Exercises

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1 Chapter 1

2 Chapter 2

2.1

2.2

2.3

2.4

2.5

2.6

2.7

2.8

(a) Row reduce the augmented matrix $[A|I]$ to determine whether there exists an inverse for A

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 4 & 5 & 6 & 0 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

Since the third column of the augmented matrix is not a pivot column, A^{-1} does not exist

(b) Row reduce the augmented matrix $[A|I]$ to determine whether there exists an inverse for A

$$[A|I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -2 \end{array}\right]$$
$$\therefore A^{-1} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -2 \end{bmatrix}$$

2.9

(a) To prove A is a subpace of $\mathbb{R}^3$, we need to prove that A is not an empty set and A is closed with respect to both inner and outer

1. Let $\vec{v} \in A$. When $\lambda = 0$, $\mu = 0$, $\vec{v} = (0, 0, 0)$. This means that $\mathbf{0} \in A$ and thus $A \neq \emptyset$
2. Closure of A
    - With respect to the outer operation: $\forall c \in \mathbb{R}, \forall \vec{x} \in A : c\vec{x} \in A$
        - $c\vec{x} = (c\lambda, c(\lambda + \mu^3), c(\lambda - \mu^3)) \in A$
    - With respect to the inner operation: $\forall \vec{x}, \vec{y} \in A : \vec{x} + \vec{y} \in A$
        - Let $\vec{x} = (\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) \in A$
        - Let $\vec{y} = (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) \in A$
        - $\vec{x} + \vec{y} = (\lambda_1 + \lambda_2, \lambda_1 + \mu_1^3 + \lambda_2 + \mu_2^3, \lambda_1 - \mu_1^3 + \lambda_2 - \mu_2^3)$
        - Let $\lambda_3 = \lambda_1 + \lambda_2$
        - Then: $\vec{x} + \vec{y} = (\lambda_3, \lambda_3 + \mu_1^3 + \mu_2^3, \lambda_3 - \mu_1^3 - \mu_2^3)$
        - Let $\mu_1^3 + \mu_2^3 = \mu_3^3$
        - Then: $\vec{x} + \vec{y} = (\lambda_3, \lambda_3 + \mu_3^3, \lambda_3 - \mu_3^3) \in A$

(b) To prove B is a subpace of $\mathbb{R}^3$, we need to prove that B is not an empty set and B is closed with respect to both inner and outer

- While B is not empty ($\vec{x} = (0, 0, 0) \in B$), B is not closed with respect to outer operation
- Let $\vec{x} = (1, -1, 0) \in B$ and $c = -1$. Then $c\vec{x} = (-1, 1, 0)$. Since there is no number in $\mathbb{R}$ such that its square is less than 0. (only complex numbers are possible).
- Therefore $c\vec{x} \notin B$

(c) To prove C is a subpace of $\mathbb{R}^3$, we need to prove that C is not an empty set and C is closed with respect to both inner and outer

1. Let $\vec{v} \in C$. When $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$, $\vec{v} = (0, 0, 0) with \gamma = 0$. This means that $\mathbf{0} \in C$ and thus $C \neq \emptyset$
2. Closure of C
    - With respect to the outer operation: $\forall c \in \mathbb{R}, \forall \vec{x} \in A : c\vec{x} \in A$

- $c\vec{x} = c(\xi_1, \xi_2, \xi_3) = (\xi_1', \xi_2', \xi_3')$ with $\xi_1' = c\xi_1, \xi_2' = c\xi_2, \xi_3' = c\xi_3, \xi_1' + \xi_2' + \xi_3' = c\gamma = \gamma'$
    - Therefore closed wrt outer operation
- With respect to the inner operation: $\forall \vec{x}, \vec{y} \in C : \vec{x} + \vec{y} \in C$
    - Since of addition of a number in $\mathbb{R}$ will always yield a number in $\mathbb{R}$, $\vec{x} + \vec{y} \in C$ if $\vec{x}, \vec{y} \in C$

(d) To prove D is a subpace of $\mathbb{R}^3$, we need to prove that D is not an empty set and D is closed with respect to both inner and outer

