Algebraic Geometry Homeworks

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Homework 1

All sheaves in HW1 take values in abelian groups(Ab).

2.6 A in Vakil

If $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$ is a sheaf morphism, then $\ker(\mathcal{F} \to \mathcal{G})_p \cong \ker(\mathcal{F}_p \to \mathcal{G}_p)$

Proof. It is known that the presheaf $\ker(\mathcal{F} \to \mathcal{G})$ is in fact a sheaf, restriction maps are induced by those of the sheaf $\mathcal{F}$.

By definition of the ker-sheaf, we already see, set-theoretically, LHS is contained in RHS, and that the inclusion is an injection.

To see this is surjective, note that, by definition of stalks and maps between them, a germ in RHS necessarily comes from a section in $\mathcal{F}(U)$ which is in the kernel of $\phi_{|_U}$, for some open set U containing p. The direct limit of this section by the restriction maps, i.e, this germ, then necessarily lies in LHS. $\square$

2.6 B in Vakil

If $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$ is a sheaf morphism, then $\operatorname{coker}(\mathcal{F} \to \mathcal{G})_p \cong \operatorname{coker}(\mathcal{F}_p \to \mathcal{G}_p)$

Proof. Cokernel in an abelian category, such as the category of abelian groups Ab, is defined to be the colimit of the following diagram:

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \\ 0 & \end{tikzcd}$$

On the other hand, stalk is defined to be the direct limit of the global sections over open sets containing the given point with restriction maps, which is a colimit in Ab.

Colimit commutes with colimit(1.6.12 of Vakil), hence taking coker and taking stalk commute.This gives the isomorphism.

$\square$

2.6.C in Vakil

If $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$ is a sheaf morphism, then $\operatorname{im}\phi$ in the category of (Ab-valued) sheaves is the sheafification of the image preseahf.

Proof. It is easy to verified that ker sheaf and coker sheaf of a sheaf morphism satisfy appropriate universal properties.Therefore, $\ker(\operatorname{coker}\phi)$(recall ker and coker are actually defined with morphism from/to them respectively), the image sheaf defined according to the rules of abelian category exists. We show it is in fact isomorphic to the sheafification of image presheaf.

The universal property of im in Ab, applying to the groups of sections over open sets, already gives isomorphisms between their groups of sections. The universal property of sheafification ensures these maps uniquely determine the sheaf morphism, which is isomorphism at the level of stalks.

By problem 1 and problem 2, a sheaf morphism is an isomorphism iff it is isomorphism at the level of stalks. The two sheaves are therefore isomorphic. □

2.6 D in Vakil

Taking stalk is an exact functor.

Proof. The claim is that exactness of the sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0\ ,$$

in the category of sheaves implies exactness of the sequence,

$$0 \longrightarrow \mathcal{F}_p \longrightarrow \mathcal{G}_p \longrightarrow \mathcal{H}_p \longrightarrow 0\ ,$$

in the Ab category.

Problem 1,2,3 show that it is indeed the case,passing to the level of stalks. □

Homework 2

Chapter II of Hartshorne, Exercise 1.2

Suppose $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$ is a sheaf morphism.

a. $(\ker\phi)_p \cong \ker\phi_p$ and $(\operatorname{im}\phi)_p \cong \operatorname{im}\phi_p$.

b. ϕ is injective or surjective $\iff$ ϕ_p is injective or surjective at every point p.

c. A sequence of sheaf morphisms is exact $\iff$ it is exact at the level of stalks.

Sol: a. This is exactly the contents of 2.6 A,B,C in HW1.

b. By the previously worked-out relation between sheaf-morphism and the induced morphism at the level of stalks, we can infer injectivity and surjectivity stalk-wise. The claim then follows from a.

c. This is immediate from a and b.

□

Homework 3

3.4 I in Vakil

Let I be an ideal of B, S a multiplicative subset of B.

a. 1.Spec$(B/I) \hookrightarrow$ Spec(B) has closed image. 2. Spec$(B_S) \hookrightarrow$ Spec(B) has open image when $S = \{1, f, f^2, ...\}$.3.Show by counter-example Spec(B_S) is neither closed nor open in Spec(B) in general.

b. 1. The topology of Spec(B/I) is the same as the one given by subspace topology. 2.The topology of Spec(B_S) is the same as the one given by subspace topology.

Sol: a. 1. Denote by p the natural projection $B \to B/I$. The notations $()^e$ and $()^c$ will be used consistently in the standard way as in commutative algebra(i.e, contraction and extension by the ring morphism). The map of topological space for the associated specs is then given by $()^c$. It is easy to see that the image of Spec(B/I) is just $V(I)$, which is clearly closed.

2.Denote by r the map $B \to B_S$. By definition of localization, it is clear that the image of Spec(B_S) by $()^c$is nothing but $D(f)$, which is a basic open set.

3.It is a standard fact that $\mathbb{Q}$ is the quotient field of $\mathbb{Z}$. Recall also that $\mathbb{Z}$ is an integral ideal. $\{(0)\}$, the image of Spec$(\mathbb{Q})$ in Spec$(\mathbb{Z})$, is a generic point.Therefore, it is not closed.On the other hand, this observation also shows Spec$(\mathbb{Z}) \setminus \{(0)\}$ has closure Spec$(\mathbb{Z})$, which means $\{(0)\}$ is not open either.

b. 1. By the correspondence theorem for CRing, we can lift a prime ideal in the quotient by I to a prime ideal containing I, with isomorphism between the quotients.

