

# MA522 Lecture Notes

Instructor: Betsy Stovall

Note taken by: Yujia Bao

**Definition.** Let  $(X, d)$  be a metric space, let  $K \subseteq X$ , then

- An open cover of  $K$  (in  $X$ ) is a set  $\mathcal{G} = \{G_\alpha\}_{\alpha \in \mathcal{A}}$  where each  $G_\alpha$  is an open subset of  $X$  and  $K \subseteq \cup_{\alpha \in \mathcal{A}} G_\alpha$ .
- $K$  is compact if every open cover of  $K$  contains a finite subcover of  $K$ , i.e. if for every open cover  $\mathcal{G} = \{G_\alpha\}_{\alpha \in \mathcal{A}}$ , there  $\exists \alpha_1, \dots, \alpha_N \in \mathcal{A}$ , s.t.  $K \subseteq \cup_{j=1}^N G_{\alpha_j}$ .

**Example.** Show that  $(0, 1]$  is not compact.

*Proof.* Let  $\mathcal{G} = \{(1/n, 2) : n \in \mathbb{N}\}$ . Then  $\mathcal{G}$  is an open cover of  $(0, 1]$ . But if  $\{n_1, \dots, n_N\}$  is any finite set,  $\cup_{j=1}^N (1/n_j, 2) = (1/\max n_j, 2) \not\subseteq (0, 1]$ .  $\square$

**Example.** Show that  $\mathbb{R}$  is not compact.

*Proof.* Let  $\mathcal{G} = \{(-n, n) : n \in \mathbb{N}\}$ .  $\square$

**Theorem 1** (Heine-Borel). *A subset of  $\mathbb{R}^n$  is compact if and only if it is closed and bounded.*

**Example.** Show that  $\{0\} \cup \{1/n : n \in \mathbb{N}\}$  is compact.

*Proof.* Let  $\mathcal{G} = \{G_\alpha\}_{\alpha \in \mathcal{A}}$  be any open cover. Then  $0 \in G_{\alpha_0}$  for some  $\alpha_0 \in \mathcal{A}$ . Since  $G_{\alpha_0}$  is open,  $\exists \epsilon > 0$  such that  $B_\epsilon(0) \subseteq G_{\alpha_0}$ . For  $N = \lceil 1/\epsilon \rceil$ , we have  $1/n \in B_\epsilon(0) \subseteq G_{\alpha_0}$  for all  $n > N$ . Choose  $\alpha_n$  such that  $1/n \in G_{\alpha_n}$  for each  $n \leq N$ . Thus  $\{G_{\alpha_j}\}_{j=0}^N$  is a finite subcover of the origin set. So the origin set is compact.  $\square$

**Definition.**  $U \subseteq X$  is precompact if  $\bar{U}$  is compact. (Here  $\bar{U}$  stands for the closure of  $U$ .)

**Example.** By Theorem 1, every Borel subset of  $\mathbb{R}^n$  is precompact.

Question: Definition of Borel set.

**Definition.**  $K \subseteq X$  is sequentially compact if every sequence  $\{x_n\}$  contains a subsequence  $\{x_{n_k}\}$  that converges to a limit in  $K$ .

**Definition.**  $U \subseteq X$  is totally bounded if  $\forall \epsilon > 0$ ,  $U$  is covered by a finite collection of  $\epsilon$ -balls, i.e.,  $\exists x_1, \dots, x_{N_\epsilon} \in X$ , s.t.  $U \subseteq \cup_{j=1}^{N_\epsilon} B_\epsilon(x_j)$ .

**Example.** Every bounded subset of  $\mathbb{R}^n$  is totally bounded.

**Example.** In discrete metric space,

$$\delta(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then, every set is both open and closed. For  $\epsilon < 1$ , the  $\epsilon$  Ball becomes a single ball. Infinite sets are bounded but not totally bounded.

**Definition.**  $K \subseteq X$  is complete if every Cauchy sequence in  $K$  converges to some limit in  $K$ .

**Definition.**  $x$  is an accumulation point of  $E \subseteq X$  if  $\forall \epsilon > 0$ ,  $(B_\epsilon(x) \setminus \{x\}) \cap E \neq \emptyset$ . This is equivalent to say  $\forall \epsilon > 0$ ,  $B_\epsilon(x) \cap E$  contains infinitely many points.

**Theorem 2.** Let  $(X, d)$  be a metric space, then TFAE

1.  $K$  is compact.
2.  $K$  has the Bolzano-Weierstrass property (Every infinity subset of  $K$  has an accumulation point in  $K$ ).
3.  $K$  is sequentially compact.
4.  $K$  is complete and totally bounded.

*Proof.*  $1 \Rightarrow 2$ : Assume  $K$  is compact. Let  $E \subseteq K$  be an infinite set. If there doesn't exist such  $E$ , then Bolzano-Weierstrass property holds trivially. Now suppose  $E$  has no accumulation point in  $K$ . That means for every  $x$  in  $K$ ,  $\exists$  neighbour  $U_x$  of  $x$  (i.e.  $x \in U_x$  and  $U_x$  is open), that contains no points of  $E$  other than (possibly)  $x$  itself. Since  $K = \cup_{x \in K} \{x\} \subseteq \cup_{x \in K} U_x$ ,  $\{U_x : x \in K\}$  is an open cover of  $K$ . By compactness,  $\exists x_1, \dots, x_N$ , s.t.  $K \subseteq \cup_{j=1}^N U_{x_j}$ . But  $\cup_{j=1}^N U_{x_j}$  contains at most  $N$  points of  $E$ . So it cannot contain all points of  $E$  because  $E$  is an infinity set. Contradiction! Since  $E \subseteq K$ ,  $\cup_{j=1}^N U_{x_j}$  should be a cover of  $E$ .

$2 \Rightarrow 3$ : Assume  $K$  has the Bolzano-Weierstrass property. Let  $\{x_n\}$  be a sequence in  $K$ . We need to show that it has a convergent subsequence. Let  $E = \{x_n : n \in \mathbb{N}\}$ .

**Case I:**  $E$  is a finite set. By the pigeonhole principle,  $\exists x \in E \subseteq K$ , s.t.  $x_n = x$  for infinitely many  $n$ . Here  $\{x_n\}$  has a constant subsequence which only takes the value  $x$ . This is a subsequence converge to  $x \in K$ .

**Case II:**  $E$  is an infinite set. By Bolzano-Weierstrass property,  $E$  has an accumulation point  $x \in K$ . Thus every ball centered at  $x$  contains infinitely many  $x_n$ s. Choose  $n_1$  such that  $x_{n_1} \in B_1(x)$ ; Choose  $n_2 > n_1$  such that  $x_{n_2} \in B_{1/2}(x)$ ; and so on. Proceeding by induction, we may find  $n_1 < n_2 < \dots < n_k < \dots$ , s.t.  $x_{n_k} \in B_{1/k}(x)$  for all  $k$ . Then  $\{x_{n_k}\}$  is a subsequence of  $\{x_n\}$  and  $x_{n_k} \rightarrow x$ .

$3 \Rightarrow 4$ : Assume  $K$  is sequentially compact.

**Completeness:** Let  $\{x_n\}$  be a Cauchy sequence in  $K$ . By sequentially compactness, there exists a convergent subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$ , say  $x_{n_k} \rightarrow x \in K$ . Then we claim that  $x_n \rightarrow x$ . Since  $x_{n_k} \rightarrow x$ , let  $\epsilon > 0$ ,  $\exists M$  s.t.  $\forall k \geq M$ ,  $d(x_{n_k}, x) < \epsilon$ . Since  $\{x_n\}$  is a

Cauchy sequence,  $\exists N$  s.t.  $\forall n, m \geq N, d(x_n, x_m) < \epsilon$ . Now fix  $k_0 \geq \max\{M, N\}$  and let  $n \geq N$ . Then

$$d(x_n, x) \leq d(x_n, x_{n_{k_0}}) + d(x_{n_{k_0}}, x).$$

Since  $k_0 \geq M$ , we have  $d(x_{n_{k_0}}, x) < \epsilon$ . Since  $n_{k_0} \geq k_0 \geq N$ , we have  $d(x_n, x_{n_{k_0}}) < \epsilon$ . Then  $d(x_n, x) < 2\epsilon$ . So  $\{x_n\}$  does converge.

**Totally boundedness:** Suppose not. Then  $\exists \epsilon > 0$  s.t.  $K$  cannot be covered by a finite union of  $\epsilon$ -balls. Thus, we may (inductively) construct a sequence  $\{x_n\}$  in  $K$  such that  $\forall n \geq 2, x_n \notin \cup_{j=1}^{n-1} B_\epsilon(x_j)$ . Let  $\{x_{n_k}\}$  be any subsequence of  $\{x_n\}$ . Pick any  $k_1, k_2$  with  $k_1 < k_2$ . Then  $x_{k_2} \notin B_\epsilon(x_{k_1})$  which means  $d(x_{n_{k_1}}, x_{n_{k_2}}) \geq \epsilon$ . So  $\{x_{n_k}\}$  is not Cauchy and hence not convergent. This contradicts with sequential compactness of  $K$ .

4  $\Rightarrow$  3: Assume  $K$  is complete and totally bounded. Let  $\{x_n\}$  be a sequence in  $K$ . We want to find a convergent subsequence. By totally boundedness,  $K$  is covered by a finite number of 1-balls,  $K \subseteq \cup_{j=1}^N B_1(y_j)$ . By the pigeonhole principle, there must exist an  $B_1(y_j)$  which contains  $x_n$  for infinitely many  $n$ . Denote that  $y_j$  as  $z_1$ . So there exists a subsequence  $\{x_{n_k^1}\}$  contained in  $B_1(z_1)$ . By the same argument,  $\exists z_2$  s.t.  $B_{1/2}(z_2)$  contains a subsequence  $\{x_{n_k^2}\}$  of  $\{x_{n_k^1}\}$ , and so on. So for each  $m \in \mathbb{N}$ , we find  $z_m$  and a subsequence  $\{x_{n_k^m}\}$  of  $\{x_{n_k^{m-1}}\}$  s.t.  $\{x_{n_k^m}\}$  is contained in  $B_{1/m}(z_m)$ . Now we define  $x_{n_k} = x_{n_k^k}$  (diagonalization).

**Claim 1:**  $\{x_{n_k}\}$  is a subsequence of  $\{x_n\}$ .

This is because  $n_k = n_k^k \geq n_k^{k-1} > n_{k-1}^{k-1} = n_{k-1}$ , where the first inequality comes from the fact that  $\{x_{n_j^k}\}$  is a subsequence of  $\{x_{n_j^{k-1}}\}$ .

**Claim 2:**  $\{x_{n_k}\}$  is a Cauchy sequence.

For  $k_1, k_2 \geq M$ ,  $x_{n_{k_1}}$  and  $x_{n_{k_2}}$  are both terms in the sequence  $\{x_{n_j^M}\}$ . So both of them lie in  $B_{1/M}(z_M)$  which means  $d(x_{n_{k_1}}, x_{n_{k_2}}) \leq 2/M$ .

By completeness of  $K$ ,  $\{x_{n_k}\}$  converges in  $K$ .

4  $\Rightarrow$  2: Assume  $K$  is complete and totally bounded. Let  $E \subseteq K$  be an infinite subset. Since  $K$  is totally bounded,  $K$  can be covered by finitely many 1-balls. By pigeonhole principle,  $\exists x_1 \in K$ , s.t.  $B_1(x_1) \cap E =: E_1$  is an infinite set. By induction, for each  $n \in \mathbb{N}^+$ ,  $\exists x_n \in K$ , s.t.  $B_{1/n}(x_n) \cap E_{n-1} =: E_n$  is an infinite set.

**Claim 1:**  $\{x_n\}$  is a Cauchy sequence.

Notice that  $\forall n, m$ ,

$$B_{1/n}(x_n) \cap B_{1/m}(x_m) \supseteq E_{\max\{m, n\}} \neq \emptyset.$$

This implies

$$d(x_n, x_m) < \frac{1}{n} + \frac{1}{m} \leq \frac{2}{\min\{n, m\}}.$$

Then as long as  $n, m > 2/\epsilon$ , we have  $d(x_n, x_m) < \epsilon$ . So  $\{x_n\}$  is a Cauchy sequence.

By completeness of  $K$ ,  $\{x_n\}$  converges, say  $x_n \rightarrow x_0$ .

**Claim 2:**  $x_0$  is an accumulation point for  $E$ .

Let  $\epsilon > 0$ . Choose  $N$  sufficiently large, such that  $\forall n \geq N$ ,  $d(x_0, x_n) < \epsilon/2$  and  $1/n < \epsilon/2$ . For any  $y \in B_{1/n}(x_n)$ , we have  $d(y, x_n) < 1/n$ . Then

$$d(x_0, y) \leq d(x_0, x_n) + d(x_n, y) < \frac{\epsilon}{2} + \frac{1}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

So we have  $y \in B_\epsilon(x_0)$ . Then  $B_\epsilon(x_0) \supseteq B_{1/n}(x_n) \supseteq E_n$ . Since  $E_n$  is an infinite subset of  $E$  and  $\epsilon$  is selected arbitrary,  $x_0$  is an accumulation point of  $E$ .

4, 3  $\Rightarrow$  1: Assume  $K$  is complete, totally bounded and sequentially compact. Let  $\mathcal{G} := \{G_\alpha\}_{\alpha \in \mathcal{A}}$  be an open cover of  $K$ . Then  $\forall x \in K$ ,  $\exists \alpha \in \mathcal{A}$ , s.t.  $x \in G_\alpha$ . Since  $G_\alpha$  is open,  $\exists r > 0$ , s.t.  $B_r(x) \subseteq G_\alpha$ . Thus we define

$$\epsilon(x) := \sup\{r > 0 : \exists \alpha \in \mathcal{A}, \text{ s.t. } B_r(x) \subseteq G_\alpha\}$$

By definition,  $\epsilon(x) > 0$ . If  $\epsilon(x) = +\infty$  for some  $x \in K$ , then since  $K$  is bounded (because  $K$  is totally bounded), there must be one  $G_\alpha$  containing all of  $K$ . Now we assume  $\epsilon(x)$  is finite for all  $x \in K$ .

**Claim:**  $\exists \epsilon_0 > 0$ , s.t.  $\epsilon(x) \geq \epsilon_0$ ,  $\forall x \in K$ .

Suppose such  $\epsilon_0$  doesn't exist. Then  $\exists \{x_n\}$ , s.t.  $\epsilon(x_n) \rightarrow 0$ . By sequential compactness of  $K$ , there exists a subsequence  $x_{n_k} \rightarrow x_0 \in K$ . As  $\epsilon(x_0) > 0$ , we can choose some  $r > 0$  and  $\alpha$  s.t.  $B_r(x_0) \subseteq G_\alpha$ . But for  $k$  sufficiently large,  $\epsilon(x_{n_k}) < r/2$  and  $d(x_0, x_{n_k}) < r/2$ . Then

$$B_{r/2}(x_{n_k}) \subseteq B_r(x_0) \subseteq G_\alpha,$$

and this implies  $r/2 \leq \epsilon(x_{n_k}) < r/2$  which leads to contradiction.

Since  $K$  is totally bounded, so  $\exists x_1, \dots, x_N$ , s.t.

$$K \subseteq \bigcup_{j=1}^N B_{\epsilon_0/2}(x_j) \subseteq \bigcup_{j=1}^N B_{\epsilon(x_j)/2}(x_j)$$

Furthermore, as  $\epsilon(x_j)/2 < \epsilon(x_j)$  (because  $\epsilon(x_j) > 0$ ),  $\exists r > 0$ , s.t.  $r > \epsilon(x_j)/2$  and  $B_r(x_j) \subseteq G_{\alpha_j}$  for some  $\alpha_j \in \mathcal{A}$ . Finally,  $\cup_{j=1}^N G_{\alpha_j}$  is a finite subcover of  $K$ . Thus  $K$  is compact.  $\square$

**Corollary.** *A subset of a complete metric space is compact if and only if it is closed and totally bounded.*

**Corollary.** *A subset of a complete metric space is precompact if and only if it is totally bounded.*

**Example.** A closed subset of a compact set is compact.

*Proof.* Suppose  $K \subseteq E$ . Since  $E$  is compact, by Bolzano-Weierstrass property, every infinite subset of  $K$  has an accumulation point. Since  $K$  is closed,  $K$  contains all its accumulation points. So  $K$  also satisfies B-W property which means  $K$  is compact.  $\square$

**Definition.** A subset  $E$  of a metric space  $X$  is not connected if  $\exists$  sets  $A, B$  s.t.

$$\begin{cases} E = A \cup B, & A \neq \emptyset, B \neq \emptyset \\ \bar{A} \cap B = \emptyset, & A \cap \bar{B} = \emptyset \end{cases}$$

We say  $A$  and  $B$  form a separation of  $E$ , or  $A$  and  $B$  separate  $E$ .

**Definition.**  $E$  is connected if it is not disconnected.

**Theorem 3.** Let  $E$  be a subset of  $X$ . Then TFAE:

1.  $E \subseteq X$  is not connected.

2.  $\exists$  open sets  $U, V \subseteq X$ , s.t.

$$E \subseteq U \cup V, E \cap U \neq \emptyset, E \cap V \neq \emptyset, U \cap V = \emptyset.$$

3.  $\exists A \subseteq E$ , s.t.

$$A \neq \emptyset, A \neq E, A = E \cap F = E \cap G,$$

where  $F$  is closed and  $G$  is open.

*Proof.*  $2 \Rightarrow 3$ : Assume 2 holds. We pick  $A = E \cap U$ . Then  $A \neq \emptyset$ . Let  $G = U$ . Then  $A = E \cap G$ . Let  $F = V^c$ . Then  $F$  is closed since  $V$  is open.

**Claim 1:**  $A = E \cap F$ .

Let  $a \in A$ . Then  $a \in E$  and  $a \in U \subseteq V^c = F$ . So  $a \in E \cap F$ . Let  $x \in E \cap F$ . Then  $x \in E$  and  $x \notin V$ . Since  $E \subseteq U \cup V$ ,  $x \in U$ . So  $x \in E \cap U = A$ .

**Claim 2:**  $A \neq E$ .

This is true because  $E \setminus A = E \setminus (E \cap F) = E \cap V$  and we know  $E \cap V \neq \emptyset$  by 2.

$3 \Rightarrow 1$ : Assume 3 holds. We define  $A$  as in 3 and let  $B = E \setminus A$ . Then by definition,  $A \cup B = E$ ,  $A \neq \emptyset$  and  $B \neq \emptyset$  (since  $A \neq E$ ). Note that

$$\bar{A} \cap B = \overline{E \cap F} \cap (E \cap F^c) \subseteq \bar{F} \cap F^c = F \cap F^c = \emptyset,$$

since  $F$  is closed. Also

$$A \cap \bar{B} = E \cap G \cap \overline{E \cap G^c} \subseteq G \cap \overline{G^c} = G \cap G^c = \emptyset,$$

since  $G^c$  is closed. Then  $E$  is not connected.

$1 \Rightarrow 2$ : Assume 1 holds. Let  $a \in A$ . Then  $a \notin \bar{B}$ , since  $A \cap \bar{B} = \emptyset$ . So  $\exists r(a) > 0$ , s.t.  $B_{r(a)}(a) \cap B = \emptyset$ . Likewise, if  $b \in B$ ,  $\exists r(b) > 0$ , s.t.  $B_{r(b)}(b) \cap A = \emptyset$ . Now define  $U = \cup_{a \in A} B_{r(a)/2}(a)$  and  $V = \cup_{b \in B} B_{r(b)/2}(b)$ . Then  $U$  and  $V$  are open sets since they are unions of open sets. Also  $E \subseteq A \cup B \subseteq U \cup V$  and  $E \cap U \supseteq A \neq \emptyset$ ,  $E \cap V \supseteq B \neq \emptyset$ . Now we just need to show  $U \cap V = \emptyset$ . If it is not true,  $\exists x \in U \cap V$ . By our construction of  $U$  and  $V$ ,  $\exists a \in A$ ,  $b \in B$ , s.t.  $x \in B_{r(a)/2}(a) \cap B_{r(b)/2}(b)$ . So

$$d(a, b) \leq d(a, x) + d(b, x) < \frac{r(a)}{2} + \frac{r(b)}{2} \leq \max\{r(a), r(b)\}$$

Then if  $r(a) \geq r(b)$ , we have  $b \in B_{r(a)}(a)$ . If  $r(b) \geq r(a)$ , we have  $a \in B_{r(b)}(b)$ . Both of them contradicts with our definition of  $r(a)$  or  $r(b)$ . So  $U \cap V = \emptyset$ .

□

**Example.**  $(-\infty, 0) \cup (0, +\infty)$  is not connected.

**Example.**  $E := \{(x, y) : y \in [-1, 1] \text{ with } x = 0 \text{ or } y = \sin(1/x) \text{ with } x > 0\}$  is connected.

**Proposition.** *If  $E$  is connected and  $f : E \rightarrow Y$  is continuous, then  $f(E)$  is connected.*

*Proof.* Assume  $f(E)$  is not connected. Then  $\exists U, V$  open with  $f(E) \subseteq U \cup V$ ,  $U \cap V = \emptyset$ ,  $f(E) \cap U \neq \emptyset$ ,  $f(E) \cap V \neq \emptyset$ . Note that  $f^{-1}(U)$  is open, nonempty. Likewise for  $f^{-1}(V)$ . Moreover  $f^{-1}(U) \cap f^{-1}(V) = \emptyset$  (If  $\exists x \in f^{-1}(U) \cap f^{-1}(V)$ , then  $f(x) \in U \cap V = \emptyset$ . Contradiction!) and  $f^{-1}(U) \cup f^{-1}(V) \supseteq E$ . By Theorem 3,  $E$  is not connected. Contradiction!  $\square$

**Definition.** Let  $(X, D)$  be a metric space. (Assume  $X \neq \emptyset$ .) Say  $\Phi : X \rightarrow X$  is a contraction if  $\exists r < 1$ , s.t.  $\forall x, y \in X$ ,

$$d(\Phi(x), \Phi(y)) \leq r \cdot d(x, y)$$

**Example.** By mean value theorem, if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable and  $\exists r < 1$ , s.t.  $\forall x, |f'(x)| \leq r$ . Then  $f$  is a contraction.

**Proposition.** *If  $\Phi$  is a contraction, then  $\Phi$  is continuous.*

**Theorem 4** (Contraction Mapping Theorem). *If  $(X, d)$  is a nonempty complete metric space and  $\Phi : X \rightarrow X$  is a contraction. Then  $\Phi$  has a unique fixed point, i.e.  $\exists! x_0 \in X$ , s.t.  $\Phi(x_0) = x_0$ .*

*Proof.* Let  $\Phi$  be a contraction with shrinking constant  $r < 1$ .

**Uniqueness:** Suppose  $x_1, x_2$  are both fixed points of the contraction  $\Phi$ . Then

$$d(x_1, x_2) = d(\Phi(x_1), \Phi(x_2)) \leq r \cdot d(x_1, x_2).$$

Thus  $d(x_1, x_2) = 0$  and  $x_1 = x_2$ , since  $r < 1$ .

**Existence:** Let  $x \in X$ . Define a sequence  $\{x_n\}$  inductively by setting  $x_0 = x$  and  $x_n = \Phi(x_{n-1})$  for  $n \in \mathbb{N}^+$ .

**Claim 1:**  $\{x_n\}$  is a Cauchy sequence and hence convergent.

Suppose  $m, n \in \mathbb{N}^+$ ,  $m > n$ . Then

$$d(x_m, x_n) = d(\Phi(x_{m-1}), \Phi(x_{n-1})) \leq r d(x_{m-1}, x_{n-1}).$$

By induction,

$$\begin{aligned} d(x_m, x_n) &\leq r^n d(x_{m-n}, x_0) \\ &\leq r^n (d(x_{m-n}, x_{m-n-1}) + d(x_{m-n-1}, x_{m-n-2}) + \cdots + d(x_1, x_0)) \\ &\leq r^n (r^{m-n-1} d(x_1, x_0) + r^{m-n-2} d(x_1, x_0) + \cdots + d(x_1, x_0)) \\ &\leq r^n d(x_1, x_0) \sum_{j=0}^{\infty} r^j \\ &= \frac{r^n}{1-r} d(x_1, x_0) \end{aligned}$$

Given any  $\epsilon > 0$ , choose  $N$  such that  $r^N d(x_1, x_0)/(1-r) < \epsilon$ . Then by the preceding calculation,  $\forall n, m \geq N$ ,  $d(x_n, x_m) < \epsilon$ .

**Claim 2:**  $\lim_{n \rightarrow \infty} x_n$  is a fixed point of  $\Phi$ .

Since  $\Phi$  is a contraction,  $\Phi$  is continuous. So we can exchange  $\Phi$  with the limit operation. Then

$$\Phi(\lim_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} \Phi(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} x_n.$$

□

**Corollary.** If  $\exists r < 1$ , s.t.  $|f'(x)| \leq r$ ,  $\forall x \in \mathbb{R}$ . Then  $\exists! x_0$ , s.t.  $f(x_0) = x_0$ .

**Example.**  $\Phi(x) = x + 1$  on  $\mathbb{R}$  is not a contraction.

**Example.**  $X = (-\infty, 0) \cup (0, \infty)$  is not complete. For  $\Phi(x) = \frac{1}{2}x$ , it is a contraction but the fixed point  $x_0 = 0$  is not in  $X$ .

**Definition.** Let  $E \subseteq X$ .

- $E$  is dense in  $X$  if  $\bar{E} = X$ .
- $E$  is nowhere dense in  $X$  if  $\bar{E}$  has empty interior.

**Example.**  $\mathbb{Z} \subseteq \mathbb{R}$  is nowhere dense in  $\mathbb{R}$ .  $\mathbb{Q} \cap (-2, 2)$  is neither dense nor nowhere dense in  $\mathbb{R}$ .

**Definition.** Let  $E \subseteq X$ , where  $X$  is some metric space.

- $E$  is meager if  $E$  can be written as a countable union of nowhere dense sets.
- $E$  is generic if  $E^c$  is meager.

**Theorem 5** (Baire Category Theorem). A nonempty complete metric space cannot be written as a countable union of nowhere dense sets.

This is equivalent to say that a complete metric space cannot be meager and is also equivalent to say that a subset of a complete metric space cannot be both meager and generic. In particular, generic subsets of complete metric space are nonempty.

*Proof.* Let  $(X, d)$  be a complete metric space and  $\{F_n\}$  be a collection of nowhere dense subsets of  $X$ . Suppose for contradiction that  $X = \cup_n F_n$ . Then  $X = \cup_n \bar{F}_n$  and  $\bar{F}_n$  are also nowhere dense. Without loss of generality, we assume each  $F_n$  is closed.

Since  $X \not\subseteq F_i$  for any  $i$  (Otherwise,  $\text{interior}(\bar{F}_i) = X \neq \emptyset$ ),  $\exists x_1 \in F_1^c$  and  $\exists r_1 > 0$ , s.t.  $\bar{B}_{r_1}(x_1) \subseteq F_1^c$ . Since  $\text{interior}(F_2) = \emptyset$ ,  $B_{r_1}(x_1) \not\subseteq F_2$ . Then  $\exists x_2$ , s.t.  $x_2 \in F_2^c \cap B_{r_1}(x_1)$ . Since  $F_2^c$  and  $B_{r_1}(x_1)$  are both open, their intersection is open.  $\exists r_2 > 0$ , s.t.  $\bar{B}_{r_2}(x_2) \subseteq F_2^c \cap B_{r_1}(x_1)$  and  $r_2 \leq r_1/2$ . By induction, we obtain a sequence  $\{x_n\}$  in  $X$  and  $\{r_n\}$  in  $(0, +\infty)$ , s.t. for all  $n$ ,

$$B_{r_n}(x_n) \subseteq F_n^c \cap B_{r_{n-1}}(x_{n-1}), \quad r_n \leq \frac{1}{2}r_{n-1}.$$

So  $r_n \rightarrow 0$ .

**Claim 1:**  $\{x_n\}$  is a Cauchy sequence.

For  $n, m \geq M$ ,  $x_n, x_m \in B_{r_M}(x_M)$ . So  $d(x_n, x_m) < r_M$  and  $r_M \rightarrow 0$ .

