General form of divide and conquer algorithm runtime: $T(n) = aT(n/b) + f(n)$, where a is the number of subproblems, b is how much the problem is reduced by and $f(n)$ is the time to divide and combine.

Theorem 1 (Master Theorem)**.** *Let $a \geq 1$ and $b > 1$ be constants and let $T(n)$ be the recurrence defined by*

$$T(n) = aT(n/b) + f(n).$$

Then $T(n)$ has the following bounds:

1. *If $f(n) = O(n^{\log_b(a)-\epsilon})$ for some constant $\epsilon > 0$, then*

$$T(n) = \Theta(n^{\log_b(a)}).$$

2. *If $f(n) = \Theta(n^{\log_b(a)})$, then*

$$T(n) = \Theta(n^{\log_b(a)} \log n).$$

3. *If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then*

$$T(n) = \Theta(f(n)).$$

Example 1. *The runtime of MergeSort is given by the following recurrence*

$$T(n) = 2T(n/2) + cn.$$

Using the Master Theorem, give a bound for $T(n)$.

Proof. We first show that $cn = \Theta(n^{\log_2(2)}) = \Theta(n)$.

Then we may apply the Master Theorem case 2, so we see that

$$T(n) = \Theta(n \log n).$$

$\square$

Example 2. *Let $T(n) = 9T(n/3) + n$. Give bounds on the recurrence $T(n)$ using the Master Theorem.*

Proof. We see that $a = 9$ and $b = 3$, and therefore

$$log_b(a) = log_3(9) = 2.$$

Let $\epsilon = 1$, then

$$n = O(n^{2-\epsilon}) = O(n^{2-1}) = O(n).$$

Therefore we may apply the first case of the Master Theorem, and so

$$T(n) = \Theta(n^{\log_3(9)}) = \Theta(n^2).$$

$\square$

Example 3. *Let $T(n) = 3T(n/4) + n\log(n)$. Give bounds on the recurrence $T(n)$ using the Master Theorem.*

Proof. We see that $a = 3$, $b = 4$ and $f(n) = n \log n$. Let $\epsilon = .0075$. Then

$$n^{(\log_4(3)+\epsilon)} \leq n^{(.7925+\epsilon)} = n.$$

Therefore

$$n\log(n) = \Omega(n^{(\log_4(3)+\epsilon)}).$$

Let $c = 3/4$. Then

$$\begin{aligned}
af(n/b) &= 3(\frac{n}{4}\log(n/4)) \\
&= \frac{3}{4}n\log(n) - \frac{3}{4}n\log(4) \\
&\leq cf(n).
\end{aligned}$$

Therefore we may apply case three of the Master Theorem, and so

$$T(n) = \Theta(n\log(n)).$$

$\square$