It is easy to see that $V(C) = V((C))$, by a similar argument as in classical algebraic geometry(where the coordinate rings are polynomials). It is then immediate that the image of $V((\bar{C}))$ is $V((C))$, where (C) is the lift of $(\bar{C})$, which is obviously closed.

2. Regardless of whether $S = \{1, f, f^2, ...\}$, the map between specs given by $()^c$ gives a bijection between {prime ideals in B_S} and {prime ideals in B disjoint from S}. The latter set will be denoted by $D(S)$.

The map $B \to B_S$ descends to $B/P^c \to B_S/P$.

Unwinding the set-theoretical notations, we see the image of $V((C))$ in $\mathrm{Spec}(B)$ is $V((C)^c) \cap D(S)$.

The explicit expressions of images of basic closed sets demonstrate they both have the subspace topologies.

□

3.4 J in Vakil

I is an ideal of B. An element f vanishes on $V(I) \iff f \in \sqrt{I}$.

Proof. It is a standard commutative algebra fact that $\bigcap_{P \supset I \text{prime}} P = \sqrt{I}$.

$P \in V(I) \iff P \supset I$. f vanishes on $V(I) \iff f \in \sqrt{I}$.

□

3.4 H in Vakil

The Spec map $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ induced by the ring morphism $f^* : A \to B$ is continuous.

Proof. Let $r \in A$, we look at $f^{-1}(D(r))$. This is the set of form of prime ideals which give rise the maps between quotients

$A/P^c \to B/P$,

such that r is non-zero in A/P^c.

This means $f^*(r)$ is non-zero in B/P, or else r would be 0 in A/P^c by definition of P^c.

This suggests that $f^{-1}(D(r))$=$D(f^*(r))$. Although the above computation only gives $\subset$, $\supset$ is immediate. □

Homework 4

4.4 A in Vakil

Given the following data:

- A collection of schemes X_i indexed by an index set I.
- $X_{ij} \subset X_i$ open subschemes with $X_{ii} = X_i$
- $f_{ij} : X_{ij} \to X_{ji}$ isomorphisms with $f_{ii} = \mathrm{id}_{X_i}$
- The cocycle condition is satisfied.

Then they "glue along X_{ij}'s by f_{ij}"to form a scheme.

Proof. As a topological space, this is just $\sqcup_{i \in I} X_i$ quotiented by the maps f_{ij}. The only place which deserves attention is where they are glued. Since the maps are isomorphisms of schemes, we can just define the local sections by the isomorphism classes given by f_{ij}. From this, it is clear the result is a locally-ringed space.

To check it is locally-affine, just pick a point where the gluing happens(as argued earlier,it is obvious for other points), choose a representative belonging to one of the open subschemes in the initial collection of schemes, and an open-affine of that. Since f_{ij}'s are isomorphisms, this open-affine has an image which is also open-affine. □

4.5 A in Vakil

Define the scheme specified by the condition "$\{x_0^2 + x_1^2 - x_2^2 = 0\}$" in $\mathbb{P}_k^2$.

Proof. $k[x_0, x_1, x_2]$ has the natural grading by degrees, and "$x_0^2 + x_1^2 - x_2^2$" is a homogeneous element of degree 2.

As a set, $V((x_0^2 + x_1^2 - x_2^2))$ is just the set of homogeneous prime ideals containing $x_0^2 + x_1^2 - x_2^2$ and not containing the irrelevant ideal. The structure sheaf is given by gluing the affine opens($x_i \to x_{i/j}$ as coordinate j-th affine chart), as described in Vakil's book or in class. □

4.5 B in Vakil

Define the scheme "cut out by homogeneous elements f_i in $\mathbb{P}_A^n$"

Proof. This is formally just like 4.5 A, instead of containing just one homogeneous element, the homogeneous ideals are required to contain all homogeneous elements f_i's. □

Homework 5

4.5 H in Vakil

Fix a graded ring $S_\bullet$.

a. $I \subset S_\bullet$ is a homogeneous ideal in $S_\bullet$, and f is a homogeneous element with deg > 0. $V(I) \subset V(f) \iff f^n \in I$ for some $n > 0$.

b. $Z \subset \operatorname{Proj}(S_\bullet)$ is a subset, $I(Z) \subset S_+$ can be defined, which is homogeneous. Moreover $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.

c. $Z \subset \operatorname{Proj}(S_\bullet)$. $V(I(Z)) = \operatorname{cl} Z$.

Proof. a)It is a standard fact that the radical of a homogeneous ideal is homogeneous. The given inclusion $\iff$ f lies in the radical of I.

b)The definition can be suggested by experience of classical algebraic geometry. Simply define it to be the ideal generated by homogeneous elements with positive degrees which are contained in Z. The set theoretic equality for ideal of union is immediate from this definition. Note that $I(Z)$ is always a radical.

c) $Z \subset V(I(Z))$ is immediate. Since $V(I(Z))$ is closed, we have $\operatorname{cl} Z \subset V(I(Z))$. Conversely, write $\operatorname{cl} Z = V((C)) \supset Z$, where (C) is a homogeneous ideal generated by a set of homogeneous elements C. We have $(C) \subset I(V((C))) \subset I(Z)$.Taking V again gives $V((C)) \supset V(I(Z))$. We can now conclude from the two reverse inclusions we obtained that $\operatorname{cl} Z = V(I(Z))$.