**Claim 2:**  $x_\infty := \lim_{x \rightarrow \infty} x_n \notin \cup_n F_n$ .

Since  $\{x_n\}_{n \geq M}$  is a sequence in  $B_{r_M}(x_M)$ ,  $x_\infty \in \overline{B_{r_M}(x_M)} \subseteq F_M^c$ . As  $M$  was arbitrary,

$$x_\infty \in \bigcap_M F_M^c = \left( \bigcup_M F_M \right)^c.$$

Then we have  $x_\infty \notin \cup_n F_n = X$ . However,  $X$  is a complete metric. Contradiction.  $\square$

**Proposition.**  $\mathbb{R}^n$  cannot be written as a countable union of hyperplanes.

*Proof.* Let  $P$  be any hyperplane. So  $P = \{x \in \mathbb{R}^n : \langle x, a \rangle = d\}$  for fixed  $a, d$ , where  $a \neq 0$ .

**Claim 1:** Hyperplane  $P$  is closed.

$P$  can also be defined by  $f^{-1}(d)$  where  $f(x) = \langle x, a \rangle$ . Since  $d$  is closed,  $f$  is continuous, the preimage  $f^{-1}$  of a closed set  $d$  is also closed.

**Claim 2:** Hyperplane  $P$  has empty interior.

Let  $x_0 \in P$ . For any  $r > 0$ , the point  $x_0 + (r/2|a|) \cdot a$  is inside  $B_r(x_0)$ . However,

$$\langle x_0 + \frac{ra}{2|a|}, a \rangle = d + \frac{r|a|}{2} \neq d.$$

So  $P$  has empty interior.

Then by definition, all hyperplanes are nowhere dense in  $\mathbb{R}^n$ . So  $\mathbb{R}^n$  cannot be written as a countable union of hyperplanes.  $\square$

**Proposition** (Well-approximable numbers). *Let*

$$\Lambda_n = \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < \frac{1}{q^n} \text{ for infinitely many } \frac{p}{q} \in \mathbb{Q} \right\}.$$

$\Lambda_n$  is generic. Thus by Theorem 5,  $\exists$  well-approximable irrationals, since  $\mathbb{Q}$  is meager (Every countable set is meager).

*Proof.* By definition

$$\Lambda_n^c = \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| \geq \frac{1}{q^n} \text{ for all but finitely many } \frac{p}{q} \in \mathbb{Q} \right\}.$$

If we can show that  $\Lambda_n^c$  is meager, then since  $\mathbb{R}$  is a complete metric space,  $\Lambda_n$  is generic. Now define

$$F_q := \left\{ x \in \mathbb{R} : \forall p \in \mathbb{Z}, \left| x - \frac{p}{q} \right| \geq \frac{1}{q^n} \right\}, \quad E_q := \bigcap_{q' \geq q} F_{q'}$$

Since  $F_{q'}$  is closed for all  $q'$ ,  $E_q$  is also closed for all  $q$ . Then

$$\Lambda_n^c = \bigcup_{q \in \mathbb{N}} E_q = \bigcup_{q \in \mathbb{N}} \bigcap_{q' \geq q} \left\{ x \in \mathbb{R} : \forall p \in \mathbb{Z}, \left| x - \frac{p}{q'} \right| \geq \frac{1}{(q')^n} \right\}.$$

Now we need to show that  $E_q$  is nowhere dense in  $\mathbb{R}$ . We know  $\bar{E}_q = E_q$ . But  $E_q \cap \{p/q' : p, q' \in \mathbb{Z}, q' > q\} = \emptyset$ . Since the latter set is a dense set in  $\mathbb{R}$ , then  $E_q^c$  contains a dense set which implies the interior of  $E_q$  is empty.  $\square$



Furthermore, we can express  $\Lambda_n$  in another way,

$$\begin{aligned}\Lambda_n &= (\Lambda_n^c)^c \\ &= \bigcap_{q \in \mathbb{N}} \bigcup_{q' \geq q} \left\{ x \in \mathbb{R} : \exists p \in \mathbb{Z}, \left| x - \frac{p}{q'} \right| < \frac{1}{(q')^n} \right\} \\ &= \bigcap_{q \in \mathbb{N}} \bigcup_{q' \geq q} \bigcup_{p \in \mathbb{Z}} \left( \frac{p}{q'} - \frac{1}{(q')^n}, \frac{p}{q'} + \frac{1}{(q')^n} \right)\end{aligned}$$

The inner part is the union of intervals of width  $2/(q')^n$  with spacing  $1/q'$ . So heuristically,

$$\begin{aligned}\Pr \left( x \in \bigcup_{p \in \mathbb{Z}} \left( \frac{p}{q'} - \frac{1}{(q')^n}, \frac{p}{q'} + \frac{1}{(q')^n} \right) \right) &\leq \frac{2/(q')^n}{1/q} = \frac{2}{(q')^{n-1}}, \\ \Pr \left( x \in \bigcup_{q' \geq q} \bigcup_{p \in \mathbb{Z}} \left( \frac{p}{q'} - \frac{1}{(q')^n}, \frac{p}{q'} + \frac{1}{(q')^n} \right) \right) &\leq \sum_{q' \geq q} \frac{2}{(q')^{n-1}} \\ &\leq \frac{2}{q^{n-\alpha}} \sum_{q' \geq q} \frac{(q')^{n-\alpha}}{(q')^{n-1}} \\ &\leq \frac{2}{q^{n-\alpha}} \sum_{q' \geq q} \frac{1}{(q')^{\alpha-1}} \\ &\leq \frac{C_n}{q^{n-\alpha}},\end{aligned}$$

as long as  $n > 2$  ( $\alpha$  is also greater than 2). So as  $n \rightarrow +\infty$ , the probability that  $x \in \Lambda_n$  goes to zero.

**Definition.** Let  $X$  be a nonempty set and  $(Y, d_y)$  be a metric space. Let  $\{f_n\}$  be a sequence of functions from  $X$  to  $Y$ , and let  $f$  be a function from  $X$  to  $Y$ .

- Say  $f_n \rightarrow f$  pointwise if  $\forall x \in X$  and  $\forall \epsilon > 0$ ,  $\exists N = N(\epsilon, x)$ , s.t.  $\forall n \geq N$ ,  $d_y(f_n(x), f(x)) < \epsilon$ .
- Say  $\{f_n\}$  is pointwise Cauchy if  $\forall x \in X$ ,  $\forall \epsilon > 0$ ,  $\exists N = N(\epsilon, x)$ , s.t.  $\forall n, m > N$ ,  $d_y(f_n(x), f_m(x)) < \epsilon$ .
- Say  $f_n \rightarrow f$  uniformly if  $\forall \epsilon > 0$ ,  $\exists N = N(\epsilon)$ , s.t.  $\forall n > N$ ,  $\forall x \in X$ ,  $d_y(f_n(x) - f(x)) < \epsilon$ .
- Say  $\{f_n\}$  is uniformly Cauchy if  $\forall \epsilon > 0$ ,  $\exists N = N(\epsilon)$ , s.t.  $\forall n, m > N$ ,  $\forall x \in X$ ,  $d_y(f_n(x), f_m(x)) < \epsilon$ . That is to say  $\lim_{n \rightarrow \infty} \sup_{x \in X} d(f_n(x), f(x)) = 0$ .

**Note.** Uniform convergence is much better than pointwise convergence.

**Definition.**  $f : X \rightarrow Y$  is a bounded function if  $f(X)$  is a bounded subset of  $Y$ . (i.e. if  $f(x)$  is contained in some metric ball  $B_{r_f}(y_f)$  in  $Y$ )

**Example.** The pointwise limit of a sequence of bounded functions need not be bounded. For example,

$$f_n(x) = \begin{cases} x & \text{for } |x| \leq N \\ N & \text{for } x \geq N \\ -N & \text{for } x \leq -N. \end{cases}$$

Each  $f_n$  is bounded in  $[-N, N]$ . However, its pointwise limit is  $f(x) = x$ , which is unbounded.

**Definition.** Let  $X, Y$  be two nonempty metric spaces. Let

$$\mathcal{B}(X, Y) := \{f : X \rightarrow Y : f(X) \text{ is a bounded set}\}.$$

**Proposition.** *The uniform limit of a sequence of bounded function is bounded.*

*Proof.* Let  $\{f_n\}$  be a sequence in  $\mathcal{B}(X, Y)$  and assume that  $f_n \rightarrow f$  uniformly. By uniform convergence,  $\exists n \in \mathbb{N}$ , s.t.  $\forall x \in X$ ,  $d_Y(f(x), f_N(x)) < 1$ . Since  $f_N$  is a bounded function,  $\exists y_0, k$ , s.t.  $f_N(x) \in B_k(y_0)$  for every  $x$  in  $X$ . So  $f(x) \in B_{k+1}(y_0)$  for every  $x$  in  $X$ . Thus  $f$  is bounded.  $\square$

**Proposition.**  $\mathcal{B}(X, Y)$  is a metric space with metric  $d_{\mathcal{B}}(f, g) = \sup_{x \in X} d(f(x), g(x))$ .

1. If  $f, g$  are bounded functions, then  $d_{\mathcal{B}}(f, g)$  is finite.
2.  $d_{\mathcal{B}}$  is a metric on  $\mathcal{B}(X, Y)$ .
3. Uniform convergent of a sequence in  $\mathcal{B}(X, Y)$  is equivalent to metric convergence with respect to  $d_{\mathcal{B}}$ .
4. If  $Y$  is complete, then so is  $\mathcal{B}(X, Y)$ .

**Definition.** Let  $X, Y$  be two nonempty metric spaces. Let

$$\mathcal{C}(X, Y) := \{\text{continuous functions from } X \text{ to } Y\}.$$

**Example.** The pointwise limit of a sequence of continuous functions need not to be continuous. Let  $f_n : [0, 1] \rightarrow \mathbb{R}$ ,  $f_n(x) = x^n$ . But

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0, & \text{if } x \in [0, 1) \\ 1, & \text{if } x = 1 \end{cases}$$

is not continuous.

**Proposition** (Honors HW). *If  $\{f_n\}$  is a sequence of functions on  $\mathbb{R}$  and  $f_n \rightarrow f$  pointwise, then the set of continuity points for  $f$  is generic.*

**Theorem 6.** *If  $\{f_n\}$  is a sequence in  $\mathcal{C}(X, Y)$  and  $f_n \rightarrow f$  uniformly. Then  $f \in \mathcal{C}(X, Y)$ .*

*Proof.* Assume  $f_n \rightarrow f$  uniformly. Let  $x_0 \in X$  and  $\epsilon > 0$ . By uniform convergence,  $\exists N \in \mathbb{N}$ , s.t.  $\forall x$ ,  $d_Y(f(x), f_N(x)) < \epsilon$ . Since  $f_N$  is continuous,  $\exists \delta > 0$ , s.t.  $\forall x$  with  $d(x, x_0) < \delta$ , we have  $d_Y(f_N(x), f_N(x_0)) < \epsilon$ . Finally, by triangle inequality,  $\forall x$  with  $d(x, x_0) < \delta$ ,

$$d_Y(f(x), f(x_0)) \leq d_Y(f(x), f_N(x)) + d_Y(f_N(x), f_N(x_0)) + d_Y(f_N(x_0), f(x_0)) < 3\epsilon.$$

$\square$

**Definition.** Let  $X, Y$  be two nonempty metric spaces. Define

$$\mathcal{C}^0(X, Y) := \mathcal{C}(X, Y) \cap \mathcal{B}(X, Y).$$

Then  $\mathcal{C}^0(X, Y)$  is a metric space with metric  $d_{\mathcal{C}^0}(f, g) := d_{\mathcal{B}}(f, g)$ .

**Definition.** For  $Y = \mathbb{R}$ ,  $X$  being any metric space, let  $\mathcal{C}^0(X) := \mathcal{C}^0(X, \mathbb{R})$  and define the norm (Question: why it is a norm?)

$$\|f\|_{\mathcal{C}^0(X)} = \sup_{x \in X} |f(x)|.$$

**Proposition.** Let  $X, Y$  be two metric spaces.  $(\mathcal{C}^0(X, Y), d_{\mathcal{C}^0})$  is a metric space which is complete if  $Y$  is complete.

*Proof.* Let  $\{f_n\}$  be a Cauchy sequence in  $\mathcal{C}^0(X, Y)$ , then  $\{f_n\}$  is uniformly Cauchy in  $\mathcal{C}^0(X, Y)$ . Thus  $\forall \epsilon > 0, \exists N \in \mathbb{N}$ , s.t.  $\forall n, m \geq N, d_{\mathcal{C}^0}(f_n, f_m) < \epsilon$ . Notice that  $d_{\mathcal{C}^0}(f_n, f_m) = \sup_{x \in X} d_Y(f_n(x), f_m(x))$ . So in particular, for any  $x \in X, d_Y(f_n(x), f_m(x)) < \epsilon$ . Now fix  $x \in X$ . Then  $\{f_n(x)\}$  is a Cauchy in  $Y$ . As  $Y$  is complete,  $\forall x \in X, \exists f(x) := \lim_{n \rightarrow \infty} f_n(x)$ .

**Claim:**  $f_n \rightarrow f$  uniformly.

Let  $\epsilon > 0$ . By uniform Cauchyness, we may choose  $N \in \mathbb{N}$ , s.t.  $\forall n, m \geq N, \forall x \in X, d_Y(f_n(x), f_m(x)) < \epsilon$ . Now fix  $n \geq N, x \in X$ . Choose  $m_x$  s.t.  $m_x \geq N$  and  $d_Y(f_{m_x}(x), f(x)) < \epsilon$ . Then

$$d_Y(f_n(x), f(x)) \leq d_Y(f_n(x), f_{m_x}(x)) + d_Y(f_{m_x}(x), f(x)) < 2\epsilon.$$

Note, it is okay that  $m_x$  depends on  $x$ , since it doesn't appear on either side of the inequality. Since  $d_Y(f_n(x), f(x)) < 2\epsilon$  for all  $x \in X$ ,

$$d_{\mathcal{C}^0}(f_n(x), f) \leq 2\epsilon < 3\epsilon.$$

Thus,  $f_n \rightarrow f$  in  $\mathcal{C}^0$ .

□

**Theorem 7.** There exists a nowhere differentiable (not differentiable at any point) continuous function  $f \in \mathcal{C}^0([0, 1])$ .

*Proof.* Since  $\mathbb{R}$  is complete,  $\mathcal{C}^0([0, 1])$  is a complete metric space and thus it is not meager. It suffices to prove that

$$F := \{f \in \mathcal{C}^0([0, 1]) : \exists x_0, \text{ s.t. } f'(x_0) \text{ exists}\}$$

is meager, i.e. a countable union with nowhere dense sets. It then suffices to prove that  $F$  is contained in a countable union of nowhere dense sets.

**Claim 1:**  $F \subseteq \bigcup_{n=1}^{\infty} F_n$ , where

$$F_n := \{f \in \mathcal{C}^0([0, 1]) : \exists x_0 \in [0, 1], \text{ s.t. } |f(x) - f(x_0)| \leq n|x - x_0|, \forall x \in [0, 1]\}.$$

If  $f \in F$ , then  $\exists x_0$ , s.t.

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. Thus, there exists  $\delta > 0$ , s.t.  $\forall x \in [0, 1]$  with  $|x - x_0| < \delta$ ,

$$|f(x) - f(x_0)| \leq (|f'(x_0)| + 1)|x - x_0|.$$

For  $x \in [0, 1]$  with  $|x - x_1| \geq \delta$ , we have

$$|f(x) - f(x_0)| \leq |f(x)| + |f(x_0)| \leq 2\|f\|_{\mathcal{C}^0} \cdot \frac{|x - x_0|}{\delta} \leq \frac{2\|f\|_{\mathcal{C}^0}}{\delta} \cdot |x - x_0|.$$

Finally, for any  $n \geq |f'(x_0)| + 1 + 2\|f\|_{\mathcal{C}^0}/\delta$ , we have  $f \in F_n$ .

**Claim 2:**  $F_N$  is closed.

**Claim 3:**  $F_N$  is nowhere dense.

Since  $F_N$  is closed, it suffices to prove that  $F_N$  has an empty interior, i.e. that  $\forall f \in F_N$ ,  $\forall \epsilon > 0$ ,  $\exists g \in \mathcal{C}^0([0, 1])$ , s.t.  $\|f - g\|_{\mathcal{C}^0([0, 1])} < \epsilon$  and  $g \notin F_N$ . Let  $f \in F_N$  and  $\epsilon > 0$ . The idea is to find  $g$ , piecewise linear, such that the slopes of linear parts of  $g$  has absolute value  $> N$ .

Since  $f$  is continuous and  $[0, 1]$  is compact,  $f$  is uniformly continuous. So there exists  $\delta > 0$ , s.t.  $|x - y| \leq \delta$ ,  $|f(x) - f(y)| \leq \epsilon$ . Choose  $n \in \mathbb{N}$ , s.t.  $1/n < \delta$  and  $2\epsilon/(1/n) > 1000N$ . Set  $x_j = j/n$ ,  $0 \leq j \leq n$ . Define

$$g(x_j) := f(x_j) + (-1)^j \epsilon,$$

and make  $g$  linear in  $x \in (x_j, x_{j+1})$ , for  $j = 0, \dots, n$ . Note, if  $x \in [x_j, x_{j+1}]$ , then  $x = (1 - \theta)x_j + \theta x_{j+1}$  for some  $0 \leq \theta \leq 1$ , and this implies  $g(x) = (1 - \theta)g(x_j) + \theta g(x_{j+1})$ .

**Subclaim 1:**  $\|g - f\|_{\mathcal{C}^0([0, 1])} < 3\epsilon$ .

Suffices to show that  $\forall j$  and  $\forall x \in [x_j, x_{j+1}]$ ,  $|g(x) - f(x)| < 3\epsilon$ . Write  $x = (1 - \theta)x_j + \theta x_{j+1}$ , with  $0 \leq \theta \leq 1$ . Then

$$\begin{aligned} |g(x) - f(x)| &= |(1 - \theta)(g(x_j) - f(x_j)) + (1 - \theta)(f(x_j) - f(x)) \\ &\quad + \theta(g(x_{j+1}) - f(x_j)) + \theta(f(x_j) - f(x))| \\ &\leq (1 - \theta)|g(x_j) - f(x_j)| + (1 - \theta)|f(x_j) - f(x)| \\ &\quad + \theta|g(x_{j+1}) - f(x_j)| + \theta|f(x_j) - f(x)| \\ &\leq (1 - \theta)\epsilon + (1 - \theta)\epsilon + \theta\epsilon + \theta\epsilon \\ &= 2\epsilon < 3\epsilon \end{aligned}$$

**Claim 2:** The slopes of  $g$  have absolute value greater than  $N$ :

Suffices to prove

$$\frac{|g(x_{j+1}) - g(x_j)|}{1/n} > N.$$

Indeed,

$$|g(x_{j+1}) - g(x_j)| \geq 2\epsilon - |f(x_{j+1}) - f(x_j)| \geq \epsilon,$$

because  $|x_{j+1} - x_j| \leq \delta$ . Since  $n\epsilon > 500N > N$ , done.

Finally, observe that  $g \notin F_N$  since any  $x_0$  belongs to some  $[x_j, x_{j+1}]$  and  $|g(x) - g(x_0)| > N|x - x_0|$  for  $x_0 \neq x \in [x_j, x_{j+1}]$ .

□

## Uniform Convergence and Integration

**Definition.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function.

- A partition of  $[a, b]$  is a finite set

$$P = \{a = x_0 < x_1 < \dots < x_N = b\}.$$

- Define intervals  $I_j := [x_{j-1}, x_j]$  for  $j = 1, \dots, N$ , with lengths  $\Delta x_j := x_j - x_{j-1}$ .
- Upper Riemann sums:

$$U(f, p) := \sum_{j=1}^N M_j(f, P) \Delta x_j,$$

where  $M_j(f, p) = \sup_{x \in I_j} f(x)$ .

- Lower Riemann sums:

$$L(f, p) := \sum_{j=1}^N m_j(f, P) \Delta x_j,$$

where  $m_j(f, p) = \inf_{x \in I_j} f(x)$ .

**Theorem 8.**  *$f$  is Riemann integrable if and only if  $\forall \epsilon > 0$ , there exists an partition  $P$ , s.t.  $U(f, P) - L(f, P) < \epsilon$ . In this case,*

$$\int_a^b f(x) dx = \inf_P U(f, P) = \sup_P L(f, P).$$

**Example.** Pointwise limit of Riemann integrable functions need not be Riemann integrable. Let  $f_n(x)$  be a function from  $[0, 1]$  to  $\mathbb{R}$  defined as following

$$f_n(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0, 1] \text{ with denominator of } x \text{ at most } n \\ 0, & \text{otherwise} \end{cases}$$

$f_n(x)$  is Riemann integrable since it is piecewise linear. Consider the limit

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0, & \text{if } x \in [0, 1] \setminus \mathbb{Q} \end{cases}$$

For any partition  $P$ ,  $U(f, P) = 1$  and  $L(f, P) = 0$ . We see the limit  $f(x)$  is not Riemann integrable.

**Theorem 9.** *Let  $\{f_n\}$  be a sequence of Riemann integrable functions on  $[a, b]$  and assume  $f_n \rightarrow f$  uniformly. Then  $f$  is Riemann integrable and*

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx.$$

**Note.** Under the assumption of pointwise convergence, the above formula can fail even if the limit  $f$  is Riemann integrable.

Under the assumption of uniform convergence, this is a total mess if  $[a, b]$  is replaced by  $[a, \infty)$  and  $\int_a^b$  is replaced by  $\int_a^\infty$ .

# Uniform Convergence and Differentiation

**Example.** Consider the sequence

$$f_n(x) = \sqrt{\frac{1}{n} + x^2}, \quad x \in \mathbb{R}.$$

**Claim:**  $f_n \rightarrow f$  uniformly, where  $f = |x|$ , on  $\mathbb{R}$ .

Let  $\epsilon > 0$ . Let  $N = \lceil 1/\epsilon \rceil$ . Then  $\forall n > N$  and  $\forall x$ , we have

$$|f_n(x) - |x|| < \frac{1}{n} < \epsilon.$$

Furthermore, each  $f_n$  is differentiable and

$$f'_n(x) = \frac{x}{\sqrt{\frac{1}{n} + x^2}}.$$

However,  $f'_n \rightarrow g$  pointwise, where

$$g(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0, \\ 0, & x = 0. \end{cases}$$

Also we observe that the limiting function  $f$  is not differentiable.

**Definition.** Let  $I$  be an interval with nonempty interior. Let

$$\mathcal{C}^k(I) := \{f : I \rightarrow \mathbb{R} : f \text{ is } k\text{'-times' differentiable and } f^{(j)} \in \mathcal{C}^0(I), 0 \leq j \leq k\}.$$

Define the norm

$$\|f\|_{\mathcal{C}^k(I)} := \sum_{j=0}^k \|f^{(j)}\|_{\mathcal{C}^0(I)}.$$

**Proposition (HW).**  $\exists A = A(I)$ , s.t.  $\|f\|_{\mathcal{C}^k(I)} \leq A(\|f\|_{\mathcal{C}^0(I)} + \|f^{(k)}\|_{\mathcal{C}^0(I)})$ . Further, this  $A$  can be independent of  $I$ .

**Theorem 10.**  $\mathcal{C}^k(I)$  is a complete metric space.

*Proof.* Prove by induction. First, we know that  $\mathcal{C}^0(I)$  is complete. We want to deduce completeness of  $\mathcal{C}^{k+1}(I)$  from completeness of  $\mathcal{C}^k(I)$ .

Now we assume  $\mathcal{C}^k(I)$  is complete and let  $\{f_n\}$  be a Cauchy sequence in  $\mathcal{C}^{k+1}(I)$ . Notice  $\forall n, m$ ,

$$\|f_n - f_m\|_{\mathcal{C}^k(I)} + \|f_n^{(k+1)} - f_m^{(k+1)}\|_{\mathcal{C}^0(I)} = \|f_n - f_m\|_{\mathcal{C}^{k+1}(I)},$$

so  $\{f_n\}$  is a Cauchy sequence in  $\mathcal{C}^k(I)$ . Then by hypothesis,  $\exists f \in \mathcal{C}^k(I)$ , s.t.  $\|f_n - f\|_{\mathcal{C}^k(I)} \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $\{f_n^{(k+1)}\}$  is Cauchy in  $\mathcal{C}^0(I)$ ,  $\exists g \in \mathcal{C}^0(I)$ , s.t.  $f_n^{(k+1)} \rightarrow g$  uniformly on  $I$ . If we can show that  $f \in \mathcal{C}^{k+1}(I)$  and  $f^{(k+1)} = g$ , then

$$\begin{aligned} \|f_n - f\|_{\mathcal{C}^{k+1}(I)} &= \|f_n - f\|_{\mathcal{C}^k(I)} + \|f_n^{(k+1)} - f^{(k+1)}\|_{\mathcal{C}^0(I)} \\ &= \|f_n - f\|_{\mathcal{C}^k(I)} + \|f_n^{(k+1)} - g\|_{\mathcal{C}^0(I)} \end{aligned}$$

goes to 0 as  $n \rightarrow \infty$ . For this, it suffices to prove that  $f^{(k)}$  is differentiable and  $(f^{(k)})' = g$ . Fix  $x_0 \in I$ . Then  $\forall x \in I$ , by the fundamental theorem of calculus, we have

$$f_n^{(k)}(x) = f_n^{(k)}(x_0) + \int_{x_0}^x (f_n^{(k)})'(y) dy.$$

Since  $f_n \rightarrow f$  in  $\mathcal{C}^k$ ,  $\lim_{n \rightarrow \infty} f_n^{(k)}(x) = f^{(k)}(x)$  for all  $x \in I$ . Since  $f_n^{(k+1)} = (f_n^{(k)})' \rightarrow g$  uniformly, we know

$$\lim_{n \rightarrow \infty} \int_{x_0}^x (f_n^{(k)})'(y) dy = \int_{x_0}^x \lim_{n \rightarrow \infty} (f_n^{(k)})'(y) dy = \int_{x_0}^x g(y) dy.$$

Finally, by linearity of limits,

$$f^{(k)}(x) = f^{(k)}(x_0) + \int_{x_0}^x g(y) dy.$$

Since  $g$  is continuous, the fundamental theorem of calculus says  $\int_{x_0}^x g(y) dy$  is differentiable with derivative  $g(x)$ . Then we can conclude that  $f^{(k)}$  is differentiable and  $f^{(k)} = g$ .  $\square$

**Proposition (HW).** *Let  $\{f_n\}$  be a sequence in  $\mathcal{C}^k(I)$ . Assume  $\{f_n^{(k)}\}$  is Cauchy in  $\mathcal{C}^0(I)$  and  $\exists x_0 \in I$ , s.t.  $\forall j = 0, \dots, k-1$ ,  $\{f_n^{(j)}(x_0)\}$  is a Cauchy sequence. Then  $\{f_n\}$  is convergent in  $\mathcal{C}^k(I)$ .*

**Theorem 11 (7.17).** *Suppose  $\{f_n\}$  is a sequence of functions, differentiable on  $[a, b]$  and such that  $\{f_n(x_0)\}$  converges for some point  $x_0$  on  $[a, b]$ . If  $\{f'_n\}$  converges uniformly on  $[a, b]$ , then  $\{f_n\}$  converges uniformly on  $[a, b]$ , to a function  $f$ , and*

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x) \quad (a \leq x \leq b).$$

**Example.** Let

$$f(x) = \begin{cases} x^2 \sin 1/x, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

It has derivative

$$f'(x) = \begin{cases} 2x \sin 1/x - \cos 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and  $f'(x)$  is discontinuous. So there exists some functions which has derivative at every point, but the derivative is not continuous.

**Recall.** Let  $(X, d)$  be a complete metric space. A subset  $K \subseteq X$  is compact if and only if  $K$  is closed and totally bounded.

**Corollary.**  $\mathcal{F} \subseteq \mathcal{C}^0(X)$  is compact if and only if  $\mathcal{F}$  is closed and totally bounded.

But what does totally bounded mean for  $\mathcal{C}^0(X)$ ? We want a simpler characterization of totally boundedness.

**Definition.** Let  $\mathcal{F} \subseteq \mathcal{C}(X)$ .

- $\mathcal{F}$  is pointwise bounded if  $\forall x \in X$ ,  $\exists M_x$ , s.t.  $\forall f \in \mathcal{F}$ ,  $|f(x)| \leq M_x$  (i.e.  $\forall x \in X$ ,  $\{f(x) : f \in \mathcal{F}\}$  is a bounded set).

