Deformation Quantization and A Generalization of the Index Theorem

Yujia Yin

Abstract

This note grows out of an essay for the Index Theory course in the 2020 fall semester at UofT. It highlights some basic concepts and results related to those considered in the book of Fedosov[2] on deformation quantization and a generalization of the Atiyah-Singer Index Theorem in that setting.

Introduction

It is a well-known fact that many "geometric theories" can be interpreted as the studies of certain subcategories of bundles, whose restricted morphisms are usually specified by imposing some compatibility conditions with distinguished sections.

For instance, differential topology studies tangent bundles of smooth manifolds, and the morphisms are the bundle maps induced by the smooth maps: the differentials.

On the other hand, a Riemannian metric g and the associated Riemannian connection $\bigtriangledown$ on a smooth manifold M, which are fundamental in Riemannian geometry, are specified by global sections on $T^{\star}M \otimes T^{\star}M$ and $TM \otimes T^{\star}M \otimes T^{\star}M$ respectively, which possess various properties(e.g. positive-definiteness) imposed by geometric considerations.

The case of complex geometry is similar: it is given by an almost complex structure J on M, which is a section on $TM \otimes T^{\star}M$ satisfying the equation $J^2 = -\mathbb{1}$, that is integrable. Here $\mathbb{1}$ is the identity map of the tangent bundle of M.

In general, the "rigidity" of underlying geometric structure is reflected by the partial differential equations imposed on the distinguished sections and the morphisms compatible with them. The "smaller" is the solution space, the more rigid is the structure.

The Atiyah-Singer Index theorem is a theorem about bundles. It is an equality between the analytic index of an elliptic differential operator between vector bundles over a closed manifold and the topological index of this elliptic differential operator.

In view of the above discussions, the analytic side often has geometric meaning. Therefore, the Atiyah-Singer Index Theorem can be seen as a revelation of the tension between the geometry and the topology of a (closed) manifold, a theme of central importance in differential geometry.

Symplectic Structures, Poisson Structures, And Deformations

One of the most natural structures to consider in mathematical physics is symplectic structure. It is a pair (M, ω), where M is a smooth maniofld and ω is a closed non-degenerate 2-form,called the symplectic form. Here non-degenerate means ω is non-degenerate pointwise as a skew-symmetric bilinear form on the tangent plane. It follows from the definition that M has even dimension.

Another important structure relevant for mathematical physics is the Poisson structure. This is a pair $(M, \{,\})$, where $\{,\}$ is a Lie algebra on $C^\infty(M)$, the $\mathbb{C}$-algebra of complex-valued smooth functions on M, which satisfies the additional identity $\{f, gh\} = \{f, h\}g + \{f, g\}h$, where f, g, h are smooth functions on M.

The physical root of symplectic structure is the idea of phase space([3],chapter 1 section 1). This is $\mathbb{R}^{2n} = \{(p, q) : p \in \mathbb{R}^n, q \in \mathbb{R}^n\}$ with the standard symplectic form $\omega = \sum_{i=1}^n dp_i \wedge dq_i$. The notations are suggestive. In the context of classical mechanics, p is the momentum at a point and q is the position. The physical content of this structure is based on the interpretation that observables are functions on the phase space. The symplectic form associates, to each observable f, a vector field called the Hamiltonian X_f of f via the pairing $\omega(Y, X_f) = df(Y)$. According to the classical picture, this completely determines the system. The restricted morphisms are those preserving the symplectic form.

While the definition of Poisson structure does not require a symplectic form, it comes for free when a symplectic form is available. To illustrate this fact, one can easily check, in the case of the standard phase space, $\{f, g\} = \omega(X_f, X_g)$ is a Poisson bracket.