□

4.5 I in Vakil

Fix a graded ring $S_\bullet$ and a homogeneous ideal I. TFAE:

a. $V(I) = \emptyset$.

b. $\forall \{f_i\}_{i\in J}$ with $(f_i)_{i\in J} = I$, $\bigcup D(f_i) = \operatorname{Proj}(S_\bullet)$.

c. $\sqrt{I} \supset S_+$.

Proof. a $\iff$ c: $I(V(I)) = \sqrt{I}$ and $\operatorname{cl} \emptyset = \emptyset$. So $\emptyset = \operatorname{cl} V(I) = V(\sqrt{I}) = \emptyset \iff \sqrt{I} \supset S_+$.

a $\iff$ b: The statement in b is equivalent to $\bigcap_{i\in J} V(f_i) = \emptyset$ by taking complement. LHS is just $V(I)$, so the conclusion follows. □

4.5 P in Vakil

Suppose $S_\bullet$ is generated in deg 1 and $f \in S_+$ is homogeneous. Define $V(f)$ and $V(I)$ for general homogeneous ideal I.

Proof. Simply define $V(f)$ to be the set of homogeneous prime ideals containing f. In general, $V(I)$ is defined to be the set of homogeneous prime ideals containing I. The definition is just as in the case of classical algebraic geometry for a hypersurface in projective space. □

Homework 6

6.3 A in Vakil

Show that morphisms of locally ringed spaces glue.

Proof. The procedure is basically described in 4.4 A. We can think of the locally ringed space X as being glued by its open subsets along inclusion maps. The map from it to a locally ringed space Y can be obtained by gluing compatible morphisms from these open subsets to Y. To finish up, just note that whether a map is morphism of locally ringed spaces can be checked stalk-locally. The compatible morphisms do indeed glue. □

Vakil 6.3 B

a. Spec A is locally ringed.

b. A ring morphism induces a morphism of locally ringed spaces.

Proof. The fact that Spec(A) is locally ringed follows from the fact that A_P is a local ring, being localization at the prime ideal.

Whether a map is a morphism of locally ringed spaces can be checked stalk-locally,assuming all the commutative diagrams have been checked. $A \to B$ induces naturally a map $\text{Spec}(B) \to \text{Spec}(A)$, for spaces and sheaves.

If a prime ideal Q of B gets mapped to $P = Q^c$. Vacuously, the stalk-map sends P into Q. $\square$

Vakil 6.3 C

A morphism of schemes is a morphism of locally ringed spaces which locally looks like morphism of affine schemes.

Proof. Let $(\pi, \pi^\sharp)$ be a morphism of scheme $X \to Y$, which by definition is just a morphism of locally ringed spaces between schemes.

Cover Y with affine-opens V_j's and $\pi^{-1}(V_j)$ by affine opens U_{ij}, where i and j are indices.

The restrictions $U_{ij} \to V_j$ are obviously morphisms of affine schemes:the sheaf-morphisms satisfy the obvious commutative-diagrams, and they are local. This is because the morphisms of affine schemes are tailored to satisfy certain local and stalk-local conditions. $\square$

Vakil 6.3 E

Make precise phrase the canonical map "$A_k^{n+1} \setminus \{0\} \to \mathbb{P}_k^n$" is a morphism of schemes.

Proof. The idea is to write $A_k^{n+1} \setminus \{0\}$ as union of open affines of A_k^{n+1}.The obvious choices are $D(x_i)$,$i = 0, ..., n+1$.

Construct morphism from $D(x_i)$ to i-th standard chart of $\mathbb{P}_k^n$.This is induced by the CRing morphism from $k[y_0, ..., y_{n-1}]$ to $k[x_0, ..., x_n]_{x_i}$ specified by $y_k \to \frac{x_k}{x_i}$. The morphisms are tailored to be compatible(which was checked when constructed projective schemes).They therefore glue. $\square$

Hartshorne Chap II 2.19

TFAE:

i Spec A is disconnected.

ii $\exists\, e_1, e_2 \neq 0$ such that $e_1 e_2 = 0, e_1^2 = e_1, e_2^2, e_1 + e_2 = 1$.

iii $A \cong A_1 \times A_2$, neither of the factors is the zero ring.

Proof. i $\implies$ ii:If $\text{Spec}(A)$ is disconnected, we can find disjoint open sets U_1 and U_2 which cover $\text{Spec}(A)$. Construct global sections e_i(i=1,2) as follows:e_i takes value $1 \in O_{\text{Spec}\,A}(U_i)$ and 0 on the other open set. This obviously glues to a global section because compatibility is trivially true as the open sets are disjoint.