- $\mathcal{F}$  is equicontinuous if  $\forall \epsilon > 0, \forall x \in X, \exists \delta = \delta(x, \epsilon) > 0$ , s.t.  $\forall f \in \mathcal{F}$  and  $\forall y \in \mathcal{B}_\delta(x)$ ,  $|f(x) - f(y)| < \epsilon$ .
- $\mathcal{F}$  is uniformly equicontinuous if  $\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0$ , s.t.  $\forall f \in \mathcal{F}$  and  $\forall x, y \in X$  with  $d(x, y) < \delta$ ,  $|f(x) - f(y)| < \epsilon$ .

**Example.**  $\mathcal{F} := \{f \in \mathcal{C}^0([0, 1]) : f \text{ is differentiable on } (0, 1) \text{ and } |f'(x)| \leq 1, \forall x \in (0, 1)\}$  is uniformly equicontinuous.

*Proof.*  $\forall f \in \mathcal{F}, x, y \in [0, 1], |f(x) - f(y)| \leq |x - y|$  by mean value theorem. So  $\forall \epsilon > 0$ , pick  $\delta = \epsilon$ . Then  $|f(x) - f(y)| < \epsilon$  for all  $f \in \mathcal{F}$  and for all  $x, y \in [0, 1]$  with  $|x - y| < \delta$ .  $\square$

**Example.**  $f_n(x) = x^n$  defined on  $[0, 1]$ . Then  $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$  is a pointwise bounded, but not a equicontinuous subset of  $\mathcal{C}^0([0, 1])$ .

*Proof.* For any  $x \in [0, 1], \{f(x) : f \in \mathcal{F}\} \subseteq [0, 1]$ . So  $\mathcal{F}$  is pointwise bounded. Fix  $x = 1$  and  $\epsilon = 0.5$ . For any  $\delta > 0$ , there exists  $N$  sufficiently large such that for all  $n > N, f_n(x - \delta/2) < 0.5$ . Thus,  $f_n(x)$  is not equicontinuous.  $\square$

**Proposition.** If  $K$  is compact, then  $\mathcal{C}^0(K) = \mathcal{C}(K)$ . ( $\mathcal{C}^0$  means bounded continuous functions, while  $\mathcal{C}$  means continuous functions)

**Example.** Show by example that a pointwise bounded subset of  $\mathcal{C}^0(K)$  need not be uniformly pointwise bounded (i.e. a bounded subset of  $\mathcal{C}^0(K)$ ).

$$f_n(x) = \begin{cases} n^2x, & 0 \leq x \leq 1/n \\ 2n - n^2x, & 1/n < x \leq 2/n \\ 0 & 2/n < x \leq 1 \end{cases}$$

**Proposition.** If  $K$  is compact, then  $\mathcal{F} \subseteq \mathcal{C}^0(K)$  is equicontinuous if and only if  $\mathcal{F}$  is uniformly equicontinuous.

*Proof.*  $\Leftarrow$ : is immediate.

$\Rightarrow$ : Assume  $K$  is compact and  $\mathcal{F} \subseteq \mathcal{C}^0(K)$  is equicontinuous. Let  $\epsilon > 0$ . Then  $\forall x \in K, \exists \delta_x > 0$ , s.t.  $\forall y \in B_{\delta_x}(x), \forall f \in \mathcal{F}, |f(x) - f(y)| < \epsilon/2$ . Since  $K \subseteq \cup_{x \in K} B_{\delta_x}(x)$ ,  $\exists x_1, \dots, x_N$ , s.t.  $K \subseteq \cup_{j=1}^N B_{\delta_{x_j}}(x_j)$ .

**Claim:**  $\exists \delta > 0$ , s.t.  $\forall y, z \in K$ , if  $d(y, z) < \delta$ , then  $y, z \in B_{\delta_{x_j}}(x_j)$ .

Suppose not. Then there exists sequences  $\{y_n\}$  and  $\{z_n\}$  such that  $d(y_n, z_n) \rightarrow 0$ , but  $y_n$  and  $z_n$  never belong to the same  $B_{\delta_{x_j}}(x_j)$ . Each  $z_n$  lives in some  $B_{\delta_{x_j}}(x_j)$  and since there are only finitely many such balls, there must be a ball that contains infinitely many  $z_n$ . Passing to a subsequence, we may assume  $\exists j_0$ , s.t.  $z_n \in B_{\delta_{x_{j_1}}}(x_{j_1})$  for all  $n$ . Similarly, passing to a further subsequence, we may assume  $y_n \in B_{\delta_{x_{j_2}}}(x_{j_2})$  for all  $n$ . Since  $\{z_n\}$  is in  $K$ , which is compact, passing to a further subsequence, we may assume  $z_n \rightarrow z$ . Since  $d(z_n, y_n) \rightarrow 0$ ,  $y_n \rightarrow z$ . Notice  $\exists j$ , s.t.  $z \in B_{\delta_{x_j}}(x_j)$  ( $z$  is not necessarily in  $B_{\delta_{x_{j_1}}}(x_{j_1})$ ). For  $n$  sufficiently large,  $y_n$  and  $z_n$  are both in  $B_{\delta_{x_j}}(x_j)$ . Contradiction.

Then we have

$$|f(y) - f(z)| \leq |f(y) - f(x_j)| + |f(x_j) - f(z)| < \epsilon$$

for any  $f \in \mathcal{F}$ .  $\square$



**Theorem 12** (Arzela-Ascoli Theorem). *If  $K$  is a compact metric space, then  $\mathcal{F} \subseteq \mathcal{C}^0(K)$  is totally bounded if and only if  $\mathcal{F}$  is pointwise bounded and equicontinuous.*

*Proof.*  $\Rightarrow$ : Assume  $\mathcal{F} \subseteq \mathcal{C}^0(K)$  is totally bounded.

**Pointwise bounded:** Since  $\mathcal{F}$  is totally bounded,  $\exists N$  and  $f_1, \dots, f_N \in \mathcal{F}$ , s.t.  $\mathcal{F} \subseteq \bigcup_{j=1}^N B_1(f_j)$ . Then  $\forall x$  and  $\forall f \in \mathcal{F}$ ,

$$|f(x)| < \sum_{j=1}^N |f_j(x)| + 1 \leq \sum_{j=1}^N \|f_j\|_{\mathcal{C}^0(K)} + 1.$$

**Equicontinuous:** Let  $\epsilon > 0$ . Since  $\mathcal{F}$  is totally bounded,  $\exists N$  and  $f_1, \dots, f_N \in \mathcal{F}$ , s.t.  $\mathcal{F} \subseteq \bigcup_{j=1}^N B_{\epsilon/3}(f_j)$ . Since each  $f_j$  is uniform continuous (being continuous on a compact set),  $\exists \delta_j > 0$ , s.t.  $\forall x, y \in K$  with  $d(x, y) < \delta_j$ , we have  $|f_j(x) - f_j(y)| < \epsilon/3$ . Let  $\delta := \min\{\delta_1, \dots, \delta_N\}$ . Let  $f \in \mathcal{F}$ . Then  $\exists j$ , s.t.  $\|f - f_j\|_{\mathcal{C}^0} < \epsilon/3$ . Finally, if  $d(x, y) < \delta$ ,

$$|f(x) - f(y)| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| < \epsilon.$$

Thus  $\mathcal{F}$  is uniform equicontinuous.

$\Leftarrow$ : Assume  $\mathcal{F}$  is pointwise bounded and equicontinuous. Let  $\epsilon > 0$ . By proposition,  $\mathcal{F}$  is uniformly equicontinuous, so  $\exists \delta > 0$ , s.t.  $\forall f \in \mathcal{F}, \forall x, y \in K$  with  $d(x, y) < \delta$ ,  $|f(x) - f(y)| < \epsilon/4$ . Since  $K$  is compact,  $K$  is totally bounded. So  $\exists N$  and  $x_1, \dots, x_N \in K$ , s.t.  $K \subseteq \bigcup_{j=1}^N B_\delta(x_j)$ . The idea is to discretize the functions in  $\mathcal{F}$ . Consider  $P := \{(f(x_1), f(x_2), \dots, f(x_N)) : f \in \mathcal{F}\} \subseteq \mathbb{R}^N$ . By pointwise boundedness, for each  $j$ ,  $\exists M_j$ , s.t.  $\{f(x_j) : f \in \mathcal{F}\} \subseteq [-M_j, M_j]$ . Therefore

$$P \subseteq [-M_1, M_1] \times [-M_2, M_2] \times \dots \times [-M_N, M_N],$$

which is a bounded (hence totally bounded) subset of  $\mathbb{R}^N$ . So  $P$  is totally bounded. Then  $\exists L$  and  $y_1, \dots, y_L \in P$ , s.t.  $P \subseteq \bigcup_{j=1}^L B_{\epsilon/4}(y_j)$ . (Note:  $B_{\epsilon/4}(y_j)$  is the ball in Euclidean space.) For each  $y_j$ , by definition,  $\exists f_j \in \mathcal{F}$ , s.t.  $(y_j)_i = f_j(x_i)$ ,  $i = 1, \dots, N$ . Thus,  $\forall f \in \mathcal{F}$ ,  $\exists j$ , s.t.  $1 \leq j \leq L$  and

$$|f(x_i) - f_j(x_i)| \leq |(f(x_1), \dots, f(x_N)) - (f_j(x_1), \dots, f_j(x_N))| < \epsilon/4, \quad i = 1, \dots, N$$

For any  $x \in K$ , by total boundedness of  $K$ ,  $\exists i \in 1, \dots, N$ , s.t.  $x \in B_\delta(x_i)$ . So

$$|f(x) - f_j(x)| \leq |f(x) - f(x_i)| + |f(x_i) - f_j(x_i)| + |f_j(x_i) - f_j(x)| < 3\epsilon/4.$$

Taking the supremum over  $x$ ,  $\|f_j - f\|_{\mathcal{C}^0(K)} \leq 3\epsilon/4 < \epsilon$ . Thus,  $f \in B_\epsilon(f_j)$  and we have  $\mathcal{F} \subseteq \bigcup_{j=1}^L B_\epsilon(f_j)$ . □

**Corollary.** *Let  $K$  be a compact metric space.*

1.  $\mathcal{F} \subseteq \mathcal{C}^0(K)$  is compact if and only if  $\mathcal{F}$  is closed, pointwise bounded and equicontinuous.
2.  $\mathcal{F} \subseteq \mathcal{C}^0(K)$  is precompact if and only if  $\mathcal{F}$  is pointwise bounded and equicontinuous.
3. Let  $\{f_n\}$  be a pointwise bounded equicontinuous sequence in  $\mathcal{C}^0(K)$ . Then  $\{f_n\}$  has a uniformly convergent subsequence.

**Proposition.** For  $k \in \mathbb{N}$  and  $I$  compact, a subset of  $\mathcal{C}^k(I)$  is totally bounded if and only if each of its  $i$ th derivatives,  $i = 0, 1, \dots, k$ , is pointwise bounded and equicontinuous.

**Theorem 13** ((Baby) Stone-Weierstrass Theorem). Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. There exists a sequence  $\{P_n\}$  of polynomials, s.t.  $P_n \rightarrow f$  uniformly on  $[a, b]$ .

Note that even if  $f$  has derivatives of all orders, it probably doesn't work if we just take  $P_n$  equal to the  $n$ -th Taylor polynomial. Also, unless  $f$  itself is a polynomial, the degree of  $P_n$  must tend to infinity.

*Proof.* First, we do some reductions:

1. We assume  $[a, b] = [0, 1]$ . If this is not true, we can approximate  $g(x) := f(\frac{x-a}{b-a})$  and replace  $P_n(x)$  by  $P_n((b-a)x + a)$ .
2. We assume  $f(0) = f(1) = 0$ . If this is not the case, we approximate  $g(x) = f(x) - f(0) - x(f(1) - f(0))$  and replace  $P_n(x)$  by  $P_n(x) + f(0) + x(f(1) - f(0))$ .

Now, we extend  $f$  to all of  $\mathbb{R}$  by setting  $f(x) = 0$  off of  $[0, 1]$ . (Warning: We are only approximating  $f$  on  $[0, 1]$ , even though the domain of  $f$  is larger.) Then  $f$  is continuous and thus uniformly continuous on  $\mathbb{R}$ . Define

$$Q_n(x) := c_n(1 - x^2)^n,$$

where  $c_n$  is chosen s.t.

$$\int_{-1}^1 Q_n(x) dx = 1,$$

i.e.  $c_n = \left(\int_{-1}^1 (1 - x^2)^n dx\right)^{-1}$ . Let

$$P_n(x) := \int_0^1 f(t)Q_n(x-t) dt = f * Q_n(x) = \int_{x-1}^x f(x-t)Q_n(t) dt.$$

We need to prove that  $P_n$  is a polynomial and  $P_n \rightarrow f$  uniformly on  $[0, 1]$ .

**Claim 1:**  $P_n$  is a polynomial of degree  $\leq 2n$ .

*Subproof.* Expand the expression for  $Q_n(x-t)$ ,

$$\begin{aligned} Q_n(x-t) &= c_n(1 - (x-t)^2)^n = \sum_{j=0}^n c_n \binom{n}{j} (-1)^j (x-t)^{2j} \\ &= \sum_{j=0}^n \sum_{k=0}^{2j} c_n \binom{n}{j} \binom{2j}{k} (-1)^{j+k} x^{2j-k} t^k \\ &= \sum_{m=0}^{2n} \left( \sum_{m \leq 2j \leq 2n} c_n \binom{n}{j} \binom{2j}{2j-m} (-1)^{j-m} x^m t^{2j-m} \right) \end{aligned}$$

So  $P_n(x) = \sum_{m=0}^{2n} a_{n,m} x^m$ , where

$$a_{n,m} = \sum_{m \leq 2j \leq 2n} c_n \binom{n}{j} \binom{2j}{2j-m} (-1)^{j-m} \int_0^1 f(t) t^{2j-m} dt.$$

□

*Subproof (An alternative approach).* For  $k \in \mathbb{N}$ ,  $P_n$  is  $k$  times differentiable and

$$P_n^{(k)}(x) = \int_0^1 f(t) Q_n^{(k)}(x-t) dt.$$

In particular, since  $Q_n$  is a polynomial of degree  $2n$ ,

$$p_n^{(2n+1)}(x) = \int_0^1 f(t) \cdot 0 dt = 0.$$

So  $P_n$  is a polynomial of degree at most  $2n$ . By induction, it suffices to prove the following proposition, which sometimes is of independent interest.  $\square$

**Proposition.** *Let  $g \in \mathcal{C}^1(\mathbb{R})$  and let  $f \in \mathcal{C}^0(\mathbb{R})$  with  $f = 0$  out of  $[0, 1]$ . Then the convolution*

$$h(x) := \int_0^1 f(t)g(x-t) dt$$

*is also in  $\mathcal{C}^1(\mathbb{R})$ , where its derivative is*

$$h'(x) = \int_0^1 f(t)g'(x-t) dt.$$

*Note: Since  $f(x) = 0$  off of  $[0, 1]$ , we can also replace the proper integral  $\int_0^1$  by the improper integral  $\int_{-\infty}^{\infty}$ .*

*Proof of Proposition.* First, observe that  $h$  is bounded in  $\mathcal{C}^1(\mathbb{R})$ . This is because

$$|h(x)| = \left| \int_0^1 f(t)g(x-t) dx \right| \leq \int_0^1 |f(t)g(x-t)| dt \leq \|f\|_{\mathcal{C}^0} \|g\|_{\mathcal{C}^1}.$$

By a similar argument, we observe that

$$|h'(x)| \leq \|f\|_{\mathcal{C}^0} \|g\|_{\mathcal{C}^1}.$$

In addition, we claim that  $h'(x)$  is continuous. Let  $x_n \rightarrow x$ . Define

$$\varphi_n(t) := f(t)g'(x_n - t) \quad \text{and} \quad \varphi(t) := f(t)g'(x - t).$$

Without loss of generality, assume  $x_n \in [x-1, x+1]$  for all  $n$ . Observe that every  $\varphi_n$  and  $\varphi$  lives on  $[0, 1]$ . For any  $t \in [0, 1]$ ,  $x_n - t \in [x-2, x+1]$ . Since  $g'$  is continuous, it is uniform continuous on  $[x-2, x+1]$ . Hence  $\forall \epsilon > 0$ ,  $\exists N$ , s.t.  $\forall n \geq N$ ,  $\forall t \in [0, 1]$ ,  $|g'(x_n - t) - g'(x - t)| < \epsilon$  (because  $|x_n - x| < \text{some } \delta$ ). Thus,  $\varphi_n \rightarrow \varphi$  uniformly on  $[0, 1]$ . Then we have

$$\lim_{x_n \rightarrow x} h'(x_n) = \lim_{x_n \rightarrow x} \int_0^1 \varphi_n(t) dt = \int_0^1 \varphi(t) dt = h'(x).$$

$\square$

**Claim 2:**  $P_n \rightarrow f$  uniformly on  $[0, 1]$ .

**Subclaim 1:**  $c_n \leq \sqrt{n}$ ,  $\forall n$ .

Let  $g(y) = (1 - y)^n$ . Then  $g''(y) = n(n - 1)(1 - y)^{n-2} \geq 0$  for  $y \in [0, 1]$ . By Taylor's theorem, for  $y \in [0, 1]$ ,

$$\begin{aligned} g(y) &= g(0) + g'(0)y + \frac{1}{2}g''(ty)y^2, \quad \text{for some } t \in [0, y] \subseteq [0, 1] \\ &\geq g(0) + g'(0)y \\ &= 1 - ny. \end{aligned}$$

Apply the above result by setting  $y$  to  $x^2$ , then we have

$$\begin{aligned} \int_{-1}^1 (1 - x^2)^n dx &\geq \int_{-1/\sqrt{n}}^{1/\sqrt{n}} (1 - x^2)^n dx \\ &\geq \int_{-1/\sqrt{n}}^{1/\sqrt{n}} 1 - nx^2 dx = \frac{4}{3\sqrt{n}}. \end{aligned}$$

Hence

$$c_n \leq \frac{3\sqrt{n}}{4} \leq \sqrt{n}.$$

**Subclaim 2:**  $\forall \delta > 0$ ,  $Q_n \rightarrow 0$  uniformly on  $\{x : \delta \leq |x| \leq 1\}$ .

On  $\delta \leq |x| \leq 1$ ,

$$Q_n(x) = c_n(1 - x^2)^n \leq \sqrt{n}(1 - \delta^2)^n \rightarrow 0,$$

since  $1 - \delta^2 < 1$ .

Now let's prove the claim. Let  $x \in [0, 1]$ .

$$\begin{aligned} P_n(x) - f(x) &= \int_0^1 f(t)Q_n(x - t) dt - f(x) \int_{-1}^1 Q_n(s) ds \\ &= \int_{x-1}^x f(x - s)Q_n(s) ds - \int_{-1}^1 f(x)Q_n(s) ds \\ &= \int_{-1}^1 f(x - s)Q_n(s) ds - \int_{-1}^1 f(x)Q_n(s) ds. \end{aligned}$$

since  $f = 0$  off of  $[0, 1]$  (so  $f(x - s)$  vanishes for  $s \notin [x - 1, x]$ ). Thus

$$P_n(x) - f(x) = \int_{-1}^1 (f(x - s) - f(x))Q_n(s) dx = \int_{-\delta}^{\delta} + \int_{-1}^{-\delta} + \int_{\delta}^1 =: I_1 + I_2 + I_3$$

For  $I_1$ , we have

$$\begin{aligned} I_1 &\leq \int_{-\delta}^{\delta} |f(x - s) - f(x)| \cdot |Q_n(s)| ds \\ &\leq \max_{|s| \leq \delta} |f(x - s) - f(x)| \int_{-1}^1 Q_n(s) ds. \\ &= \max_{|s| \leq \delta} |f(x - s) - f(x)| \end{aligned}$$

For  $I_2$  and  $I_3$ , we have

$$I_2 + I_3 \leq 2 \cdot \max_{s \in \mathbb{R}} |f(x-s) - f(x)| \cdot \max_{\delta \leq |s| \leq 1} |Q_n(s)| \leq 4 \cdot \|f\|_{C^0} \cdot \max_{\delta \leq |s| \leq 1} |Q_n(s)|$$

Now, let  $\epsilon > 0$ . Since  $f$  is uniformly continuous,  $\exists \delta > 0$ , s.t. whenever  $|s| \leq \delta$ ,  $|f(x-s) - f(x)| < \epsilon/2$ ,  $\forall x \in \mathbb{R}$ . So  $I_1 < \epsilon/2$ ,  $\forall n$ . Also, by the subclaim,  $\exists N$ , s.t.  $\forall n \geq N$ ,  $\max_{\delta \leq |s| \leq 1} |Q_n(s)| < \epsilon/(4\|f\|_{C^0})$ . So  $\forall n \geq N$  and  $x \in [0, 1]$ ,

$$|P_n(x) - f(x)| \leq |I_1| + |I_2 + I_3| < \epsilon.$$

□

**Definition.** Let  $E$  be a nonempty set.

$$\mathcal{F}(E) := \mathcal{F}(E; \mathbb{R}) := \{f : E \rightarrow \mathbb{R}\}.$$

**Definition.** A family  $\mathcal{A} \subseteq \mathcal{F}(E)$  is an algebra if  $\forall c \in \mathbb{R}$ , and  $f, g \in \mathcal{A}$ ,

$$cf, f + g, fg \in \mathcal{A}.$$

**Definition.** Let  $\mathcal{A} \subseteq \mathcal{F}(E)$ .

- $\mathcal{A}$  separates points in  $E$  if  $\forall x_1, x_2 \in E$  with  $x_1 \neq x_2$ ,  $\exists f \in \mathcal{A}$ , s.t.  $f(x_1) \neq f(x_2)$ .
- $\mathcal{A}$  is nonvanishing on  $E$  if  $\forall x \in E$ ,  $\exists f \in \mathcal{A}$ , s.t.  $f(x) \neq 0$ .

**Example.** The set  $P$  of polynomials on  $\mathbb{R}$  is an algebra, which separates points in  $\mathbb{R}$  and vanishes nowhere.

**Example.** The set  $P_{\text{odd}}$  is not an algebra and is not nonvanishing (because it is always 0 at  $x = 0$ ). The set  $P_{\text{even}}$  is an algebra, which is nonvanishing on  $\mathbb{R}$  but it doesn't separate points.

**Proposition.** Let  $\mathcal{A} \subseteq \mathcal{F}(E)$  be an algebra that separates points and vanishes nowhere. Then  $\forall x_1 \neq x_2 \in E$  and  $\forall c_1, c_2 \in \mathbb{R}$ ,  $\exists f \in \mathcal{A}$ , s.t.  $f(x_1) = c_1$  and  $f(x_2) = c_2$ .

*Proof.* By definition,  $\exists g, h, k \in \mathcal{A}$ , s.t.  $g(x_1) \neq g(x_2)$ ,  $h(x_1) \neq 0$ ,  $k(x_2) \neq 0$ . Now let

$$\begin{aligned} u &= (g - g(x_1))k = gk - g(x_1)k \in \mathcal{A}, \\ v &= (g - g(x_2))h = gh - g(x_2)h \in \mathcal{A}. \end{aligned}$$

Then  $u(x_1) = 0, u(x_2) \neq 0, v(x_1) \neq 0, v(x_2) = 0$ . Finally, let

$$f = c_1 \frac{v}{v(x_1)} + c_2 \frac{u}{u(x_2)} \in \mathcal{A}.$$

□

**Theorem 14** (Full Stone Weierstrass). Let  $K$  be a compact set and  $\mathcal{A} \subseteq C^0(K)$  be an algebra that separates points and vanishes nowhere. Then  $\mathcal{A}$  is dense in  $C^0(K)$ . In other words,  $\forall f \in C^0(K)$ , there exists a sequence  $\{f_n\}$  in  $\mathcal{A}$ , s.t.  $f_n \rightarrow f$  uniformly.

*Proof.* Let's  $\mathcal{C} = \bar{\mathcal{A}}$ . Claim  $\mathcal{C}$  is an algebra (HW). We now need to show that  $\mathcal{C} = C^0(K)$ .

**Claim 1:** If  $f \in \mathcal{C}$ , then so is  $|f|$ .

Let  $a := \|f\|_{\mathcal{C}^0(K)}$ . By baby Stone Weierstress (aka Weierstress preparation lemma), there exists a polynomial  $P$  on  $\mathbb{R}$  such that  $|P(y) - |y|| < \epsilon$  for any  $y \in [-a, a]$ . Define  $g := P \circ f$ . Then  $g(x) = \sum_{n=0}^N a_n f(x)^n$ , where the  $a_n$ 's are the coefficients of  $P$ . Since  $\mathcal{C}$  is an algebra,  $g \in \mathcal{C}$ . Furthermore,  $\forall x \in K$ ,  $f(x) \in [-a, a]$ . So

$$|g(x) - |f(x)|| = |P(y) - |y|| < \epsilon.$$

Thus  $\| |f| - g \|_{\mathcal{C}^0(K)} \leq \epsilon$ . Since  $\epsilon$  was arbitrary,  $|f| \in \bar{\mathcal{C}} = \mathcal{C}$ .

**Claim 2:** If  $f_1, \dots, f_n \in \mathcal{C}$ , then so are  $\max\{f_1, \dots, f_n\}$  and  $\min\{f_1, \dots, f_n\}$ .

If  $N = 2$ , this follows from Claim 1 and

$$\max\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2}, \quad \min\{f, g\} = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

For larger  $N$ , by induction, we have

$$\begin{aligned} \max\{f_1, \dots, f_{N+1}\} &= \max\{\max\{f_1, \dots, f_N\}, f_{N+1}\}, \\ \min\{f_1, \dots, f_{N+1}\} &= \min\{\min\{f_1, \dots, f_N\}, f_{N+1}\}. \end{aligned}$$

**Claim 3:** Let  $f \in \mathcal{C}^0(K)$ ,  $\epsilon > 0$  and  $x_0 \in K$ . Then  $\exists g_{x_0} \in \mathcal{C}$ , s.t.  $g_{x_0}(x_0) = f(x_0)$  and  $g_{x_0}(x) > f(x) - \epsilon$  for all  $x \in K$  (approximate  $f$  from not too far below).

Let  $x_1 \in K$ . Then  $\exists h_{x_1} \in \mathcal{C}$ , s.t.  $h_{x_1}(x_0) = f(x_0)$  and  $h_{x_1}(x_1) = f(x_1)$ . For  $y \in K$ , define

$$G_y := \{x \in K : h_y(x) > f(x) - \epsilon\}.$$

Then  $G_y$  is open since  $h_y$  is continuous. Also  $y \in G_y$  since  $h_y(y) = f(y)$ . Thus,  $\{G_y : y \in K\}$  is an open cover of  $K$ , so  $\exists y_1, \dots, y_N \in K$ , s.t.  $K \subseteq \cup_{n=1}^N G_{y_n}$ . Now, let  $g_{x_0} = \max\{h_{y_1}, \dots, h_{y_N}\}$ . By Claim 2,  $g_{x_0} \in \mathcal{C}$ . Furthermore,  $g_{x_0}(x_0) = f(x_0)$ . Finally, for  $x \in G_{y_n}$ ,  $g_{x_0}(x) \geq h_{y_n}(x) > f(x) - \epsilon$ . Thus,  $g_{x_0}$  is the function we want.

**Claim 4:**  $\forall f \in \mathcal{C}^0(K)$  and  $\forall \epsilon > 0$ ,  $\exists g \in \mathcal{C}$ , s.t.  $\forall x$ ,  $f(x) - \epsilon < g(x) < f(x) + \epsilon$ .

Let  $x_0 \in K$ . By Claim 3,  $\exists g_{x_0} \in \mathcal{C}$ , s.t.  $g_{x_0} = f(x_0)$  and  $g_{x_0}(x) > f(x) - \epsilon$ ,  $\forall x \in K$ . For  $y \in K$ , define

$$H_y := \{x \in K : g_y(x) < f(x) + \epsilon\}.$$

Then  $H_y$  is open since  $g_y$  is continuous. Again,  $y \in H_y$  since  $g_y(y) = f(y)$ . So  $\{H_y : y \in K\}$  is an open cover of  $K$ . Then  $\exists y_1, \dots, y_N \in K$ , s.t.  $K \subseteq \cup_{n=1}^N H_{y_n}$ . Finally, define  $g = \min\{g_{y_1}, \dots, g_{y_N}\}$ . Then  $\forall x$ ,  $g(x) > f(x) - \epsilon$ . Also  $\exists n$ , s.t.  $x \in H_{y_n}$ . Then  $g(x) \leq g_{y_n}(x) < f(x) + \epsilon$ .