- While D is not empty ($\vec{x} = (0,0,0) \in B$), D is not closed with respect to outer operation
- Let $\vec{x} = (0,1,0) \in D$ and $c = 0.5$. Then $c\vec{x} = (0, 0.5, 0)$.
- Since $0.5 \notin \mathbb{R}$, $c\vec{x} \notin D$

2.10

(a) Let $A = [x_1, x_2, x_3]$ and row reduce it to see if all columns are pivots

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Since the third column does not have a pivot, the three vectors are not linearly independent
- From the row reduced matrix form, we can find that $x_3 = 2x_1 + (-1x_2)$

(b) Let $A = [x_1, x_2, x_3]$ and row reduce it to see if all columns are pivots

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Since all columns of A have a pivot, the three vectors are linearly independent

2.11

Let $A = [x_1, x_2, x_3]$. We must find x such that $Ax = y$. We will row reduce the augmented matrix $[A|y]$.

$$
\begin{aligned}
[A|y] &= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]
\end{aligned}
$$

We can verify that $-6x_1 + 3x_2 + 2x_3 = y$

2.12

When a vector $\vec{v} \in U_1 \cap U_2$, then there must be $\vec{x}$ and $\vec{y}$ such that $U_1\vec{x} = U_2\vec{y} = \vec{v}$. It is clear that $\vec{v}$ will be a linear combination of the basis of $U_1 \cap U_2$. We can get that $U_1\vec{x} - U_2\vec{y} = \vec{0}$. Let $A = [U_1 - U_2]$. Let's solve for the homogenous equation $A\vec{x'} = \vec{0}$:

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 2 & -1 & 1 & -2 & 3 \\ 1 & -1 & 1 & 2 & 2 & -6 \\ -3 & 0 & -1 & -2 & 0 & 2 \\ 1 & -1 & 1 & -1 & 0 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 9 & 0 & 3 & 0 & -4 & 8 \\ 0 & 9 & -6 & 0 & -10 & 20 \\ 0 & 0 & 0 & 3 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

We can easily find that the nullspace of A is

$$
span\left(\begin{bmatrix} -1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 10 \\ 0 \\ -6 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ -20 \\ 0 \\ 21 \\ 0 \\ 9 \end{bmatrix}\right)
$$

Now, the interesting part of these three vectors is that, $\vec{x}$ is a linear combination of the upper half of the 3 vectors in the basis and $\vec{y}$ is the linear combination of the lower half of the 3 vectors in the basis. To get the basis of $U_1 \cap U_2$, we simply need to compute the result of multiplying U_1 and 3 vectors that are the upper half of the 3 vectors in the basis of nullspace of A.

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -3 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -3 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ -6 \\ -12 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -3 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -8 \\ -20 \\ 0 \end{bmatrix} = \begin{bmatrix} -48 \\ 12 \\ 24 \\ 12 \end{bmatrix}$$

This means that:

$$U_1 \cap U_2 = span\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 24 \\ -6 \\ -12 \\ -6 \end{bmatrix}, \begin{bmatrix} -48 \\ 12 \\ 24 \\ 12 \end{bmatrix} \right)$$

Therefore the basis of $U_1 \cap U_2$ is:

$$\begin{bmatrix} 4 \\ -1 \\ -2 \\ -1 \end{bmatrix}$$

2.13

Before approaching the subquestions, we should solve for U_1 and U_2

$$[A_1|\vec{0}] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -2 & -1 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus U_1 is:

$$\text{span}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right)$$

$$[A_2|\vec{0}] = \left[\begin{array}{ccc|c} 3 & -3 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 7 & -5 & 2 & 0 \\ 3 & -1 & 2 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus U_2 is also:

$$\text{span}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right)$$

(a) dimensions of U_1 and U_2 are both 1

(b) bases of U_1 and U_2 are both shown above

(c) since U_1 and U_2 have the same basis, the basis of $U_1 \cap U_2$ is:

$$\text{span}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right)$$

2.14

So for some reason, exercises 2.13 and 2.14 are the same???

