2.3. 非決定性動的計画の応用
— Egg Dropping Problem —

2.3.1. 有限段非決定性動的計画（最大加法型）

目的関数

$$V(x_0;\pi) = r(x_0,u_0) + \bigvee_{x_1\in T_1}\Big\{r(x_1,u_1)+\bigvee_{x_2\in T_2}\Big\{r(x_2,u_2)+ \cdots + \bigvee_{x_{N-1}\in T_{N-1}}\Big\{r(x_{N-1},u_{N-1})+\sum_{x_N\in T_N} k(x_N)\Big\}\cdots\Big\}\Big\}$$

$$\left[V(x_0,\pi) = \bigvee_{(x_1,x_2,\dots,x_N)\in X(x_0,\pi)} \Big(r(x_0,u_0)+r(x_1,u_1)+\cdots+r(x_{N-1},u_{N-1})+k(x_N)\Big)\right]$$

Nondeterministic DP Problem (Max-Add)

Nondeterministic Dynamic Programming Problem :

$$\mathrm{P}(x_0) \qquad \text{Minimize} \quad V(x_0;\pi) \quad \text{subject to} \quad \pi\in\Pi$$

Let $v(x_0)$ be the optimal value of the problem $\mathrm{P}(x_0)$ $(x_0 \in X\setminus X_G)$ and

$$v(x_0) = k(x_0), \qquad x_0\in X_G.$$

Thus, we have the following recursive equation.

Corollary 2.1.(Max-Add Criterion)

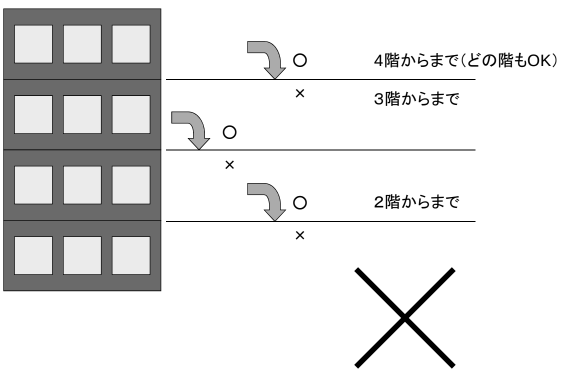
$$v(x) = k(x), \qquad x\in X_G$$
$$v(x) = \min_{u\in U(x)}\Big[r(x,u)+\bigvee_{y\in T(x,u)} v(y)\Big], \qquad x\in X\setminus X_G.$$

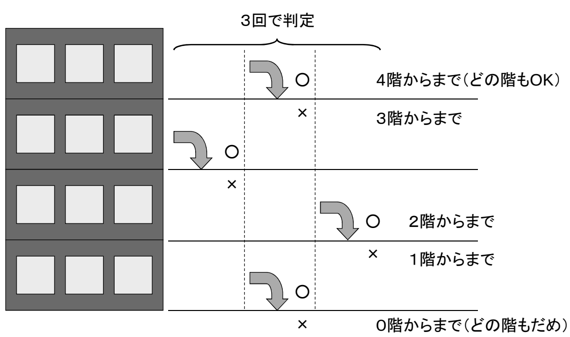
2.3.2. Egg Dropping Problem

Egg Dropping Problem

Suppose that we wish to know which windows in a 36-story building are safe to drop eggs from, and which will cause the eggs to break on landing.
Suppose 2 eggs are available.
What is the least number of egg-droppings that is guaranteed to work in all cases?

Moshe Sniedovich,　INFORMS 2007(Puerto Rico) 講演資料より





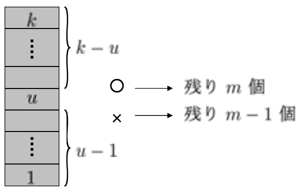
2.3.3. 非決定性DPとしての定式化

状態　$x = (m, k) \longleftarrow$ たまご：m 個，　ビル：k 階

決定　$u \longleftarrow$ u 階からたまごを落とす

状態推移

$$x = (m,k) \xRightarrow{\;u\;} \begin{matrix} (m, k-u) \\ (m-1, u-1) \end{matrix}$$