□

**Theorem 15** (Picard's theorem). Let  $t_0 \in \mathbb{R}$  and  $y_0 \in \mathbb{R}^k$ . Let  $a, b \in \mathbb{R}$  and define

$$B := \{y \in \mathbb{R}^k : |y - y_0| \leq b\},$$

and

$$R := [t_0 - a, t_0 + a] \times B.$$

Let  $F : R \rightarrow \mathbb{R}^k$  be a bounded, continuous function and let  $M := \|F\|_{\mathcal{C}^0(R)}$ . Assume that  $\exists C \in \mathbb{R}$ , s.t.  $\forall t \in (t_0 - a, t_0 + a)$ ,  $\forall u, y \in B$ ,  $|F(t, u) - F(t, y)| \leq C|u - y|$ . Then,  $\exists!$  function  $g : (t_0 - \tilde{a}, t_0 + \tilde{a}) \rightarrow B$ , s.t.  $g$  is differentiable and  $g$  solves the initial value problem

$$\begin{cases} g(t_0) &= y_0 \\ g'(t) &= F(t, g(t)), \quad \forall t \in (t_0 - \tilde{a}, t_0 + \tilde{a}). \end{cases}$$

Here,  $\tilde{a} = \min\{a, b/M\}$ .

Warning: The  $C$  in the assumption is crucial for this theorem. Consider  $k = 1$ ,  $F(t, y) = y^{1/3}$  and the initial value problem

$$\begin{cases} g(0) &= 0 \\ g'(t) &= (g(t))^{1/3} \end{cases}$$

Here is one solution,  $g(t) = 0$  for all  $t$ . Here is another solution,

$$g(t) = \begin{cases} ct^{3/2}, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

where  $c^2 = 8/27$ . Actually, there are infinitely many distinct solutions.

*Proof.* Observe that  $g$  solves our initial value problem if and only if

- $g$  is continuous;
- $|g(t) - y_0| \leq b$ ,  $\forall t \in I$ .
- $g(t) = y_0 + \int_{t_0}^t F(s, g(s)) \, ds$ ,  $\forall t \in I := [t_0 - \tilde{a}, t_0 + \tilde{a}]$ .

Here we are using the fact that differentiable functions are continuous and equal the integral of their derivative plus some constant. On the other hand, that condition of  $g$  implies continuity of the integrand and then we can use the fundamental theorem of calculus.

Consider the set

$$\mathcal{M} := \{g \in \mathcal{C}^0(I; \mathbb{R}^k) : g(t) \in B, \forall t \in I; g(t_0) = y_0\}.$$

Consider the map  $\Phi : \mathcal{M} \rightarrow \mathcal{C}^0(I, \mathbb{R}^k)$ ,

$$[\Phi(g)](t) := y_0 + \int_{t_0}^t F(s, g(s)) \, ds.$$

By the fundamental theorem of calculus,  $\Phi(g)$  is differentiable and hence it is continuous.

Now, we want to show that there exists a unique fixed point  $g$  of  $\Phi$  in  $\mathcal{M}$ . The idea is to apply contraction mapping theorem. Observe that  $\mathcal{M}$  is closed since it is the intersection of two closed sets (careful here). Also since  $\mathcal{C}^0(I, \mathbb{R})$  is complete,  $\mathcal{M}$  is complete. Since the function  $g(t) = y_0$ ,  $\forall t \in I$  is in  $\mathcal{M}$ ,  $\mathcal{M} \neq \emptyset$ . Thus, by the contraction mapping theorem, any contraction on  $\mathcal{M}$  has a unique fixed point  $p$ . Now we want to show  $\Phi$  is a contraction on  $\mathcal{M}$ .

Compute

$$\begin{aligned} |\Phi(g)(t) - y_0| &= \left| \int_{t_0}^t F(s, g(s)) \, ds \right| \leq \left| \int_{t_0}^t |F(s, g(s))| \, ds \right| \\ &\leq \left| \int_{t_0}^t M \, ds \right| = |t - t_0| M \leq \tilde{a} M \leq b. \end{aligned}$$

Let  $g_1, g_2 \in \mathcal{M}$ .

$$\begin{aligned} |\Phi(g_1)(t) - \Phi(g_2)(t)| &= \left| \int_{t_0}^t F(s, g_1(s)) - F(s, g_2(s)) \, ds \right| \\ &\leq \left| \int_{t_0}^t |F(s, g_1(s)) - F(s, g_2(s))| \, ds \right| \\ &\leq \left| \int_{t_0}^t C |g_1(s) - g_2(s)| \, ds \right| \\ &\leq |t - t_0| \cdot C \cdot \|g_1 - g_2\|_{C^0(I; \mathbb{R}^k)} \\ &\leq \tilde{a} C \|g_1 - g_2\|_{C^0(I; \mathbb{R}^k)} \end{aligned}$$

So  $\Phi$  is a contraction if and only if  $\tilde{a}C < 1$ . We have two approaches to fix this.

**Approach 1.** By contraction mapping theorem and above computation, we can uniquely solve the initial value problem for a shorter time, say on  $I_0 := [t_0 - a_0, t_0 + a_0]$ , where  $a_0 := \min\{a, b/M, 1/(2C)\}$ . Now define  $t_{-1} := t_0 - a_0$  and  $t_1 := t_0 + a_0$ . Look at the new initial value problem on  $[t_{-1} - a_0, t_1 + a_0]$ . Define  $y_{\pm 1} := g_0(t_{\pm 1})$ . The new initial value problem can be written as

$$\begin{cases} g_{\pm 1}(t_{\pm 1}) = y_{\pm 1} \\ g'_{\pm 1}(t) = F(t, g(t)) \end{cases}$$

We find a unique solution on

$$I_{\pm 1} := [t_{\pm 1} - b_1, t_{\pm 1} + b_1],$$

where  $b_1 = \min\{a - a_0, \frac{b - a_0 M}{M}, 1/(2C)\}$ . Furthermore, by uniqueness (in contraction mapping theorem),  $g_{\pm 1}$  equals  $g$  on  $I_{\pm 1} \cap I_0$ . Thus there exists a unique solution on

$$I_1 \cup I_0 \cup I_{-1} = [t_0 - a_0 - b_1, t_0 + a_0 + b_1].$$

Let  $a_1 := a_0 + b_1$ . Then  $a_1 \geq \{a, b/M, 2/(2C)\}$ . Iterate this process will finish the proof.

**Approach 2.** Find a better (but equivalent) metric on  $\mathcal{M}$ .

**Definition.** Two metrics  $d_1$  and  $d_2$  are equivalent if there exists positive constants  $c_1, c_2$ , s.t.  $\forall f, g$ ,

$$c_1 d_2(f, g) \leq d_1(f, g) \leq c_2 d_2(f, g).$$

**Proposition (HW).** *Two equivalent metrics yield the same open and closed set, the same continuous functions, the same Cauchy and convergent sequences.*



Now define

$$d_C(f, g) := \sup_{t \in I} e^{-2C|t-t_0|} |g(t) - f(t)|.$$

Then

$$e^{-2C\tilde{a}} \|f - g\|_{C^0(I, \mathbb{R}^k)} \leq d_C(f, g) \leq \|f - g\|_{C^0(I, \mathbb{R}^k)}.$$

So the metrics are equivalent and  $\mathcal{M}$  is complete with respect to  $d_C$ . Finally, for  $t \in I$ ,

$$\begin{aligned} e^{-2C|t-t_0|} |\Phi(g_1)(t) - \Phi(g_2)(t)| &= e^{-2C|t-t_0|} \left| \int_{t_0}^t F(s, g_1(s)) - F(s, g_2(s)) \, ds \right| \\ &\leq e^{-2C|t-t_0|} \left| \int_{t_0}^t C e^{2C|s-t_0|} e^{-2C|s-t_0|} |g_1(s) - g_2(s)| \, ds \right| \\ &\leq e^{-2C|t-t_0|} \left| \int_{t_0}^t C e^{2C|s-t_0|} d_C(g_1, g_2) \, ds \right| \\ &\leq e^{-2C|t-t_0|} \cdot C \cdot d_C(g_1, g_2) \cdot \frac{1}{2C} e^{2C|t-t_0|} \\ &= \frac{1}{2} d_C(g_1, g_2) \end{aligned}$$

So  $\Phi$  is a contraction on  $\mathcal{M}$  with respect to  $d_C$  and hence has unique fixed point. □

## Analytic Functions

**Definition.**  $\mathbb{C}$  is the complex field defined by

$$\mathbb{C} := \{x + iy : x, y \in \mathbb{R}\}.$$

Define  $\overline{x + iy} := x - iy$ . Then  $(x + iy)(x - iy) = x^2 + y^2$ . Define  $|x + iy| := \sqrt{x^2 + y^2}$  and  $d(z, w) = |z - w|$ . Note that  $\mathbb{C}$  is a complete metric space. Recall that for a field, if  $x_1 + iy_1, x_2 + iy_2 \in \mathbb{C}$ , then

- $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \in \mathbb{C}$ .
- $(x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1 y_1 - x_2 y_2) + i(x_1 y_2 + x_2 y_1) \in \mathbb{C}$ .
- $-(x + iy) = -x - iy \in \mathbb{C}$ .
- If  $x_2 + iy_2 \neq 0$  (i.e.  $x_2 \neq 0$  or  $y_2 \neq 0$ ), then

$$\frac{(x_1 + iy_1)}{(x_2 + iy_2)} = \frac{(x_1 + iy_1)}{(x_2 + iy_2)} \cdot \frac{(x_2 - iy_2)}{(x_2 - iy_2)} = \frac{(x_1 x_2 + y_1 y_2)}{x_2^2 + y_2^2} + i \frac{(-x_1 y_2 + x_2 y_1)}{x_2^2 + y_2^2}.$$

**Definition.** If  $\{c_n\}$  is a sequence of complex numbers,  $\sum_{n=0}^{\infty} c_n$  converges if the sequence  $\{s_n\}$  of partial sums  $s_n := \sum_{n=0}^N c_n$  converges. Furthermore, by completeness,  $\{s_n\}$  converges if and only if  $\{s_n\}$  is Cauchy, in which case, we say  $\{s_n\}$  satisfies the Cauchy criterion, that  $\forall \epsilon, \exists N$ , s.t.  $\forall n > m \geq N$ ,

$$\left| \sum_{j=m+1}^n c_j \right| < \epsilon.$$

**Definition.**  $\sum_{n=0}^{\infty} c_n$  converges absolutely if  $\sum_{n=0}^{\infty} |c_n|$  converges.

**Proposition.** If  $\sum_{n=0}^{\infty} c_n$  converges absolutely, then  $\sum_{n=0}^{\infty} c_n$  converges.

*Proof.* Use the Cauchy criterion and triangle inequality.  $\square$

**Definition.** Let  $X$  be a nonempty set and  $\{f_n\}$  be a sequence of functions mapping from  $X$  into  $\mathbb{C}$ . Say that  $\sum_{n=1}^{\infty} f_n$  converges (uniformly) if the sequence  $s_n := \sum_{j=0}^n f_j$  converges (uniformly).

**Proposition.**  $\sum_{n=0}^{\infty} f_n$  converges uniformly if and only if  $\sum_{n=0}^{\infty} f_n$  satisfies a uniform Cauchy criterion, which means  $\forall \epsilon > 0, \exists N, \text{ s.t. } \forall n \geq m \geq N \text{ and } \forall x \in X,$

$$\left| \sum_{j=m}^n f_j(x) \right| < \epsilon.$$

**Note.** Usually, it is much easier to show the convergence a sequence by showing that it is Cauchy.

**Theorem 16** (Weierstress M test). Let  $\{M_n\}$  be a sequence in  $[0, \infty)$ . Let  $f_n$  be a sequence of functions mapping from  $X \neq \emptyset$  into  $\mathbb{C}$ . Assume that  $\forall n \in \mathbb{N}$  and  $x \in X, |f_n(x)| \leq M_n$ . Then, if  $\sum_{n=0}^{\infty} M_n$  converges, then  $\sum_{n=0}^{\infty} f_n$  converges uniformly on  $X$ .

*Proof.* Show  $\sum_{n=0}^{\infty} f_n$  satisfies the uniform Cauchy criterion:

$$\left| \sum_{j=m}^n f_j(x) \right| \leq \sum_{j=m}^n |f_j(x)| \leq \sum_{j=m}^n M_j.$$

Since  $\sum_{n=0}^{\infty} M_n$  is Cauchy, we are done.  $\square$

**Theorem 17** (Root test). Let  $\{c_n\}$  be a complex sequence in  $\mathbb{C}$ . Let  $L := \limsup |c_n|^{1/n}$ . If  $L < 1$ , then  $\sum c_n$  converges absolutely. If  $L > 1$ , then  $\sum c_n$  diverges (badly).

**Recall.** Let  $\{c_n\}$  be a sequence in  $\mathbb{R}$ . Then

$$\limsup s_n = \sup\{\text{subsequence limits of } \{s_n\}\} = \lim_{n \rightarrow \infty} \sup\{s_k : k \geq n\}.$$

*Proof.* If  $L < 1$ , then  $\frac{L+1}{2} > L$ . So  $\exists N, \text{ s.t. } \forall n \geq N, |c_n|^{1/n} < \frac{L+1}{2}$ . Thus  $|c_n| < \left(\frac{L+1}{2}\right)^n$  for  $n \geq N$  and  $\sum \left(\frac{L+1}{2}\right)^n$  converges since  $\frac{L+1}{2} < 1$ . By comparison,  $\sum |c_n|$  converges.

If  $L > 1$ , define  $L' := \min\{\frac{L+1}{2}, 2\}$ . Since  $L' < L$ ,  $|c_n|^{1/n} > L'$  infinitely often. Thus  $|c_n| > (L')^n$  infinitely often and since  $L' > 1$ ,  $c_n$  cannot converge to zero. So  $\sum c_n$  diverges.  $\square$

**Definition.** Let  $\{c_n\}$  be a complex sequence. The radius of convergence of the power series  $\sum_{n=0}^{\infty} c_n(z - z_0)^n$  is

$$R := \liminf |c_n|^{-1/n} = \left( \limsup |c_n|^{1/n} \right)^{-1}.$$

**Theorem 18.** Let  $R$  denote the radius of convergence of the complex power series  $\sum_{n=0}^{\infty} c_n(z - z_0)^n$ . Then  $\forall R' < R$ ,  $\sum_{n=0}^{\infty} c_n(z - z_0)^n$  converges absolutely uniformly on  $\{z \in \mathbb{C} : |z - z_0| \leq R'\}$  and  $\sum_{n=0}^{\infty} c_n(z - z_0)^n$  diverges on  $\{z \in \mathbb{C} : |z - z_0| > R\}$ .

*Proof.* If  $|z - z_0| \leq R' < R$ ,

$$\limsup |c_n(z - z_0)^n|^{1/n} = \limsup |c_n|^{1/n} \cdot |z - z_0| \leq \frac{1}{R} \cdot R'.$$

So eventually,  $|c_n| \cdot |z - z_0|^n < \alpha^n$  for some  $\alpha \in (R'/R, 1)$ . Outside  $\{|z - z_0| \leq R\}$ , root test shows divergence.  $\square$

**Lemma.** *The series*

$$f^{(k)}(y) = \sum_{n=k}^{\infty} c_n n(n-1) \cdots (n-k+1)(x-a)^{n-k},$$

*has the same radius of convergence as  $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ .*

*Proof.* We want to show

$$\lim_{n \rightarrow \infty} (n(n-1) \cdots (n-k+1))^{1/n} = 1.$$

Since  $\lim_{n \rightarrow \infty} (n(n-1) \cdots (n-k+1))^{1/n} \geq 1$ , it suffices to show that the lim sup is  $\leq 1$ . But

$$\lim_{n \rightarrow \infty} \sup (n(n-1) \cdots (n-k+1))^{1/n} \leq \lim_{n \rightarrow \infty} n^{k/n} = \left( \lim_{n \rightarrow \infty} n^{1/n} \right)^k = 1.$$

$\square$

**Theorem 19.** *Let  $R$  be the radius of convergence of the power series  $\sum_{n=0}^{\infty} c_n(x-a)^n$ . Assume  $R > 0$  and define  $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$  on  $I := (a-R, a+R)$ . Then  $f$  is indefinitely differentiable on  $I$  and we can differentiate it term by term:*

$$f^{(k)}(y) = \sum_{n=k}^{\infty} c_n n(n-1) \cdots (n-k+1)(x-a)^{n-k}.$$

*Proof.* Let  $g_N(x) = \sum_{n=0}^N c_n(x-a)^n$ . Let  $0 < R' < R$ . By the lemma and previous theorem,  $\{g_N\}$  is a Cauchy sequence in  $\mathcal{C}^k((a-R', a+R'))$ . Thus,  $\{g_N\}$  converges in  $\mathcal{C}^k((a-R', a+R'))$ . By uniqueness of limits, the limit must be  $f$ . Thus  $f$  is differentiable to order  $k$  on  $(a-R', a+R')$  and

$$f^{(k)}(x) = \lim_{N \rightarrow \infty} g_N^{(k)}(x) = \lim_{N \rightarrow \infty} \sum_{n=0}^N c_n(n \cdots (n-k+1))(x-a)^{n-k}.$$

Since  $k$  is arbitrary and  $R'$  is arbitrary, we are done.  $\square$

**Corollary.** *Under the hypotheses of the theorem,  $f^{(k)}(a) = k!c_k$ , i.e.  $c_k = \frac{1}{k!}f^{(k)}(a)$ .*

**Theorem 20.** *Let  $I$  denote the set of all points  $x$  at which  $\sum_{n=0}^{\infty} c_n(x-a)^n$  converges. Then*

$$f(x) := \sum_{n=0}^{\infty} c_n(x-a)^n$$

*is continuous on  $I$ .*

*Proof.* First, we do some reductions:

- By replacing  $f(x)$  with  $f(x + a)$ , we can assume  $a = 0$ .
- Also, without loss of generality, we just need to prove for  $0 < R < \infty$ . This is because  $R = 0$ , then there is only one point at which the infinity sum converges. If  $R = \infty$ , then  $I = \mathbb{R}$ . So we can replace  $f(x)$  with  $f(Rx)$  and assume  $R = 1$ .
- It suffices to prove continuity at each interval and replacing  $f(x)$  with  $f(-x)$ , it suffices to prove the following theorem.

□

**Theorem 21.** *If  $\sum c_n$  converges, then  $\sum c_n x^n$  converges  $\forall |x| < 1$  and  $\lim_{x \rightarrow 1^-} \sum c_n x^n = \sum c_n$ .*

*Proof.* Define  $s_{-1} := 0$ ,  $s_n := \sum_{j=0}^n c_j$ ,  $s := \sum_{j=0}^{\infty} c_j$ . Define

$$f(x) := \sum_{n=0}^{\infty} c_n x^n, \quad x \in (-1, 1].$$

Then we can write

$$\sum_{n=0}^m c_n x^n = \sum_{n=0}^m (s_n - s_{n-1}) x^n = \sum_{j=0}^{m-1} s_j (x^j - x^{j+1}) + s_m x^m.$$

If  $|x| < 1$ ,  $s_m x^m \rightarrow 0$  as  $x \rightarrow \infty$  since  $s_m$  is bounded. Thus, for  $|x| < 1$ ,

$$f(x) = \sum_{j=0}^{\infty} s_j (x^j - x^{j+1}) = (1 - x) \sum_{j=0}^{\infty} s_j x^j.$$

Since  $\sum_{j=0}^{\infty} x^j = 1/(1 - x)$ , we can write

$$\sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n = f(x) - f(1) = (1 - x) \sum_{j=0}^{\infty} s_j x^j - s(1 - x) \sum_{j=0}^{\infty} x^j = (1 - x) \sum_{j=0}^{\infty} (s_j - s) x^j.$$

Let  $\epsilon > 0$ . Choose  $N$ , s.t.  $\forall j \geq N$ ,  $|s_j - s| < \epsilon$ . Thus,

$$\left| (1 - x) \sum_{j=N}^{\infty} (s_j - s) x^j \right| < (1 - x) \sum_{j=N}^{\infty} \epsilon |x|^j < \epsilon \cdot \frac{1 - x}{1 - |x|} < \epsilon,$$

if  $0 \leq x \leq 1$ . Since  $\sum c_n$  converges,  $\sum c_n x^n$  also converges. Furthermore, let  $\delta := \epsilon / \sum_{j=0}^{N-1} |s - s_j|$ . Then if  $1 - \delta < x < 1$ ,

$$\left| (1 - x) \sum_{j=0}^{N-1} (s - s_j) x^j \right| \leq (1 - x) \sum_{j=0}^{N-1} |s - s_j| < \delta \cdot \sum_{j=0}^{N-1} |s - s_j| = \epsilon.$$

Combining with the previous result, we obtain

$$|f(x) - f(1)| = \left| (1 - x) \sum_{j=0}^{\infty} (s_j - s) x^j \right| < 2\epsilon.$$

□

**Definition.** The Cauchy product of the two series  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$  is the series  $\sum_{n=0}^{\infty} c_n$ , where  $c_n := a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$ .

**Theorem 22.** Assume  $\sum a_n$  converges absolutely and  $\sum b_n$  converges. Then their Cauchy product  $\sum c_n$  converges. Furthermore, it converges to  $\sum a_n \cdot \sum b_n$ . If  $\sum b_n$  converges absolutely as well, then  $\sum c_n$  converges absolutely.

*Proof.* Some notations:  $A_N := \sum_{n=0}^N a_n$ ,  $B_N := \sum_{n=0}^N b_n$ ,  $C_N := \sum_{n=0}^N c_n$ ,  $A := \sum a_n$ ,  $B := \sum b_n$ ,  $\bar{A} = \sum |a_n|$ ,  $M := \sup B_N$ .

$$\begin{aligned} C_N &= c_0 + \cdots + c_N \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_N + \cdots + a_N b_0) \\ &= a_0 B_N + a_1 B_{N-1} + \cdots + a_N B_0 \\ &= A_N B + a_0 (B_N - B) + \cdots + a_N (B_0 - B) \end{aligned}$$

Define  $\Gamma_N := a_0 (B_N - B) + \cdots + a_N (B_0 - B)$ . If we can show that  $\lim_{N \rightarrow \infty} \Gamma_N = 0$ , it follows that  $\lim_{n \rightarrow \infty} C_n = AB$ .

Let  $\epsilon > 0$ . By absolute convergence of  $\sum a_n$ ,  $\exists N$ , s.t.  $\forall n \geq m \geq N_1$ ,  $\sum_{j=m}^n |a_j| < \epsilon/2M$ . Since  $B_n \rightarrow B$ ,  $\exists N_2$ , s.t.  $\forall n > N_2$ ,  $|B_n - B| < \epsilon/\bar{A}$ . Finally, if  $N \geq N_1 + N_2$ ,

$$\begin{aligned} |\Gamma_N| &\leq \sum_{j=0}^N |a_j| \cdot |B_{N-j} - B| \\ &= \sum_{j=0}^{N_1} |a_j| \cdot |B_{N-j} - B| + \sum_{j=N_1+1}^N |a_j| \cdot |B_{N-j} - B| \\ &< \sum_{j=0}^{N_1} |a_j| \cdot \frac{\epsilon}{\bar{A}} + \sum_{j=N_1+1}^N |a_j| \cdot 2M \\ &< \bar{A} \cdot \frac{\epsilon}{\bar{A}} + \frac{\epsilon}{2M} \cdot 2M \\ &= 2\epsilon \end{aligned}$$

Now we want to show that  $\sum c_n$  converges absolutely if  $\sum b_n$  converges absolutely. Note,

$$\begin{aligned} \sum_{n=0}^N |c_n| &= |c_0| + \cdots + |c_N| \\ &= |a_0 b_0| + |a_0 b_1 + a_1 b_0| + \cdots + |a_0 b_N + \cdots + a_N b_0| \\ &\leq |a_0| \cdot |b_0| + (|a_0| \cdot |b_1| + |a_1| \cdot |b_0|) + \cdots + (|a_0| \cdot |b_N| + \cdots + |a_N| \cdot |b_0|) \end{aligned}$$

Since  $\sum |a_n|$  and  $\sum |b_n|$  are both convergent, we are done. □

**Example.** Consider the Cauchy product of the two absolutely convergent power series

$$\begin{aligned} \sum_{n=0}^{\infty} a_n \quad \text{with} \quad a_n &= \frac{1}{n!} x^n, \\ \sum_{n=0}^{\infty} b_n \quad \text{with} \quad b_n &= \frac{1}{n!} y^n. \end{aligned}$$

By definition, we have

$$\begin{aligned}
c_n &= a_0 b_n + \cdots + a_n b_0 \\
&= \sum_{j=0}^n a_j b_{n-j} \\
&= \sum_{j=0}^n \frac{1}{j!} x^j \frac{1}{(n-j)!} y^{n-j} \\
&= \frac{1}{n!} \sum_{j=0}^n \frac{n!}{j!(n-j)!} x^j y^{n-j} \\
&= \frac{1}{n!} (x+y)^n
\end{aligned}$$

Upshot:

$$\left( \sum_{n=0}^{\infty} \frac{1}{n!} x^n \right) \left( \sum_{n=0}^{\infty} \frac{1}{n!} y^n \right) = \sum_{n=0}^{\infty} \frac{1}{n!} (x+y)^n,$$

i.e.  $e^x e^y = e^{x+y}$ .

## Unordered Series

**Definition.** Let  $S$  be a set and  $\{a_s\}_{s \in S}$  be a function from  $S$  into  $\mathbb{R}$ . We say that the unordered series  $\sum_{s \in S} a_s$  converges to  $b \in \mathbb{R}$  if  $\forall \epsilon > 0$ ,  $\exists$  a finite set  $S_\epsilon \subseteq S$ , s.t.  $\forall$  finite set  $S'$  with  $S_\epsilon \subseteq S' \subseteq S$ ,

$$\left| \sum_{s \in S'} a_s - b \right| < \epsilon.$$

**Proposition.** *An unordered series can have at most one sum.*

**Theorem 23.** *The following are equivalent:*

1. *The unordered series  $\sum_{s \in S} a_s$  converges.*
2.  *$\forall \epsilon > 0$ ,  $\exists$  a finite set  $S_\epsilon \subseteq S$ , s.t.  $\forall$  finite set  $S' \subseteq S \setminus S_\epsilon$ ,  $\sum_{s \in S'} |a_s| < \epsilon$ .*
3.  *$\sum_{s \in S} |a_s|$  converges absolutely.*
4.  *$\sup\{\sum_{s \in S'} |a_s| : S' \subseteq S \text{ is finite}\} < \infty$ .*

*Proof.* 1  $\Rightarrow$  2: Assume  $\sum_{s \in S} a_s$  converges to  $b$  and let  $\epsilon > 0$ . Then  $\exists$  a finite set  $S_\epsilon \subseteq S$ , s.t.  $\forall$  finite set  $S'$  with  $S_\epsilon \subseteq S' \subseteq S$ ,  $|\sum_{s \in S'} a_s - b| < \epsilon$ . Let  $S'' \subseteq S \setminus S_\epsilon$  be a finite set. Now let

$S''_+ := \{s \in S'' : a_s > 0\}$  and  $S''_- := \{s \in S'' : a_s < 0\}$ . Then

$$\begin{aligned}
\left| \sum_{s \in S''} |a_s| \right| &= \left| \sum_{s \in S''_+} a_s - \sum_{s \in S''_-} a_s \right| \\
&= \left| \sum_{s \in S''_+ \cup S_\epsilon} a_s - \sum_{s \in S_\epsilon} a_s - \sum_{s \in S''_- \cup S_\epsilon} a_s + \sum_{s \in S_\epsilon} a_s \right| \\
&\leq \left| \sum_{s \in S''_+ \cup S_\epsilon} a_s - b \right| + 2 \left| \sum_{s \in S_\epsilon} a_s - b \right| + \left| \sum_{s \in S_\epsilon \cup S''_-} a_s - b \right| \\
&< 4\epsilon.
\end{aligned}$$

2  $\Rightarrow$  4:  $\sup\{\sum_{s \in S'} |a_s| : S' \subseteq S \text{ is finite}\} \leq \epsilon + \sum_{s \in S_\epsilon} |a_s| < \infty$ .

