596 Homework 3 Yujia Fan yf198

Problem 1

(a) 99% of the time: $F_1(X_1, X_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dxdy$

1% of the time: $F_2(X_1, X_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} dxdy$

$f_{x,y} = f_1(x_1, x_2)\cdot P_1 + f_2(x_1, x_2)\cdot P_2$

$$= 0.99\cdot\frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} + 0.01\cdot\frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

(b) $f_1(x) = \int_{-\infty}^{+\infty} f_{x,y}\, dy = 0.99\cdot\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy + 0.01\cdot\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\cdot\int_{-\infty}^{+\infty} e^{-\frac{y^2}{2\sigma^2}} dy$

$= 0.99\cdot\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + 0.01\cdot\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$, doesn't contain y

In the same logic,

$f_2(y) = 0.99\cdot\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} + 0.01\cdot\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}$, doesn't contain x

So they are independent.

(c) For two random variables x, y such that $x^2+y^2=1$

uncorrelated: $E(X) = E(Y) = 0$, $E(XY) = E(E(XY|X)) = E(XE[Y|X]) = 0$

$\Rightarrow Cov(X,Y) = E(XY) - E(X)E(Y) = 0$

And obviously, they are not independent.

Problem 2

(a) $P(\zeta \le z) = P(|x+s| \le z) = P(-z-s \le x \le z-s)$

$= F_x(z-s) - F_x(-z-s)$

because $z \geqslant 0$, $f_\zeta(z) = \frac{dP(\zeta \le z)}{dz} = f_x(z-s) - (-1)\cdot f_x(-z-s)$

$= f_x(z-s) + f_x(-z-s)$

(b) s is a random variable.

$$s = \begin{cases} 0, & P=0.2 \\ 1, & P=0.8 \end{cases}$$

$f_\zeta(z) = f_\zeta(z|s=0)\cdot P(s=0) + f_\zeta(z|s=1)\cdot P(s=1)$

$= [f_x(z) + f_x(-z)]\cdot 0.2 + [f_x(z-1) + f_x(-z-1)]\cdot 0.8$

(c) $P(s=0|\zeta=z) = \dfrac{P(\zeta=z|s=0)\cdot P(s=0)}{P(\zeta=z|s=0)\cdot P(s=0) + P(\zeta=z|s=1)\cdot P(s=1)}$

$= \dfrac{P(\zeta=z|s=0)\cdot 0.2}{P(\zeta=z|s=0)\cdot 0.2 + P(\zeta=z|s=1)\cdot 0.8}$

$= \dfrac{P(\zeta=z|s=0)}{P(\zeta=z|s=0) + 4P(\zeta=z|s=1)}$

Problem 3

(a) $\mathbb{P} = p^{N_h} \cdot (1-p)^{N-N_h}$

(b) $H(p) = N_h \ln p + (N-N_h)\ln(1-p)$

$$\frac{dH(p)}{dp} = \frac{N_h}{p} - \frac{N-N_h}{p} = 0 \quad \Rightarrow \quad (1-p)N_h - p(N-N_h) = 0 \quad \text{so } \tilde{p} = \frac{N_h}{N}$$

(c) Average:

$$\mathbb{E}(\tilde{p}) = \mathbb{E}\left[\frac{N_H}{N}\right] = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} 1_{\{x_i=H\}}\right] = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[1_{\{x_i=H\}}] = \frac{1}{N}\cdot N(1*p+0*(1-p)) = p$$

Mean Square Error:

$$\mathbb{E}(\tilde{p}-p)^2 = \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N} 1_{\{x_i=H\}} - p\right)^2 = \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}(1_{\{x_i=H\}}-p)\right)^2$$

$$= \frac{1}{N^2}\mathbb{E}\left[\sum_{i=1}^{N}\sum_{j=1}^{N}(1_{\{x_i=H\}}-p)(1_{\{x_j=H\}}-p)\right]$$

$$= \frac{1}{N^2}\mathbb{E}\left[\sum_{i=1}^{N}(1_{\{x_i=H\}}-p)^2\right]$$

$$= \frac{1}{N^2}\mathbb{E}\left[\sum_{i=1}^{N}(1^2_{\{x_i=H\}} - 2p1_{\{x_i=H\}} + p^2)\right]$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\left[\mathbb{E}(1_{\{x_i=H\}} - 2p1_{\{x_i=H\}} + p^2)\right]$$

$$= \frac{1}{N^2}\cdot N\cdot(p-2p^2+p^2)$$

$$= \frac{p(1-p)}{N} \leq \frac{0.25}{N}$$

As $N \to +\infty$, the mean square error $\to 0$, which mean the estimator of p will nearly not oscillate and obtain the real p.

Problem 4

(a) Because x_1 and w_2 are Gaussian, $x_2 = \alpha x_1 + w_2$ is also Gaussian.