A fundamental way in which quantum mechanics is different from classical mechanics is that observables are no longer functions, but operators acting on some Hilbert space. As such, multiplication of observables is not necessarily commutative. This motivates the idea of a deformation of the algebra $C^\infty(M)$. Suppose $\hbar \in I$ for some open interval I containing 0[1], and fix a Poisson structure $\{,\}$ on M. A strict deformation quantization in the direction of $\{,\}$ is a family of associative unital products $\star_\hbar$ on $C^\infty(M)$ with the same unit 1,which varies "smoothly" in $\hbar$, such that $\star_0$ is the usual

[1]Other domains are also considered, depending on the specific problems.

product, and $\frac{d}{d\hbar} f \star_\hbar g|_{\hbar=0} = \{f, g\}$ for all pair of functions $f, g \in C^\infty(M)$. The quotation mark is meant to indicate that a mathematical rigorous definition requires a discussion involving the point-set topology of $C^\infty(M)$(cf. section 1 of [7]). It is expected that "classical limits" can be recovered by letting $\hbar$ go to 0.

Experience has shown that strict deformation quantizations are difficult to work with([6] p6,[1] p25 and [5] section 4). A much more manageable variant is formal deformation quantization, which is often referred to simply as deformation quantization in the literature and is defined as follows:

Definition 0.1 (Formal Deformation Quantization([2],chapter 5))**.** Let $Z = C^\infty(M)[[\hbar]]$ be the ring of formal power series formed by the ring $C^\infty(M)$ and the formal variable $\hbar$. A deformation quantization on M is a unital(with the unit element being 1) associative product $*$ making Z into a $\mathbb{C}$-algebra,such that the followings hold:

1. If $a(x, \hbar) = \sum_{i=0}^\infty a_i(x)\hbar^i$ and $b(x, \hbar) = \sum_{j=0}^\infty b_j(x)\hbar^j$ are elements in Z, then $a * b = \sum_k^\infty c_k \hbar^k$, where c_k depends on a_i,b_j, $\partial^\alpha a_i$ and $\partial^\beta b_j$, with $i + j + |\alpha| + |\beta| \le k$.

2. In the same notation as above, $c_0 = a_0 b_0$.

3. The commutator $[a, b]$ with respect to $*$ is $-i\hbar\{a, b\}$ with higher order terms in $\hbar$.

An algebra $(C^\infty(M)[[\hbar]], *)$ satisfying 1$\sim$3 agrees with the intuition for a deformation in parameter $\hbar$, but it is easier to work with: it was proved by Kontsevich in [4] that any finite dimensional Poisson manifold possesses a "canonical" deformation quantization. The word "canonical" is used in a technical and precise sense, see the abstract of [4] and the discussion therein. This existence result is a corollary of his formality theorem proved in the same paper.

To proceed further with index theory, it is necessary to restrict our discussion to a sympletic manifold with the Poisson structure induced by its symplectic structure.

Quantum Algebras and Index Theorem

Fedosov's genralization formally resembles the original Atiyah-Singer Theorem. The word "formal" here can be taken literally. It concerns with the index of an elliptic element in the bundle of deformed algebra $W_D(E^0, E^1)$ of the quantum observables with coefficient in $\mathrm{Hom}(E^0, E^1)$, where E^0 and E^1 are complex vector bundles over M of the same rank.

The definition of $W_D(E^0, E^1)$ involves a number of other concepts which are important in their own right.

$W_D(E^0, E^1)$ is first of all a bundle of formal Weyl algebra.

Weyl algebras already made their appearance in some expositions of the original Atiyah-Singer Index Theorem.

Definition 0.2 (Σ^∞ ([2] definition 3.1.6)). Fix $h \in]0,1]$.A function $a(x,h) \in C^\infty(\mathbb{R}^{2n}, \hbar)$ is in Σ^∞ if it admits expansion $a(x,h) = a_0(x) + a_1(x)h + ... + a_N(x)h^N, N < +\infty$, where all the $a_i(x,h)$'s are smooth with partials of all orders being of polynomial growth.

$\Sigma^\infty(E^0, E^1)$ is just the set of matrices with entries in Σ^∞, here E^0 and E^1 are Hermitian vector spaces of equal (complex) dimensional n(the same as in the definition) with fixed bases. $2n$ real dimension is needed for the Weyl calculus. This refers to,in the same notation as in the previous section, the following transformation([2],3.1.2):

$$\hat{a}u(q) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} \exp(\frac{i}{h}q(p-p'))a(\frac{p+p'}{q})u(p')dp'dq$$

Here u is in Schwartz space. Operators such as $\hat{a}$ are called Weyl operators([2],3.1.2). For reasons pointed out in the second section, this passage from functions to operators can be interpreted as a kind of quantization, called the Weyl quantization. It is therefore necessary to consider their composition law, a topic which is already discussed extensively in the literature(cf.[2],3.2).

