Algebraic Geometry 2 Homeworks

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Problem Set 1

Vakil 6.3 M(Wrong question, this is the 6.3M in the 2015 version of Rising Sea)

Let (A, m) be a local ring and X be a scheme, and p is a point in X. Suppose $\mathrm{Spec}(A) \xrightarrow{\pi} X$ is a morphism of schemes. Show that any open set containing p contains the image of π. Show also that there is a bijection between $\mathrm{Mor}\,(\mathrm{Spec}\, A, X)$ and $\{p \in X : \mathcal{O}_{X,p} \to A \text{ a morphism of local rings}\}$.

Proof. Simply observe that any open set of $\mathrm{Spec}\, A$ containing m is necessarily the whole $\mathrm{Spec}\, A$ because the ring is local. Therefore, by continuity of π, we see the preimage of any open set containing p is necessarily the whole $\mathrm{Spec}\, A$, which means any such open set necessarily contains of image of $\mathrm{Spec}\, A$ under π.

The bijection can be established if we show there is a 1-1 correspondence between pointed morphism sending m to p and morphism of local ring from $\mathcal{O}_{X,p}$ into A.

This correspondence is clear from the observation made in the beginning,because it is obviously sufficient to consider morphism from $\mathrm{Spec}\, A$ into open-affines containing p. Combining the direct image interpretation of $\mathcal{O}_{X,p}$ and the adjunction relation between morphism of schemes and morphism of rings of global sections gives the desired correspondence. $\square$

Vakil 6.3 M(2017 version)

1. Let B be a ring, and $X_{/B}$ a scheme over B. Let $f_i (i = 0, ..., n)$ be $n + 1$ functions on X without common zeros. Show that $[f_0 : ... : f_n]$ defines a function from X to $\mathbb{P}^n$.

2. Suppose g is a non-vanishing function on B, show that $[gf_0 : ... : gf_n]$ is the same as $[f_0 : ... : f_n]$.

This was discussed in class. It is not hard to convince oneself the validity of this argument by thinking geometrically(think about map into $\mathbb{CP}^n$). Even the proof carries over, because only local-affine and gluing constructions were used.

Vakil 6.3 J

a. Given $p \in X$, define a canonical morphism of schemes $\operatorname{Spec} \mathcal{O}_{X,p} \to X$.

b. Define a canonical morphism of schemes $\operatorname{Spec} k(p) \to X$.

a. *Proof.* By the direct image interpretation of $\mathcal{O}_{X,p}$, there are canonical morphisms from $\operatorname{Spec} \mathcal{O}_{X,p}$ to any open affine containing p, which satisfy obvious compatible condition. These then glue to a morphism into X because of compatibility. This is canonical since no special choices were made.

   No generality was lost by this construction, because it was shown in 6.3 M any possible morphism necessarily sends its image to an open affine containing p anyway. □

b. *Proof.* The quotient map $\mathcal{O}_{X,p} \to k(p)$ induces a morphism $\operatorname{Spec} k(p) \to \operatorname{Spec} \mathcal{O}_{X,p}$. Composing with the morphism constructed in a gives a canonical morphism of scheme from $\operatorname{Spec} k(p)$ into X. □

Vakil 9.1 C

Prove the Yoneda's lemma, then describe how it specializes to the case of schemes.

Proof. There are too many variants of what people call the "Yoneda's Lemma"[1], the one that is most relevant for us is $h_A \cong h_B \iff A \cong B$. The variants are either consequences of this one by switching arrows, or implied by the intermediate steps of the proof.

Let F be an isomorphism of functors from h_A to h_B. Denote by $\phi \in h_B(A)$ the image of 1_A under F. There is also a morphism $\rho \in h_A(B)$ which gets mapped to 1_B.

Consider now the following diagram given by the isomorphism F(in the horizontal direction):

$$\begin{tikzcd} h_A(A) \arrow[r] \arrow[d, "\rho^*"] & h_B(A) \arrow[d, "\rho^*"] \\ h_A(B) \arrow[r] \arrow[d, "\phi^*"] & h_B(B) \arrow[d, "\phi^*"] \\ h_A(A) \arrow[r] & h_B(A) \end{tikzcd}$$

□

Start with 1_A at the upper left corner and go to $h_B(B)$ on the commutative diagram along the indicated arrows.We get $\phi\rho = 1_B$.

Next, trace the paths of ρ from $h_A(B)$ to $h_B(A)$ along the indicated arrows. By commutativity of the diagram and the fact that F is an isomorphism, we get $\rho\phi = 1_A$.

[1] For example, this is also called Yoneda's lemma: for a functor F and an object A,$\mathrm{Mor}^{Fun}(h_A, F) \cong F(A)$. The proof is done by considering similar diagrams and exploit naturality.

Therefore $A \cong B$ because ϕ and ρ are explicit isomorphisms between them.

Specializing to functor of points for schemes we get the slogan "a scheme is determined by the morphisms into it".

Vakil 9.1 D*(Need to say it more clearly and concisely)

If X,$Y \to Z$ are schemes, show that $h_X \underset{h_Z}{\times} h_Y$ is a Zariski sheaf.

Proof. We verify by unwinding the definitions. To say that this functor is a Zariski sheaf is to say that it is a sheaf for an arbitrary scheme.

h_X, h_Y and h_Z are Zariski sheaves being representable functors.

Now, pick any open cover $\mathfrak{U}$ a scheme X. Injectivity property for sheaves is clearly satisfied by examining the projection maps. The equalizer property is also clear. $\square$

Vakil 9.1 F(THE GEOMETRIC NATURE OF THE NOTION OF "OPEN SUBFUNCTOR"(kind of tedious))

a. Show that an open subfunctor of an open subfunctor is also an open subfunctor.

b. Suppose $h^{'} \to h$ and $h^{''} \to h$ are two open subfunctors of h. Define the intersection of these two open subfunctors, which should also be an open subfunctor of h.

c. Suppose U and V are two open subschemes of a scheme X, so $h_U \to h_X$ and $h_V \to h_X$ are open subfunctors. Show that the intersection of these two open subfunctors is $h_{U \cap V}$.

Observation: We observe the following commutative diagram which will be used heavily throughout this question. This is nothing but consequence of general nonsense:

$$\begin{tikzcd}
A_7 \arrow[rr] \arrow[dr] \arrow[drr] \arrow[dd] & & A_6 \arrow[dl] \arrow[d] \\
& A_5 \arrow[dl] \arrow[dr] & A_4 \arrow[d, hook] \\
A_2 \arrow[dr] & & A_3 \arrow[dl] \\
& A_1 &
\end{tikzcd}$$

Here $A_5 = A_2 \underset{A_1}{\times} A_3$. The arrow from A_4 into A_3 is a monomorphism. $A_6 = A_4 \underset{A_3}{\times} A_4$. $A_7 = A_2 \underset{A_1}{\times} A_4$.

The arrows from A_7 to A_5 and A_6 are the unique ones produced by universal property of fibre products.