一般に M 個のたまごで K 階のビルを調べる場合について考える。
$(M > 0,\ K > 0)$

状態空間：$X = \{0, 1, 2, \ldots, M\} \times \{0, 1, 2, \ldots, K\}$

初期状態：$x_0 = (M, K)$

終了状態：$X_G = \{0, 1, 2, \ldots, M\} \times \{0\}$

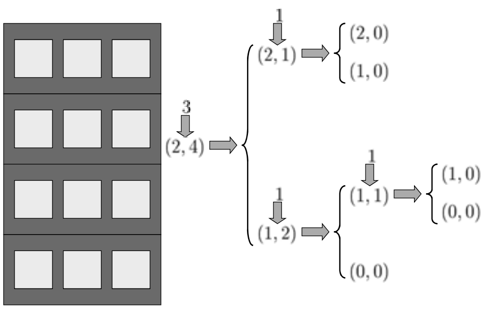
決定空間：$U = \{1, 2, \ldots, K\}$

決定制約：$U(x) = \begin{cases} \{1, 2, \ldots, k\} & m \geq 2,\ k \geq 1 \\ \{1\} & m = 1,\ k \geq 1 \end{cases}$　，$x = (m, k) \in X \backslash X_G$

非決定性推移：

$T(x, u) = \{(m, k-u),\ (m-1, u-1)\}, \quad x = (m, k) \in X \setminus X_G,\ u \in U(x)$

（注：$x = (m, k) \in X \setminus X_G$ に対し、$k > 0$ のときは必ず $m > 0$ となる）



利得関数：$r(x, u) = 1$，　$x \in X \setminus X_G,\ u \in U(x)$

終端利得関数：$k(x) = 0, \quad x \in X_G$

$\Downarrow$

非決定性動的計画問題：

$\mathrm{P}(x_0) \qquad \text{Minimize} \quad V(x_0; \pi) \quad \text{subject to} \quad \pi \in \Pi(0)$

$$\left(V(x_0, \pi) = \bigvee_{(x_1, x_2, \ldots, x_N) \in X(x_0, \pi)} \Big(r(x_0, u_0) + r(x_1, u_1) + \cdots + r(x_{N-1}, u_{N-1}) + k(x_N) \Big) \right)$$

2.3.4. 再帰式

再帰式

$$\begin{aligned} v(x) &= k(x), & x \in X_G \\ v(x) &= \min_{u \in U(x)} \Big[r(x, u) + \bigvee_{y \in T(x,u)} v(y) \Big], & x \in X \setminus X_G. \end{aligned}$$

任意の $m = 1, 2, \ldots, M$ に対し

$$\begin{aligned} v(m, 1) &= \min_{u \in \{1\}} \Big[r((m,1), u) + \bigvee_{y \in T((m,1),u)} v(y) \Big] \\ &= r((m,1),1) + \bigvee_{y \in T((m,1),1)} v(y) \\ &= 1 + \Big(v(m, 0) \vee v(m-1, 0) \Big) \\ &= 1 + (0 \vee 0) \\ &= 1 \end{aligned}$$

また、任意の $k = 1, 2, \ldots, K$ に対し

$$\begin{aligned} v(1, k) &= \min_{u \in \{1\}} \Big[r((1,k), u) + \bigvee_{y \in T((1,k),u)} v(y) \Big] \\ &= r((1,k),1) + \bigvee_{y \in T((1,k),1)} v(y) \\ &= 1 + \Big(v(1, k-1) \vee v(0, 0) \Big) \\ &= 1 + v(1, k-1) \;=\; \cdots \;=\; k + v(1, 0) \\ &= k \end{aligned}$$

最後に $m, k \geq 2$ のときには

$$\begin{aligned} v(m, k) &= \min_{u \in \{1,2,\ldots,k\}} \Big[r((m,k), u) + \bigvee_{y \in T((m,k),u)} v(y) \Big] \\ &= 1 + \min_{u \in \{1,2,\ldots,k\}} \Big[v(m, k-u) \vee v(m-1, u-1) \Big] \end{aligned}$$