4  $\Rightarrow$  3: Let  $B := \sup\{\sum_{s \in S'} |a_s| : S' \subseteq S \text{ is finite}\}$ . We want to show that  $\sum_{s \in S} |a_s| = B$ . Let  $\epsilon > 0$ . By the definition of sup, there exists a finite subset  $S_\epsilon \subseteq S$  such that  $\sum_{s \in S_\epsilon} |a_s| > B - \epsilon$ . So if  $S'$  is a finite set with  $S_\epsilon \subseteq S' \subseteq S$ ,

$$B - \epsilon < \sum_{s \in S_\epsilon} |a_s| \leq \sum_{s \in S'} |a_s| \leq B.$$

Thus, by definition, the series  $\sum_{s \in S} |a_s|$  converges.

3  $\Rightarrow$  2: The argument is similar as the argument for showing 1  $\Rightarrow$  2.

3  $\Rightarrow$  1: Suppose for contradiction that the unordered series  $\sum_{s \in S} a_s$  does not converge. This means for all  $b \in \mathbb{R}$ ,  $\exists \epsilon > 0$ , s.t.  $\forall$  finite  $S_\epsilon \subseteq S$ ,  $\exists$  a finite set  $S'$  with  $S_\epsilon \subseteq S' \subseteq S$ , s.t.

$$\left| \sum_{s \in S'} a_s - b \right| > \epsilon.$$

This implies

$$\sum_{s \in S'} a_s - b > \epsilon, \quad \text{or} \quad \sum_{s \in S'} a_s - b < -\epsilon,$$

which further implies

$$b - \epsilon > \sum_{s \in S'} a_s > b + \epsilon.$$

Now, let's choose  $b$  to be the limit of  $\sum_{s \in S} |a_s|$ . We have

□

**Lemma.** If  $\sum_{s \in S} |b_s|$  converges and  $|a_s| \leq |b_s|$  for all  $s$ . Then  $\sum_{s \in S} |a_s|$  converges.

*Proof.* Let  $\epsilon > 0$ . If  $\sum_{s \in S} |b_s|$  converges, by the theorem, there exists a finite set  $S_\epsilon \subseteq S$ , s.t.  $\forall$  finite set  $S' \subseteq S \setminus S_\epsilon$ ,  $\sum_{s \in S'} |b_s| < \epsilon$ . Then  $\forall$  finite set  $S' \subseteq S \setminus S_\epsilon$ ,  $\sum_{s \in S'} |a_s| \leq \sum_{s \in S'} |b_s| < \epsilon$ . Then  $\sum_{s \in S} |a_s|$  converges. □

**Corollary.** The unordered series  $\sum_{n \in \mathbb{N}} a_n$  converges if and only if  $\sum_{n=1}^{\infty} a_n$  converges absolutely.

**Proposition (HW).** Show directly (without the theorem), that if  $\lambda \in \mathbb{R}$  and  $\sum_{s \in S} a_s, \sum_{s \in S} b_s$  converges, then

$$\sum_{s \in S} \lambda a_s = \lambda \sum_{s \in S} a_s$$

and

$$\sum_{s \in S} a_s + b_s = \sum_{s \in S} a_s + \sum_{s \in S} b_s$$

**Proposition.** The unordered series  $\sum_{(i,j) \in \mathbb{N}^2} a_{i,j}$  converges if and only if  $\sum_{j=1}^{\infty} (\sum_{i=1}^{\infty} |a_{i,j}|)$  converges. In this case,

$$\sum_{(i,j) \in \mathbb{N}^2} a_{i,j} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j}.$$

*Proof.* Idea:  $\Leftrightarrow$  follows directly from  $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |a_{i,j}| = \lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} \sum_{j=1}^N \sum_{i=1}^M |a_{i,j}|$ . For the identity, consider the positive and negative parts separately.  $\square$

**Definition.** Let  $G \subseteq \mathbb{R}$  be an open set and let  $f : G \rightarrow \mathbb{R}$ . Say that  $f$  is analytic on  $G$  if  $\forall a \in G, \exists \epsilon > 0$ , s.t.  $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$ , on  $(a-\epsilon, a+\epsilon)$  (i.e.  $f$  has a power series representation on  $(a-\epsilon, a+\epsilon)$ ).

**Theorem 24.**  $f$  is analytic on the open set  $G \subseteq \mathbb{R}$  if and only if  $G$  can be written as a union of open intervals at which  $f$  has a power series representation.

**Note.** Every open subset of  $\mathbb{R}$  is a countable union of disjoint open intervals.

**Theorem 25.** Suppose that  $f(x) = \sum_{n=0}^{\infty} c_n x^n$  on  $\{|x| < R\}$ , some  $R > 0$ . If  $|a| < R$ , then  $f$  has a power series representation centered at  $a$  converging on  $|x-a| < R-|a|$ .

*Proof.* Notice that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a+a)^n = \sum_{n=0}^{\infty} \sum_{j=0}^n c_n \binom{n}{j} a^{n-j} (x-a)^j.$$

Observe that

$$\sum_{n=0}^{\infty} \sum_{j=0}^n \left| c_n \binom{n}{j} a^{n-j} (x-a)^j \right| = \sum_{n=0}^{\infty} |c_n| (|x-a| + |a|)^n.$$

Converges on  $|x-a| + |a| < R$ . So by proposition, we can switch the order of summation,

$$f(x) = \sum_{j=0}^{\infty} \left( \sum_{n=j}^{\infty} c_n \binom{n}{j} a^{n-j} \right) (x-a)^j, \quad |x-a| < R-|a|,$$

and  $\sum_{n=j}^{\infty} c_n \binom{n}{j} a^{n-j}$  is the coefficient  $c_j(a)$ . In particular, the sum representing  $c_j(a)$  converges absolutely  $\forall |a| < R$  and

$$c_j(a) = \frac{1}{j!} f^{(j)}(a).$$

$\square$



**Theorem 26.** *If  $f$  and  $g$  are analytic on the open interval  $I$ , then so are  $f + g$  and  $f \cdot g$ . In particular, if  $f$  and  $g$  both have power series representations centered at  $a$  on  $(a - R, a + R)$ , then so do  $f + g$  and  $f \cdot g$ .*

**Theorem 27.** *If  $f$  is analytic on the open interval  $I$ ,  $g$  is analytic on the open interval  $J$ , and  $g(J) \subseteq I$ . Then  $f \circ g$  is analytic on  $J$ .*

*Proof.* Let  $a \in J$ . By translating and adding a constant to  $g$ , we can assume  $a = g(a) = 0$ . Now we expand

$$f(y) = \sum_{n=0}^{\infty} b_n y^n, \quad g(x) = \sum_{k=0}^{\infty} c_k x^k, \quad \text{on } |y| < \epsilon, |x| < \delta.$$

Define  $\bar{g}(x) := \sum_{k=0}^{\infty} |c_k| x^k$ . The  $\bar{g}(x)$  is continuous on  $|x| < \delta$ . So by shrinking  $\delta$  if needed, we may assume that  $|\bar{g}(x)| < \epsilon$ ,  $|x| < \delta$ .

$$f \circ g(x) = \sum_{n=0}^{\infty} b_n (g(x))^n.$$

By previous theorem and induction, for each  $n$ ,

$$(g(x))^n = \sum_{k=0}^{\infty} a_k^{(n)} x^k, \quad (\bar{g}(x))^n = \sum_{k=0}^{\infty} \bar{a}_k^{(n)} x^k, \quad \text{on } |x| < \delta.$$

Furthermore,  $|a_k^{(n)}| \leq \bar{a}_k^{(n)}$ . Therefore,

$$f \circ g(x) = \sum_{n=0}^{\infty} b_n \sum_{k=0}^{\infty} a_k^{(n)} x^k.$$

We want to switch the order of summation, but it requires absolute convergence. Now

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |b_n a_k^{(n)} x^k| \leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |b_n| \cdot \bar{a}_k^{(n)} \cdot |x|^k = \sum_{n=0}^{\infty} |b_n| (\bar{g}(|x|))^n,$$

which converges since  $\bar{g}(|x|) < \epsilon$ . So

$$f \circ g(x) = \sum_{k=0}^{\infty} \left( \sum_{n=0}^{\infty} b_n a_k^{(n)} \right) x^k.$$

□

**Corollary.** *If  $f$  and  $g$  are analytic on the open interval  $I$  and  $g \neq 0$  on  $I$ . Then  $f/g$  is analytic on  $I$ .*

*Proof.* Since  $1/x$  is analytic on  $\mathbb{R} \setminus \{0\}$ , by the previous theorem,  $1/g$  is analytic. And so is  $f \cdot 1/g = f/g$ . □

**Theorem 28.** *Let  $E \subseteq \mathbb{R}$ . Then  $[x, y] \subseteq E$  for all  $x, y \in E$  if and only if  $E = \emptyset$ ,  $E$  is a singleton (i.e.  $E$  is a single point), or  $E$  is an interval.*

*Proof.*  $\Leftarrow$ : This direction is trivial.

$\Rightarrow$ : Assume  $\forall x, y \in E, [x, y] \subseteq E$ . We may assume  $E$  has at least 2 points. Thus,

$$\alpha := \inf E < \sup E =: \beta.$$

Claim that  $(\alpha, \beta) \subseteq E$ . If so, we're done, since  $E \subseteq [\alpha, \beta]$ .

Let  $x \in (\alpha, \beta)$ . Then  $\exists y \in E$ , s.t.  $y < x$  and  $\exists z \in E$ , s.t.  $z > x$ . By hypothesis,  $x \in [y, z] \subseteq E$ . □

**Theorem 29.**  $E \subseteq \mathbb{R}$  is connected if and only if  $E$  is empty,  $E$  is singleton, or  $E$  is an interval.

Notice, by the last theorem, the R.H.S. is equivalent to  $[x, y] \subseteq E$  for all  $x, y \in E$ .

*Proof.*  $\Rightarrow$ : It suffices to prove the contrapositive: If  $E \neq \emptyset, E \neq \{x\}$  and  $E$  is not an interval, then  $\exists x, y \in E$  and  $z$  with  $x < z < y$  s.t.  $z \notin E$ . Then  $E \cap (-\infty, z)$  and  $E \cap (z, \infty)$  form a separation of  $E$ . So  $E$  is not connected.

$\Leftarrow$ : Assume that  $E = \emptyset$  or  $E \neq \{x\}$  or  $E$  is an interval and  $E$  is not connected. Since a set that isn't connected must contain at least 2 points,  $E$  is an interval. Now we fix a separation  $E = A \cup B$ . By definition,  $A, B \neq \emptyset$  and  $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ . Fix  $x \in A$  and  $y \in B$ . Since  $A \cap B = \emptyset$ ,  $x \neq y$ . So we may assume  $x < y$  (otherwise, just rename  $A$  and  $B$ ). Define  $z := \sup A \cap [x, y]$ . Then,  $z \geq x$  because  $x \in A \cap [x, y]$  and  $z \leq y$  because  $y$  is an upper bound of  $A$ . So  $z \in [x, y] \subseteq E = A \cup B$ . Furthermore  $z \in \bar{A} \cap [x, y] \subseteq \bar{A}$ . Since  $\bar{A} \cap B = \emptyset$ ,  $z \notin B$ . Then  $z \in A$ . Since  $A \cap \bar{B} = \emptyset$ ,  $z \notin \bar{B}$ . So  $\exists r > 0$ , s.t.  $(z - r, z + r) \cap B = \emptyset$ . But  $y \geq z$  and  $y \in B$ , so  $z + r \leq y$ . So  $z + r/2 \in [z, y] \cap B^c \subseteq [x, y] \cap A$ . But  $z$  was an upper bound for  $[x, y] \cap A$  and  $z + r/2 > z$ . Contradiction! □

**Theorem 30.** Let  $I$  be an open interval and assume that  $f$  and  $g$  are analytic on  $I$ . Let  $E := \{x \in I : f(x) = g(x)\}$ . If  $E$  has an accumulation point in  $I$ , then  $E = I$ , i.e.  $f \equiv g$  on  $I$ .

Note, it is extremely important for  $I$  to be an open interval and the assumption that  $E$  has an accumulation point in  $I$ .

*Proof.* By replacing  $f$  with  $f - g$ , we may assume that  $g \equiv 0$  on  $I$ . Since  $f$  is continuous,  $E$  is closed in  $I$ . Let  $E'$  be a set of accumulation points of  $E$ . Therefore  $E' \cap I \subseteq E$ . Then  $E' \cap I$  is closed in  $I$ .

**Claim:**  $E' \cap I$  is open.

Let  $a \in E' \cap I$ . Then for some  $\epsilon > 0$ ,  $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x - a)^n$ , on  $|x - a| < \epsilon$ . Since  $a \in E' \cap I \subseteq E$ ,  $f(a) = f^{(0)}(a) = 0$ . Suppose  $f(a) = f'(a) = \dots = f^{(k)}(a) = 0$  and  $f^{(k+1)}(a) \neq 0$  for some  $k \geq 0$ . By Taylor's theorem with remainder,

$$f(x) = \frac{1}{(k+1)!} f^{(k+1)}(a)(x - a)^{k+1} + \frac{1}{(k+2)!} f^{(k+2)}(t_{x,a})(x - a)^{k+2},$$

for some  $t_{x,a}$  between  $a$  and  $x$ . By continuity of  $f^{(k+2)}$ , there exists  $\delta > 0$ , s.t.  $\forall |x - a| < \delta$  and  $|t - a| < \delta$ ,

$$\frac{1}{(k+2)!} |f^{(k+2)}(t)| \cdot |x - a| < \frac{1}{2} \cdot \frac{1}{(k+1)!} \cdot |f^{(k+1)}(a)|.$$

Thus for  $0 < |x - a| < \delta$ ,

$$|f(x)| \geq \frac{1}{2} \cdot \frac{1}{(k+1)!} \cdot |f^{(k+1)}(a)| \cdot |x - a|^{k+1} \neq 0.$$

So  $a \notin E'$ . Contradiction. So  $f^{(n)}(a) = 0$  for all  $n$ . So  $f \equiv 0$  on  $(a - \epsilon, a + \epsilon)$ . So  $(a - \epsilon, a + \epsilon) \subseteq E' \cap I$ . Since  $a$  was arbitrary,  $E' \cap I$  is open.

Then we conclude that  $E' \cap I$  is closed and open. Since  $I$  is connected ( $I$  is an open interval),  $E' \cap I = \emptyset$  or  $E' \cap I = I$ . In the latter case,  $E = I$ .  $\square$

## The exponential function

**Properties.** Let  $E(z) := \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ ,  $z \in \mathbb{C}$ . By root test, its radius of convergence is  $\infty$  and it has the following properties:

1.  $E(z) \cdot E(w) = E(z + w)$  for every  $z, w \in \mathbb{C}$ .
2.  $E(z) \neq 0$  and  $E(-z) = \frac{1}{E(z)}$  for every  $z \in \mathbb{C}$ .
3.  $E(x) > 0$  for every  $x \in \mathbb{R}$ .
4.  $E'(x) = E(x)$  for every  $x \in \mathbb{R}$ .

*Proof.* 1. By Cauchy products.

2.  $E(z)E(-z) = E(0) = 1$ .
3. For  $x \geq 0$ ,  $E(x) \geq 1$  by inspection. For  $x < 0$ ,  $E(x) = \frac{1}{E(-x)} > 0$ .
4. Differentiate term by term.

$\square$

**Definition.**  $e := E(1)$ .

**Proposition.**  $E(x) = e^x$ ,  $\forall x \in \mathbb{R}$ .

*Proof.* For  $x = n \in \mathbb{N}$ , this follows from property 1 and induction.  $E(n) = E(1 + \cdots + 1) = (E(1))^n = e^n$ . For  $x = n \in \mathbb{Z}$ , this follows from preceding and property 2. For  $x = \frac{n}{m}$  with  $n \in \mathbb{Z}$ ,  $m \in \mathbb{N}$ , we know that

$$E\left(\frac{n}{m}\right)^m = E(n)$$

by property 1. So by property 2 and uniqueness of  $m$ -th roots,  $E(n/m) = E(n)^{1/m} = e^{n/m}$ . So the conclusion holds for all rationals. Finally, for  $x \in \mathbb{R}$ ,  $e^x$  is (by definition, since  $e > 1$ )  $\sup\{e^p : p \in \mathbb{Q} \text{ and } p \leq x\}$ . So the general case of proposition follows from continuity of  $E$ .  $\square$

**Theorem 31.** 1.  $e^x$  is continuous and differentiable on  $\mathbb{R}$  and  $\frac{d}{dx}e^x = e^x$ .

2.  $e^x$  is positive and strictly increasing on  $\mathbb{R}$ .

3.  $\lim_{x \rightarrow \infty} e^x = \infty$  and  $\lim_{x \rightarrow -\infty} e^x = 0$ .

4. For fixed  $n$ ,  $\lim_{x \rightarrow \infty} e^x x^{-n} = \infty$  and  $\lim_{x \rightarrow \infty} e^{-x} x^n = 0$ .

*Proof.* 1. Proved by the 4th statement in the proposition.

2. Combine the 3th statement in the proposition and the fact that  $E(x) = e^x$ ,  $e^x > 0$ . Also, since  $E'(x) = E(x) > 0$ , so  $e^x$  is strictly increasing.

3. It is clear that  $\lim_{x \rightarrow \infty} e^x = \lim_{x \rightarrow \infty} E(x) = \infty$ . Also by inspection,

$$\lim_{x \rightarrow -\infty} e^x = \lim_{x \rightarrow \infty} e^{-x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0.$$

4. For  $x \geq 0$ ,  $e^x \geq \frac{1}{(n+1)!} x^{n+1}$ . So  $e^x x^{-n} \geq \frac{1}{(n+1)!} x \rightarrow \infty$ . On the other hand,  $\lim_{x \rightarrow \infty} e^{-x} x^n = \lim_{x \rightarrow \infty} \frac{1}{e^x x^{-n}} = 0$ .

□

**Lemma.** Let  $I$  and  $J$  be intervals. Let  $f : I \rightarrow J$  be a continuous bijection onto  $J$ . Then  $f$  has a continuous inverse. Furthermore, if  $y \in \text{int}(J)$  and  $f$  is differentiable at  $f^{-1}(y)$  with  $f'(f^{-1}(y)) \neq 0$ . Then  $f^{-1}$  is differentiable at  $y$  and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

**Note.** The conclusion is not true if  $I$  is an open interval and  $J$  is a subset of an arbitrary metric space. For example, consider  $I = \mathbb{R}$  and  $J = 8$ . Map  $\mathbb{R}$  into a bounded open interval (s.t. arctan).

**Remarks.** If  $K$  is compact and  $f : K \rightarrow J$  is a continuous bijection, then  $f^{-1}$  is continuous.

*Proof.*  $f$  is either strictly increasing or strictly decreasing. Without loss of generality, we may assume  $f$  is strictly increasing. Let  $g := f^{-1}$ . Then  $g$  is strictly increasing.

**Claim 1:**  $g$  is continuous.

Let  $y \in J$ . Then both  $g(y^+) = \lim_{t \rightarrow y^+} g(t)$  and  $g(y^-) = \lim_{t \rightarrow y^-} g(t)$  exists (modify as needed if  $y$  is an end point). Furthermore, we know  $g(y^-) \leq g(y) \leq g(y^+)$ . If  $g(y^-) = g(y) = g(y^+)$ , then  $g$  is continuous at  $g$ . Let's suppose  $g(y^-) < g(y)$ . Because  $g$  is increasing, then  $g(t) \leq g(y^-)$  for all  $t < y$  and  $g(t) \geq g(y)$  for all  $t \geq y$ . So  $g$  omits in  $(g(y^-), g(y)) \subseteq I$ . But  $I$  is the domain of  $f$ , which is the range of  $g$ . Contradiction. So  $g(y^-) = g(y)$ . Similarly,  $g(y^+) = g(y)$ .

**Clam 2:** If  $y \in \text{int}(J)$  and  $f$  is differentiable at  $f^{-1}(y)$  with  $f'(f^{-1}(y)) \neq 0$ . Then  $f^{-1}$  is differentiable at  $y$ .

We want to evaluate

$$\lim_{h \rightarrow 0} \frac{g(y+h) - g(y)}{y+h-y} = \lim_{h \rightarrow 0} \frac{g(y+h) - g(y)}{f(g(y+h)) - f(g(y))}.$$

Define

$$\varphi(t) = \begin{cases} \frac{t - g(y)}{f(t) - f(g(y))}, & t \neq g(y), \\ \frac{1}{f'(g(y))}, & t = g(y). \end{cases}$$

Then  $\frac{1}{\varphi}$  is continuous on  $I$  and  $\frac{1}{\varphi(g(y))} = f'(g(y)) \neq 0$ . So  $\varphi$  is continuous on a neighbourhood of  $g(y)$ . So  $\varphi \circ g$  is continuous at  $y$ , i.e.

$$\lim_{h \rightarrow 0} \frac{g(y+h) - g(y)}{f(g(y+h)) - f(g(y))} = \lim_{h \rightarrow 0} \varphi(g(y+h)) = \varphi(g(y)) = \frac{1}{f'(g(y))}.$$

□

**Definition.** Since  $E : x \mapsto e^x$  is continuous, differentiable, strictly increasing and maps  $\mathbb{R}$  onto  $(0, \infty)$ . Thus it has an inverse, which we call  $\log$ , that is continuous, differentiable, strictly increasing and maps  $(0, \infty)$  onto  $\mathbb{R}$ .

**Properties.** 1.  $e^{\log y} = y$  for all  $y > 0$  and  $\log(e^x) = x$  for all  $x \in \mathbb{R}$ .

2.  $\frac{d}{dx} \log y = \frac{1}{y}, \forall y > 0$ .

3.  $\log(1) = 0$ .

4.  $\log(y) = \int_1^y 1/s \, dx, \forall y > 0$ .

5.  $\log(uv) = \log u + \log v, \forall u, v > 0$ .

6.  $\log(y) \rightarrow +\infty$  as  $y \rightarrow +\infty$ .  $\log y \rightarrow -\infty$  as  $y \rightarrow 0$ .

7. For  $u > 0$  and  $\alpha \in \mathbb{R}$ ,  $\log(u^\alpha) = \alpha \log(u)$  and  $u^\alpha = e^{\alpha \log u}$ .

*Proof.* 1. Done.

2. By the lemma,  $\log$  is differentiable and  $\frac{d}{dy} \log(y) = \frac{1}{E'(\log(y))} = \frac{1}{E(\log(y))} = \frac{1}{y}$ .

3. Because  $E(0) = 1$ , done.

4. By property two and three, done.

5.  $\log(uv) = \log(e^{\log u} \cdot e^{\log v}) = \log(e^{\log u + \log v}) = \log u + \log v$ .

6. Done.

7.  $\log(u^\alpha) = \log((e^{\log u})^\alpha) = \log(e^{\alpha \log u}) = \alpha \log u$ .

□

**Proposition.** For any  $a > 0$ ,  $x \mapsto a^x$  is differentiable on  $\mathbb{R}$  and

$$\frac{d}{dx} a^x = (\log a) \cdot a^x.$$

*Proof.*  $a^x = e^{x \log a}$ . By the chain rule, done.

□

**Proposition (HW).** If  $\alpha \neq -1$ , then  $x \mapsto x^\alpha$  has  $x \mapsto \frac{x^{\alpha+1}}{\alpha+1}$  as an antiderivative.

**Proposition.**  $\forall \epsilon > 0, 0 = \lim_{x \rightarrow \infty} x^{-\epsilon} \log x = \lim_{x \rightarrow 0+} x^\epsilon \log x$ .

This is saying  $\log$  goes to infinity very slow.

*Proof.* Notice that

$$\lim_{x \rightarrow 0+} x^\epsilon \log x = \lim_{y \rightarrow \infty} \left(\frac{1}{y}\right)^\epsilon \log \left(\frac{1}{y}\right) = - \lim_{y \rightarrow \infty} y^{-\epsilon} \log y.$$

So it suffices to prove the first equation.

$$\lim_{x \rightarrow \infty} x^{-\epsilon} \log x = \lim_{t \rightarrow \infty} (e^t)^{-\epsilon} \log(e^t) = \lim_{t \rightarrow \infty} e^{-\epsilon t} t = 0.$$

□

## Trigonometric Functions

**Definition.** Define

$$C(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \quad \text{and} \quad S(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1},$$

where  $z \in \mathbb{C}$  (radius of convergence is  $+\infty$ ).

**Definition.** Let  $z := x + iy$  where  $x, y \in \mathbb{R}$ . Recall that  $\bar{z} = x - iy$ . Define  $\operatorname{Re}(z) := x$  and  $\operatorname{Im}(z) := y$ . Then

- $\operatorname{Re}(z) = (z + \bar{z})/2$ ,
- $\operatorname{Im}(z) = (z - \bar{z})/(2i)$ ,
- $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$ ,
- $\overline{z + w} = \bar{z} + \bar{w}$ .

**Note.** For all  $x \in \mathbb{R}$

$$\overline{E(ix)} = \overline{\sum_{n=0}^{\infty} \frac{1}{n!} (ix)^n} = \sum_{n=0}^{\infty} \overline{\frac{1}{n!} (ix)^n} = E(\overline{ix}) = E(-ix).$$

By this, we have

$$C(x) = \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!} = \operatorname{Re}(E(ix)) = \frac{1}{2} (E(ix) + E(-ix)).$$

$$S(x) = \frac{1}{i} \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!} = \operatorname{Im}(E(ix)) = \frac{1}{2i} (E(ix) - E(-ix)).$$

**Properties.** Trigonometric functions have the following properties:

1.  $C(x)^2 + S(x)^2 = 1, \forall x \in \mathbb{R}$ .
2.  $C(0) = 1$  and  $S(0) = 0$ .

3.  $C'(x) = -S(x)$ ,  $S'(x) = C(x)$ ,  $\forall x \in \mathbb{R}$ .

*Proof.* 1.  $C(x)^2 + S(x)^2 = |E(ix)|^2 = E(ix) \cdot \overline{E(ix)} = E(ix)E(-ix) = E(0) = 1$ .

2. Done.

3.  $C'(x) + iS'(x) = \frac{d}{dx}E(ix) = iE'(ix) = iE(ix) = iC(x) + i^2S(x)$ . So we have  $C'(x) = -S(x)$ ,  $S'(x) = C(x)$ . □

**Proposition.**  $C$  has a positive zero.

*Proof.* Since  $C(0) = 1 \neq 0$  and since  $C(x) = C(-x)$ , it suffices to show  $C$  has a zero. Suppose not. Since  $C(0) > 0$  and  $C$  is continuous,  $C(x) > 0$ ,  $\forall x \in \mathbb{R}$ . Then  $S$  is strictly increasing since  $S' = C$ . So for  $0 < x < y$ ,

$$S(x)(y - x) = \int_x^y S(x) dt < \int_x^y S(t) dt = C(x) - C(y).$$

Since  $C(x) \leq 1$  (because  $C^2 + S^2 = 1$ ), and  $C(y) > 0$ , so  $S(x) < \frac{1}{y-x}$ ,  $\forall 0 < x < y$ . Fix  $x$  and let  $y \rightarrow \infty$ , we see  $S(x) = 0$ . But  $S$  strictly increasing and  $S(0) = 0$ , so  $S(x) > 0$ . Contradiction! So  $C$  must have a zero. □

**Definition.** Define  $\pi := 2 \inf\{x > 0 : C(x) = 0\}$ . By continuity and  $C(0) \neq 0$ ,  $\pi > 0$  and the infimum is a minimum, i.e.  $C(\pi/2) = 0$ .

**Theorem 32.** The following statements hold:

1.  $C(k\pi) = (-1)^k$ ,  $S(k\pi) = 0$ ,  $C(k\pi + \pi/2) = 0$  and  $S(k\pi + \pi/2) = (-1)^k$ ,  $k \in \mathbb{Z}$ .
2.  $E(z + 2k\pi i) = E(z)$ ,  $\forall k \in \mathbb{Z}$  and consequently  $C$  and  $S$  are periodic with period  $2\pi$ .

*Proof.* 1. Since  $C(\pi/2) = 0$  and  $C^2 + S^2 = 1$ , we see  $S(\pi/2) = \pm 1$ . By definition of  $\pi$  and continuity of  $C$  and  $C(0) > 0$ ,  $S$  is increasing on  $[0, \pi/2]$  and since  $S(0) = 0$ ,  $S(\pi/2) = 1$ . So  $E(i\pi/2) = C(\pi/2) + iS(\pi/2) = 0 + i$ . So  $E(ik\pi/2) = i^k$  and all identities in 1 hold.