The observation is that the arrow $A_7 \to A_6$ is an isomorphism by exploiting universal property.[2]

a. *Proof.* $h'' \subset h' \subset h$, each is an open subfunctor of the functor to the right.

Pick any representable functor h_X, we have the following diagram.

$$\begin{tikzcd}
h'' \arrow[rr] & & h' \arrow[dr] & \\
h_V \arrow[u] \arrow[r, "open"] & h_U \arrow[ur] \arrow[dr, "open"] & & h \\
& & h_X \arrow[ur] &
\end{tikzcd}$$

Here U and V are two schemes representing h_U and h_V. h_V is the fibre product of h_U and h'' over h', which we observed is isomorphic to fibre product of h'' and h_X over h.

Since composition of open immersions is open immersion, we see $h_V \to h_X$ obtained by composing the two arrows in the diagram is induced by an open immersion. This shows h'' is also an open subfunctor of h by arbitrariness of X. $\square$

b. *Proof.* Consider the following diagram:

$$\begin{tikzcd}
& A_4 \arrow[ddl] \arrow[ddrr] & & \\
& A_6 \arrow[u] \arrow[d] \arrow[dr] & & \\
A_2 \arrow[dr] & h_X \arrow[l] & A_5 \arrow[l] \arrow[r] & A_3 \arrow[dll] \\
& A_1 & &
\end{tikzcd}$$

Here A_2 and A_3 are two open subfunctors of A_1. A_4 is the fibre product of A_2 and A_3 over A_1, defined object by object. A_4 will be the definition of "intersection" of A_2 and A_3 in A_1.

We verify it is an open subfunctor.

Indeed, suppose h_X in the diagram is a representable functor. A_5 is the fibre product of h_X and A_3 over A_1, and A_6 is the fibre product of A_4 and h_X over A_2. The arrow $h_X \leftarrow A_5$ is induced by an open immersion because A_3 is an open subfunctor of A_1. The arrow from A_6 to A_5 is an isomorphism by observation made in the beginning. Their composition, which is the vertical arrow from A_6 to h_X, is therefore open immersion.

This shows A_4 is an open subfunctor of A_2, which means it is an open subfunctor of A_1 by a. $\square$

[2] The way to remember it is to remember associativity of intersection in SET.

c. *Proof.* We verify $h_U \underset{h_X}{\times} h_V \cong h_{U\cap V}$.

Note that $U \cap V$ is a fibre product in the category of schemes.

It is easy to produce a morphism of functors between the two using universal properties[3], and it is equally easy to see this is in fact an isomorphism of functors. □

Vakil 9.1 I(Actually pretty tricky to do the gluing carefully)

If a functor h is a Zariski sheaf that has an open cover by representable functors ("is covered by schemes"), then h is representable.

Proof. Consider the following diagram:

$$\begin{tikzcd}
& & & & h_{X\cap Y} \arrow[dl] \arrow[dr] & & & & \\
\dots & \dots \arrow[drrr] & \dots \arrow[drr] & h_X \arrow[dr] & & h_Y \arrow[dl] & \dots \arrow[dll] & \dots \arrow[dlll] & \dots \\
& & & & h & & & &
\end{tikzcd}$$

For any pair h_X and h_Y, there is an open subfunctor $h_{X\cap Y}$. The arrows from it to h_X and h_Y correspond to gluing morphisms. Since this is true for any pair, and all the gluing maps are obviously compatible by universal properties of fibre products, the schemes represented by these representable open subfunctors glue to a scheme H. The idea is to show that $h \cong h_H$.

$$\begin{tikzcd}
& & & & h_{X\cap Y} \arrow[dl] \arrow[dr] & & & & \\
\dots & \dots \arrow[drrr] & \dots \arrow[drr] & h_X \arrow[dr] & & h_Y \arrow[dl] & \dots \arrow[dll] & \dots \arrow[dlll] & \dots \\
& & & & h & & & & \\
& & & & h_H \arrow[u] \arrow[uull] \arrow[uul] \arrow[uur] \arrow[uurr] & & & &
\end{tikzcd}$$

The Zariski sheaf conditions are used in the following way:

The identity property is used to invert the arrow at each open subfunctor. These arrows are compatible by construction of H. Then the gluing property of a sheaf produces a single arrow whenever a scheme gets plugged into the two functors. Tracing through the arrows and invoking universal properties, it is not difficult(but somewhat tedious) to check the morphism of functors produced this way is in fact produces an inverse, and is therefore isomorphism of functors. □

[3] Do it object by object

Vakil 12.1 J

Find the dimension of the Zariski tangent space at the point $[(2, 2i)]$ of $\mathbb{Z}[2i] \cong \mathbb{Z}[x]/(x^2+4)$. Find the dimension of the Zariski tangent space at the point $[(2, x)]$ of $\mathbb{Z}[\sqrt{-2}] \cong \mathbb{Z}[x]/(x^2+2)$. Think about what happens when $\mathbb{Z}$ is replaced by $\mathbb{C}$ and 2 by y, then consider $\mathbb{C}[x,y]/\left(x^2+y^2\right)$ and $\mathbb{C}[x,y]/\left(x^2+y\right)$.

Vakil 12.3 F

Find all the singular closed points of the following plane curves. Here we work over $k = \bar{k}$ of characteristic 0 to avoid distractions.

a. $y^2 = x^2 + x^3$. This is an example of a node.

b. $y^2 = x^3$. This is called a cusp.

c. $y^2 = x^4$. This is called a tacnode.

These can be done directly using the Jacobian criterion. Call the polynomial determining the curve F, and its Jacobian $J(F)$. It will be an interesting exercise to compute the dimensions of the Zariski tangents at the singular points.

a. $J(F) = (3x^2 + 2x, -2y)$. The only singular closed points are $(0,0)$ and $(-\frac{2}{3}, 0)$. Dimensions of the tangent spaces at the singular points are both 2.

b. $J(F) = (3x^2, -2y)$, whose only singular closed point is $(0,0)$. Dimension of the tangent space at the singular point is 2.

c. $J(F) = (4x^3, -2y)$, whose only singular closed point is $(0,0)$. Dimension of the tangent space at the singular point is 2.

Vakil 12.3 K

Suppose $X \subset \mathbb{P}^n_k$[4] is a degree d hypersurface cut out by $f = 0$, and L is a line not contained in X. Exercise 8.2.E showed that X and L meet at d points, counted "with multiplicity". Suppose L meets X "with multiplicity at least 2" at a k-valued point $p \in L \cap X$, and that p is a regular point of X. Show that L is contained in the tangent plane to X at p.

Proof. This is clear geometrically. The mental picture is the tangent to a plane curve at a smooth point[5]. I will do the algebra carefully when I have time. □

Starting from problem set 2, *Problems are the more important ones.

[4] In this section, k is assume to be algebraically closed.

[5] This picture is not entirely accurate, but I still haven't internalized the "normal" condition in geometric terms.