Thus,they correspond to elements in A, the ring of global sections of $\text{Spec}(A)$. It is trivial to verify they behave as in ii.

ii $\implies$ iii: a$\in$ A can be written as $ae_1 + ae_2 = a$. So $a \to (ae_1, ae_2)$ is easily verified to be an ring isomorphism from A to $A_1 \times A_2$.

iii $\implies$ i:This follows from $\text{Spec}(A_1 \times A_2) = \text{Spec}(A_1) \sqcup \text{Spec}(A_2)$. The right hand side is interpreted as disjoint union of topological spaces, which is clearly disconnected. □

Vakil 7.3 H

If X $\to$ $\text{Spec}(k)$ is a finite morphism,then X is a finite union of points with discrete topology, whose residual field at each point is finite extension of k.

Proof. By the very definition of finite morphism, the pre-image of open-affine $\text{Spec}(B)$ in the base is open affine which happens to be isomorphic to be $\text{Spec}(A)$ of a finitely generated $B-$algebra A, which is also finitely generated as $B-$module. So $X \cong \text{Spec}(A)$, where A is a finitely generated $k-$algebra.

Now, since finite $k-$ algebra which happens to be an integral domain is automatically a field, we conclude all prime ideals of A are maximal. Hence, all points are closed,because they correspond to $\text{Spec}(A/P)$.

The residual field $K(P)$ at each point P is a field which is also a finitely generated $k-$algebra, hence a finite extension of k by Zariski lemma.

Discreteness and finiteness will follow if we can show the closed points are in fact open, because $\text{Spec}(A)$ is Q.C.

It is a standard fact that a finite $k-$algebra is necessarily Artinian(i.e, satisfying the descending chain condition,or D.C.C,which is easily seen by viewing ideals as k vector subspaces). Artin rings are known to have only finitely many maximal ideals. Applying it to our case shows we have finitely many closed points. This proves finiteness. Discreteness follows from structure theorem of Artin rings(A is Artinnian $\iff$ A is direct product of finitely many local Artin rings). □

Comment: I don't seem to be able to find a proof for discreteness and finiteness without quoting "fancy theorems" as Vakil suggested. The statement in this question is essentially equivalent to these facts without using the technical jargon,unless the case of finite k-algebras is considerably easier than general Artin rings.

Vakil 7.3 K

Finite morphisms have finite fibres.

Proof. Finite morphisms are preserved under base change,see Vakil 9.4B. Therefore, $X \to Y$ is finite $\implies$ $X_P \to \mathrm{Spec}(K(P))$ is also finite, where $P \in Y$ and $K(P)$ is its residue field. Therefore, by 7.3 H of Vakil, we conclude X_P is finite. $\square$

Vakil 7.3 O

$X \xrightarrow{\pi} Y$ is locally of finite type if Y is covered by affine opens $\{\mathrm{Spec}(B_i)\}_{i\in I}$ and $\pi^{-1}(\mathrm{Spec}(B_i))$ is of locally finite type over B_i.

Proof. This is sometimes taken as definition.

Pick any $\mathrm{Spec}(B)$ open-affine in the base.By quasi-compactness, only finitely many $\mathrm{Spec}(B_i)$ is needed to cover it.Moreover, it is possible, by Q.C, to cover $\mathrm{Spec}(B)$ with finitely many basic open sets of $\mathrm{Spec}(B_i)$, call them $\mathrm{Spec}((B_i)_{f_j})$.

Since $\mathrm{Spec}((B_i)_{f_j})) \hookrightarrow \mathrm{Spec}(B)$ is injective, $B \to (B_i)_{f_j}$ is onto. A finitely generated $(B_i)_{f_j}$-algebra is also finitely generated as B-algebra.

The fact that the morphism is of locally finite type over $\mathrm{Spec}(B_i)$ now implies $X \to Y$ is locally of finite type. $\square$

Homework 7

Vakil 8.1.J

a. Define the finite union and arbitrary intersection of closed subschemes.

b. Describe the scheme-theoretic intersection of $V(y - x^2)$ and $V(y)$ in $\mathbb{A}^2$. Describe their scheme-theoretic union.

c. Show that the underlying set of a finite union of closed subschemes is the finite union of the underlying sets, and similarly for arbitrary intersections.

d. Describe the scheme-theoretic intersection of $V(y^2 - x^2)$ and $V(y)$ in $\mathbb{A}^2$. Hence show that if X, Y, and Z are closed subschemes of W, $(X \cap Z) \cup (Y \cap Z) \neq (X \cup Y) \cap Z)$ in general. In particular, not all properties of intersection and union carry over from sets to schemes.

Solutions:

a. *Proof.* Let X be a scheme, and $\{Z_i\}_{(i\in I)}$ a collection of its closed subschemes. They correspond to quasi-coherent ideal sheaves $\{\mathcal{J}_i\}_{i\in I}$.

   For finite index set I, the ideal sheaf,$\sum_{i\in I} \mathcal{J}_i$,locally defined by ideals generated by combinations of elements from $\mathcal{J}_i$'s, is quasi-coherent, and hence defines a closed-subscheme, which we define to be their scheme-theoretic intersection.

   If I is arbitrary, then $\bigcap_{i\in I} \mathcal{J}_i$,locally defined by literal intersection of ideals, is quasi-coherent. This defines a closed-subscheme which we call their scheme-theoretic union.

Everything works just as in classical algebraic geometry.

□

b. *Proof.* It is easy to see that the ideal sheaf corresponding to the union is generated by $(y - x^2, y)$. For the sake of concreteness let's consider what happens in $\mathbb{A}^2_k$, where k is a field.