2.  $E(z + 2\pi i k) = E(z)E(2\pi i)^k = E(z)$  by part 1. □

**Lemma.** If  $E(it) = 1$  and  $t \in [0, 2\pi)$ , then  $t = 0$ .

*Proof.* Suppose  $t \in (0, 2\pi)$  and  $E(it) = 1$ . By the proof of part 1 of the previous theorem,  $S \neq 0$  on  $(0, \pi/2]$ . Since  $S(\pi/2 - t) = S(\pi/2 + t)$  (Exercise) so  $S > 0$  on  $(0, \pi)$ . Since  $S(t) = -S(-t) = -S(2\pi - t)$ ,  $S < 0$  on  $(\pi, 2\pi)$ . So  $t = \pi$ . But  $E(i\pi) = -1$ . □

**Theorem 33.** If  $z \in \mathbb{C}$  with  $|z| = 1$ , then  $\exists! t \in [0, 2\pi)$ , s.t.  $E(it) = z$ .

*Proof.* The above lemma proves the uniqueness of  $z$ . It remains to prove the existence. Let  $z \in \mathbb{C}$  with  $|z| = 1$ . Write  $z = x + iy$ ,  $x, y \in \mathbb{R}$ . Thus  $x^2 + y^2 = 1$ .

**Case 1:**  $x \geq 0, y \geq 0$ .

Then  $0 \leq x \leq 1$  and since  $\cos 0 = 1$  and  $\cos \pi/2 = 0$ , by the intermediate value theorem,  $\exists t \in [0, \pi/2]$ , s.t.  $\cos t = x$ . Furthermore,  $y = \sqrt{1 - x^2} = \sqrt{1 - \cos^2 t} = \sqrt{\sin^2 t} = |\sin t|$ . Since  $\sin$  is increasing on  $[0, \pi/2]$  (because  $\sin' = \cos$  is nonnegative) and  $\sin(0) = 0$ ,  $\sin t \geq 0$  and  $|\sin t| = \sin t$ .

**Case 2:**  $x \geq 0, y > 0$ .

Then  $\bar{z}$  is in the first quadrant. By Case 1, there exists  $t \in (0, \pi/2]$ , s.t.  $e^{it} = \bar{z}$ . Therefore,  $z = e^{-it} = e^{i(2\pi-t)}$  and  $2\pi - t \in [3\pi/2, 2\pi)$ .

**Case 3:**  $x < 0$ .

Then by Case 1 and Case 2,  $\exists t \in (-\pi/2, \pi/2)$ , s.t.  $-z = e^{it}$ . Therefore  $z = (-1)e^{it} = e^{i(\pi+t)}$  and  $\pi + t \in (\pi/2, 3\pi/2)$ .

□

**Corollary.** *The circumference of the unit circle is  $2\pi$ .*

*Proof.*

$$\text{Circumference} = \int_0^{2\pi} |E'(it)| dt = \int_0^{2\pi} |E(it)| dt = \int_0^{2\pi} C^2(t) + S^2(t) dt = 2\pi.$$

□

**Corollary.** *If  $z \in \mathbb{C}$  with  $z \neq 0$ , then  $\exists! t \in [0, 2\pi)$ , s.t.  $z = |z|e^{it}$ .*

**Theorem 34** (Algebraic Completeness of  $\mathbb{C}$ ). *Let  $P(z) = a_0 + a_1z + \cdots + a_nz^n$  be a complex polynomial with  $a_n \neq 0$  and  $n \geq 1$ . Then there exists  $z_0 \in \mathbb{C}$  such that  $P(z_0) = 0$ .*

*Proof.* Let  $\mu = \inf_{z \in \mathbb{C}} |P(z)|$ . We claim that  $\mu$  is a minimum (it is achieved). Indeed,  $|P(z)| \geq |a_n| \cdot |z|^n - \sum_{j=0}^{n-1} |a_j| \cdot |z|^j$ . So  $\exists R$ , s.t.  $\forall |z| > R$ ,  $|P(z)| \geq \mu + 1$ . Therefore by the continuity of  $|P(z)|$ ,  $\mu = \inf_{|z| \leq R} |P(z)| = \min_{|z| \leq R} |P(z)|$ . Thus  $\exists z_0 \in \mathbb{C}$ , s.t.  $|P(z_0)| = \mu$ . If  $\mu = 0$ , we are done.

So now suppose  $\mu > 0$ . Define  $Q(z) = \frac{P(z+z_0)}{P(z_0)}$ . Then  $Q$  is a polynomial.  $Q(0) = 1$  and  $|Q(z)| \geq 1$  for all  $z$  (because  $P(z_0)$  is the minimum). Thus

$$Q(z) = 1 + \sum_{j=k}^n b_j z^j, \quad \text{with } b_k \neq 0.$$

By the previous theorem,  $\exists \theta \in [0, 2\pi/k)$ , s.t.  $e^{ik\theta} = -\frac{|b_k|}{b_k}$ . Thus for  $r > 0$ ,

$$\begin{aligned} |Q(re^{i\theta})| &= \left| 1 + |b_k|r^k \cdot \frac{e^{ik\theta}b_k}{|b_k|} + \sum_{k+1}^n b_j r^j e^{ij\theta} \right| \\ &= \left| 1 - |b_k|r^k + \sum_{k+1}^n b_j r^j e^{ij\theta} \right| \\ &\leq 1 - |b_k|r^k + \sum_{j=k+1}^n |b_j|r^j \end{aligned}$$



Notice that  $|b_j|r^j \leq \frac{1}{2}|b_k|r^k$  for sufficiently small  $r$ . So we further have

$$|Q(re^{i\theta})| \leq 1 - \frac{1}{2}|b_k|r^k < 1.$$

Contradiction! Since  $|Q(x)| \geq 1$  for all  $x$ . Tracing back, we see  $\mu = 0$ . So there exists  $z_0 \in \mathbb{C}$  such that  $P(z_0) = 0$ .  $\square$

**Corollary.** Let  $P(z) = a_0 + a_1z + \cdots + a_nz^n$  be a complex polynomial with  $a_n \neq 0$  and  $n \geq 1$ . There exists  $z_1, \dots, z_n \in \mathbb{C}$ , s.t.

$$P(z) = a_n(z - z_1) \cdots (z - z_n).$$

*Proof.* By the theorem, there exists  $z_n \in \mathbb{C}$  s.t.  $P(z_n) = 0$ . By long division algorithm,  $P(z) = (z - z_n)Q(z) + \text{constant}$ , where  $Q$  is a polynomial with degree  $n - 1$ . Evaluating both sides at  $z = z_n$ , we see that the constant is zero, i.e.  $z - z_n | P$ . Now repeat this procedure and we're done.  $\square$

## Banach Spaces

**Definition.**  $X$  is a real (or complex) vector space if  $\forall x, y \in X$  and  $\alpha, \beta \in \mathbb{R}$  (or  $\mathbb{C}$ ),  $\alpha x + \beta y \in X$  and some axioms hold.

**Definition.** A norm on  $X$  is a function  $\|\cdot\|_X : X \rightarrow \mathbb{R}$  satisfying

- $\forall x \in X$ ,  $\|x\|_X \geq 0$  and  $\|x\|_X = 0$  if and only if  $x = 0$ .
- $\forall \alpha \in \mathbb{R}$  (or  $\mathbb{C}$ ),  $\forall x \in X$ ,  $\|\alpha x\|_X = |\alpha| \cdot \|x\|_X$ .
- $\forall x, y \in X$ ,  $\|x + y\|_X \leq \|x\|_X + \|y\|_X$ .

**Definition.** The normed vector space  $(X, \|\cdot\|)$  is a Banach space if  $X$  is a complete metric space with respect to the distance  $d(x, y) = \|x - y\|$ .

**Example.**  $\mathbb{R}^k$  with euclidean metric is a Banach space. For any interval  $I \subseteq \mathbb{R}$ ,  $\mathcal{C}^k(I)$  is a Banach space. For any metric space  $X$ ,  $\mathcal{C}^0(X)$  is a Banach space.

**Definition.** Define

$$l^\infty := l^\infty(\mathbb{N}) := \{\text{bounded sequences } \{x_n\}_{n \in \mathbb{N}} \text{ in } \mathbb{R}\}.$$

Define  $\|\{x_n\}\|_{l^\infty} := \sup_n |x_n|$ .

**Definition.** For  $1 \leq p < \infty$ ,

$$l^p := l^p(\mathbb{N}) := \{\text{real sequences } \{x_n\}_{n \in \mathbb{N}} \text{ with } \|\{x_n\}\|_{l^p} < \infty\},$$

where

$$\|\{x_n\}\|_{l^p} := \left( \sum |x_n|^p \right)^{1/p}.$$

**Note.** In fact,  $(l^\infty(\mathbb{N}), \|\cdot\|_{l^\infty}) = (\mathcal{C}^0(\mathbb{N}), \|\cdot\|_{\mathcal{C}^0(\mathbb{N})})$ . So we have already seen that  $l^\infty$  is a Banach space.

**Theorem 35** (Holders inequality). *Let  $1 \leq p, q \leq \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$  ( $1/\infty = 0$ ). Let  $\{a_n\} \in l^p$  and  $\{b_n\} \in l^q$ . Then  $\{a_n b_n\} \in l^1$  and*

$$\|\{a_n b_n\}\|_{l^1} = \sum_{n=1}^{\infty} |a_n b_n| \leq \|\{a_n\}\|_{l^p} \cdot \|\{b_n\}\|_{l^q}.$$

*Proof.* If  $\{a_n\} = \{0\}$  or  $\{b_n\} = \{0\}$ , the inequality holds trivially. Now suppose  $p = \infty$  and  $q = 1$  (the argument also works for  $p = 1$  and  $q = \infty$ ). We have

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \sum_{n=1}^{\infty} \|\{a_n\}\|_{l^\infty} \cdot |b_n| = \|\{a_n\}\|_{l^\infty} \cdot \|\{b_n\}\|_{l^1}.$$

Now consider  $p, q \neq \infty$ . Replacing  $\{a_n\}$  with  $\{a_n / \|\{a_n\}\|_{l^p}\}$  and  $\{b_n\}$  with  $\{b_n / \|\{b_n\}\|_{l^q}\}$  if needed, we may assume  $\|\{a_n\}\|_{l^p} = \|\{b_n\}\|_{l^q} = 1$ .

**Claim:** For  $1 < p, q < \infty$  and  $x, y \geq 0$ ,  $xy \leq x^p/p + y^q/q$ .

Define

$$f_y(x) = \frac{x^p}{p} + \frac{y^q}{q} - xy.$$

Then  $f'_y(x) = x^{p-1} - y$  and  $f''_y(x) = (p-1)x^{p-2} \geq 0$ . So  $f_y$  has a global minimum at the zero of  $f'_y$ , i.e.  $x = y^{1/(p-1)}$ . Remember that  $1/q = (p-1)/p$ , so

$$\begin{aligned} f_y(y^{1/(p-1)}) &= \frac{y^{p/(p-1)}}{p} + \frac{y^q}{q} - y^{p/(p-1)} \\ &= y^q \left( \frac{1}{p} + \frac{1}{q} - 1 \right) \\ &= 0. \end{aligned}$$

Then we see

$$\begin{aligned} \sum_{n=0}^{\infty} |a_n b_n| &\leq \sum_{n=0}^{\infty} \frac{|a_n|^p}{p} + \frac{|b_n|^q}{q} \\ &= \frac{1}{p} \sum_{n=1}^{\infty} |a_n|^p + \frac{1}{q} \sum_{n=1}^{\infty} |b_n|^q \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1. \end{aligned}$$

□

**Proposition.** *Let  $1 \leq p, q \leq \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . Then for every real sequence  $\{a_n\}$ ,*

$$\left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} = \sup_{\{b_n\} \in l^q, \|\{b_n\}\|_{l^q}=1} \sum_{n=1}^{\infty} |a_n b_n|.$$

*Proof.* By Holders inequality,  $\text{RHS} \leq \text{LHS}$ . If  $\{a_n\} \in l^p$ , take  $b_n = |a_n|^{p-1}$ . Then

$$\|\{b_n\}\|_{l^q} = \left( \sum_{n=1}^{\infty} (|a_n|^{p-1})^q \right)^{1/q} = \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/q} = \|\{a_n\}\|_{l^p}^{p/q}.$$

So  $\{b_n\} \in l^q$ . Divide  $\{b_n\}$  by  $\|\{b_n\}\|_{l^q}$  to make  $\|\{b_n\}\|_{l^q}=1$  and we have

$$\sum_{n=1}^{\infty} |a_n b_n| = \sum_{n=1}^{\infty} \frac{|a_n|^p}{\|\{a_n\}\|_{l^p}^{p/q}} = \frac{\|\{a_n\}\|_{l^p}^p}{\|\{a_n\}\|_{l^p}^{p/q}} = \|\{a_n\}\|_{l^p}.$$

If  $\{a_n\} \notin l^p$ , take

$$b_n^N = \begin{cases} |a_n|^{p-1}/\text{normalizing factor}, & n \leq N, \\ 0, & n > N. \end{cases}$$

Then

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N |a_n b_n^N| = \lim_{N \rightarrow \infty} \left( \sum_{n=1}^N |a_n|^p \right)^{1/p} = \infty.$$

□

**Theorem 36** (Triangle inequality). *For  $\{a_n\}, \{b_n\} \in l^p$ ,  $\{a_n + b_n\} \in l^p$  and*

$$\|(a_n + b_n)\|_{l^p} \leq \|\{a_n\}\|_{l^p} + \|\{b_n\}\|_{l^p}.$$

*Consequently,  $l^p$  is a normed vector space.*

*Proof.* Choose  $q$  such that  $\frac{1}{q} + \frac{1}{p} = 1$ . Then by the above proposition, we see

$$\begin{aligned} \|\{a_n + b_n\}\|_{l^p} &= \sup_{\{c_n\} \in l^q, \|\{c_n\}\|_q=1} \sum_{n=1}^{\infty} |c_n| \cdot |a_n + b_n| \\ &\leq \sup_{\|\{c_n\}\|_q=1} \left( \sum_{n=1}^{\infty} |c_n a_n| + \sum_{n=1}^{\infty} |c_n b_n| \right) \\ &\leq \|\{a_n\}\|_{l^p} + \|\{b_n\}\|_{l^p}. \end{aligned}$$

□

**Definition.**  $\sum x_n$  converges absolutely if  $\sum \|x_n\|$  converges.  $\sum x_n$  converges if the sequence  $s_n := x_1 + \dots + x_n$  of partial sums converges.

**Example.** Define

$$l_K^\infty = \{\text{real sequences } \{x_n\}, \text{ s.t. } x_n = 0, \text{ for all except finitely many } n\},$$

where  $K$  stands for compact support. Let  $x_n := (0, \dots, 0, 2^{-n}, 0, \dots)$  where  $2^{-n}$  appears at the  $n$ th position. Then  $\sum x_n$  converges absolutely but it doesn't converge in  $l_K^\infty$ .

**Theorem 37.** *The normed vector space  $(X, \|\cdot\|)$  is a Banach space if and only if every absolutely convergence series in  $X$  converges.*

*Proof.*  $\Rightarrow$ : If  $X$  is a Banach space and  $\sum x_n$  converges absolutely, then by triangle inequality, for  $n \geq m$ ,

$$\|s_n - s_{m-1}\| = \|a_m + \cdots + a_n\| \leq \sum_{j=m}^n \|a_j\|.$$

So the sequence of partial sums is Cauchy. Since  $X$  is complete, the sequence of partial sums converge and thus  $\sum x_n$  converges.

$\Leftarrow$ : Assume that every absolute convergence series in  $X$  converges. Let  $\{x_n\}$  be a Cauchy sequence in  $X$ . Recall that a Cauchy sequence with a convergence subsequence must converge (in any metric space). We need to prove that  $\{x_n\}$  has a convergent subsequence. The idea is to use telescoping series.

By Cauchyness, there exists a subsequence  $\{x_{n_k}\}$  s.t.  $\|x_{n_{k+1}} - x_{n_k}\| \leq 2^{-(k+1)}$ ,  $\forall k \in \mathbb{N}$ . Then  $\sum x_{n_{k+1}} - x_{n_k}$  converges absolutely, so it converges by hypothesis. Therefore,

$$\lim_{k \rightarrow \infty} x_{n_k} = x_{n_1} + \sum_{k=1}^{\infty} (x_{n_{k+1}} - x_{n_k}),$$

which implies  $\{x_{n_k}\}$  is a convergent subsequence of  $\{x_n\}$ . □

**Theorem 38.**  $l^p$  is a Banach space for  $1 \leq p \leq \infty$ .

*Proof.* We've already checked the case when  $p = \infty$ . Let  $1 \leq p < \infty$  and assume  $\sum \|\mathbf{x}_n\|_{l^p} \leq \infty$ , where  $\mathbf{x}_n$  is a sequence for each  $n$ . If we can show  $\sum \mathbf{x}_n$  converges, then by the above theorem,  $l^p$  is a Banach space. For each  $k \in \mathbb{N}$ , the  $k$ -th coordinate satisfy

$$\sum_n |x_{n,k}| \leq \sum_n \left( \sum_k |x_{n,k}|^p \right)^{1/p} = \sum_n \|\mathbf{x}_n\|_{l^p} < \infty.$$

So we can define a sequence  $\{y_k\}$  by  $y_k = \sum_n x_{n,k}$ . We need to show  $\{y_k\} \in l^p$  and  $\sum \mathbf{x}_n = \{y_k\}$  with convergence in  $l^p$ . Choose  $q$  with  $\frac{1}{q} + \frac{1}{p} = 1$  and pick any  $\{a_k\} \in l^q$  with  $\|\{a_k\}\|_{l^q} = 1$ . We see

$$\sum_k |a_k y_k| = \sum_k \left| a_k \sum_n x_{n,k} \right| \leq \sum_k \sum_n |a_k x_{n,k}| \leq \sum_n \sum_k |a_k x_{n,k}| \leq \sum_n \|\mathbf{x}_n\|_{l^p} \|\mathbf{a}_k\|_{l^q} = \sum_n \|\mathbf{x}_n\|_{l^p} < \infty.$$

Thus,  $\|\{y_k\}\|_{l^p} < \infty$ . Question. To show  $\sum_n \mathbf{x}_n = \{y_n\}$  is similar. Show  $\sum_{n=N}^{\infty} \mathbf{x}_n \rightarrow 0$  in  $l^p$  as  $N \rightarrow \infty$  using a similar argument. □

## Bounded linear operators

**Definition.** If  $X, Y$  are real vector spaces, a map  $T : X \rightarrow Y$  is linear if  $\forall \alpha_1, \alpha_2 \in \mathbb{R}, \forall x_1, x_2 \in X$ ,  $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$ .

**Definition.** Let  $T : X^{\text{normed v.s.}} \rightarrow Y^{\text{normed v.s.}}$  be linear. We say  $T$  is a bounded linear operator (or  $T$  is bounded) if  $\exists C$ , s.t.  $\forall x \in X$ ,  $\|T(x)\|_Y \leq C \cdot \|x\|_X$ .

**Example.**  $\frac{d}{dx} : \mathcal{C}^k(I) \rightarrow \mathcal{C}^{k-1}(I)$  is a bounded linear operator.

**Example.**  $X = \mathcal{C}^0([a, b])$  and  $Y = \mathbb{R}$ ,  $T(f) = \int_a^b f(x) dx$  is a bounded linear operator.

For convention, we write  $T(x)$  as  $Tx$ .

**Theorem 39.** *Let  $X, Y$  be normed vector spaces and  $T : X \rightarrow Y$  being a linear operator. Then the following are equivalent:*

1.  $T$  is a bounded linear operator;
2.  $T$  is uniformly continuous on  $X$ ;
3.  $T$  is continuous on  $X$ ;
4.  $T$  is continuous at 0.

*Proof.*  $1 \Rightarrow 2$  : By 1, we have  $\forall x, y, \|Tx - Ty\| = \|T(x - y)\| \leq C\|x - y\|$ . So  $T$  is Lipschitz, which implies 2.

$2 \Rightarrow 3$  : Done by definition.

$3 \Rightarrow 4$  : Done by definition.

$4 \Rightarrow 1$  : Assume  $T$  is continuous at 0. Then  $\exists \delta > 0$ , s.t.  $\|x\| \leq \delta$  implies  $\|Tx\| \leq 1$ . Now if  $x = 0$ ,  $\|Tx\| = 0$ . For  $x \neq 0$ , since  $\|\delta x / \|x\|\| = \delta$ , so  $\|T(\delta x / \|x\|)\| \leq 1$ . By linearity and arithmetic,  $\|Tx\| \leq \|x\|/\delta$ . So  $T$  is a bounded linear operator ( $C = 1/\delta$ ).

□

**Definition.** Define

$$\mathcal{L}(X, Y) := \{\text{bounded linear operators from } X \text{ to } Y\}.$$

Then  $\mathcal{L}(X, Y)$  is a vector space, since

$$\begin{aligned} \|(\alpha T + \beta S)x\|_Y &= \|\alpha(Tx) + \beta(Sx)\|_Y \\ &\leq |\alpha| \cdot \|Tx\|_Y + |\beta| \cdot \|Sx\|_Y \\ &\leq (|\alpha|C_1 + |\beta|C_2)\|x\|_X. \end{aligned}$$

**Definition.** Define the norm

$$\|T\|_{X \rightarrow Y} := \sup_{0 \neq x \in X} \frac{\|Tx\|_Y}{\|x\|_X}.$$

**Proposition.** *The following properties holds,*

1.  $\|T\| = \sup_{\|x\|=1} \|Tx\|$ ;
2.  $\|T\| = \min \{C : \|Tx\| \leq C\|x\|, \forall x \in X\}$ , in particular,  $\forall x, \|Tx\| \leq \|T\| \cdot \|x\|$ ;
3.  $\|\cdot\|$  is a norm;
4. If  $Y$  is a Banach space, so is  $\mathcal{L}(X, Y)$ ;

5. If  $X, Y, Z$  are normed vector spaces and  $T \in \mathcal{L}(X, Y)$ ,  $S \in \mathcal{L}(Y, Z)$ , then  $S \circ T \in \mathcal{L}(X, Z)$  and  $\|S \circ T\| \leq \|S\| \cdot \|T\|$ .

*Proof.* 1. By definition

$$\|T\| = \sup_{0 \neq x \in X} \frac{\|Tx\|}{\|x\|} = \sup_{0 \neq x \in X} \left\| T \frac{x}{\|x\|} \right\| = \sup_{\|y\|=1} \|Ty\|.$$

2. For any  $C < \|T\|$ , by definition of  $\|T\|$ ,  $\exists x \neq 0$ , s.t.  $\|Tx\|/\|x\| > C$ . So  $\|Tx\| > C\|x\|$ . Also check that  $\|T\| \geq \|Tx\|/\|x\|$ , which implies  $\|Tx\| \leq \|T\| \cdot \|x\|$ .

3. Done.

4. Assume that  $\{T_n\}$  is a Cauchy sequence. Then  $\forall x$ ,  $\{T_n x\}$  is a Cauchy sequence (because  $\|T_n x - T_m x\| \leq \|T_n - T_m\| \cdot \|x\|$ ). Since  $Y$  is complete,  $\{T_n x\}$  converges. Define  $Tx := \lim T_n x$ . We need to show that  $T \in \mathcal{L}(X, Y)$ . We see  $T$  is linear by linearity of limits and the  $T_n$ 's. For the boundedness of  $T$ , we see  $\|Tx\| = \lim \|T_n x\| \leq (\limsup \|T_n\|)\|x\|$ . Note that  $\{\|T_n\|\}$  is Cauchy in  $\mathbb{R}$  since

$$\|T_m\| - \|T_n\| \leq \|T_m - T_n\|$$

and hence  $\{\|T_n\|\}$  is convergent and bounded. So  $T$  is bounded.

5. Apply 2, we have

$$\|S \circ T(x)\| \leq \|S\| \cdot \|Tx\| \leq \|S\| \cdot \|T\| \cdot \|x\|.$$

□

**Definition.** Define the following norms on  $\mathbb{R}^n$ :

1.  $\|x\|_{l_\infty} := \max_{1 \leq i \leq n} |x_i|$
2.  $\|x\|_{l_p^n} := (\sum_{i=1}^n |x_i|^p)^{1/p}$ ,  $1 \leq p \leq \infty$ .

A linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  takes the form  $Tx = Ax$ , where  $A = (a_{ij})$  is a  $m \times n$  matrix.

**Proposition.** In the above notation,

1.  $\|T\|_{l_n^\infty \rightarrow l_m^\infty} = \max_{i=1, \dots, m} \sum_{j=1}^n |a_{ij}|$ ;
2.  $\|T\|_{l_n^1 \rightarrow l_m^1} = \max_{j=1, \dots, n} \sum_{i=1}^m |a_{ij}|$ .

*Proof.* 1. Define  $C := \max_{i=1, \dots, m} \sum_{j=1}^n |a_{ij}|$ . We need to first show  $\|Tx\|_{l_m^\infty} \leq C\|x\|_{l_n^\infty}$ ,  $\forall x$ . This is true since

$$\max_{i=1, \dots, m} |(Tx)_i| = \max_{i=1, \dots, m} \left| \sum_j a_{ij} x_j \right| \leq \max_{i=1, \dots, m} \sum_j |a_{ij}| \cdot \|x\|_{l_n^\infty}.$$

Now we need to show if  $C' < C$ , then  $\exists x$ , s.t.  $\|Tx\|_{l_m^\infty} > C'\|x\|_{l_n^\infty}$ . This is enough to find  $x \neq 0$ , s.t.  $\|Tx\| = C\|x\|$ . This is always possible in finite dimensional vector space (may not be possible in infinity dimensional vector space). Choose  $i$  to maximize  $\sum_j |a_{ij}|$ . Take

$$x_j = \text{sign}(a_{ij}) := \begin{cases} 1 & \text{if } a_{ij} \geq 0 \\ -1 & \text{if } a_{ij} < 0. \end{cases}$$

Then  $\|Tx\|_{l_m^\infty} = \sum_j |a_{ij}| = C\|x\|_{l_n^\infty}$ .

□

## The open mapping and closed graph theorem

**Definition.** Let  $X, Y$  be metric spaces. A function  $f : X \rightarrow Y$  is an open map if  $f(U)$  is open in  $Y$  whenever  $U$  is open.

**Theorem 40** (Open Mapping Theorem). *Let  $X$  and  $Y$  be Banach spaces. Then a surjective map  $T \in \mathcal{L}(X, Y)$  is also an open map.*

*Proof.* Assume  $T \in \mathcal{L}(X, Y)$  is surjective. Then,  $Y = \cup_{n=1}^{\infty} T(B_n(0))$ . By the Baire Category Theorem ( $Y$  is complete), for some  $n$ ,  $\overline{T(B_n(0))}$  has nonempty interior. Since  $\overline{T(B_n(0))} = n\overline{T(B_1(0))}$  by linearity of  $T$ ,  $\overline{T(B_1(0))}$  has nonempty interior. Suppose  $y_0 \in \text{int}(\overline{T(B_1(0))})$ . Then  $\exists r > 0$ , s.t.  $B_r(y_0) \subseteq \overline{T(B_1(0))}$ .

**Claim 1:**  $B_{2r}(0) \subseteq B_r(y_0) - B_r(y_0) := \{y - y' : y, y' \in B_r(y_0)\}$

If  $\|x\| < 2r$ ,  $y_0 + x/2, y_0 - x/2 \in B_r(y_0)$ . If  $y, y' \in B_r(y_0)$ , then  $\|y - y'\| \leq \|y - y_0\| + \|y_0 - y'\| < 2r$ .

So  $B_{2r}(0) \subseteq \overline{T(B_1(0))} - \overline{T(B_1(0))} = \overline{T(B_2(0))}$ . So  $B_r(0) \subseteq \overline{T(B_1(0))}$ .