Problem Set 2

Vakil 8.2G

Show that if $S_\bullet$ s generated (as an A-algebra) in degree 1 by n+1 elements $x_0, \ldots, x_n$ then $\operatorname{Proj} S_\bullet$ may be described as a closed subscheme of $\mathbb{P}^n_A$ as follows. Consider $A^{\oplus(n+1)}$ as a free module with generators $\mathrm{t}_0, \ldots, \mathrm{t}_\pi$ associated to $x_0, \ldots, x_n$ The surjection of $A^{\oplus(n+1)} = A\left[t_0, t_1, \ldots, t_n\right] \longrightarrow S_\bullet$, given by $\mathrm{t_i} \longmapsto \mathrm{x_i}$ implies $\mathrm{S}_\bullet = \mathrm{A}\left[\mathrm{t}_0, \mathrm{t}_1, \ldots \mathrm{t_n}\right]/\mathrm{I}$ where I is a homogeneous ideal.

Vakil 8.2H*

Suppose $S_\bullet$ is a finitely generated graded ring over an algebraically closed field k , generated in degree 1 by $x_0, \ldots, x_n$ inducing closed embeddings $\operatorname{Proj} S_\bullet \hookrightarrow \mathbb{P}^n$ and $\operatorname{Spec} S_\bullet \hookrightarrow A^{n+1}$. Give a bijection between the closed points of Proj $S_\bullet$ and the "lines through the origin" in $\operatorname{Spec} S_\bullet \subset \mathbb{A}^{n+1}$.

Proof. This is done by identifying $S_\bullet$ with a quotient of $k[\underline{X}]$ via first isomorphism theorem. This is exactly the kind of construction which appears in classical algebraic geometry. □

Vakil 13.1D

If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is a locally free sheaf. If $\mathcal{F}$ is an invertible sheaf, show that $\mathcal{F} \otimes \mathcal{F}^\vee \cong \mathcal{O}_X$.

Vakil 13.11.F

I think the question was meant to be 13.1.F. Here is what 13.1.F asks:

If $\mathscr{E}$ is a locally free sheaf of finite rank, and $\mathscr{F}$ and $\mathscr{G}$ are $\mathscr{O}_{\mathrm{X}}$-modules, show that $\mathcal{H}om(\mathscr{F}, \mathscr{G} \otimes \mathscr{E}) \cong \mathcal{H}om(\mathscr{F} \otimes \mathscr{E}^\vee, \mathscr{G})$.

Vakil 13.1.I*

Suppose s is a section of a locally free sheaf $\mathscr{F}$ on a scheme X. Define the notion of the subscheme cut out by $s = 0$ denoted $V(s)$. Check it is well-defined.

Proof. There is a coordinate free way to do this question, which is more elegant. I will write that up once I manage to incorporate it into a note on geometric interpretations of all the variants of Nakayama's lemma.

For now, I just give a solution that works.

Pass to affine opens where the modules of sections are free over the coordinate rings.

Choose bases, and write s in these bases, then $s = 0$ gives rise to ideal I_U in each affine open U, the closed subscheme structure is given by taking Spec of the quotient of the coordinate ring by I_U. The union is definitely a scheme.

Well-definedness of this construction follows from the compatibility of the restriction maps. □

Vakil 13.3.H

Vakil put too many things in one questions.

1. Show that any $t \in \Gamma(X_s, \mathcal{F})$ gets extended to a global section after multiplying high enough powers of s.
2. Describe a natural map $\oplus_{n\geq 0}\Gamma(X, \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^{\otimes n}) \to \Gamma(X_s, \mathcal{F})$ and show that it is an isomorphism.

Vakil 13.5.F

Suppose $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of locally free sheaves. Show that for any r, there is a filtration of $\wedge^{\mathrm{r}}\mathscr{F}$:

$$\wedge^{\mathrm{r}}\mathscr{F} = \mathscr{G}^0 \supset \mathscr{G}^1 \supset \cdots \supset \mathscr{G}^{\mathrm{r}} \supset \mathscr{G}^{\mathrm{r}+1} = 0$$

with subquotients

$$\mathscr{G}^{\mathrm{p}}/\mathscr{G}^{\mathrm{p}+1} \cong \left(\wedge^{\mathrm{p}}\mathscr{F}'\right) \otimes \left(\wedge^{\mathrm{r}-\mathrm{p}}\mathscr{F}''\right)$$

for each p. In particular, if the sheaves have finite rank, then $\det \mathscr{F} = (\det \mathscr{F}') \otimes (\det \mathscr{F}'')$.

Vakil 13.7.D*

Show that the support of a finite type quasicoherent sheaf on a scheme X is a closed subset.Show that the support of a quasicoherent sheaf need not be closed.

Proof. This is the same as proving the set $\{p : s_p = 0\}$ is open.

Apply the geometric Nakayama's lemma to produce a finite set of generator $\{m_1, ..., m_n\}$ in an affine open of p where the germ of the section s at it is 0.

Write $s_p = r_{1,p}m_1 + ... + r_{n,p}m_n$, where $r_{i,p} \in R_p$, and R_p is the localization of the coordinate ring of the affine open at p.

Pick the open set to be the intersection of non-vanishing set of the "denominators" of the $r_{i,p}$(because we want to be able to invert them at other points), which is an open set because it is finite intersection of basic open sets. It is then clear the germs will be zero at any point in this open neighbourhood.

Consider now the $\mathbb{Z}$-module obtained by taking direct product of countably many copies of $\mathbb{Z}/2\mathbb{Z}$, indexed by the prime numbers. Consider then the section with "1" in all entries. Obviously, (2) is the only element in the support of this section in Spec($\mathbb{Z}$). This is not an open set because any open neighbourhood of (2) necessarily contains the generic point (0).

□

Vakil 13.7.J

If $\mathscr{F}$ is a finite type quasicoherent sheaf on X, show that rank($\mathscr{F}$) is an upper semicontinuous function on X.

Vakil 13.7.K*

a. i. If X is reduced, $\mathscr{F}$ is a finite type quasicoherent sheaf on X, and the rank is constant, show that $\mathscr{F}$ is locally free.

   ii. Show that finite type quasicoherent sheaves on an integral scheme are locally free on a dense open set.

b. Show that part (a) can be false without the condition of X being reduced.

a. i. *Proof.* Use the geometric Nakayama lemma. The only thing that needs to be check is linear independence of the lift of basis. Reducedness[6] is needed to ensure there can't be a non-trivial combination resulting zero, by passing to the stalk at a well-chosen point. □

   ii. *Proof.* A simple and crucial observation is that, locally, the rank can only drop because of the Geometric Nakayama's lemma. Therefore, $U := \{p \in X : \text{p has an open neighbourhood containing only points with rank as p}\}$ is an open set. Moreover any open neighbourhood of a point in the complement of U necessarily intersects with it by an infinite descent argument on the rank.[7] Finally, conclude by i. [8] □

b. *Proof.* Consider the fat point $\mathrm{Spec}(k[\epsilon]/(\epsilon^2))$, and the sheaf generated by the principal ideal (ϵ),considered as a module. The geometry then reduces to just commutative algebra. Clearly, this module cannot possibly be free. □

Problem Set 3

Vakil 16.2.C*

Suppose $\pi : X \to Y$ is a finite morphism of locally Noetherian schemes. If $\mathscr{F}$ is a coherent sheaf on X, show that $\pi_*\mathscr{F}$ is a coherent sheaf.

