

Exercises

Yuji Takai

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1 Chapter 1

2 Chapter 2

2.1

2.2

2.3

2.4

2.5

2.6

2.7

2.8

- (a) Row reduce the augmented matrix $[A|I]$ to determine whether there exists an inverse for A

$$\begin{aligned} [A|I] &= \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 4 & 5 & 6 & 0 & 0 & 1 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array} \right] \end{aligned}$$

Since the third column of the augmented matrix is not a pivot column, A^{-1} does not exist

- (b) Row reduce the augmented matrix $[A|I]$ to determine whether there exists an inverse for A

$$\begin{aligned} [A|I] &= \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \\ &\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -2 \end{array} \right] \\ \therefore A^{-1} &= \left[\begin{array}{cccc} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -2 \end{array} \right] \end{aligned}$$

2.9

(a) To prove A is a subspace of \mathbb{R}^3 , we need to prove that A is not an empty set and A is closed with respect to both inner and outer

1. Let $\vec{v} \in A$. When $\lambda = 0, \mu = 0, \vec{v} = (0, 0, 0)$. This means that $\mathbf{0} \in A$ and thus $A \neq \emptyset$
2. Closure of A
 - With respect to the outer operation: $\forall c \in \mathbb{R}, \forall \vec{x} \in A : c\vec{x} \in A$
 - $c\vec{x} = (c\lambda, c(\lambda + \mu^3), c(\lambda - \mu^3)) \in A$
 - With respect to the inner operation: $\forall \vec{x}, \vec{y} \in A : \vec{x} + \vec{y} \in A$
 - Let $\vec{x} = (\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) \in A$
 - Let $\vec{y} = (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) \in A$
 - $\vec{x} + \vec{y} = (\lambda_1 + \lambda_2, \lambda_1 + \mu_1^3 + \lambda_2 + \mu_2^3, \lambda_1 - \mu_1^3 + \lambda_2 - \mu_2^3)$
 - Let $\lambda_3 = \lambda_1 + \lambda_2$
 - Then: $\vec{x} + \vec{y} = (\lambda_3, \lambda_3 + \mu_1^3 + \mu_2^3, \lambda_3 - \mu_1^3 - \mu_2^3)$
 - Let $\mu_1^3 + \mu_2^3 = \mu_3^3$
 - Then: $\vec{x} + \vec{y} = (\lambda_3, \lambda_3 + \mu_3^3, \lambda_3 - \mu_3^3) \in A$

(b) To prove B is a subspace of \mathbb{R}^3 , we need to prove that B is not an empty set and B is closed with respect to both inner and outer

- While B is not empty ($\vec{x} = (0, 0, 0) \in B$), B is not closed with respect to outer operation
- Let $\vec{x} = (1, -1, 0) \in B$ and $c = -1$. Then $c\vec{x} = (-1, 1, 0)$. Since there is no number in \mathbb{R} such that its square is less than 0. (only complex numbers are possible).
- Therefore $c\vec{x} \notin B$

(c) To prove C is a subspace of \mathbb{R}^3 , we need to prove that C is not an empty set and C is closed with respect to both inner and outer

1. Let $\vec{v} \in C$. When $\xi_1 = 0, \xi_2 = 0, \xi_3 = 0, \vec{v} = (0, 0, 0)$ with $\gamma = 0$. This means that $\mathbf{0} \in C$ and thus $C \neq \emptyset$
2. Closure of C
 - With respect to the outer operation: $\forall c \in \mathbb{R}, \forall \vec{x} \in A : c\vec{x} \in A$

$$- c\vec{x} = c(\xi_1, \xi_2, \xi_3) = (\xi'_1, \xi'_2, \xi'_3) \text{ with } \xi'_1 = c\xi_1, \xi'_2 = c\xi_2, \xi'_3 = c\xi_3, \xi'_1 + \xi'_2 + \xi'_3 = c\gamma = \gamma'$$

– Therefore closed wrt outer operation

• With respect to the inner operation: $\forall \vec{x}, \vec{y} \in C : \vec{x} + \vec{y} \in C$

– Since of addition of a number in \mathbb{R} will always yield a number in \mathbb{R} , $\vec{x} + \vec{y} \in C$ if $\vec{x}, \vec{y} \in C$

(d) To prove D is a subspace of \mathbb{R}^3 , we need to prove that D is not an empty set and D is closed with respect to both inner and outer

• While D is not empty ($\vec{x} = (0, 0, 0) \in B$), D is not closed with respect to outer operation

• Let $\vec{x} = (0, 1, 0) \in D$ and $c = 0.5$. Then $c\vec{x} = (0, 0.5, 0)$.