**Claim 2:**  $B_{r/2}(0) \subseteq T(B_1(0))$ .

Let  $y_1 \in B_{r/2}(0)$ . Then  $\exists x_1 \in B_{1/2}(0)$ , s.t.  $\|y_1 - Tx_1\| < r/4$ . Let  $y_2 := y_1 - Tx_1$ . In general, given  $y_n \in B_{2^{-n}r}(0) \subseteq \overline{T(B_{2^{-n}}(0))}$ ,  $\exists x_n \in B_{2^{-n}}(0)$ , s.t.  $\|y_n - Tx_n\| < 2^{-(n+1)}r$ . Set  $y_{n+1} = y_n - Tx_n$ . Then  $y_{n+1} \in B_{2^{-(n+1)}r}(0)$ . So we repeat. We obtain sequences  $\{y_n\}$  in  $Y$  and  $\{x_n\}$  in  $X$ , s.t.  $\|x_n\| < 2^{-n}$ ,  $\forall n$  and  $\|y_n - Tx_n\| < 2^{-(n+1)}r$ . Notice that

$$\|y_n - Tx_n\| = \|y_{n-1} - Tx_{n-1} - Tx_n\| = \cdots = \|y_1 - \sum_{j=1}^n Tx_j\| = \|y_1 - T \sum_{j=1}^n x_j\|.$$

Now  $\sum x_n$  converges absolutely (since  $\|x_n\| < 2^{-n}$ ). Since  $X$  is complete,  $\exists x \in X$ , s.t.  $\sum x_n = x$ . Moreover,  $\|x\| \leq \sum \|x_n\| < \sum 2^{-n} = 1$ . So  $x \in B_1(0)$ . Finally,

$$y_1 = \lim_{n \rightarrow \infty} T \sum_{j=1}^n x_j = T \left( \lim_{n \rightarrow \infty} \sum_{j=1}^n x_j \right) = T(x) \in T(B_1(x)).$$

Let  $U$  be open and let  $y \in T(U)$ . Then  $\exists x \in U$ , s.t.  $Tx = y$ . Since  $U$  is open,  $\exists \epsilon > 0$ , s.t.  $B_\epsilon(x) \subseteq U$ . Notice that

$$T(B_\epsilon(0)) + Tx = T(B_\epsilon(0) + x) = T(B_\epsilon(x)) \subseteq T(U),$$

and  $Tx = y$ . Finally, by Claim 2,

$$B_{\epsilon r/2}(y) = B_{\epsilon r/2}(0) + y \subseteq \epsilon T(B_1(0)) + y \subseteq T(U).$$

□

**Corollary.** *If  $X$  and  $Y$  are Banach spaces and  $T \in L(X, Y)$  is a bijection. Then  $T^{-1}$  is also a bounded linear operator.*

*Proof.* It suffices to show that  $T^{-1}$  is continuous. Let  $U \subseteq X$  be an open set. Then the preimage of  $U$  with respect to  $T^{-1}$ , which is  $T(U)$ , is open (since  $T$  is an open map by the open mapping theorem). So  $T^{-1}$  is continuous.  $\square$

**Theorem 41** (Closed Graph Theorem). *Let  $X, Y$  be Banach spaces. The map  $T : X \rightarrow Y$  is a bounded linear operator if and only if the graph*

$$\Gamma_T := \{(x, Tx) \in X \times Y : x \in X\}$$

*is a closed linear subspace of  $X \times Y$ . Here  $X \times Y$  is the Banach space with operations  $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$  and  $\alpha(x, y) = (\alpha x, \alpha y)$ ,  $\|(x, y)\| = \|x\|_X + \|y\|_Y$ .*

*Proof.*  $\Rightarrow$ : Homework.

$\Leftarrow$ : Assume that  $\Gamma_T$  is a closed linear subspace.

**Claim 1:**  $T$  is linear.

Let  $x, x' \in X$  and  $\alpha, \alpha' \in \mathbb{R}$ , then  $(x, Tx), (x', Tx') \in \Gamma_T$ . So  $(\alpha x + \alpha' x', \alpha Tx + \alpha' Tx') \in \Gamma_T$ . Thus by definition of  $\Gamma_T$ ,  $\alpha Tx + \alpha' Tx' = T(\alpha x + \alpha' x')$ .

**Claim 2:**  $T$  is continuous.

We know  $\Gamma_T$  is a Banach space. We consider the projections  $P_x : \Gamma_T \rightarrow X$  with  $(x, y) \mapsto x$  and  $P_y : \Gamma_T \rightarrow Y$  with  $(x, y) \mapsto y$ . Then  $P_x$  and  $P_y$  are bounded, since  $\|y\|, \|x\| \leq \|(x, y)\|$ . Furthermore,  $P_x$  is a bijection. So  $P_x^{-1} : X \rightarrow \Gamma_T$ ,  $x \mapsto (x, Tx)$ , is a bounded linear operator (by the previous corollary). Finally,  $T = P_y \circ P_x^{-1}$ .

$\square$

## Invertible linear operators and the van Neumann series

**Theorem 42.** *Let  $X$  and  $Y$  be Banach spaces and let  $\Omega(X, Y)$  denote the set of bijections in  $\mathcal{L}(X, Y)$ . Then  $\Omega(X, Y)$  is an open subset of  $\mathcal{L}(X, Y)$  and the inversion map  $A \mapsto A^{-1}$  is a continuous bijection of  $\Omega(X, Y)$  onto  $\Omega(Y, X)$*

*Proof.* By corollary of the open mapping theorem, we know  $A \in \Omega(X, Y)$  implies  $A^{-1} \in \mathcal{L}(Y, X)$ . Since  $A$  is a bijection,  $A^{-1}$  is also a bijection. So  $A^{-1} \in \Omega(Y, X)$ . Thus the inversion map  $A \mapsto A^{-1}$  is a bijection. It remains to prove that  $\Omega(X, Y)$  is open and  $A \mapsto A^{-1}$  is continuous.

**Lemma:** If  $A \in \Omega(X, Y)$  and  $\|B - A\|_{X \rightarrow Y} < 1/\|A^{-1}\|_{Y \rightarrow X}$ . Then  $B \in \Omega(X, Y)$  and

$$B^{-1} = B^{-1}AA^{-1} = (A^{-1}B)^{-1}A^{-1} = (I - (I - A^{-1}B))^{-1}A^{-1} = \sum_{n=0}^{\infty} (I - A^{-1}B)^n A^{-1},$$

with convergence (indeed absolute convergence) in  $\mathcal{L}(Y, X)$  and

$$\|B^{-1} - A^{-1}\|_{Y \rightarrow X} \leq \frac{\|A^{-1}\|^2 \cdot \|A - B\|}{1 - \|A^{-1}\| \cdot \|A - B\|}$$



*Proof of the lemma.* Observe that

$$\|(I - A^{-1}B)^n A^{-1}\| \leq \|I - A^{-1}B\|^n \cdot \|A^{-1}\| = \|A^{-1}(A - B)\|^n \cdot \|A^{-1}\| \leq \|A^{-1}\|^{n+1} \cdot \|A - B\|^n.$$

By hypothesis, we know  $\|A^{-1}\| \cdot \|A - B\| < 1$ . So by geometric series test,  $\sum_{n=0}^{\infty} (I - A^{-1}B)^n A^{-1}$  converges absolutely. And since  $L(Y, X)$  is a Banach space, the series converges. Now we compute

$$\begin{aligned} B \cdot \sum_{n=0}^{\infty} (I - A^{-1}B)^n A^{-1} &= \sum_{n=0}^{\infty} B(I - A^{-1}B)^n A^{-1} \\ &= \sum_{n=0}^{\infty} A(I - (I - A^{-1}B))(I - A^{-1}B)^n A^{-1} \\ &= \sum_{n=0}^{\infty} A(I - A^{-1}B)^n A^{-1} - \sum_{n=1}^{\infty} A(I - A^{-1}B)^n A^{-1} \\ &= A(I - A^{-1}B)^0 A^{-1} \\ &= I. \end{aligned}$$

For the other direction, we compute

$$\begin{aligned} \left( \sum_{n=0}^{\infty} (I - A^{-1}B)^n A^{-1} \right) B &= \sum_{n=0}^{\infty} (I - A^{-1}B)^n A^{-1} B \\ &= \sum_{n=0}^{\infty} (I - A^{-1}B)^n (I - (I - A^{-1}B)) \\ &= \sum_{n=0}^{\infty} (I - A^{-1}B)^n - \sum_{n=1}^{\infty} (I - A^{-1}B)^n \\ &= (I - A^{-1}B)^0 \\ &= I. \end{aligned}$$

Note the first equality is due to the continuity (or equivalently, boundedness) of  $B$ .

**Aside:**  $S_N = \sum_{n=0}^N (I - A^{-1}B)^n A^{-1}$ . We know  $\lim S_N = S$ . We want  $B \cdot S = \lim B S_N$ .

$$\|BS - BS_N\| \leq \|B\| \cdot \|S - S_N\| \rightarrow 0.$$

So  $B \in \Omega(X, Y)$  and  $B^{-1}$  is indeed the given sum. Then finally, we have

$$\begin{aligned} \|B^{-1} - A^{-1}\|_{Y \rightarrow X} &= \left\| \sum_{n=1}^{\infty} (I - A^{-1}B)^n A^{-1} \right\| \\ &\leq \sum_{n=1}^{\infty} \|I - A^{-1}B\|^n \cdot \|A^{-1}\| \\ &\leq \sum_{n=1}^{\infty} \|A^{-1}\|^{n+1} \cdot \|A - B\|^n \\ &= \frac{\|A^{-1}\|^2 \cdot \|A - B\|}{1 - \|A^{-1}\| \cdot \|A - B\|}. \end{aligned}$$

□

□

## Multivariable Calculus

**Definition.** Let  $(V, \|\cdot\|_V)$ ,  $(W, \|\cdot\|_W)$  be normed vector spaces. Let  $E$  be an open subset of  $V$  and  $f : E \rightarrow W$ . We say  $f$  is differentiable at  $v \in E$  if  $\exists L_v \in \mathcal{L}(V, W)$  s.t.

$$\lim_{h \in V, h \rightarrow 0} \frac{\|f(v+h) - f(v) - L_v(h)\|}{\|h\|} = 0.$$

We call  $L_v$  the total derivative of  $f$  at  $v$  and write

$$Df_v = Df(v) = L_v \in \mathcal{L}(V, W).$$

**Proposition.** *Total derivatives, if they exist, are unique.*

*Proof.* If  $L_v$  and  $L'_v$  both satisfies the equation above. Then by triangle inequality, we have

$$\lim_{h \rightarrow 0} \frac{\|L_v(h) - L'_v(h)\|_W}{\|h\|_V} = 0.$$

So  $\forall \epsilon > 0$ ,  $\exists \delta > 0$ , s.t.  $\|h\| < \delta$  implies

$$\frac{\|L_v(h) - L'_v(h)\|}{\|h\|} < \epsilon.$$

Since

$$\frac{\|L_v(h) - L'_v(h)\|}{\|h\|} = \frac{\| \|h\| \cdot (L_v(h/\|h\|) - L'_v(h/\|h\|)) \|}{\|h\|} = \|L_v - L'_v\|,$$

we have  $\|L_v - L'_v\| < \epsilon$ . Since  $\epsilon$  is arbitrary,  $L_v = L'_v$ . □

**Proposition.** *If  $f$  is differentiable at  $v$ , then  $f$  is continuous at  $v$ .*

*Proof.* If  $f$  is differentiable at  $v$ , look at

$$\|f(v+h) - f(v)\| = o(\|h\|) + \|L_v(h)\| \leq o(\|h\|) + \|L_v\| \cdot \|h\| \rightarrow 0.$$

□

**Example.** If  $f \in \mathcal{L}(V, W)$ , then  $f$  is differentiable at each point  $v \in V$  and  $Df_v = f$ .

*Proof.* By linearity,  $f(v+h) - f(v) - f(h) \equiv 0$  and we are done. □

**Example.** Let  $V = \mathbb{R}^k$ ,  $W = \mathbb{R}$ . Let  $f(x) = \|x\|_2^2 = \sum_{j=1}^k x_j^2$ . Then  $\forall v \in \mathbb{R}^k$ ,  $f$  is differentiable at  $v$  and  $Df_v(h) = 2v^T h$ . In matrix form,  $Df_v = [2v_1, 2v_2, \dots, 2v_k]$ .

*Proof.*

$$\lim_{h \rightarrow 0} \frac{\|f(v+h) - f(v) - 2v^T h\|}{\|h\|} = \lim_{h \rightarrow 0} \frac{\|h\|^2}{\|h\|} = 0.$$

□

**Example.** Let  $V$  be an arbitrary space but  $V \neq \{0\}$ ,  $W = \mathbb{R}$ . Let  $f(x) = \|x\|$  (any norm). Then  $f$  is not differentiable at 0.

*Proof.* Let  $L_v \in \mathcal{L}(V, \mathbb{R})$ . Let  $e \in V$  with  $\|e\| = 1$ . By definition,

$$0 = \lim_{h \in \mathbb{R}, h \rightarrow 0} \frac{|f(0 + he) - f(0) - L_v(he)|}{|h|} = \lim_{h \in \mathbb{R}, h \rightarrow 0} \frac{||h| - hL_v(e)|}{|h|} = \lim_{h \in \mathbb{R}, h \rightarrow 0} \left| 1 - \frac{h}{|h|} L_v(e) \right|.$$

If  $L_v(e) = 0$ , this is false because the limit is 1. If  $L_v(e) \neq 0$ , this is false because the left and right limits of the RHS disagree.  $\square$

**Example.** Let  $V = \mathcal{C}^0([a, b])$ ,  $W = \mathbb{R}$ . Then  $F(g) = \int_a^b (g(x))^2 dx$  is differentiable at  $g$ ,  $\forall g \in \mathcal{C}^0([a, b])$  and  $DF_g(f) = \int_a^b 2f(x)g(x) dx$ .

*Proof.*

$$\|F(g + h) - F(g) - DF_g(h)\| = \left\| \int_a^b h^2(x) dx \right\| \leq (b - a)\|h\|^2 = o(\|h\|).$$

$\square$

**Example.** Let  $X, Y$  be Banach spaces. Define  $V = \mathcal{L}(X, Y)$ ,  $W = \mathcal{L}(Y, X)$  and  $E = \Omega(X, Y)$  (the set of all bijections from  $X$  to  $Y$ ),  $F(T) = T^{-1}$  for  $T \in \Omega(X, Y)$ . Then  $F$  is differentiable at each point of  $E$  and

$$DF_T(h) = -T^{-1}hT^{-1}$$

(Remember that  $DF_T \in \mathcal{L}(V, W)$ ,  $h \in \mathcal{L}(X, Y)$  and  $DF_T(h) \in \mathcal{L}(Y, X)$ .)

*Proof.* Let  $T \in \Omega(X, Y)$ . For  $h$  small, we can write

$$F(T + h) = (T + h)^{-1} = (T(I + T^{-1}h))^{-1} = (I + T^{-1}h)^{-1}T^{-1} = \sum_{n=0}^{\infty} (-1)^n (T^{-1}h)^n T^{-1}.$$

Now

$$F(T + h) - F(T) + T^{-1}hT^{-1} = \sum_{n=2}^{\infty} (-1)^n (T^{-1}h)^n T^{-1},$$

since  $F(T)$  cancels out the zeroth term and  $T^{-1}hT^{-1}$  cancels out the first term. So

$$\|F(T + h) - F(T) + T^{-1}hT^{-1}\| \leq \frac{\|T^{-1}h\|^2 \cdot \|T^{-1}\|}{1 - \|T^{-1}h\|} \leq \frac{\|T^{-1}\|^3 \cdot \|h\|^2}{1 - \|T^{-1}\| \cdot \|h\|} = \mathcal{O}(\|h\|^2).$$

$\square$

**Theorem 43** (Chain rule). *Let  $V, W, X$  be normed vector spaces and assume that  $f : V \rightarrow W$  is differentiable at  $v \in V$  and  $g : W \rightarrow X$  is differentiable at  $f(v)$ . Then  $g \circ f$  is differentiable at  $v$  and  $D(g \circ f)_v = Dg_{f(v)} \circ Df_v \in \mathcal{L}(V, X)$ .*

Special case: If  $V, W, X$  are finite dimensional vector spaces, then we can think of  $Dg_{f(v)}$ ,  $Df_v$  as matrices and we can think of the composition  $Dg_{f(v)} \circ Df_v$  as matrix multiplication.

*Proof.* Notation:

$$\begin{aligned}\epsilon(h) &= \|h\|^{-1} (f(v+h) - f(v) - Df_v(h)), \\ \delta(k) &= \|k\|^{-1} (g(f(v)+k) - g(f(v)) - Dg_{f(v)}(k)).\end{aligned}$$

We know that

$$\lim_{h \rightarrow 0} \epsilon(h) = 0_W, \quad \lim_{k \rightarrow 0} \delta(k) = 0_X.$$

Let  $k = f(v+h) - f(v)$ , we have

$$\begin{aligned}& g(f(v+h)) - g(f(v)) - Dg_{f(v)} \circ Df_v(h) \\ &= g(f(v) + (f(v+h) - f(v))) - g(f(v)) - Dg_{f(v)}(f(v+h) - f(v)) + Dg_{f(v)}(f(v+h) - f(v) - Df_v(h)) \\ &= \|k\| \cdot \delta(k) + Dg_{f(v)}(\|h\| \cdot \epsilon(h)) =: T_1 + T_2.\end{aligned}$$

Notice that  $\|T_2\| \leq \|Dg_{f(v)}\| \cdot \|h\| \cdot \|\epsilon(h)\|$ , so

$$\lim_{h \rightarrow 0} \frac{\|T_2\|}{\|h\|} = 0.$$

On the other hand,

$$\|k\| = \|(\|h\|\epsilon(h) + Df_v(h))\| \leq \|h\| \cdot (\|\epsilon(h)\| + \|Df_v\|).$$

By continuity of  $f$  at  $v$ ,  $\lim_{h \rightarrow 0} \|f(v+h) - f(v)\| = 0$ . So  $\lim_{h \rightarrow 0} \delta(f(v+h) - f(v)) = 0$ . So

$$\lim_{h \rightarrow 0} \frac{\|T_1\|}{\|h\|} \leq \lim_{h \rightarrow 0} (\|\epsilon(h)\| + \|Df_v\|) \cdot \|h\| = 0.$$

□

Henceforth,  $V = \mathbb{R}^k$ ,  $W = \mathbb{R}^m$ , all norms are Euclidean.

**Proposition.** *If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable at  $x_0$ , then all first order partial derivatives of  $f$  at  $x_0$  exist and moreover,*

$$Df_{x_0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \cdots & \frac{\partial f_m}{\partial x_n}(x_0) \end{bmatrix}.$$

*Proof.* Notice that

$$\lim_{h \rightarrow 0, h \in \mathbb{R}} \frac{\|f(x_0 + he_j) - f(x_0) - Df_{x_0}(he_j)\|}{\|h\|} = 0.$$

Also we have

$$\lim_{h \rightarrow 0, h \in \mathbb{R}} \frac{\|f(x_0 + he_j) - f(x_0) - Df_{x_0}(he_j)\|}{\|h\|} = \left\| \frac{\partial f}{\partial x_j}(x_0) - Df_{x_0}e_j \right\|.$$

Since  $\frac{\partial f}{\partial x_j}(x_0)$  is the  $j$ th column of the Jacobian matrix and  $Df_{x_0}(e_j)$  is the  $j$ th column of total derivative. So we are done. □

**Theorem 44** (Single variable mean value theorem). *If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then  $\exists c \in (a, b)$ , s.t.*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

How can we generalize this result to  $f : E \rightarrow \mathbb{R}^m$ , where  $E \subseteq \mathbb{R}^n$ ? First, we consider the case  $n \geq 1, m = 1$ .

**Definition.** Let  $V$  be a vector space. The set  $E \subseteq V$  is convex if  $\forall x, y \in E$ , and  $0 \leq t \leq 1$ ,  $(1 - t)x + ty \in E$ . In other words, the line segment connecting  $x$  and  $y$  is in  $E$ .

**Theorem 45.** *Let  $E \subseteq \mathbb{R}^n$  be open and convex. Let  $f : E \rightarrow \mathbb{R}$  be differentiable. Let  $x, y \in E$ . Then  $\exists t \in [0, 1]$ , s.t.*

$$f(y) - f(x) = \text{D}f((1 - t)x + ty)(y - x).$$

*Note, here  $\text{D}f((1 - t)x + ty)$  is an  $1 \times n$  matrix and  $y - x$  is an  $n \times 1$  matrix.*

*Proof.* Define  $g(t) = f((1 - t)x + ty)$ . Then  $g$  is continuous on  $[0, 1]$ . By the chain rule,  $g$  is differentiable on  $(0, 1)$  and  $g'(t) = \text{D}f((1 - t)x + ty) \cdot (y - x)$ . Finally, by the mean value theorem,  $\exists t \in [0, 1]$ , s.t.  $g'(t) = g(1) - g(0) = f(y) - f(x)$ . We are done.  $\square$

Now consider  $m > 1$ . The direct analogue of mean value theorem is false.

**Example.** Let  $\gamma(t) = (\cos(t), \sin(t))$ . Then,  $\gamma(0) = \gamma(2\pi)$ , i.e.  $\gamma(0) - \gamma(2\pi) = 0$ . But  $\gamma'(t) = (-\sin(t), \cos(t))$  is never 0.

**Theorem 46.** *Let  $E \subseteq \mathbb{R}^n$  be open, convex and  $f : E \rightarrow \mathbb{R}^m$  be differentiable. If  $\|\text{D}f(x)\|_{\mathcal{L}(l_n^2, l_m^2)} \leq M$  (this norm is not the 2-norm of the matrix),  $\forall x \in E$ , then*

$$\|f(x) - f(y)\| \leq M\|x - y\|$$

*for all  $x, y \in E$ .*

*Proof.* Fix  $x_0, y_0 \in E$ . Define  $v := f(y_0) - f(x_0)$  and  $g(x) := v^T f(x)$ . By the previous theorem,  $\exists t \in [0, 1]$ , s.t.

$$\|v\| \cdot \|f(x_0) - f(y_0)\| = g(y_0) - g(x_0) = v^T \text{D}f((1 - t)x_0 + ty_0)(y_0 - x_0) \leq \|v\| \cdot M \cdot \|y_0 - x_0\|.$$

$\square$

Both theorems are false if  $E$  is not convex.

**Example.** Define

$$E := \{(x, y) \in \mathbb{R}^2 : |x| < 1, |y| < 1, y = 0 \Rightarrow x = 0\}.$$

Let

$$f(x, y) = \begin{cases} 0 & \text{if } x \leq 0, \\ 0 & \text{if } x > 0, y < 0, \\ e^{-1/x} & \text{if } x > 0, y > 0. \end{cases}$$

**Theorem 47.** *Let  $E \subseteq \mathbb{R}^n$  be open connected and  $f : E \rightarrow \mathbb{R}^m$  be differentiable with  $\text{D}f \equiv 0$  on  $E$ . Then  $f$  is a constant function on  $E$ .*

*Proof.* Let  $x_0 \in E$ . Let  $F := \{x \in E : f(x) = f(x_0)\}$ . Then  $F$  is closed (relative to  $E$ ) and  $f$  is continuous. Let  $y_0 \in F$ . Then  $\exists r > 0$ , s.t.  $B_r(y_0) \subseteq E$ . Since  $B_r(y_0)$  is convex, by the previous theorem,  $f(x) - f(y_0) = 0$ ,  $\forall x \in B_r(y_0)$ . So  $B_r(y_0) \in F$ . Thus  $F$  is open. Since  $E$  is connected,  $F$  closed and open implies  $F = E$  or  $F = \emptyset$ . But  $x_0 \in F$ , so  $F \neq \emptyset$ . Thus  $F = E$ .  $\square$

Question: Is there any relationship between partial derivatives and differentiability of  $f$ ?

Answer: No.

**Example.** Let

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0) \end{cases}$$

Then all partial derivatives of  $f$  of all orders exist. But  $f$  is not differentiable, since it is not continuous at 0.

**Definition.** Let  $E^{\text{open}} \subseteq \mathbb{R}^n$  and  $f : E \rightarrow \mathbb{R}^m$ . Say  $f$  is continuously differentiable on  $E$  if  $f$  is differentiable on  $E$  and  $Df$  is continuous on  $E$  (i.e. as a map from  $E$  into  $\mathcal{L}(l_n^2, l_m^2)$ ).

**Definition.** Say  $f \in \mathcal{C}^1(E)$  if  $f$  is continuously differentiable on  $E$  and

$$\|f\|_{\mathcal{C}^1(E)} := \|f\|_{\mathcal{C}^0(E)} + \|Df\|_{\mathcal{C}^0(E)}.$$

**Definition.** Say  $f \in \mathcal{C}_{loc}^1(E)$  if  $f$  is continuously differentiable on  $E$ . Warning:  $\mathcal{C}_{loc}^1(E)$  is not a normed (not even a normable) space.

**Theorem 48.** Let  $E$  be a nonempty open subspace of  $\mathbb{R}^n$  and  $f : E \rightarrow \mathbb{R}^m$ . Then  $f$  is continuously differentiable on  $E$  if and only if all first order partial derivatives of  $f$  exist and are continuous on  $E$ .

*Proof.*  $\Rightarrow$ : Immediate follows from the proposition earlier.

$\Leftarrow$ : By the proposition, we just need to show that

$$\lim_{h \rightarrow 0} \frac{\|f(x_0 + h) - f(x_0) - Df(x_0)h\|}{\|h\|} = 0, \quad \forall x_0 \in E.$$

Note, here  $Df(x_0)$  is the Jacobian matrix. Considering each component separately, it suffices to consider the case  $m = 1$ . Fix  $x_0 \in E$  and let  $\epsilon > 0$ . Choose  $\delta > 0$ , s.t.  $B_\delta(x_0) \subseteq E$  and  $x \in B_\delta(x_0) \Rightarrow \|\nabla f(x) - \nabla f(x_0)\| < \epsilon$ . Let  $\vec{h} \in \mathbb{R}^n$  and assume  $\|\vec{h}\| < \delta$ . Set  $\vec{k}_0 := (0, \dots, 0)$  and  $\vec{k}_j := (h_1, h_2, \dots, h_j, 0, \dots, 0)$ . Therefore,

$$f(x_0 + h) - f(x_0) = \sum_{j=1}^n f(x_0 + k_j) - f(x_0 + k_{j-1}).$$

By the mean value theorem for one variable, for each  $j$ , there exists  $\theta_j \in [0, 1]$ , such that

$$f(x_0 + k_j) - f(x_0 + k_{j-1}) = \left( \frac{\partial f}{\partial x_j}(x_0 + k_{j-1} + \theta_j h_j e_j) \right) h_j.$$

Then

$$f(x_0 + h) - f(x_0) - \nabla f(x_0) \cdot h = \sum_{j=1}^n \left( \frac{\partial f}{\partial x_j}(x_0 + k_{j-1} + \theta_j h_j e_j) - \frac{\partial f}{\partial x_j}(x_0) \right) h_j.$$

So

$$|f(x_0 + h) - f(x_0) - \nabla f(x_0) \cdot h| \leq \left| \sum_{j=1}^n \left( \frac{\partial f}{\partial x_j}(x_0 + k_{j-1} + \theta_j h_j e_j) - \frac{\partial f}{\partial x_j}(x_0) \right) h_j \right| < n\epsilon \|h\|.$$

□

**Theorem 49** (Inverse function theorem). *Let  $E$  be an open subset of  $\mathbb{R}^n$ . Let  $f : E \rightarrow \mathbb{R}^n$  be continuously differentiable on  $E$ . Assume  $Df(a)$  is invertible (as a linear operator) for some  $a \in E$ . Then there exists an open neighbourhood  $U$  of  $a$  and  $V$  of  $f(a)$  such that  $f$  is a bijection of  $U$  onto  $V$  and if  $g : V \rightarrow U$  is the local inverse of  $f$ . Then  $g$  is continuously differentiable on  $V$  and*

$$Dg(y) = Df(g(y))^{-1}, \quad y \in V.$$

Warning:  $Df$  invertible throughout  $E$  and  $E$  is connected does not imply that  $f$  has a global inverse, unless  $n = 1$ .

*Proof.* First we do some reductions. We may assume that  $a = f(a) = 0$  and  $Df(a) = I_n$ . Otherwise, replace  $f$  with  $Df(a)^{-1}(f(x + a) - f(a))$ .