2.15

(a)
- Proving F is a subspace of $\mathbb{R}^3$
  - When x, y, z are all 0, the vector $\vec{v} \in F$ would be $\vec{0}$. Therefore, $F \neq 0$
  - If a vector $\vec{v} = (x, y, z) \in F$, then $x + y - z = 0$. $\forall \lambda \in \mathbb{R}$, $\lambda\vec{v} = (\lambda x, \lambda y, \lambda z)$. Since $x+y-z = 0$, $\lambda x+\lambda y-\lambda z = \lambda(x+y-z) = \lambda \cdot 0 = 0$. Thus, $\lambda\vec{v} \in F$ and F is closed with respect to outer operation
  - Let $\vec{v_1} = (x_1, y_1, z_1) \in F$ and $\vec{v_2} = (x_2, y_2, z_2) \in F$. Also, let $\vec{v_3} = \vec{v_1}+\vec{v_2} = (x_1+x_2, y_1+y_2, z_1+z_2) = (x_3, y_3, z_3)$. Since both vectors are in F, $x_1+y_1-z_1 = 0$ and $x_2+y_2-z_2 = 0$. This means that $x_3+y_3-z_3 = (x_1+x_2)+(y_1+y_2)-(z_1+z_2) = (x_1+y_1-z_1)+(x_2+y_2-z_2) = 0+0 = 0$. Therefore, $\vec{v_3} \in F$ and F is closed with respect to inner operation
  - Since F is not empty and is closed with respect to the inner and outer operations, F is a subspace of $\mathbb{R}^3$
- Proving G is a subspace of $\mathbb{R}^3$
  - When a, b are both 0, the vector $\vec{v} \in F$ would be $\vec{0}$. Therefore, $F \neq 0$
  - Let $\vec{v} = (a - b, a + b, a - 3b) \in G$. $\forall \lambda \in \mathbb{R}$, $\lambda\vec{v} = (\lambda(a - b), \lambda(a + b), \lambda(a - 3b)) = (\lambda a - \lambda b, \lambda a + \lambda b, \lambda a - 3\lambda b)$. If we let $a' = \lambda a \in \mathbb{R}$ and $b' = \lambda b \in \mathbb{R}$, then $\lambda\vec{v} = (a' - b', a' + b', a' - 3b')$ which means $\lambda\vec{v} \in G$.

(b) When a vector $\vec{v} = (x + y - z)$ is in $F \cap G$, then $x + y - z = 0$ and $x = a - b, y = a + b, z = a - 3b$ for some $a, b \in \mathbb{R}$. This means that $(a - b) + (a + b) - (a - 3b) = a + 3b = 0$. Therefore, $a = -3b$. This means that $\vec{v} = (-4b, -2b, -6b) = -2b(2, 1, 3)$. Therefore, the basis of $F \cap G$ is:

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

(c)
- For F, since $x + y - z = 0$, y and z are both free variables so a basis of F is

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

- For $\vec{v} = (x, y, z) \in G$, since $x = (a - b), y = (a + b), z = (a - 3b)$, a basis for G is:

$$\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \right)$$

- If a vector $\vec{v} \in F \cap G$, then there must be $\vec{x}$ and $\vec{y}$ such that $\vec{v} = F\vec{x} = G\vec{y}$. This means $F\vec{x} - G\vec{y} = \vec{0}$. To solve for $\vec{x}$ and $\vec{y}$, solve the homegeneous equation $Ax = 0$ where A is augmented matrix $A = [F, -G]$

$$\begin{aligned} A &= \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{bmatrix} \end{aligned}$$

This means that basis of nullspace of A is

$$\begin{bmatrix} -2 \\ -6 \\ -3 \\ 1 \end{bmatrix}$$

This means that $\vec{x}$ is any scalar multiple of $(-2, -6)$ and $\vec{y}$ is any scalar multiple of $(-3, 1)$. Therefore, $\vec{v}$ would be any scalar multiple of:

$$F\vec{x} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -6 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ -6 \end{bmatrix}$$

This basis is equal to the one found in (b) since $(-4, -2, -6) = -2(2, 1, 3)$. So there are the same!

2.16

2.17

2.18