Proof. This can be reduced to commutative algebra.

The statement is equivalent to the following claim: if A and B are two Noetherian rings, and $A \to B$ gives B a finitely generated A-module structure. Then a coherent B-module M is also coherent as an A-module.

This is immediate from the definition of coherence and the fact B is finitely generated as A-module. [9] □

[6] Here is a recap of input from commutative algebra: The nilradical, which is the intersection of all prime ideals, is the zero ideal in a reduced ring. Therefore, given any non-zero element, there must be a prime ideal not containing it.

[7] In retrospect, unwinding the propositional logic might be more convoluted than what I actually wrote down.

[8] While Noetherian property was not used in an essential way, I don't think reducedness is all that is needed. It is necessary to impose a finite cover by affine opens condition to make the argument work, or a uniform bound on the rank.

[9] It may appear that this solution does not use the condition that the rings are Noetherian, but it is necessary, because proposition 13.6.1 in Vakil's book is needed. The proposition asserts, for a module over a Noetherian ring, finitely generated is equivalent to coherent.

Vakil 16.3.F

Verify that the following is an example showing that pullback is not left-exact: consider the exact sequence of sheaves on $\mathbb{A}^1_k$ where $i : p \hookrightarrow \mathbb{A}^1_k$ is the origin [10]

$$0 \to \mathscr{O}_{\mathbb{A}^1_k}(-p) \to \mathscr{O}_{\mathbb{A}^1_k} \to i_*\mathscr{O}\Big|_p \to 0$$

Vakil 16.3.H

Suppose $\pi : X \to Y$ is quasicompact and quasiseparated, and $\mathscr{F}$ and $\mathscr{G}$ are quasicoherent sheaves on X and Y respectively.

a. Describe a natural morphism $(\pi_*\mathscr{F}) \otimes \mathscr{G} \to \pi_*(\mathscr{F} \otimes \pi^*\mathscr{G})$.

b. If $\mathscr{G}$ is locally free, show that this natural morphism is an isomorphism.

c. If π is affine, then show that this natural morphism is an isomorphism.

I have the feeling that these three questions are consequences of general nonsenses if the diagrams are set up correctly.

Vakil 6.4.A*

Suppose that $\phi : S_\bullet \longrightarrow R_\bullet$ is a morphism of ($\mathbb{Z}^{\geq 0}$)-graded rings. Show that this induces a morphism of schemes $(\operatorname{Proj} R_\bullet) \backslash V(\phi(S_+)) \to \operatorname{Proj} S_\bullet$.

Vakil 6.4.D

Show that the map of graded rings[11] $S_{n\bullet} \hookrightarrow S_\bullet$ induces an isomorphism $\operatorname{Proj} S_\bullet \to \operatorname{Proj} S_{n\bullet}$.

Vakil 8.2.D

Show that an injective linear map of k-vector spaces $V \hookrightarrow W$ induces a closed embedding $\mathbb{P}V \hookrightarrow \mathbb{P}W$

Vakil 8.2.G

This was assigned in problem set 2.

Vakil 8.2.H*

This was assigned in problem set 2.

[10] The notation $\mathscr{O}_{\mathbb{A}^1_k}(-p)$ is a little bit funny. In general, when X is a normal scheme, and D is a Weil divisor, $\mathscr{O}(D)$ is the sheaf associated to D. The definition is in 14.2.2. In our case, $\mathscr{O}_{\mathbb{A}^1_k}(-p)$ just means sheaf of functions vanishing at the point p.

[11] Vakil's notation $S_{n\bullet}$, which is pretty standard, refers to the nth Veronese subring $\oplus_{j=0}^{\infty} S_{nj}$. The subscript is the product of n and i.

Problem Set 4

Vakil 15.3.C

Suppose $\mathscr{F}$ is a finite type quasicoherent sheaf on X.

a. Show that $\mathscr{F}$ is globally generated at p if and only if "the fiber of $\mathscr{F}$ is generated by global sections at p" i.e., the map from global sections to the fiber $\mathscr{F}_p/\mathfrak{m}\mathscr{F}_p$ is surjective, where $\mathfrak{m}$ is the maximal ideal of $\mathscr{O}_{\mathrm{X,p}}$. [12]

b. Show that if $\mathscr{F}$ is globally generated at p, then "$\mathscr{F}$ is globally generated near p$''$: there is an open neighborhood U of p such that $\mathscr{F}$ is globally generated at every point of U.

c. Suppose further that X is a quasicompact scheme. Show that if $\mathscr{F}$ is globally generated at all closed points of X, then $\mathscr{F}$ is globally generated at all points of X.

Vakil 15.3.E

An invertible sheaf $\mathscr{L}$ on X is globally generated if and only if for any point $p \in X$, there is a global section of $\mathscr{L}$ not vanishing at p.

Vakil 15.4.A

Describe a morphism of S_0 -modules $\mathrm{M}_n \rightarrow \Gamma\left(\mathrm{Proj}\ \mathrm{S}_\bullet, \tilde{\mathrm{M}}(\mathrm{n})_\bullet\right)$ extending the $n = 0$ case of Exercise 15.1.D.

Vakil 15.4.B

Show that $\Gamma_\bullet(\mathscr{F})$ is a graded $S_\bullet$ -module.[13]

Vakil 15.4.C

Show that the map $\mathrm{M}_\bullet \rightarrow \Gamma_\bullet\left(\widetilde{\mathrm{M}}_\bullet\right)$ arising from the previous two exercises is a map of $S_\bullet$ -modules.

Vakil 15.4.H

Show that each closed subscheme of Proj $S_\bullet$ arises from a homogeneous ideal $I_\bullet \subset S_\bullet$. This fulfills promises made in Exercises 8.2.B and 15.1.E.

[12] It's probably better to use $\mathfrak{m}_p$ to avoid confusion.

[13] The notation $\Gamma_\bullet(\mathscr{F})$ denotes the graded abelian groups with component $\Gamma_n(\mathscr{F}) = \Gamma_\bullet(\mathscr{F}(n))$. The details can be found on page 417, 15.4.1. The point of this question is to show it is a graded $S_\bullet$-module.

Vakil 15.4.K

Suppose $S_\bullet$ is a finitely generated graded ring over a field k, so $X = \text{Proj } S_\bullet$ is a projective k -scheme. Show that $(\oplus_{n\geq 0}\Gamma(X, \mathscr{O}(n)))$ is a finitely generated k-algebra.

