Class Field Theory: Tate's Thesis

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Abstract

This is based on the talks by Yiwang Chen. We explain the contents of Tate's thesis, and its relation to $GL(1)$-Langlands. We also discuss how $GL(1)$-Langlands is related to Class field theory.

Contents

1 Introduction

The main references for technical details are Tate's original thesis published in [2](p305) and the survey article by Steve Kudla from a more modern viewpoint in [1](Chapter 6).

Recall the reciprocity problem asks for identifications of extrinsic data(more precisely, the Galois data) of a number field K with intrinsic data of K. The intrinsic data used in Class field theory are the class groups. Originally, it was the ray class groups that were used, and the theory was later rewritten by Chevalley using idele class groups.

In the Langlands program, the intrinsic data are automorphic representations. The crudest identifications are the identifications of L-functions: every Artin L-function is conjecturally an automorphic L-function. This identifies every Galois representation with an automorphic representation.

For 1-dimensional Galois representation, which is just a fancier way to call a character $\rho : \Gamma_K \to GL(1, \mathbb{C}) \cong \mathbb{C}^\times$(so we have the factorization $\Gamma_K \xrightarrow{p} \Gamma_K^{ab} \xrightarrow{r} \mathbb{C}^\times$), it is not a conjecture but a theorem that the Artin L-function $L(s, \rho)$ associated to ρ is automorphic. More precisely, by the isomorphy theorem, it can be proved that $L(s, \rho)$ is a Hecke L-function associated to a character of the idele class group C_K(a Hecke character).

One implication of Tate's thesis is that Hecke L-functions can be interpreted as automorphic L-functions of K for the group $GL(1)$. Therefore, "Class field theory"+"Tate's thesis" $\implies$ $GL(1)$-Langlands.

2 Notations and Definitions

3 Facts from Abstract Harmonic Analysis

4 Examples and Computations

5 The Statements

6 The Proofs

7 Towards The Langlands Program

References

[1] Joseph Bernstein, Daniel Bump, SS Kudla, E de Shalit, D Gaitsgory, and JW Cogdell. *An introduction to the Langlands program*, volume 140. Springer Science & Business Media, 2003.

[2] John William Scott Cassels and Albrecht Fröhlich. *Algebraic number theory*. London Mathematical Society London, 2010.