The computation of the ideal sheaf implies that the intersection is $\mathrm{Spec}(\frac{k[x]}{(x^2)})$. □

c. *Proof.* This can be verified affine-locally.The underlying Sp's of corresponding schemes are what we claim they are almost by definition. □

d. *Proof.* By a similar analysis as in (a), we see the intersection of $V(y^2 - x^2)$ and $V(y)$ is as in (a).

Observe $(x-y)$ and $(x+y)$ are co-maximal, and their intersection is (y^2-x^2). Therefore, scheme-theoretic union of $V(x-y)$ and $V(x+y)$ is $V(y^2-x^2)$. Its intersection with $V(y)$ was worked out.

On the other hand, $V(x-y)$ intersects $V(y)$ is nothing but $V(x, y)$, which is also $V(x+y)$ intersects $V(y)$.

By definition in a, their union is again $V(x, y)$.

This gives an example demonstrating scheme-theoretic operations do not always behave as their set-theoretic analogues. □

Vakil 9.2.C

1. Interpret the intersection of two closed embeddings into X as their fibered product over X.
2. Show that "locally closed embeddings" are preserved by base change.
3. Define the intersection of n locally closed embeddings $X_i \hookrightarrow Z$ the fibered product of the X_i over Z (mapping to Z). Show that the intersection of (a finite number of) locally closed embeddings is also a locally closed embedding.

1. *Proof.* Let $Z_1 \overset{i}{\hookrightarrow} X$ and $Z_2 \overset{j}{\hookrightarrow} X$ be two closed subschemes.

   It is a commutative algebra fact that $\frac{A}{\mathfrak{a}} \otimes_A \frac{A}{\mathfrak{b}} \cong \frac{A}{\mathfrak{a}+\mathfrak{b}}$ as a ring, where $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of A.

   Cover X by local-affines and apply this isomorphism repeatedly will imply the two definitions of scheme-theoretic union agree. □

2. *Proof.* Suppose $X \xrightarrow{f} S$ is locally closed, and base change to $X_{S'}$ along $S' \xrightarrow{g} S$. Cover S by open-affines $\{S_i\}$ so that the restrictions $f^{-1}(S_i) \xrightarrow{f|_{f^{-1}(S_i)}} S_i$ are close embeddings. Finally, cover $g^{-1}(S_i)$'s by open affines. □

3. *Proof.* By induction, it is enough to prove this for a pair.

   Cover the base Z by open-affines whose restrictions to preimages of both locally closed embedding are all closed embeddings. It is clear then the restrictions to preimages in the fibre-products are closed embedding by part 1. $\square$

Vakil 9.2.H

Prove that $\mathbb{A}^n_A \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(A)$ and $\mathbb{P}^n_A \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(A)$Thus affine space and projective space are pulled back from their "universal manifestation" over the final object $\mathrm{Spec}\,\mathbb{Z}$.

Proof. Simply observe that $\mathbb{Z}[\underline{X}] \otimes_{\mathbb{Z}} A \cong A[\underline{X}]$. Therefore, $\mathbb{A}^n_A \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(A)$.

The case for $\mathbb{P}^n_A$ follows by applying the above argument to standard affine-charts and then glue them up. $\square$

Vakil 9.4.B(b,d)

Show that the following properties of morphisms are preserved by base change:

b. quasi-separated

d. finite

Solutions:

b. *Proof.*

$$\begin{tikzcd} X & X \times_S S' \arrow[l] \arrow[d, "f'"] \\ S \arrow[from=u, "f"] & S' \arrow[l] \end{tikzcd}$$

   Consider the diagram as drawn.

   Since QS is affine-local on the target, it suffices to consider the case when both S and $S^{'}$ are affine schemes.

   Since f is QS, X is necessarily QS(this is not vacuous,because in the general situation X is just the pre-image of a affine-open of a morphism from a bigger scheme).

   By one of the criteria of QS, and property of fibre product, we will show QS if we can show intersection of open affines of the form $U \times_S S^{'}$ where U is open affine of X is QC. On the other hand, this is clear, because that is the same as fibre product of the intersection of U_i's with $S^{'}$ over S. Now, $U_1 \cap U_2$ can be written as union of finitely many open-affines. Its fibre product with $S^{'}$ over S is then a finite union of affine-schemes(hence,QC open sets), and is therefore QC. $\square$

d. *Proof.* Finiteness is also affine-local on the target. We may proceed as in b, with the same diagram and notations.

Since f is finite, we may assume X is in fact affine. Now, it is immediate $X \times_S S^{'}$ is an affine scheme.

Finiteness of $f^{'}$follows from the fact if the ring morphism $A \to B$ equips B with a finitely generated module structure, and C is an A-module, then $A \otimes_A C \to B \otimes_A C$ also gives rise to a finitely generated $A \otimes_A C$- module structure. $\square$

Comments:
1.It appears that the fact that these morphisms are "affine-local on the target" was used in an essential way. Their proofs are not entirely straightforward.
2.The arguments would have broken down at various stages if either S or $S^{'}$ are not both affine-schemes. Thanks to the "affine-local on the target" property, this is the only case that needs to be considered.
3. A high concept proof probably exists if one uses the diagonal morphism explicitly in the definitions.