Vakil 16.4.F

(For this exercise, we work over a field k.) Suppose we have a morphism $\pi : X \to \mathbb{P}^n$, corresponding to the base-point-free linear series $\Gamma(\mathbb{P}^n, \mathscr{O}(1)) \to \Gamma(X, \mathscr{L})$ (so $\mathscr{L} = \pi^*\mathscr{O}(1)$). If the scheme-theoretic image of X in $\mathbb{P}^n$ lies in a hyperplane, we say that the linear series (or X itself) is degenerate (and otherwise, nondegenerate). Show that a base-point-free linear series $\mathrm{V} \to \Gamma(\mathrm{X}, \mathscr{L})$ is nondegenerate if and only if the map $\mathrm{V} \to \Gamma(\mathrm{X}, \mathscr{L})$ is an inclusion. In particular, a complete linear series is always nondegenerate.

Vakil 16.4.G

Suppose we are given a map $\pi : \mathbb{P}^1_k \to \mathbb{P}^n_k$ where the corresponding invertible sheaf on $\mathbb{P}^1_k$ is $\mathscr{O}(\mathrm{d})$. (This can reasonably be called a degree d map, cf. Exercises 17.4.F and 18.6.I.) Show that if d < n, then the image is degenerate. Show that if d = n and the image is nondegenerate, then the image is isomorphic (via an automorphism of projective space, Exercise 16.4.B) to a rational normal curve.

Vakil 16.7.A*

Show that the Grassmannian functor is a Zariski sheaf.

Proof. Vakil rushed through the definition of the Grassmannian functor. He only define the maps of objects, but not the maps of morphisms. Once we see how the arrows are mapped, it will be clear that this functor is a Zariski sheaf.

Recall that a morphism of schemes $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ comes with a morphism of sheaf $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. This morphism can also be thought of as a map from $\mathcal{O}_Y$ to $\mathcal{O}_X$, sometimes referred to as the f-map[14].

If we trivialize in affine opens, and then extend, we will get the following map of rank k quotients:

$$\begin{tikzcd} \mathcal{O}_X^n \arrow[r, two heads] & \mathscr{L} \\ \mathcal{O}_Y^n \arrow[u] \arrow[r, two heads] & \mathscr{M} \arrow[u] \end{tikzcd}$$

These are the maps at the level of elements, which collectively determine the a map of sets of rank k quotients. The identity axiom and gluing axiom can then be deduced from the fact that $\mathcal{O}_S$ is a sheaf for any scheme S. $\square$

[14]For instance, on the Stacks Project,6.23.

Vakil 16.7.B*

a. Suppose I is a k-subset.There is an open subfunctor $\mathrm{G(k,n)_I}$ of $\mathrm{G(k,n)}$ where the k sections of $\mathscr{L}$ corresponding to I are linearly independent.

b. Show that these open subfunctors $\mathrm{G(k,n)_I}$ cover the functor $G(k,n)$ (as I runs through the k-subsets).

a. *Proof.* Vakil's hint suggests to mimick the usual construction of the standard charts of Grassmannian. But that is not quite the definition of open subfunctor! Namely the fibre product in the diagram

$$\begin{tikzcd} & h_X \underset{G}{\times} G_I \arrow[dl] \arrow[dr] & \\ h_X \arrow[dr] & & G_I \arrow[dl] \\ & G & \end{tikzcd}$$

is representable and the induced arrow into h_X is given by an open embedding.

The following observation is needed for this approach to work:Given a scheme X , and a subfunctor $F \subset h_X$, which a priori is not known to be open, suppose for a family of open subschemes $\{U_i\}_{i\in J}$ covering X, $h_{U_i} \underset{h_X}{\times} F$ is representable, and the arrow from it into h_{U_i} is induced by an open embedding for each $i \in J$, then F is a representable open subfunctor of h_X.

Choose the U_i's to be the trivializing charts as suggested in Vakil's hint, and invoke the above observation, we see G_I is an open subfunctor of G.

A more direct way to show the functor is open was discussed in the 2-23 lecture. Following Vakil's hint, we get each $\{\mathcal{O}_X^n \to \mathscr{L}\}/\sim$ in the fibre product corresponds to an open subscheme of X, and collectively, the arrow into h_X corresponds to the open embedding of their scheme-theoretic union into X. ∎

b. Obvious.

Vakil 16.7.C*

Show that $G(k,n)_{\{1\ldots k\}}$ is represented by $\mathbb{A}^{k(n-k)}$.

Proof. We prove $G_I \cong \mathbb{A}^{(n-k)k}$ by constructing explicit isomorphism of functors.

Consider the following diagram, by restricting to trivializing charts, we may assume X is affine:

$$\begin{tikzcd} \mathcal{O}_X^n \arrow[r] & \mathcal{O}_X^k \\ \tilde{\mathbb{Z}}^n \arrow[u] \arrow[r, dashed] & \tilde{\mathbb{Z}}^k \arrow[u] \end{tikzcd}$$

The morphisms in the vertical direction are given by the natural morphism of $\mathbb{Z}$-modules $\mathbb{Z} \to \Gamma(X, \mathcal{O}_X)$.

These induce the morphism in the horizontal direction. This induced morphism can be represented by a sheaf of $n \times k$ matrices, with the k columns corresponding to the index set I being independent. $\square$

16.7.A,B,C together imply $G(k, n)$ is representable by 9.1 I.

Problem Set 5

Vakil 18.2.C

Suppose $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is a short exact sequence of sheaves of abelian groups on a topological space, and $\mathscr{U}$ is a finite open cover such that on any intersection $\mathrm{U_I}$ of open subsets in $\mathscr{U}$, the map $\Gamma(\mathrm{U_I}, \mathscr{G}) \to \Gamma(\mathrm{U_I}, \mathscr{H})$ is surjective. Show that we get a "long exact sequence of cohomology for $\mathrm{H}^{\mathrm{i}}_{\mathscr{U}}$". Describe what this means in the case X is a Q.C speareted A-scheme, and the terms in the exact sequence consists of quasi-coherent sheaves.

Vakil 18.2.E*

Suppose $\pi : X \to Y$ is an affine morphism, and Y is a quasicompact and separated A-scheme (and hence X is too, as affine morphisms are both quasicompact and separated). If $\mathscr{F}$ is a quasicoherent sheaf on X, describe a natural isomorphism $H^i(Y, \pi_*\mathscr{F}) \cong H^i(X, \mathscr{F})$.

Proof. The point is that, under the hypothesis that $(\pi, \pi^\sharp)$ is affine, $\{\pi^{-1}(\mathscr{U})\}(\mathscr{U}$ taken over all covers of Y by affine opens) is cofinal in covers of X by affine opens. The separatedness condition ensures this. More explicitly, separatedness allows one to deduce $V \cap W$ is affine if V and W are affine opens in $\pi^{-1}(U)$, where U is affine open of Y(Lemma 26.21.7(1) on Stacks Project).