Because x_2 and w_3 are Gaussian, $x_3 = \alpha x_2 + w_3$ is also Gaussian.

...

In the same way, all x_n are Gaussian.

(b) $f(x_n, x_{n-1}, \cdots, x_3, x_2, x_1)$

$$= f(x_n | x_{n-1}, \cdots, x_1) \cdot f(x_{n-1} | x_{n-2} \cdots x_1) \cdot \cdots f(x_3 | x_2 x_1) \cdot f(x_2 / x_1) \cdot f(x_1)$$

$$= f(x_n | x_{n-1}) \cdot f(x_{n-1} | x_{n-2}) \cdots f(x_3 | x_2) \cdot f(x_2 / x_1) \cdot f(x_1)$$

$$= f(w_n) \cdot f(w_{n-1}) \cdots f(w_2) \cdot f(x_1)$$

$$= \frac{1}{\sqrt{2\pi \frac{1}{1-\alpha^2}}} \cdot e^{-\frac{1}{2}(1-\alpha^2)x_1^2} \cdot \frac{e^{-\frac{1}{2}(x_2 - \alpha x_1)^2}}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{1}{2}(x_3 - \alpha x_2)^2}}{\sqrt{2\pi}} \cdots \cdot \frac{e^{-\frac{1}{2}(x_n - \alpha x_{n-1})^2}}{\sqrt{2\pi}}$$

(c) $$\max \quad -\frac{1}{2}(1-\alpha^2)x_1^2 + \frac{1}{2}\log(1-\alpha^2) - \frac{1}{2}\sum_{i=2}^{N}(x_i - \alpha x_{i-1})^2$$

$$\Rightarrow \min \quad \underbrace{(1-\alpha^2)x_1^2 - \log(1-\alpha^2)}_{\text{too small } (|\alpha|<1)} + \sum_{i=2}^{N}(x_i - \alpha x_{i-1})^2$$

$$\Rightarrow \min \sum_{i=2}^{N}(x_i - \alpha x_{i-1})^2 = \sum_{i=2}^{N} x_i^2 - 2\alpha \sum_{i=2}^{N} x_i x_{i-1} + \alpha^2 \sum_{i=2}^{N} x_{i-1}^2$$

$$-2\sum_{i=2}^{N} x_i x_{i-1} + 2\alpha \sum_{i=2}^{N} x_{i-1}^2 = 0$$

$$\alpha = \frac{\sum_{i=2}^{N} x_i x_{i-1}}{\sum_{i=2}^{N} x_{i-1}^2}$$

Problem 5

(a) $L_0 = (2\pi)^{-N/2} \cdot |\Sigma_0|^{-1/2} \cdot \exp[-\frac{1}{2}(X-\mu_0)^T \Sigma_0^{-1} (X-\mu_0)]$

$L_1 = (2\pi)^{-N/2} \cdot |\Sigma_1|^{-1/2} \cdot \exp[-\frac{1}{2}(X-\mu_1)^T \Sigma_1^{-1} (X-\mu_1)]$

By N-P lemma, a critical region of size α is obtained by the following equation for k:

$$\alpha = \mathbb{P}\left(\frac{L_0}{L_1} \le k \mid H_0\right)$$

Note that, $\frac{L_0}{L_1} = \frac{|\Sigma_0|^{-\frac{1}{2}}}{|\Sigma_1|^{-\frac{1}{2}}} \cdot \exp\{-\frac{1}{2}(X-\mu_0)^T \Sigma_0^{-1} (X-\mu_0) + \frac{1}{2}(X-\mu_1)^T \Sigma_1^{-1} (X-\mu_1)\}$

By taking log on both sides of $\frac{L_0}{L_1} \le k$,

$$-\frac{1}{2}\log\frac{|\Sigma_0|}{|\Sigma_1|} - \frac{1}{2}(X-\mu_0)^T \Sigma_0^{-1} (X-\mu_0) + \frac{1}{2}(X-\mu_1)^T \Sigma_1^{-1} (X-\mu_1) \le \log k$$

(b) If $\Sigma_0 = \Sigma_1$, $|\Sigma_0| = |\Sigma_1|$, then we have

$$-\frac{1}{2}(X-\mu_0)^T \Sigma_0^{-1} (X-\mu_0) + \frac{1}{2}(X-\mu_1)^T \Sigma_1^{-1} (X-\mu_1) \le \log k \ , \ \Sigma_0 = \Sigma_1$$

(c) If additionaly $\mu_0 = \mu_1$, then

$-\frac{1}{2}(X-\mu_0)^T \Sigma_0^{-1} (X-\mu_0) + \frac{1}{2}(X-\mu_1)^T \Sigma_1^{-1} (X-\mu_1) = 0 \le \log k$, for any $k > 0$.

That means we never reject H_0, this makes sense because If H_0 is accepted, we don't reject H_0; Else if H_1 is accepted, we don't reject H_1, since $H_1 = H_0$, we don't reject H_0.