Vakil 9.3.D

What is the scheme-theoretic fiber of $\operatorname{Spec} \mathbb{Z}[i] \to \operatorname{Spec} \mathbb{Z}$ over the prime (p)?

Proof. We first look at how rational prime ideals split over $\mathbb{Z}[i]$.

1. $(0)^e = (0)$
2. $(p)^e = Q^2$,where Q is the unique prime ideal lying above (p) in $\mathbb{Z}[i]$.
3. $(p)^e = Q_1 Q_2$, where Q_i's are distinct prime ideals lying above (p).
4. p^e is a prime ideal.

These are all the possibility by elementary algebraic number theory.

Case 2 only occurs for (2) because 2 is the only ramified prime in $\mathbb{Z}[i]$.

Case 3 and 4 are completely determined by $(\frac{-1}{p})$ by translating quadratic reciprocity law into the setting of quadratic extensions.

Case 3 corresponds to $(\frac{-1}{p}) = 1$, whereas case 4 corresponds to $(\frac{-1}{p}) = -1$.

Computation of fibres is now straightforward,which only requires tautological computations of residual fields and commuting rules of localization and tensor product.

(0) has residue field $\mathbb{Q}$. So the fibre above is $\operatorname{Spec}((\mathbb{Z}[i]) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}(i))$.

For (2), we have $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{F}_2 \cong \mathbb{Z}[i]/2\mathbb{Z}[i] \cong \mathbb{Z}[i]/(1+i)^2$. Therefore the fibre above it is $\operatorname{Spec}(\frac{\mathbb{Z}[i]}{(1+i)^2})$.

If (p) is inert, then similar analysis shows the tensor product is isomorphic to $\frac{\mathbb{Z}[i]}{(p)^e}$.$(p)$ is inert $\implies$ $(p)^e$ is prime and this quotient is isomorphic to $\mathbb{F}_{p^2}$. The fibre above (p) is just $\operatorname{Spec}(\mathbb{F}_{p^2})$

If (p) splits, then by a similar analysis as above, with an additional simplification of the quotient using the Chinese remainder theorem(CRT), we see the tensor product is $\cong \mathbb{F}_p \times \mathbb{F}_p$, and the fibre above (p) is $\operatorname{Spec}(\mathbb{F}_p \times \mathbb{F}_p)$,a disjoint union of two point schemes with coordinate ring $\mathbb{F}_p$. □

Problem-Solving Session of 2020-11-14(In Progress)

All problems are from chapter 2 of Hartshorne's GTM52.

Q1 2.16

Let X be a scheme,$f \in \Gamma(X, O_X)$, define $X_f = \{x \in X : f_x \notin m_x\}$, where m_x is the maximal ideal of the local ring $O_{X,x}$.

a. $U = \operatorname{Spec}(B)$ is an open-affine, $\bar{f} \in B = \Gamma(U, O_X|_U)$ is the restriction of f on U,then $U \cap X_f = D(\bar{f})$. This implies X_f is open.

b. Suppose X is Q.C. Let $A = \Gamma(X, O_X)$, and $a \in A$ be an element whose restriction to X_f is 0. Show that $f^n a = 0$ for some $n > 0$.

c. Now, suppose X has a finite cover by open-affines $\{U_i\}$ such that all pairwise intersections are Q.C. Let $b \in \Gamma(X_f, O_{X_f})$. Show that $f^n b$ is the restriction of an element of A.

d. Under the hypothesis of c, $\Gamma(X_f, O_{X_f}) \cong A_f$.

Solutions:

a. *Proof.* X_f is defined stalk-locally, we may just as well consider the case where X is affine. But it is a tautology to check a global section vanishes at a point $\iff$ its germ at that point is contained in the maximal ideal of the local ring at the point. The equality is therefore immediate.

   The case where X is not affine is done by union all such intersections, which necessarily results an open set. □

b. *Proof.* Cover X by finitely many open-affines $\{U\}$. We see $a|_U f|_U^{n_U} = 0$ at every such open-affine, with n_U depending on U. The claim then follows by taking $n > \max\{n_U\}$. □

c. *Proof.* Just "clear denominators" all at once. The fact that all pairwise intersections are Q.C ensures there is a big enough power of f to do the job. □

d. *Proof.* By c, we know any $b \in \Gamma(X_f, O_{X_f})$ satisfies $f^{n_b} b = a|_{X_f}$. The morphism $\Gamma(X_f, O_{X_f}) \to A_f$ can be defined by $b \to \frac{a}{f^{n_b}}$. Part b ensures it is a well-defined monomorphism. Surjectivity is immediate. □

Q2 2.17

a. Let $f : X \to Y$ be a morphism of schemes, and suppose Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Conclude f is an isomorphism.

b. A scheme X is affine iff there is a finite set of elements $f_1, ..., f_r \in A = \Gamma(X, O_X)$,such that the X_{f_i}'s are affine-opens and $f_1, ..., f_r$ generate the unit ideal in A.