Therefore, the isomorphism is given by taking the limit of de Rham cohomologies with respect inverse image of covers of Y by affine opens. $\square$

Vakil 18.2.H

Suppose X is a quasicompact separated k-scheme, and $\mathscr{F}$ is a quasicoherent sheaf on X. Give an isomorphism

$$\mathrm{H}^i(\mathrm{X}, \mathscr{F}) \underset{\mathrm{k}}{\otimes} \mathrm{K} \cong \mathrm{H}^i\left(\mathrm{X} \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} K, \mathscr{F} \underset{\mathrm{k}}{\otimes} \mathrm{K}\right)$$

for all i, where K/k is any field extension. Here $\mathscr{F} \underset{\mathrm{k}}{\otimes} \mathrm{K}$ means the pullback of $\mathscr{F}$ to $X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} K$. Hence $h^i(X, \mathscr{F}) = h^i(X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} K, \mathscr{F} \underset{k}{\otimes} K)$. If $i = 0$ (taking $\mathrm{H}^0 = \Gamma$), show the result without the quasicompact and separated hypotheses.

Vakil 18.2.I

Suppose X is a scheme over a field k, $\mathscr{L}$ is an invertible sheaf on X, and K/k is a field extension. Show that $\mathscr{L}$ is base-point-free if and only if its pullback to $X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} K$ is base-point-free.

Vakil 18.3.A*

Compute the cohomology groups $\mathrm{H}^i\left(\mathbb{A}_k^2 \backslash\{(0,0)\}, \mathscr{O}\right)$. In particular, show that $\mathrm{H}^1\left(\mathbb{A}_k^2 \backslash\{(0,0)\}, \mathscr{O}\right) \neq 0$ and thus give another proof of the fact that $\mathbb{A}_{\mathrm{k}}^2 \backslash\{(0,0)\}$ is not affine.

Proof. If we can find a cover by affine opens so that the de Rham cohomology with respect to this cover has non-trivial H^1, then it necessarily has non-trivial H^1.

We can cover $\mathbb{A}_k^2$ by $D(x)$ and $D(y)$ in $\mathbb{A}_k^2$. Their intersection is $D(xy)$. These are all affine, with global sections $k[x,y]_x$, $k[x,y]_y$ and $k[x,y]_{xy}$ respectively.

We claim that $\frac{1}{xy}$ cannot be expressed as sum of two elements in $k[x,y]_x$ and $k[x,y]_y$ respectively. This will show $\frac{1}{xy}$ represents a 1-cocycle which is not a 1-coboundary.

Suppose the contrary, and we will derive from it a contradiction. Indeed, this would mean we have

$$x^{m-1}y^{n-1} = P(x,y)x^m + Q(x,y)y^n,$$

for some non-negative integer n and m, whereas P and Q are polynomials in $k[x,y]$. Moreover, P must be relatively prime to y^n and Q is necessarrily relatively prime to x^m. Since the two sides of the equations are divisible by x^{m-1}, which forces Q to be divisible by x^{m-1}. This shows $m=1$. Similarly, it can be shown $n=1$. Hence, P depends only on x and Q depends only on y.

We get a contradiction by considering degree of the two sides as a polynomial in one of the variables. $\square$

Vakil 18.3.D*

(Computing $H^i(\mathbb{P}_A^n, \mathcal{O}_{\mathbb{P}_A^n}(m))$)Prove Theorem 18.1.3 for general n:

a. $\mathrm{H}^0\left(\mathbb{P}_{\mathrm{A}}^n, \mathscr{O}_{\mathbb{P}_{\mathrm{A}}^n}(\mathrm{m})\right)$ is a free A-module of rank $\left(\begin{array}{c}\mathrm{n}+\mathrm{m} \\ \mathrm{m}\end{array}\right)$ if $\mathrm{m} \geq 0$

b. $\mathrm{H}^{\mathrm{n}}\left(\mathbb{P}_{\mathrm{A}}, \mathscr{O}_{\mathrm{P}_4}(\mathrm{m})\right)$ is a free A-module of rank $\left(\begin{array}{c}-\mathrm{m}-1 \\ -\mathrm{n}-\mathrm{m}-1\end{array}\right)$ if $\mathrm{m} \leq -\mathrm{n}-1$

c. $\mathrm{H}^i\left(\mathbb{P}_{\mathrm{A}}^{\mathrm{n}}, \mathscr{O}_{\mathrm{P}^n}(\mathrm{m})\right)=0$ otherwise.

Problem Set 6

Vakil 12.5.C*

Show that $\{0\} \cup \{x \in \mathrm{K}^\times : v(x) > 0\}$ is the unique maximal ideal of the valuation ring. Thus the valuation ring is a local ring.

Proof. Denote(suggestively) this ideal by $\mathfrak{m}$.It suffices to prove that any ideal which contains an element r which is not in $\mathfrak{m}$ is necessarily the entire ring.

Indeed, for $r \notin \mathfrak{m}$, $v(r)$ has to be 0. Therefore, $0 = v(1) = v(r\frac{1}{r}) = v(r) + v(\frac{1}{r}) \implies v(\frac{1}{r}) = 0$. Hence, $\frac{1}{r}$ is in fact an integral element. Any ideal containing r must therefore contain 1 and so it must be the whole ring. $\square$

Vakil 12.5.E

Conversely, suppose $(A, \mathfrak{m})$ is a discrete valuation ring. Show that $(A, \mathfrak{m})$ is a Noetherian regular local ring of dimension 1.

Vakil 14.2.H*

Here is a Weil divisor that is not locally principal. Let $X = \operatorname{Spec} k[x, y, z]/\left(xy - z^2\right)$, a cone, and let D be the line $z = x = 0$. (a) Show that D is not locally principal. In particular $\mathscr{O}_X(\mathrm{D})$ is not an invertible sheaf. (b) Show that $div(x) = 2\mathrm{D}$. This corresponds to the fact that the plane $x = 0$ is tangent to the cone X along D.

a. *Proof.* We show that D is not principal in any affine open neighbourhood of the vertex $[(x, y, z)]$ of the cone.

   Indeed, if it were principal, then the ideal (x, z) corresponding to the line $x = z = 0$ would be principal. However, by the calculations in 12.1.3, we know this is false. $\square$

b. *Proof.* Note that, in order for x to vanish, z must also vanish. Therefore, the only closed subscheme of codimension 1 on which x vanishes is the line $x = z = 0$. Therefore, $div(f) = Val_D(x)[D]$.

   Now, $\mathcal{O}_{X,D}$ has uniformizer z. Since $xy = z^2$ and y is invertible in this local ring, we get $Val_y(x) = 2$. Therefore, $Val_D(x) = 2$. $\square$

Vakil 14.2.K*

Here is a torsion Picard group. Suppose that Y is a hypersurface in $\mathbb{P}^n_k$ corresponding to an irreducible degree d polynomial. Show that $\operatorname{Pic}(\mathbb{P}^n_k - Y) \cong \mathbb{Z}/(\mathrm{d})$.

Proof. Recall we have the following exact sequence from 14.2.8.1:

$$\mathbb{Z} \overset{1 \to [Z]}{\longrightarrow} Cl(X) \longrightarrow Cl(X - Z) \longrightarrow 0,$$

where Z is an irreducible subscheme of codimension 1.