したがって、“Egg Dropping Problem” に対しては次の再帰式を用いれば
よい。

$$
\begin{aligned}
v(m,\ 0) &= 0 \quad (m = 0, 1, 2, \ldots, M)\\
v(m,\ 1) &= 1\ ,\ \pi^*(m,\ 1) = 1 \quad (m = 1, 2, \ldots, M)\\
v(1,\ k) &= k\ ,\ \pi^*(1,\ k) = 1 \quad (k = 1, 2, \ldots, K)\\
v(m,\ k) &= 1 + \min_{u \in \{1,2,\ldots,k\}} \Big[v(m,\ k-u) \vee v(m-1,\ u-1) \Big] \quad (m,\ k \geq 2)
\end{aligned}
$$

m \ k	0	1	$2, 3, \ldots$
0	0		
1	0	1	$2, 3, \ldots$
$2, 3, \ldots$	0	1	$v(m, k)$

2.3.5. 数值例

例 1 $x_0 = (2, 3)$ のとき $(M = 2,\ K = 3)$

$U(x_0) = \{1, 2, 3\}$ より

$$
\begin{aligned}
& v(x_0)\\
&= 1 + \min\Big[\big(v(2,2) \vee v(1,0)\big), \big(v(2,1) \vee v(1,1)\big), \big(v(2,0) \vee v(1,2)\big) \Big]\\
&= 1 + \min\Big[\big(v(2,2) \vee 0\big), \big(1 \vee 1\big), \big(0 \vee 2\big) \Big]
\end{aligned}
$$

ここで

$$
\begin{aligned}
v(2,2) &= 1 + \min\Big[\big(v(2,1) \vee v(1,0)\big), \big(v(2,0) \vee v(1,1)\big) \Big]\\
&= 1 + \min\Big[\big(1 \vee 0\big), \big(0 \vee 1\big) \Big]\\
&= 2
\end{aligned}
$$

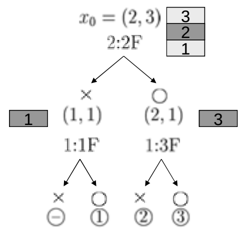
$\pi^*(2,2) = 1, 2$

したがって

$$
\begin{aligned}
v(x_0) &= 1 + \min\left[(2 \vee 0), (1 \vee 1), (0 \vee 2)\right]\\
&= 1 + \min(2, 1, 2)\\
&= 2
\end{aligned}
$$

$\pi^*(x_0) = 2$

$$
\begin{cases}
\pi^*(2,\ 3) = 2\\
\pi^*(m,\ 1) = 1
\end{cases}
$$



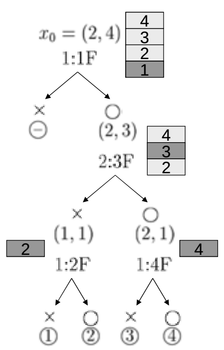
例 2 $x_0 = (2, 4)$ のとき $(M = 2,\ K = 4)$

$U(x_0) = \{1, 2, 3, 4\}$ より

$$
\begin{aligned}
v(x_0) &= 1 + \min\Big[\big(v(2,3) \vee v(1,0)\big), \big(v(2,2) \vee v(1,1)\big),\\
&\qquad \big(v(2,1) \vee v(1,2)\big), \big(v(2,0) \vee v(1,3)\big) \Big]\\
&= 1 + \min\Big[\big(v(2,3) \vee 0\big), \big(v(2,2) \vee 1\big), \big(1 \vee 2\big), \big(0 \vee 3\big) \Big]\\
&= 1 + \min\left[(2 \vee 0), (2 \vee 1), (1 \vee 2), (0 \vee 3)\right]\\
&= 1 + \min(2,\ 2,\ 2,\ 3)\\
&= 3
\end{aligned}
$$

$\pi^*(x_0) = 1, 2, 3$

$$
\begin{cases}
\pi^*(2,\ 4) = \underline{1}, 2, 3\\
\pi^*(2,\ 3) = 2\\
\pi^*(m,\ 1) = 1
\end{cases}
$$



$\underline{\underline{\pi^*(2,4) = 1,2,3 \qquad \pi^*(2,3) = 2 \qquad \pi^*(2,2) = 1,2}}$