Q3 2.18

a. Let A be a ring,$X = \text{Spec}(A)$, and $f \in A$. Show that f is nilpotent iff $D(f) = \emptyset$.

b. Let $\phi : A \to B$ be a homomorphism of rings, and $f : Y = \text{Spec}(B) \to X = \text{Spec}(A)$ the induced morphism of affine schemes. Show that ϕ is injective iif the map of sheaves $f^\sharp : O_X \to f_*O_Y$ is injective. Show furthermore in that case f is dominant,i.e.,$f(Y)$ is dense in X.

c. With the same notation, show that if ϕ is surjective, then f is homeomorphism of Y onto a closed subset of X, and $f^\sharp : O_X \to f_*(O_Y)$ is surjective.

d. Prove the converse of c:if $f : Y \to X$ is a homeomorphism onto a closed subset, and $f^\sharp : O_X \to f_*O_Y$ is surjective, then ϕ is surjective.

Solutions:

a. *Proof.* It was shown that $D(f) \cong \text{Spec}(A_f)$. " $\implies$ " is immediate, as $A_f = 0$ when f is nilpotent.

The converse is done contrapositively. If f is not nilpotent, then A_f is easily seen to be non-zero. There will be a maximal ideal(hence prime) containing a non-zero element, hence $\text{Spec}(A_f) \ncong \emptyset$. $\square$

b. *Proof.* " $\implies$ " is immediate: injectivity of sheaf-morphism can be checked stalk-locally. $A \to B$ injective clearly implies any induced map $A_{(\phi)^{-1}(P)} \to B_P$ is injective for any prime ideal P of B.

"$\impliedby$": Assuming injectivity of morphism of sheaf, we look at commutative diagram

$$\begin{tikzcd} A = \Gamma(X, O_X) \arrow[r, "\phi"] \arrow[dr] & B = \Gamma(Y, O_Y) \arrow[d] \\ & O_Y(f^{-1}(X)) \end{tikzcd}$$

A non-trivial element in the kernel of the upper horizontal ring morphism is also in the kernel of the diagonal ring morphism, which can't happen because of the given sheaf morphism is injective.

If there is a point $P \in X$ which has an open neighborhood not intersecting $f(Y)$, then we get $f_*(O_Y)_P = 0$. The only way ϕ_P is injective is if $A_P = 0$. Which means P contains the whole ring A, which means A is the zero ring. In this case, the morphism trivially has dense image(onto, in fact). If A is not the zero ring, then such P cannot exist, which means every neighbourhood of every point in X intersects $f(Y)$, implying $f(Y)$ is dense in X.

□

c. *Proof.* By the third isomorphism theorem, $A/\ker\phi \cong B$. So $\mathrm{Spec}(B) \cong \mathrm{Spec}(A/\ker\phi)$ canonically. Surjectivity of sheaf morphism is immediate consequence of the fact that the induced morphism is a closed embedding. □

d. *Proof.* Pick $b \in \Gamma(X, O_X) = B$, and WLOG assume it is non-zero.

$f^\sharp$ is surjective $\implies$ $b|_{U_i = D(f_i)}$ can be written as image of $\frac{a}{(f_i)^{n_i}}$ under ϕ. Cover X by finitely many(say n) such basic open sets.

Let $N > \max\{n_i\}$. Since $D(f_i) \subset D((f_i)^N)$, we can cover X by $D((f_i)^N)$.

Therefore, we can write $1 = \sum_{i=1}^n c_i f_i^N$, for some $\{c_i\} \subset A$,

and $b = \phi(\sum_{i=1}^n c_i a_i f_i^{N-n_i})$.

This proves surjectivity. □

Comment on d: How was the fact $f(Y)$ is closed used? This was needed to produce the finite affine covers with the properties we claimed to have.

Q4 3.5

A morphism is quasi-finite(Q.F) if for every point $y \in Y$, $f^{-1}(y)$ is a finite set.

a. Show that a finite morphism is Q.F.

b. Show that a finite morphism is closed,i.e., it sends a closed set to closed set.

c. Give an example of a surjective,finite-type,and Q.F morphism, which is not finite.

Q5 3.6

Let X be an integral scheme. Show that the local ring O_ξ of the generic point ξ of X is a field. This is called the function field of X and is denoted by $K(X)$. Show that if $U = \mathrm{Spec}(A)$ is any open-affine of X, then $K(X)$ is isomorphic to the quotient field of A.

Q6 3.7

A morphism $f : X \to Y$, with Y irreducible, is generically finite if $f^{-1}(\eta)$ is a finite set, where η is the generic point of Y. A morphism $f : X \to Y$ is dominant if $f(X)$ is dense in Y. Now, let $f : X \to Y$ be dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $U \subset Y$ such that the induced morphism $f^{-1}(U) \to U$ is finite.

Q7 3.8

A scheme is normal if all of its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \mathrm{Spec}(A)$ of X, let $\tilde{A}$ be the integral closure of A in its quotient field and let $\tilde{U} = \mathrm{Spec}(\tilde{A})$. Show that one can glue $\tilde{U}$'s to obtained a normal integral scheme $\tilde{X}$, called the normalization of X. Show that there is a morphism $\tilde{X} \to X$ with the following universal property:for every normal integral scheme Z, and for every dominant $f : Z \to X$, f factors uniquely through $\tilde{X}$. If X is of finite type over a field k, then the morphism $\tilde{X} \to X$ is a finite morphism.

