppose $f, g : \mathbb{N} \to \mathbb{N}$. We say that $f = O(g)$ (also written as $f \in O(g)$) if there are constants c, N such that

$$f(n) \leq cg(n)$$

for every $n \geq N$.

Example 1. $5n + 2 = O(n)$

Proof. Let $c = 7$ and $N = 1$. Then for every $n \geq N$,

$$\begin{aligned} f(n) &= 5n + 2 \\ &\leq 5n + 2n \\ &= 7n \\ &= cn \\ &= cg(n). \end{aligned}$$

□

Example 2. $5n^2 - 10n + 15 = O(n^2)$.

Proof. Let $c = 20$ and $N = 1$. Then for every $n \geq N$,

$$\begin{aligned} f(n) &= 5n^2 - 10n + 15 \\ &\leq 5n^2 + 0 + 15n^2 \\ &= 20n^2 \\ &= cg(n). \end{aligned}$$

□

Example 3. $5n^2 - 10n + 15 = O(n^3)$

Proof. Let $c = 20$ and $N = 1$. Then for every $n \geq N$,

$$\begin{aligned} f(n) &= 5n^2 - 10n + 15 \\ &\leq 5n^3 + 0 + 15n^3 \\ &= 20n^3 \\ &= cg(n). \end{aligned}$$

□

Example 4. *Prove that* $n^3 \neq O(n^2)$.

Proof. Suppose, for the sake of contradiction that $n^3 = O(n^2)$. Then there are c, N such that $n^3 \leq cn^2$ for every $n \geq N$. Therefore, by dividing n^2 on both sides, this means that

$$n \leq c.$$

This is false for all $n > c$. Hence, $n^3 \neq O(n^2)$. □

Example 5. $2^{n+5} = O(2^n)$.

Proof. Let $c = 32$, $N = 1$. Then

$$\begin{aligned} f(n) &= 2^{n+5} \\ &= 32 \cdot 2^n \\ &= cg(n). \end{aligned}$$

□

Definition 2. Let f, g be functions. We say that $f = \Omega(g)$ if there are positive constants c, N such that

$$cg(n) \leq f(n),$$

for every $n \geq N$.

$f = \Omega(g)$ if and only if $g = O(f)$.

Example 6. $5n + 2 = \Omega(n)$.

Proof. It suffices to show that $n = O(5n + 2)$. Let $c = 1, N = 1$.

$$\begin{aligned} n &\leq 5n + 2 \\ &= c(5n + 2). \end{aligned}$$

□

Example 7. $5n^2 - 10n + 15 = \Omega(n^2)$

Proof. It suffices to show that $n^2 = O(5n^2 - 10n + 15)$. Let $c = 1$, $N = 1$. Then we must show that

$$n^2 \leq c(5n^2 - 10n + 15),$$

for all $n \geq 1$. This is equivalent to

$$10n \leq 4n^2 + 15.$$

This is true if and only if

$$10 \leq 4n + \tfrac{15}{n},$$

which holds for all $n \geq 1$. $\square$

Definition 3. We say that $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$. Or, equivalently, $f = O(g)$ and $g = O(f)$.

Example 8. $5n^2 - 10n + 15 = \Theta(n^2)$.