• Since $0.5 \notin \mathbb{R}$, $c\vec{x} \notin D$

2.10

(a) Let $A = [x_1, x_2, x_3]$ and row reduce it to see if all columns are pivots

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Since the third column does not have a pivot, the three vectors are not linearly independent
- From the row reduced matrix form, we can find that $x_3 = 2x_1 + (-1x_2)$

(b) Let $A = [x_1, x_2, x_3]$ and row reduce it to see if all columns are pivots

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Since all columns of A have a pivot, the three vectors are linearly independent

2.11

Let $A = [x_1, x_2, x_3]$. We must find x such that $Ax = y$. We will row reduce the augmented matrix $[A|y]$.

$$\begin{aligned} [A|y] &= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right] \end{aligned}$$

We can verify that $-6x_1 + 3x_2 + 2x_3 = y$

2.12

When a vector $\vec{v} \in U_1 \cap U_2$, then there must be \vec{x} and \vec{y} such that $U_1\vec{x} = U_2\vec{y} = \vec{v}$. It is clear that \vec{v} will be a linear combination of the basis of $U_1 \cap U_2$. We can get that $U_1\vec{x} - U_2\vec{y} = \vec{0}$. Let $A = [U_1 - U_2]$. Let's solve for the homogenous equation $A\vec{x} = \vec{0}$:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & -1 & 1 & -2 & 3 \\ 1 & -1 & 1 & 2 & 2 & -6 \\ -3 & 0 & -1 & -2 & 0 & 2 \\ 1 & -1 & 1 & -1 & 0 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 9 & 0 & 3 & 0 & -4 & 8 \\ 0 & 9 & -6 & 0 & -10 & 20 \\ 0 & 0 & 0 & 3 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

We can easily find that the nullspace of A is

$$\text{span} \left(\begin{bmatrix} -1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 10 \\ 0 \\ -6 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ -20 \\ 0 \\ 21 \\ 0 \\ 9 \end{bmatrix} \right)$$

Now, the interesting part of these three vectors is that, \vec{x} is a linear combination of the upper half of the 3 vectors in the basis and \vec{y} is the linear combination of the

lower half of the 3 vectors in the basis. To get the basis of $U_1 \cap U_2$, we simply need to compute the result of multiplying U_1 and 3 vectors that are the upper half of the 3 vectors in the basis of nullspace of A.

$$\begin{aligned} \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -3 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -3 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 10 \\ 0 \end{bmatrix} &= \begin{bmatrix} 24 \\ -6 \\ -12 \\ -6 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -3 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -8 \\ -20 \\ 0 \end{bmatrix} &= \begin{bmatrix} -48 \\ 12 \\ 24 \\ 12 \end{bmatrix} \end{aligned}$$

This means that:

$$U_1 \cap U_2 = \text{span} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 24 \\ -6 \\ -12 \\ -6 \end{bmatrix}, \begin{bmatrix} -48 \\ 12 \\ 24 \\ 12 \end{bmatrix} \right)$$

Therefore the basis of $U_1 \cap U_2$ is:

$$\begin{bmatrix} 4 \\ -1 \\ -2 \\ -1 \end{bmatrix}$$

2.13

Before approaching the subquestions, we should solve for U_1 and U_2

$$\begin{aligned} [A_1|\vec{0}] &= \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -2 & -1 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

Thus U_1 is:

$$\text{span} \left(\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right) \right)$$

$$\begin{aligned} [A_2|\vec{0}] &= \left[\begin{array}{ccc|c} 3 & -3 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 7 & -5 & 2 & 0 \\ 3 & -1 & 2 & 0 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

Thus U_2 is also:

$$\text{span} \left(\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right) \right)$$

- (a) dimensions of U_1 and U_2 are both 1
- (b) bases of U_1 and U_2 are both shown above
- (c) since U_1 and U_2 have the same basis, the basis of $U_1 \cap U_2$ is:

$$\text{span} \left(\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right) \right)$$

2.14

So for some reason, exercises 2.13 and 2.14 are the same???