Since  $f$  is continuously differentiable, there exists  $\delta > 0$ , s.t.  $\|Df(x) - I_n\| < 1/2$  for all  $|x| < \delta$ . Let's fix  $y_0 \in \mathbb{R}^n$  and define  $\varphi_{y_0}(x) = x - (f(x) - y_0)$ . Then we have  $\forall x_1, x_2 \in B_\delta(0)$ .

$$\|\varphi_{y_0}(x_1) - \varphi_{y_0}(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\|.$$

Therefore,  $\varphi_{y_0}$  has at most one fixed point. In other words,  $f(x) = y_0$  has at most one solution  $x \in B_\delta(0)$ . Therefore,  $f$  is one-to-one on  $B_\delta(0)$ . Let  $U := B_\delta(0)$ ,  $V := f(U)$ . Then  $f$  is a bijection of  $U$  onto  $V$  and we need to show that  $V$  is open. Let  $x_0 \in U$  and set  $y_0 := f(x_0)$  (We need to show that  $\exists r > 0$ , s.t.  $\overline{B_r(y_0)} \subseteq V$ , i.e.  $\forall y \in \overline{B_r(y_0)}$ ,  $\exists x \in U$ , s.t.  $f(x) = y$ ).

Fix  $r > 0$ , s.t.  $\overline{B_r(x_0)} \subseteq U$ . We will prove that  $B_{r/2}(y_0) \subseteq f(\overline{B_r(x_0)})$ . Let  $y \in B_{r/2}(y_0)$ . Observe that for  $|x - x_0| \leq r$ ,

$$\begin{aligned} |\varphi_y(x) - x_0| &\leq |\varphi_y(x) - \varphi_y(x_0)| + |\varphi_y(x_0) - x_0| \\ &\leq \frac{1}{2}\|x - x_0\| + |f(x_0) - y| \\ &< \frac{1}{2}r + \frac{1}{2}r = r. \end{aligned}$$

So  $\varphi_y$  maps  $\overline{B_r(x_0)}$  into  $B_r(x_0)$ . So  $\varphi_y$  is a contraction on the complete metric space  $\overline{B_r(x_0)}$ . Therefore,  $\varphi_y$  has a unique fixed point,  $x \in \overline{B_r(x_0)}$ , in other words, there exists  $x \in \overline{B_r(x_0)}$ , s.t.  $f(x) = y$ . Therefore,  $y \in f(\overline{B_r(x_0)})$ . Since  $y$  is arbitrary,  $B_{r/2}(y_0) \subseteq f(\overline{B_r(x_0)}) \subseteq f(U) = V$ . Since  $y_0$  was arbitrary,  $V$  is open.

Let  $g$  denote the local inverse. We need to show that  $g$  is continuously differentiable. Let  $y, k$  be such that  $y, y + k \in V$ . Then let  $x := g(y)$  and  $h := g(y + k) - x$ . So  $f(x) = y$  and

$f(x+h) = y+k$ . Recall,  $\varphi_y(x) := x - (f(x) - y)$  is a contraction with shrinking constant  $1/2$  on  $U$ . Since  $\varphi_y(x+h) - \varphi_y(x) = h - k$ , we have

$$\|h - k\| \leq \frac{1}{2}\|(x+h) - x\| = \frac{1}{2}\|h\|.$$

So  $\|k\| \geq \frac{1}{2}\|h\|$ . Therefore,

$$\begin{aligned} \frac{\|g(y+k) - g(y) - Df(g(y))^{-1}k\|}{\|k\|} &= \frac{\|x+h - x - Df(x)^{-1}(f(x+h) - f(x))\|}{\|k\|} \\ &\leq \frac{\|Df(x)^{-1}\| \cdot \|f(x+h) - f(x) - Df(x)h\|}{\frac{1}{2}\|h\|}. \end{aligned}$$

Let  $\epsilon > 0$ . Then  $\exists \delta = \delta(x, \epsilon) > 0$ , s.t.  $\|h\| < \delta$  implies RHS of the above inequality is less than  $\epsilon$ . Now if  $\|k\| < \frac{1}{2}\delta$ , then  $\|h\| < \delta$ . So the LHS of the above inequality is less than  $\epsilon$ . So  $g$  is differentiable at  $y$  and  $Dg(y) = Df(g(y))^{-1}$ .

Finally, we need to show that  $g$  is continuously differentiable. We know that  $g$  is continuous since it is differentiable.  $Df$  is continuous by assumption and we proved  $A \mapsto A^{-1}$  is continuous. So  $Dg$ , the composition of these maps is continuous.  $\square$

**Example.**  $f(x, y) = (e^x \cos y, e^x \sin y)$ , for  $(x, y) \in \mathbb{R}^2$ .  $f$  does not have a global inverse, since the  $y$  variable has period  $2\pi$ .

**Theorem 50.** *Let  $E$  be an open subset of  $\mathbb{R}^n$ . If  $f : E \rightarrow \mathbb{R}^n$  is continuously differentiable and  $Df(x)$  is invertible  $\forall x \in E$ . Then  $f$  is an open map.*

*Proof.* Let  $W \subseteq E$  be open. Let  $x \in W$ . Let  $U_x$  and  $V_x$  be the sets guaranteed in the last theorem. Let  $g_x : V_x \rightarrow U_x$  be the local inverse. Then,

$$f(U_x \cap W) = g_x^{-1}(U_x \cap W)$$

is a neighbourhood of  $f(x)$  which is contained in  $f(W)$ . So  $f(x)$  is an interior point of  $f(W)$ . Since  $x \in W$  was arbitrary,  $f(W)$  is open.  $\square$

**Theorem 51** (Implicit function theorem). *Let  $E^{\text{open}} \subseteq \mathbb{R}^n \times \mathbb{R}^m$  and  $F : E \rightarrow \mathbb{R}^m$  continuously differentiable. Assume  $F(a, b) = 0$  for some  $(a, b) \in E$  and that*

$$D_y F(a, b) := \begin{bmatrix} \frac{\partial F_1}{\partial y_1}(a, b) & \cdots & \frac{\partial F_1}{\partial y_m}(a, b) \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial y_1}(a, b) & \cdots & \frac{\partial F_m}{\partial y_m}(a, b) \end{bmatrix}$$

*is nonsingular. Then there exists neighbourhoods  $U$  of  $a$  and  $V$  of  $b$  and a function  $f : U \rightarrow V$ , s.t.  $\forall x \in U, y \in V, F(x, y) = 0$  if and only if  $y = f(x)$ . Moreover,  $f$  is continuously differentiable on  $U$  and*

$$Df(x) = -D_y F(x, f(x))^{-1} D_x F(x, f(x)).$$

*Proof.* The idea is to use the inverse function theorem. Define  $G(x, y) := (x, F(x, y))$ . Then  $G$  is continuously differentiable on  $E$  and

$$DG(a, b) = \begin{bmatrix} I_n & 0 \\ D_x F(a, b) & D_y F(a, b) \end{bmatrix}$$



is nonsingular. So there exist neighbourhoods  $A$  of  $(a, b)$  and  $B$  of  $G(a, b)$  such that  $G(A) = B$  and a local inverse  $H : B \rightarrow A$ . Then  $H$  must take the form  $H(x, z) = (x, h(x, z))$  (we use  $h$  to denote the second element of  $H$ ). Let's pick  $\epsilon > 0$ , s.t.  $B_\epsilon(a) \times B_\epsilon(b) \subseteq A$ . Then  $G(B_\epsilon(a) \times B_\epsilon(b))$  is open, contains  $(a, 0)$  (which is  $G(a, b)$ ). So  $\exists \delta > 0$ , s.t.  $B_\delta(a) \times \{0\} \subseteq G(B_\epsilon(a) \times B_\epsilon(b))$ . Notice that  $B_\delta(a) \subseteq B_\epsilon(a)$ . Now define  $f : B_\delta(a) \rightarrow B_\epsilon(b)$  by  $f(x) := h(x, 0)$  (this is okay since  $(x, 0) \subseteq B$  and  $G(A) = B$ ). Let  $U = B_\delta(a)$ ,  $V = B_\epsilon(b)$ . If  $x \in U$ ,  $y \in V$ , then  $(x, y) \in B_\epsilon(a) \times B_\epsilon(b) \subseteq A$ . So

$$F(x, y) = 0 \iff G(x, y) = (x, 0) \iff y = h(x, 0) = f(x).$$

$f$  is continuously differentiable because  $H$  is. Formula for  $Df$ :

$$F(x, f(x)) \equiv 0.$$

By the chain rule, we have

$$D_x F(x, f(x)) + D_y F(x, f(x)) Df(x) = 0.$$

□

**Example** (an application of the implicit function theorem). The zero set is  $\{(x, y) : F(x, y) = 0\}$ . The theorem tells us that it can be written (locally) as  $\{(x, f(x)) : x \in U \subseteq \mathbb{R}^n\}$ .

**Definition.**  $X, Y$  are finite dimensional vector spaces. Let  $T \in \mathcal{L}(X, Y)$ . Define the null space as

$$\mathcal{N}(T) := \{x \in X : Tx = 0\}.$$

Define the range as

$$\mathcal{R}(T) := TX$$

which is a subspace of  $Y$ . Define the rank of  $T$

$$\text{rank}(T) = \dim \mathcal{R}(T) = \dim X - \dim \mathcal{N}(T).$$

If  $X = \mathbb{R}^n$ ,  $Y = \mathbb{R}^m$ ,  $T$  is then associate to a matrix  $A$ . In this case

$$\text{rank}(T) = \dim(\text{span of columns of } A) = \dim(\text{span of rows of } A).$$

If (in the Euclidean case)  $\text{rank of } T = r \leq \min(m, n)$ . Then by elementary row and column operations, we can transform  $A$  into

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

In other words,  $\exists B \in \text{GL}(m)$  and  $C \in \text{GL}(n)$ , s.t.

$$BAC = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Here  $\text{GL}(m)$  stands for invertible  $m \times m$  matrices.

The idea behind rank theorem is a nonlinear version of the above.

**Proposition.** Let  $X, Y$  be finite dimensional vector spaces.  $A \mapsto \text{rank}(A)$  is a lower semi continuous map from  $\mathcal{L}(X, Y)$  into  $\{0, 1, \dots, \min\{\dim X, \dim Y\}\}$ , i.e.

$$\inf_{T \rightarrow A} \lim \text{rank}(T) \geq \text{rank}(A).$$

i.e.  $\exists \delta > 0$ , s.t.  $\|T - A\| < \delta$  implies  $\text{rank}(T) \geq \text{rank} A$

The limit on the LHS may not exist, so we use  $\inf \lim$ .

*Proof.* Suppose  $\dim X = n$  and  $\text{rank} A = k$ . Then  $A$  have a  $k \times k$  minor (i.e. pick  $k$  rows and  $k$  columns of  $A$ ) with nonzero determinant. Determinant of a given minor is a continuous map, so there exists a neighbourhood  $U$  of  $A$  s.t.  $\forall T \in U$ , some  $k \times k$  minor has nonzero determinant. So the rank is  $\geq k$ .  $\square$

**Corollary.** Let  $E$  be an open subset of  $\mathbb{R}^n$  and  $f : E \rightarrow \mathbb{R}^m$  is continuously differentiable on  $E$ . If  $a \in E$  and  $\text{rank}(Df(a)) = \min\{m, n\}$ , then there exists a neighbourhood  $U$  of  $a$ , s.t.  $Df(x)$  has rank  $\min\{m, n\}$  throughout  $U$ .

**Theorem 52** (Rank Theorem). Let  $E$  be an open subset of  $\mathbb{R}^n$  and  $f : E \rightarrow \mathbb{R}^m$  be continuously differentiable on  $E$ .

1. Assume that  $Df(x)$  has rank  $r$  for all  $x \in E$ . Then  $\forall a \in E$ ,  $\exists$  open sets  $U, V$  with  $a \in U$ , a continuously differentiable bijection  $H : V \rightarrow U$  with  $DH$  nonsingular on  $V$  and a linear isomorphism  $G \in GL(\mathbb{R}^m)$ , s.t.

$$G \circ f \circ H(x) = (x_1, \dots, x_r, 0, \dots, 0) + (0, \dots, 0, \varphi(x_1, \dots, x_r))$$

with  $D\varphi(H(a)) = 0$ . Here  $\varphi$  maps from  $\mathbb{R}^r$  to  $\mathbb{R}^{m-r}$ .

2. If  $Df(a)$  has rank  $r$ , then  $\exists$  open sets  $U, V$  and bijection  $H : V \rightarrow U$  (continuously differentiable with continuously differentiable inverse) and an isomorphism  $G$  such that

$$G \circ f \circ H(x) = (x_1, \dots, x_r, 0, \dots, 0) + (0, \dots, 0, \psi(x_1, \dots, x_n)).$$

with  $D\psi(0) = 0$ . Here  $\psi$  maps from  $\mathbb{R}^n$  to  $\mathbb{R}^{m-r}$ .

*Proof.* 1. If  $r = 0$ .  $f$  is a constant on each connected component of  $E$ . We are done.

Assume  $r > 0$ . Replacing  $f$  with  $f(x + a) - f(a)$ , we may assume  $a = 0$  and  $f(a) = 0$ . Replacing  $f$  with  $A \circ f \circ B$  for appropriately chosen  $A \in GL(m)$ ,  $B \in GL(n)$ , we may assume

$$Df(0) = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Define an auxiliary function

$$F(x) := (f_1(x), \dots, f_r(x), x_{r+1}, \dots, x_n).$$

Then  $DF(0) = I_n$  and  $F(0) = 0$ . So  $\exists$  neighbourhoods  $U, V$  of 0, s.t.  $F$  is a bijection of  $U$  onto  $V$  and  $H := (F|_U)^{-1}$  is continuously differentiable on  $V$ . Now look at  $f \circ H$ ,

$$f \circ H(y) = (y_1, \dots, y_r, \psi(y)),$$

Now we need to show that  $\psi$  is independent of  $y_{r+1}, \dots, y_n$ . Decompose  $y$  as  $(u, v)$  where  $u \in \mathbb{R}^r$  and  $v \in \mathbb{R}^{n-r}$ .

**Claim:**  $D_v\psi \equiv 0$ .

$$D(f \circ H) = \begin{bmatrix} I_r & 0 \\ D_u\psi & D_v\psi \end{bmatrix}$$

has rank equal to  $r + \text{rank} D_v(\psi)$ . But  $D(f \circ H) = (Df) \circ (DH)$ .  $DH$  has rank  $r$ , because  $Df$  has rank  $r$  and  $DH$  is invertible. So  $D_v\psi$  has rank 0, i.e.  $D_v\psi \equiv 0$ .

Shrinking  $V$ , we may assume  $V$  is convex. Then the claim implies  $\psi$  is independent of  $V$ , i.e.

$$\psi(u, v) = \varphi(u),$$

for some function  $\varphi$  on  $\pi(v) := \{u : \exists v, \text{ s.t. } (u, v) \in V\}$

2. Use the beginning of the proof of the first part. Work up to the point where we show  $D_v\psi \equiv 0$  and we don't need that for the second part.

□

**Corollary.** *In case 1 of the rank theorem, by allowing  $G$  to be continuously differentiable with continuously differentiable inverse, we can have  $G \circ f \circ H(x) = (x_1, \dots, x_r, 0, \dots, 0)$ .*

*Proof.*

$$\text{new}G(y) = \text{old}G(y) - (0, \varphi(y_1, \dots, y_r)).$$

□

## Higher Order Derivatives

**Theorem 53.** *Let  $E$  be an open subset of  $\mathbb{R}^n$ . Let  $f : E \rightarrow \mathbb{R}$ . Assume that all second order partials of  $f$  exist and are continuous throughout  $E$ . Then  $\forall 1 \leq i, j \leq n$ ,*

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f = \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} f.$$

*Formally speaking, for  $\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f$ , we differentiate 1st w.r.t. the inner partial.*

Remark: By assumption,  $\nabla f := (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$  is continuously differentiable. So  $\nabla f$  is continuous, so  $f$  is differentiable throughout  $E$ . Furthermore,

$$D\nabla f = D^2f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}.$$

*Proof.* Reindexing suffices to consider  $i = 1, j = 2$ . Fixing all other variables, we may assume  $n = 2$ . Choose  $x, h$ , s.t.  $x, x + h \in E$  and  $h_1, h_2 \neq 0$ . Define

$$S_h f(x) := f(x + h) - f(x + (h_1, 0)) - (f(x + (0, h_2)) - f(x)).$$

Define  $g_{h_2}(x) = f(x + (0, h_2)) - f(x)$ . Then  $S_h f(x) = g_{h_2}(x + (h_1, 0)) - g_{h_2}(x)$ . By one variable mean value theorem,  $\exists \theta_1, \theta_2 \in (0, 1)$ , s.t.

$$\begin{aligned} S_h f(x) &= h_1 \frac{\partial g_{h_2}}{\partial x_1}(x + (\theta_1 h_1, 0)) \\ &= h_1 \left( \frac{\partial f}{\partial x_1}(x + (\theta_1 h_1, h_2)) - \frac{\partial f}{\partial x_1}(x + (\theta_1 h_1, 0)) \right) \\ &= h_2 h_1 \frac{\partial^2 f}{\partial x_2 \partial x_1}(x + (\theta_1 h_1, \theta_2 h_2)). \end{aligned}$$

Rearrange:

$$S_h f(x) = f(x + h) - f(x + (0, h_2)) - (f(x + (h_1, 0)) - f(x)).$$

By the mean value theorem again,  $\exists \nu_1, \nu_2 \in (0, 1)$ , s.t.

$$S_h f(x) = h_1 h_2 \frac{\partial^2 f}{\partial x_1 \partial x_2}(x + (\nu h_1, \nu h_2)).$$

So we have

$$\frac{\partial^2 f}{\partial x_2 \partial x_1}(x + (\theta_1 h_1, \theta_2 h_2)) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(x + (\nu h_1, \nu h_2)).$$

Send  $h \rightarrow 0$ . By continuity, we have

$$\frac{\partial^2 f}{\partial x_2 \partial x_1}(x) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(x).$$

□

**Corollary.** Let  $E$  be an open subset of  $\mathbb{R}^n$ . Let  $f : E \rightarrow \mathbb{R}$ . Assume all  $k$ -th order partials of  $f$  exist and are continuous throughout  $E$ . Then we can freely change order of differentiation in  $\frac{\partial}{\partial x_{i_k}} \cdots \frac{\partial}{\partial x_{i_1}} f$  without changing the value.

*Proof.* Use case  $k = 2$  and induction, with the fact that any reordering of partials can be achieved by switching two at a time. □

## Multivariate Taylor's Theorem with Remainder

**Notation.** Fix a dimension  $n$ . A multiindex  $\alpha$  is an element of  $(\mathbb{Z}_{\geq 0})^n$ . If  $\alpha$  is a multiindex,

$$D^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}.$$

If  $x \in \mathbb{R}^n$ , then

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Define  $|\alpha| = \alpha_1 + \cdots + \alpha_n$ , and  $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$ . Define multinomial coefficient,

$$\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}.$$

**Theorem 54** (Multinomial theorem). For  $b \in \mathbb{R}^n$ ,  $m \in \mathbb{N}$ , then

$$(b_1 + \cdots + b_n)^m = \sum_{\alpha: |\alpha|=m} \binom{|\alpha|}{\alpha} b^\alpha.$$

When  $n = 2$ , it is the binomial theorem,

$$\binom{|\alpha|}{\alpha} = \frac{(\alpha_1 + \alpha_2)!}{\alpha_1! \alpha_2!} = \binom{\alpha_1 + \alpha_2}{\alpha_1} = \binom{\alpha_1 + \alpha_2}{\alpha_2}.$$

*Proof.* Use induction on  $n$ . Assume case  $n - 1$  is proved. Write  $b = (b', b_n)$ , where  $b' \in \mathbb{R}^{n-1}$ . We know that

$$\begin{aligned} (b_1 + \cdots + b_n)^m &= \sum_{j=0}^m \binom{m}{j} (b_1 + \cdots + b_{n-1})^j b_n^{m-j} \\ &= \sum_{j=0}^m \sum_{|\alpha|=j} \frac{m!}{j!(m-j)!} \cdot \frac{|\alpha|!}{\alpha!} (b')^\alpha b_n^{m-j} \\ &= \sum_{j=0}^m \sum_{|\alpha|=j} \frac{m!}{j!(m-j)!} \cdot \frac{j!}{\alpha!} (b')^\alpha b_n^{m-j} \\ &= \sum_{j=0}^m \sum_{|\alpha|=j} \frac{m!}{(m-j)!} \cdot \frac{1}{\alpha!} (b')^\alpha b_n^{m-j} \end{aligned}$$

Let  $\beta := (\alpha, m - j)$ , we have

$$(b_1 + \cdots + b_n)^m = \sum_{|\beta|=m} \frac{|\beta|!}{\beta!} b^\beta.$$

□

**Theorem 55** (Multivariate Taylor's theorem with remainder). Assume  $f : B_r(a) \rightarrow \mathbb{R}$  is  $(k + 1)$ -times continuously differentiable on  $B_r(a)$ . For each  $x \in B_r(a)$ ,  $\exists \theta_x \in (0, 1)$ , s.t.

$$f(x) - \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(a) (x - a)^\alpha = \sum_{|\beta|=k+1} \frac{1}{\beta!} D^\beta f((1 - \theta_x)a + \theta_x x) (x - a)^\beta.$$

*Proof.* Let  $g(t) = f((1 - t)a + tx)$ . By the one variable Taylor's theorem,  $\exists t \in (0, 1)$ , s.t.

$$g(1) = \sum_{m=0}^k \frac{1}{m!} g^{(m)}(0) + \frac{1}{(k+1)!} g^{(k+1)}(t).$$

Now we need to simplify  $g^{(m)}(t)$ :

$$\begin{aligned} g^{(m)}(0) &= \left( \frac{d}{dt} \Big|_{t=0} \right)^m f(a + t(x - a)) \\ &= ((x_1 - a_1) \frac{\partial}{\partial y_1} + \cdots + (x_n - a_n) \frac{\partial}{\partial y_n})^m f(a) \\ &= \sum_{|\alpha|=m} \frac{m!}{\alpha!} (x - a)^\alpha D^\alpha f(a). \end{aligned}$$

□

**Definition.** Let  $E$  be an open subspace of  $\mathbb{R}^n$ .  $f : E \rightarrow \mathbb{R}$  is real analytic on  $E$  if  $\forall x_0 \in E, \exists \epsilon > 0$ , s.t.

$$f(x) = \sum_{\alpha} \frac{1}{\alpha!} D^{\alpha} f(x_0) (x - x_0)^{\alpha}, \quad |x - x_0| < \epsilon.$$

## Second derivative test for critical points

By Clairait's theorem, if  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is second order continuously differentiable, then

$$D^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

is symmetric. The symmetric  $n \times n$  matrix  $A$  is positive definite if  $\forall x \neq 0, x^T A x > 0$ .  $A$  is negative definite if  $\forall x \neq 0, x^T A x < 0$ , equivalently,  $(-A)$  is positive definite).

If  $A$  is positive or negative definite, then  $A$  is nonsingular. But  $A$  is nonsingular does not imply whether  $A$  is positive definite or negative definite.

**Example.** The identity matrix,  $I$ , is positive definite. And  $-I$  is negative definite. The matrix  $\text{diag}(1, \dots, 1, -1, \dots, -1)$  is neither positive definite nor negative definite.

**Proposition.** Let  $A$  be an  $n \times n$  matrix. For  $1 \leq k \leq n$ , let  $\Delta_k$  denote the determinant of the upper left  $k \times k$  minor of  $A$ . Then  $A$  is positive definite if and only if  $\Delta_k$  is positive for  $1 \leq k \leq n$ .  $A$  is negative definite if and only if  $(-1)^k \Delta_k > 0$ , for  $1 \leq k \leq n$ .

*Proof.* It suffices to prove the statement about positive definite matrices.

$\Leftarrow$ : By induction. If  $n = 1$ , the base case trivially holds. Assume we have proved for dimensions  $k \leq n - 1$ . Let  $A$  be an  $n \times n$  matrix and assume  $\Delta_k > 0, \forall k$ . Let  $0 \neq x \in \mathbb{R}^n$ . Look at  $x^T A x$ . Write

$$A = \begin{bmatrix} B & b \\ b^T & a \end{bmatrix},$$

where  $B$  is  $(n-1) \times (n-1)$ . If  $x_n = 0$ ,  $x^T A x = (x')^T B x' > 0$ , where  $x = (x', x_n)$ , by induction hypothesis. Now we may assume  $x_n \neq 0$ . Multiplying  $x$  by a constant, we can assume that  $x_n = 1$ . Then define

$$f(x) := x^T A x = x^T B x + 2x^T b + a.$$

Then  $\nabla f(x) = 2Bx + 2b$ . So  $f$  has a critical point at  $x = -B^{-1}b$  ( $B$  is invertible since  $\Delta_{n-1} \neq 0$  by induction hypothesis). Since  $B$  is positive definite,  $\lim_{\|x\| \rightarrow \infty} f(x) = \infty$ . So  $\min f$  is achieved at the critical. So  $f(x) \geq f(-B^{-1}b), \forall x$ . Finally,

$$\begin{aligned} f(-B^{-1}b) &= b^T B^{-1} B B^{-1} b - 2b^T B^{-1} b + a \\ &= b^T B^{-1} b + a \\ &= (\det B)^{-1} (-b^T ((\det B) B^{-1}) b + (\det B) a) \\ &= (\det B)^{-1} \det A \end{aligned}$$

$\Rightarrow$ : If  $A$  is positive definite. Then  $\Delta_k \neq 0, \forall k$ , otherwise, could found a vector  $0 \neq x = (x_1, \dots, x_k, 0, \dots, 0)$  such that  $Ax = 0$ . Considering  $x = e_1, \Delta_1 > 0$ . Suppose  $\Delta_1, \dots, \Delta_{k-1} >$

0 and  $\Delta_k < 0$ . Looking at vectors of the form  $(x_1, \dots, x_k, 0, \dots, 0)$ , we may assume  $k = n$ . In the notation of the previous direction,

$$\begin{bmatrix} -(B^{-1}b)^T & 1 \end{bmatrix} A \begin{bmatrix} -B^{-1}b \\ 1 \end{bmatrix} < 0.$$

□

**Theorem 56** (Second order derivative test). *Let  $E$  be an open subset of  $\mathbb{R}^n$ . Let  $f : E \rightarrow \mathbb{R}$  be twice continuously differentiable on  $E$ . Assume  $\nabla f(x_0) = 0$  ( $x_0$  is a critical point) and that  $D^2f(x_0)$  is nonsingular ( $x_0$  is a nondegenerate critical point),*

1. *If  $D^2f(x_0)$  is positive definite,  $f$  has a strict local minimum at  $x_0$  (Think  $f(x) = x_1^2 + \dots + x_n^2$ ).*
2. *If  $D^2f(x_0)$  is negative definite,  $f$  has a strict local maximum at  $x_0$  (Think  $f(x) = -x_1^2 - \dots - x_n^2$ ).*
3. *If  $D^2f(x_0)$  is neither, then neither local maximum nor local minimum (Think  $f(x) = x_1^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_n^2$ ).*

**Lemma.** *Let  $E$  be an open subset of  $\mathbb{R}^n$ . If  $f : E \rightarrow \mathbb{R}$  is twice continuously differentiable, then  $\forall x \in E$ ,*

$$\lim_{h \rightarrow 0} \frac{|f(x_h) - f(x) - Df(x)h - \frac{1}{2}h^T D^2f(x)h|}{\|h\|^2} = 0.$$

*Proof.* Let  $x, x_h \in E$ , with  $h \neq 0$ . Let  $\omega = h/\|h\|$ . Then  $\|\omega\| = 1$ .

$$\begin{aligned} f(x+h) - f(x) &= \int_0^{\|h\|} \frac{d}{dt} f(x+t\omega) dt \\ &= \int_0^{\|h\|} Df(x+t\omega)\omega dt \\ &= Df(x)h + \int_0^{\|h\|} \omega^T D^2f(x+t\omega)\omega(\|h\| - t) dt \\ &= Df(x)h + \frac{1}{2}h^T D^2f(x)h + \int_0^{\|h\|} \omega^T (D^2f(x+t\omega) - D^2f(x))\omega(\|h\| - t) dt \end{aligned}$$

Let  $\epsilon(\|h\|) = D^2f(x+t\omega) - D^2f(x)$ . The error term

$$\begin{aligned} \left| \int_0^{\|h\|} \omega^T (D^2f(x+t\omega) - D^2f(x))\omega(\|h\| - t) dt \right| &\leq \int_0^{\|h\|} \epsilon(\|h\|)(\|h\| - t) dt \\ &= \frac{1}{2}\|h\|^2 \epsilon(\|h\|). \end{aligned}$$

□