In the case Z is a degree d hypersurface in $\mathbb{P}^n_k$, the image of $\mathbb{Z}$ in $Cl(X)$ under the above map is the cyclic group generated by $\mathcal{O}(d)$. The result then follows from first isomorphism theorem. $\square$

Vakil 14.2.L

Keeping the same notation, assume d > 1 (so $\operatorname{Pic}(\mathbb{P}^n - Y) \neq 0$), and let $H_0, \ldots, H_n$ be the $n+1$ coordinate hyperplanes on $\mathbb{P}^n$. Show that $\mathbb{P}^n - Y$ is affine, and $\mathbb{P}^n - Y - H_i$ is a distinguished open subset of it. Show that the $\mathbb{P}^n - Y - H_i$ form an open cover of $\mathbb{P}^n - Y$. Show that $\operatorname{Pic}(\mathbb{P}^n - Y - H_i) = 0$. Then by Exercise 14.2.$T$, each $\mathbb{P}^n - Y - H_i$ is the Spec of a unique factorization domain, but $\mathbb{P}^n - Y$ is not. Thus the property of being a unique factorization domain is not an affine-local property $-$ it satisfies only one of the two hypotheses of the Affine Communication Lemma 5.3 .2 .

Vakil 14.2.R

On the cone over the smooth quadric surface $X = \operatorname{Spec} k[w,x,y,z]/(wz - xy)$, let Z be the Weil divisor cut out by $w = x = 0$. Exercise 12.1.D showed that Z is not cut out scheme-theoretically by a single equation. Show more: that if $n \neq 0$, then $n[Z]$ is not locally principal.

Problem Set 7

Vakil 14.2.O

We compute the Picard group of $\mathbb{P}^1 \times \mathbb{P}^1$. Consider

$$\mathrm{X} = \mathbb{P}^1_{\mathrm{k}} \times_{\mathrm{k}} \mathbb{P}^1_{\mathrm{k}} \cong \operatorname{Proj} \mathrm{k}[w, x, \mathrm{y}, z]/(wz - xy)$$

a smooth quadric surface. Show that $\operatorname{Pic} X \cong \mathbb{Z} \oplus \mathbb{Z}$ as follows: Show that if $L = \{\infty\} \times_k \mathbb{P}^1 \subset X$ and $M = \mathbb{P}^1 \times_k \{\infty\} \subset X$, then $X - L - M \cong \mathbb{A}^2$. This will give a surjection $\mathbb{Z} \oplus \mathbb{Z} \to \mathrm{Cl} X$. Show that $\mathscr{O}(\mathrm{L})$ restricts to $\mathscr{O}$ on L and $\mathscr{O}(1)$ on M. Show that $\mathscr{O}(\mathrm{M})$ restricts to $\mathscr{O}$ on M and $\mathscr{O}(1)$ on L.

Vakil 14.2.T*

Suppose that A is a Noetherian integral domain. Show that A is a unique factorization domain if and only if A is integrally closed and Cl Spec $A = 0$.

Proof. This is actually proved in Hartshorne chapter II 6.2[15], which is probably the most involved *-problem in this problem set. It is however not hard to convince oneselves the result. Details will be supplied later.

[15] By the way, what is in the page of Matsumura's book Hartshorne cited is a collection of claims whose proofs are left as exercises. The "hard-algebra" facts are actually very technical.

It was pointed out in the 3.24 lecture that a simpler approach would be to invoke the algebraic Hartog's lemma a couple of times.

□

Vakil 16.5.A*

In each of the following cases, prove that the morphism $\mathrm{C} \setminus \{p\} \to Y$ cannot be extended to a morphism $C \to Y$.

a. Projectivity of Y is necessary. Suppose $\mathrm{C} = \mathbb{A}^1_{\mathrm{k}}, \mathrm{p} = 0, \mathrm{Y} = \mathbb{A}^1_{\mathrm{k}}$, and $\mathrm{C}\setminus\{\mathrm{p}\} \to Y$ is given by $\mathrm{t}'' \mapsto 1/\mathrm{t}''$

b. One-dimensionality of C is necessary. Suppose $\mathrm{C} = \mathbb{A}^2_{\mathrm{k}}, \mathrm{p} = (0,0), \mathrm{Y} = \mathbb{P}^1_{\mathrm{k}}$ and $C\setminus\{p\} \to Y$ is given by $(x,y) \mapsto [x,y]$

c. Non-singularity of C is necessary. Suppose $\mathrm{C} = \operatorname{Spec} \mathrm{k}[\mathrm{x},\mathrm{y}]/\left(\mathrm{y}^2 - \mathrm{x}^3\right), \mathrm{p} = 0$, $Y = \mathbb{P}^1_k$, and $C\setminus\{p\} \to Y$ is given by $(x,y) \mapsto [x,y]$

a. *Proof.* A regular function from $\mathbb{A}^1_k$ to itself is a polynomial $p(t'')$. No one variable polynomial can agree with $1/t''$ at infinitely many points.[16] □

b. *Proof.* Such extension necessarily restricts to regular maps from lines passing through the origin. Pick any line, which is isomorphic to $\mathbb{A}^1_k$, since all points on the line, except possibly the origin, get sent to a point, it is necessarily constant. Now, different lines get sent to different points on $\mathbb{P}^1_k$, so we get a contradiction since the origin can't be sent to different points by one map. □

c. *Proof.* This has to do with the origin being a cusp.

   Consider the two parametrizations $t \to (t^2, t^3)$ and $t \to (t^2, -t^3)$, where t ranges over $\mathbb{A}^1_k$.

   If an extension exists, compose it with the two parametrization shows that the origin necessarily goes to $[1:0]$. Therefore, the image of this singular cubic lies entirely in the affine chart U_0 of $\mathbb{P}^1_k$. Hence, the map $(x,y) \to \frac{y}{x}$ is an element of $k[x,y]/(y^2 - x^3)$, the global sections of the singular cubic, which is false. □

Vakil 16.5.B

Suppose X is a Noetherian k-scheme, and Z is an irreducible codimension 1 subvariety whose generic point is a regular point of X (so the local ring $\mathscr{O}_{X,z}$ is a discrete valuation ring). Suppose $\pi : X - \to Y$ is a rational map to a projective k-scheme. Show that the domain of definition of the rational map includes a dense open subset of Z. In other words, rational maps from Noetherian k-schemes to projective k-schemes can be extended over regular codimension 1 sets.

[16] For the sake of constructing counter-example, just assume k is algebraically closed.

Vakil 16.6.A

Suppose $\pi : X \to \operatorname{Spec} A$ is proper, and $\mathscr{L}$ is an invertible sheaf on X. Show that $\mathscr{L}$ is very ample if and only if there exist a finite number of global sections $s_0, \dots s_n$ of $\mathscr{L}$, with no common zeros, such that the morphism

$$[s_0, \dots, s_n] : X \to \mathbb{P}^n_A$$

is a closed embedding.