2.15

- (a) • Proving F is a subspace of \mathbb{R}^3
- When x, y, z are all 0, the vector $\vec{v} \in F$ would be $\vec{0}$. Therefore, $F \neq 0$
 - If a vector $\vec{v} = (x, y, z) \in F$, then $x + y - z = 0$. $\forall \lambda \in \mathbb{R}$, $\lambda \vec{v} = (\lambda x, \lambda y, \lambda z)$. Since $x + y - z = 0$, $\lambda x + \lambda y - \lambda z = \lambda(x + y - z) = \lambda \cdot 0 = 0$. Thus, $\lambda \vec{v} \in F$ and F is closed with respect to outer operation
 - Let $\vec{v}_1 = (x_1, y_1, z_1) \in F$ and $\vec{v}_2 = (x_2, y_2, z_2) \in F$. Also, let $\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2) = (x_3, y_3, z_3)$. Since both vectors are in F , $x_1 + y_1 - z_1 = 0$ and $x_2 + y_2 - z_2 = 0$. This means that $x_3 + y_3 - z_3 = (x_1 + x_2) + (y_1 + y_2) - (z_1 + z_2) = (x_1 + y_1 - z_1) + (x_2 + y_2 - z_2) = 0 + 0 = 0$. Therefore, $\vec{v}_3 \in F$ and F is closed with respect to inner operation
 - Since F is not empty and is closed with respect to the inner and outer operations, F is a subspace of \mathbb{R}^3
- Proving G is a subspace of \mathbb{R}^3
- When a, b are both 0, the vector $\vec{v} \in F$ would be $\vec{0}$. Therefore, $F \neq 0$
 - Let $\vec{v} = (a - b, a + b, a - 3b) \in G$. $\forall \lambda \in \mathbb{R}$, $\lambda \vec{v} = (\lambda(a - b), \lambda(a + b), \lambda(a - 3b)) = (\lambda a - \lambda b, \lambda a + \lambda b, \lambda a - 3\lambda b)$. If we let $a' = \lambda a \in \mathbb{R}$ and $b' = \lambda b \in \mathbb{R}$, then $\lambda \vec{v} = (a' - b', a' + b', a' - 3b')$ which means $\lambda \vec{v} \in G$.
- (b) When a vector $\vec{v} = (x + y - z)$ is in $F \cap G$, then $x + y - z = 0$ and $x = a - b, y = a + b, z = a - 3b$ for some $a, b \in \mathbb{R}$. This means that $(a - b) + (a + b) - (a - 3b) = a + 3b = 0$. Therefore, $a = -3b$. This means that $\vec{v} = (-4b, -2b, -6b) = -2b(2, 1, 3)$. Therefore, the basis of $F \cap G$ is:

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

- (c) • For F , since $x + y - z = 0$, y and z are both free variables so a basis of F is

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

- For $\vec{v} = (x, y, z) \in G$, since $x = (a - b), y = (a + b), z = (a - 3b)$, a basis for G is:

$$\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \right)$$

- If a vector $\vec{v} \in F \cap G$, then there must be \vec{x} and \vec{y} such that $\vec{v} = F\vec{x} = G\vec{y}$. This means $F\vec{x} - G\vec{y} = \vec{0}$. To solve for \vec{x} and \vec{y} , solve the homogeneous equation $Ax = 0$ where A is augmented matrix $A = [F, -G]$

$$A = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This means that basis of nullspace of A is

$$\begin{bmatrix} -2 \\ -6 \\ -3 \\ 1 \end{bmatrix}$$

This means that \vec{x} is any scalar multiple of $(-2, -6)$ and \vec{y} is any scalar multiple of $(-3, 1)$. Therefore, \vec{v} would be any scalar multiple of:

$$F\vec{x} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -6 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ -6 \end{bmatrix}$$

This basis is equal to the one found in (b) since $(-4, -2, -6) = -2(2, 1, 3)$. So there are the same!

2.16

2.17

2.18

2.19

2.20

a. Done in separate place

b. Basis change vector P_1 can be found by computing the coordinates of the basis vectors of B in terms of B'

$$\begin{aligned}[B'|b_1] &= \left[\begin{array}{cc|c} 2 & 1 & 2 \\ -2 & 1 & 1 \end{array} \right] \\ &= \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{3}{2} \end{array} \right] \\ [B'|b_2] &= \left[\begin{array}{cc|c} 2 & 1 & -1 \\ -2 & 1 & -1 \end{array} \right] \\ &= \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -1 \end{array} \right]\end{aligned}$$

Therefore P_1 is:

$$\begin{bmatrix} \frac{1}{4} & 0 \\ \frac{3}{2} & -1 \end{bmatrix}$$

c. (i) If determinant of C is not equal to 0, then C is a basis of \mathbb{R}^3

$$\begin{aligned}|C| &= 1 \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \\ &= 1 \times 1 - 0 \times (-2) + 1 \times 3 \\ &= 4 \neq 0\end{aligned}$$

Therefore, C is a basis of \mathbb{R}^3

(ii)

d.