A Weyl operator is called elliptic if it is the Weyl quantization of an element in $\Sigma^\infty(E^0, E^1)$ whose "constant term" is invertible outside of a ball centerer at the origin with radius $R > 0$, whose inverse is of Schwartz class([2],definition 4.2.1). It is known that elliptic implies Fredholm.

A bundle of Weyl algebra is a bundle W over a manifold M whose fibres are $\Sigma^\infty(E^0_x, E^1_x)$.

A prototype of the index theorem for deformation quantization is that of $\mathbb{R}^{2n}$. This requires the concept of a virtual bundle of compact support and its Chern character(cf. [2],1.4).

The fact relevant for index theory is that an elliptic operator determines a virtual bundle of compact support, and its analytic index is the same as the integral of the Chern character of this virtual bundle of compact support over $\mathbb{R}^{2n}$([2], proposition 4.2.8).

A formal Weyl algebra corresponding to symplectic manifold M is the algebra of formal power series in $\hbar$ with coefficients in polynomial of TM,multiplication of coefficient is by the Weyl composition law, which requires the symplectic structure. Deformation quantization comes into this picture as the central algebra of the formal Weyl algebra([2],p140). As in the case of Weyl algebra, it makes sense to consider matrix valued formal Weyl algebras.A bundle of formal Weyl algebra is defiend similarly as that of a bundle of Weyl algebra([2],5.1).

W_D is also called a subalgebra of W of flat sections, because its elements are annihilated by the abelian connection D. An abelian connection is a connection satisfying $D^2 = 0$([2],5.2).

By appropriately extending the definition of ellipticity for flat sections, and the definition of indices, as well as the Chern character of a virtual bundle defined by an elliptic section, Fedosov proved an index theorem which takes the following form:

Theorem 0.1 (Fedosov)**.** *For any elliptic element $\Xi \in W_D$, we have the index formula*

$$ind\ \Xi = \int_M ch\xi\, exp(-\frac{\Omega}{2\pi\hbar})\hat{A}(TM)$$

Here ξ is the class of the virtual bundle determined by Ξ, Ω is the curvature form of the abelian connection D, $\hat{A}$ is the A-hat class. The orientation of integration is given by the form $(-\Omega_0)^n$, where Ω_0 is a 2-form related to the variation of symplectic structure([2],5.3).

Since the right hand side is essentially topological, Fedosov's index theorem can be regarded as a revelation of the tension between the formal geometry and the topology of a symplectic manifold.

References

[1] Chiara Esposito. *Formality Theory: From Poisson Structures to Deformation Quantization*, volume 2. Springer, 2014.

[2] Boris V Fedosov. Deformation quantization and index theory. *Mathematical topics*, 9, 1996.

[3] Gerald B Folland. *Harmonic analysis in phase space.* Number 122 in Annals of mathematical studies. Princeton university press, 1989.

[4] Maxim Kontsevich. Deformation quantization of poisson manifolds. *Letters in Mathematical Physics*, 66(3):157–216, 2003.

[5] Hideki Omori, Yoshiaki Maeda, Naoya Miyazaki, and Akira Yoshida. *Deformation quantization of Fréchet-Poisson algebras of Heisenberg type.* Keio University. Department of Mathematics RR 2001/002, 2001.

[6] Markus J Pflaum. From weyl quantization to modern algebraic index theory. *Managing Editor: Professor NJ Hitchin, Mathematical Institute, University of Oxford, 24–29 St Giles, Oxford OX1 3LB, United Kingdom*, page 84, 2007.

[7] Marc A Rieffel. Deformation quantization of heisenberg manifolds. *Communications in mathematical physics*, 122(4):531–562, 1989.