Q8 3.9

a. Let k be a field, and let $\mathbb{A}_k^1 = \mathrm{Spec}(k[x])$. Show that $\mathbb{A}_k^1 \times_{\mathrm{Spec}(k)} \mathbb{A}_k^1 \cong A_k^2$, and the underlying point-set of the product is not the product of underlying point-sets of the factors.

b. Let k be a field, let s and t be indeterminates over k. $\mathrm{Spec}(k(s))$,$\mathrm{Spec}(k(t))$,and $\mathrm{Spec}(k)$ are all one-point spaces. Describe the product scheme $\mathrm{Spec}(k(s)) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k(t))$.

Q9 3.10

1. If $f : X \to Y$ is a morphism, and $y \in Y$ a point, show that $sp(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.

2. Let $X = \mathrm{Spec}(k[s,t]/(s - t^2))$, let $Y = \mathrm{Spec}(k[s])$, and let $f : X \to Y$ be the morphism defined by sending $s \to s$. If $y \in Y$ is a point $a \in k$ with $a \neq 0$, show that the fibre X_y consists of two points, with residue field k. If $y \in Y$ corresponds to $0 \in k$, show that the fibre X_y is a nonreduced one-point scheme. If η is the generic point of Y, show that X_η is a one-point scheme, whose residue field is an extension of degree two of the residue field of η.(Assuming k is algebraically closed.)

Q10 3.11

a. Closed immersions are stable under base extension: if $f : X \to Y$ is a closed immersion, and if $X' \to X$ is any morphism, then $f' : Y \times_X X' \to X'$ is also a closed immersion.

b. If Y is a closed subscheme of an affine scheme $X = \mathrm{Spec}\, A$, then Y is also affine, and in fact Y is the closed subscheme determined by a suitable ideal $\mathfrak{a} \subset A$ as the image of the closed immersion $\mathrm{Spec}(A/\mathfrak{a}) \to \mathrm{Spec}(A)$.

c. Let Y be a closed subset of a scheme X, and give Y the reduced induced subscheme structure. If Y' is any other closed subscheme of X with the same underlying topological space, show that the closed immersion $Y \to X$ factors through Y'.

d. Let $f : Z \to X$ be a morphism. Then there is a unique closed subscheme of X with the following property: the morphism f factors through Y, and if $Y^{'}$ is any other closed subscheme of X through which f factors, then $Y \to X$ factors through $Y^{'}$ also. Y is called the scheme theoretic image of f. If Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.

Q11 3.12

a. Let $\phi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set $U = \{\mathfrak{p} \in \operatorname{Proj} T : \mathfrak{p} \not\supset \phi(S_+)\}$ is equal to $\operatorname{Proj} T$, and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is a closed immersion.

b. If $I \subset S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X = \operatorname{Proj} S$ defined as image of the closed immersion $\operatorname{Proj} S/I \to X$. Show that different homogeneous ideal can give rise to the same closed subscheme.(Hint: take $I^{'} = \bigoplus_{d \geq d_0} I_d$.)

Q12 3.13

a. A closed immersion is a morphism of finite type.

b. A Q.C open immersion is of finite type.

c. A composition of two morphisms of finite type is of finite type.

d. Morphisms of finite type are stable under base extension.

e. If X and Y are schemes of finite type over S, then $X \times_S Y$ is finite type over S.

f. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms, and if f is Q.C, and $g \circ f$ is of finite type, then f is of finite type.

g. $f : X \to Y$ is a morphism of finite type, and if Y is noetherian, then X is noetherian.

Solutions:

a. *Proof.* This is immediate by the properties of closed immersion. The pre-image of an open-affine $\operatorname{Spec} R$ is $\operatorname{Spec} \frac{R}{I}$ where I is an ideal of R. $\frac{R}{I}$ is obviously a finitely generated R-module. □

b. *Proof.* Pick any open-affine in the image $\operatorname{Spec} R$. It is possible to cover the pre-image by finitely many open-affines isomorphic to $\operatorname{Spec} R_f$. Trivially, R_f's are finitely generated R-algebras. □

c. *Proof.* This is straightforward commutative algebra. □

d. *Proof.* The argument is identical to that used in the proof of Vakil 9.4.B.b&d. □

e. *Proof.* d implies $X \times_S Y \to Y$ is a morphism of finite type.

   Now $Y \to S$ is a morphism of finite type by hypothesis. It follows from c $X \times_S Y \to S$ is also finite type.

   □

Q13 3.14

If X is a scheme of finite type over a field, show that the set of closed points of X is dense. Give an example to show that this is not true for arbitrary schemes.

Q14 3.15

Let X be a scheme of finite type over a field k(not necessarily algebraically closed).

a. TFAE(in which case we say that X is geometrically irreducible):

   i $X \times_k \bar{k}$ is irreducible, where $\bar{k}$ denotes the algebraic closure of k.

   ii $X \times k_s$ is irreducible, where k_s is the separable closure of k in $\bar{k}$.

   iii $X \times_k K$ is irreducible for every extension field K of k.

b. TFAE(in which case we say X is geometrically reduced):

   i $X \times_k \bar{k}$ is reduced.

   ii $X \times_k k_p$ is reduced, where k_p is the perfect closure of k.

   iii $X \times_k K$ is reduced for all extension fields K of k.

c. We say a scheme is geometrically integral of $X \times_k \bar{k}$ is integral. Give examples of integral schemes which are neither geometrically irreducible nor geometrically reduced.