Vakil 16.6.E

Suppose $\mathscr{L}$ and $\mathscr{M}$ are invertible sheaves on a proper A-scheme X, and $\mathscr{L}$ is ample. Show that $\mathscr{L}^{\otimes n} \otimes \mathscr{M}$ is very ample for $n \gg 0$.

Vakil 16.6.F*

Show that every line bundle on a projective A-scheme X is the difference of two very ample line bundles. More precisely, for any invertible sheaf $\mathscr{L}$ on X, we can find two very ample invertible sheaves $\mathscr{M}$ and $\mathscr{N}$ such that $\mathscr{L} \cong \mathscr{M} \otimes \mathscr{N}^\vee$.

Proof. It is not easy to prove this from scratch. The intended way is probably to appeal to 16.6 E. Choose n sufficiently large so that both the $n+1$-th and n-th tensor power of $\mathscr{L}$ tensor with $\mathcal{O}(1)$ are very ample. Set them to be $\mathscr{M}$ and $\mathscr{N}$ will give the desire isomorphism. □

Problem Set 8

Vakil 9.7.B

Show that normalizations of integral schemes exist in general.

Vakil 9.7.C*

Show that normalizations are integral and surjective.

Proof. We assume the scheme X is integral to begin with. Pick a basis of X consisting of affine opens. The normalization X' of X is defined as follows:

1.$\mathcal{O}_{X'}(U)$ is the integral closure of $\mathcal{O}_X(U)$ in its quotient field.[17]

2.The projections of $\mathcal{O}_{X'}$ are determined uniquely by those making the diagrams of the following kind commute, given by the universal property of integral closure.

$$\begin{array}{ccc} \mathcal{O}_{X'}(U) & \longrightarrow & \mathcal{O}_{X'}(V) \\ \uparrow & & \uparrow \\ \mathcal{O}_X(U) & \xrightarrow{\rho^U_V} & \mathcal{O}_X(V) \end{array}$$

[17]The open sets are all affine.

3.Glue these affine schemes.

These data give the normalization $X' \to X$. The result is independent of basis of affine opens because any one of such basis is cofinal with any other basis. It is clearly an integral scheme, which can be checked locally.

Surjectivity can be checked affine-locally, and is true by the commutative algebra fact(Atiyah-MacDonald, p62,Theorem 5.10) that for any morphism of rings $A \to B$ with B integral over A, any prime ideal of A can be obtained by contracting a prime ideal of B via this morphism. $\square$

Vakil 9.7.E*

Suppose that char $\mathrm{k} \neq 2$. Show that $\operatorname{Spec} \mathrm{k}[\mathrm{t}] \to \operatorname{Spec} \mathrm{k}[\mathrm{x}, \mathrm{y}] / \left(\mathrm{y}^2 - \mathrm{x}^2(\mathrm{x}+1)\right)$ given by $(\mathrm{x}, \mathrm{y}) \mapsto \left(\mathrm{t}^2 - 1, \mathrm{t}\left(\mathrm{t}^2 - 1\right)\right)$ is a normalization. The target curve is called the nodal cubic curve.

Vakil 17.2.H

If $\mathscr{S}_\bullet$ is finitely generated in degree 1 (Hypotheses 17.2.1), describe a canonical closed embedding

$$\begin{tikzcd} Proj(\mathscr{S}_\bullet) \arrow[rr, hook, "i"] \arrow[dr, "\beta"'] & & \mathbb{P}\mathscr{S}_1 \arrow[dl] \\ & X & \end{tikzcd}$$

and an isomorphism $Proj(\mathscr{S}_\bullet) \cong i^* \mathscr{O}_{\mathbb{P}\mathscr{S}_1}(1)$ arising from the surjection Sym $^\bullet \mathscr{S}_1 \to \mathscr{S}_\bullet$

Vakil 17.2.I*

Suppose $\mathscr{F}$ is a locally free sheaf of rank $n+1$ on X. Exhibit a bijection between the set of sections $s : X \to \mathbb{P}\mathscr{F}$ of $\mathbb{P}\mathscr{F} \to X$ and the set of surjective homomorphisms $\mathscr{F} \to \mathscr{L} \to 0$ of $\mathscr{F}$ onto invertible sheaves on X.

Vakil 17.3.B

Suppose $\pi : X \to Y$ and $\rho : Y \to Z$ are projective morphisms, and Z is quasicompact. Show that $\pi \circ \rho$ is projective.

Problem Set 9

Vakil 19.1.C

Suppose $\pi : X \to Y$ is a finite morphism whose degree at every point of Y is 0 or 1 . Show that π is injective on points. If $p \in X$ is any point, show that π induces an isomorphism of residue fields $\kappa(\pi(p)) \to \kappa(p)$. Show that π induces an injection of tangent spaces.

Vakil 19.2.B*

Suppose that k is algebraically closed, so the previous remark applies. Show that $C\backslash\{p\}$ is affine.

Vakil 19.9.B

Show that isomorphism classes of four ordered distinct (closed) points on $\mathbb{P}^1$ (over an algebraically closed field k), up to projective equivalence (automorphisms of $\mathbb{P}^1$), are classified by the cross-ratio.

Vakil 19.9.C

Explain why genus 1 curves over an algebraically closed field of characteristic not 2 are classified (up to isomorphism) by j-invariant.

Vakil 19.11.B*

Suppose $E \hookrightarrow \mathbb{P}_{\mathbb{C}}^2$ is a smooth complex plane cubic (hence genus 1), yielding a graded ring $\mathrm{S}_\bullet$. Let $\mathrm{t} : \mathrm{E} \rightarrow \mathrm{E}$ be translation by a non-torsion point. Show that t does not correspond to a map of graded rings $S_\bullet \rightarrow S_\bullet$, even after regrading (cf. Remark 16.4.7).

Vakil 21.2.E*

(JACOBIAN DESCRIPTION OF $\Omega_{A/B}$). Suppose $A = B\left[x_1, \ldots, x_n\right] / \left(f_1, \ldots, f_r\right)$. Then $\Omega_{A/B} = \left\{\oplus_i A dx_i\right\} / \left\{df_j = 0\right\}$ may be interpreted as the cokernel of the Jacobian matrix (12.1.6.1)

$$J : A^{\oplus r} \rightarrow A^{\oplus n}$$

Vakil 21.2.F

This notion of relative differentials is interesting even for finite extensions of fields.

(a) Suppose K/k is a separable algebraic extension. Show that $\Omega_{\mathrm{K/k}} = 0$.

(b) Suppose k is a field of characteristic $\mathrm{p}, \mathrm{K} = \mathrm{k}\left(\mathrm{t}^{\mathrm{p}}\right), \mathrm{L} = \mathrm{k}(\mathrm{t})$. Compute $\Omega_{\mathrm{L/K}}$ (where $\mathrm{K} \hookrightarrow \mathrm{L}$ is the "obvious" inclusion).

Vakil 21.2.P

Suppose $\pi : X \rightarrow Y$ is locally of finite type, and Y (and hence X) is locally Noetherian. Show that $\Omega_{X/Y}$ is a coherent sheaf on X.