Abstract

When studying surfaces, it is often convenient to use local coordinates which can simplify expressions of certain quantities. The existences of these coordinate systems are not always immediate. This note shows how the existence problem of these coordinates can be solved by solving very simple PDEs. Our treatments are essentially accessible to students currently taking an introductory course on PDEs.

Contents

1 Introduction

In a typical course on classical differential geometry, namely the study of smooth surfaces in $\mathbb{R}^3$, one often hears phrases which begin with " by the theory of PDEs ", followed by several lines of equations, after that, the problems in hands are magically "solved ". This is very much true even in more advanced courses on geometry.

We will try to give a flavour of the kind of PDE theory needed to tackle some interesting geometry problems. They turn out to be fairly simple.

We specifically focus on the problems concerning existence of orthogonal coordinates and isothermal coordinates. Each will take up one section. All the results are local,and cannot in general be global by topological consideration.

We will assume as much regularity as needed for the arguments to stand up. All surfaces considered sit in $\mathbb{R}^3$, although the arguments will make it clear it is enough to have a Riemannian metric and dimension 2 as a smooth(C^2 is good enough) manifold. Variables of functions will be suppressed if they don't matter.

2 Integrating Factor and Orthogonal Coordinates

We discuss in this section how real integrating factor in the plane can be used to establish existence of orthogonal system.

2.1 A Problem in ODE

One often encounters, in ODE, equations of the form

$$a(x, y)dx + b(x, y)dy = 0, \tag{2.1.1}$$

where a and b are C^1 functions in x and y. Call left hand side V for brevity.

The solution to this equation can be written implicitly if the equation is exact. In other words, if

$$dU = a(x, y)dx + b(x, y)dy \tag{2.1.2}$$

for a smooth scalar function $U(x, y)$. Because then y can be implicitly determined by x(or vice-versa) using

$$U(x, y) \equiv C, \tag{2.1.3}$$

for some constant C depending on initial condition.

Following a special case of Poincaré's Lemma , a sufficient condition for this to happen is

$$dV = 0 \tag{2.1.4}$$

when the region under consideration is simply connected(e.g.,a ball). Here d is the exterior derivative for differential forms.

It would be very nice if all the equation of the form (2.1.1) is exact. But clearly, (2.1.4) is not always satisfied, even when the region is simply connected.

An integrating factor for (2.1.1) is a non-vanishing scalar function μ such that , μV is exact.

It is actually true that such μ always exists, at least locally, as we will show using PDE. Even though this μ may be hard to find, the theoretical existence is sufficient for many interesting applications.

2.2 Reduction to Linear PDE

The idea is to consider the problem in an open ball(possibly small), and apply the Poincaré's Lemma.

Specifically, we want a μ such that $d(\mu V) = 0$. Expand $d(\mu V)$, we have,

$$\begin{aligned} d(\mu V) &= d(\mu a dx + \mu b dy) \\ &= \{(\mu b)_x - (\mu a)_y\} dx \wedge dy \\ &= \{(\mu_x b + \mu b_x) - (\mu_y a + \mu a_y)\} dx \wedge dy \\ &= \{b\mu_x - a\mu_y + (b_x - a_y)\mu\} dx \wedge dy \end{aligned}$$

Requiring this to be identically zero, we come to the linear homogeneous first order PDE:

$$b\mu_x - a\mu_y + (b_x - a_y)\mu = 0 \tag{2.2.1}$$

The only condition we put on μ is that it is non-vanishing.

Clearly, if a solution, assuming it exists, is non-zero at a point, then by continuity, it is non-zero in a neighbourhood containing that point, such solution μ is then a local integrating factor.

In order for the ODE(or the PDE coming from it) to be non-degenerate, we require a and b not to vanish simultaneously at the point. If this is the case, then the equation always has non-vanishing solution in an neighbourhood containing that point.

Theorem 1. *Assuming a and b do not vanish simultaneous in a neighbourhood of a point (x_0, y_0), then a non-vanishing solution to equation (2.2.1) exists in a (possibly smaller) neighbourhood of (x_0, y_0).*

Proof. Method of characteristics is applicable here. Choose $z_0 \neq 0$, with the ode for characteristics determines a curve, by the existence and uniqueness theorem of ode. Now, because $(b, -a) \neq 0$ at (x_0, y_0), one can choose a fixed vector (c, d) so that $\det \begin{bmatrix} b & -a \\ c & d \end{bmatrix} \neq 0$ in a neighbourhood of (x_0, y_0)(by continuity). Prescribe z along the line $(x, y) = (c, d)s + (x_0, y_0)$. s is a parameter used to determine the initial condition. Choose the initial condition along the straight-line compatible with $z_0 = z(x_0, y_0)$.The computation of the determinant shows the condition on transversality is met. Therefore, by observation before the proof, we have a non-vanishing solution in a neighbourhood of (x_0, y_0). □

2.3 Existence of Orthogonal System

Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^3$. Let $\vec{r}(u, v)$ denote position of a point on a given surface with coordinates (u, v). We define,

$$E = \langle \vec{r}_u, \vec{r}_u \rangle, \quad F = \langle \vec{r}_u, \vec{r}_v \rangle, \quad G = \langle \vec{r}_v, \vec{r}_v \rangle.$$

The first fundamental form of the surface is defined to be:

$$I = Edu^2 + 2Fdudv + Gdv^2$$

Here du and dv are, from appropriate viewpoint, linear functionals(on the tangent plane of a point on surface) defined by specifying:

$$du(\vec{r}_u) = 1, \quad du(\vec{r}_v) = 0$$

$$dv(\vec{r}_u) = 0, \quad dv(\vec{r}_v) = 1.$$

These are enough to specify the two linear functionals because $\vec{r}_u$, $\vec{r}_v$ form a basis for the tangent plane at each point. We emphasize the multiplication in the first fundamental form is multiplication of real numbers, rather than wedge product.

For the geometric contents of the maps du, dv, as well as the first fundamental form, we recommend the reader to consult any standard text on differential geometry of surfaces. We recommend Do Carmo's text [3, section 2.4,2.4]. For our purpose, we only need to know the first fundamental form is well-defined on the tangent plane, i.e., it is independent of the choice of parametrization.

A coordinate system is orthogonal if the first fundamental form has no crossing term $dudv$. Many computations are easier if they are done in an orthogonal coordinate system. One typical example is the computation of Gaussian curvature. Differential equations associated with the surface are also easier in this coordinate system, as we shall see later.

It is very easy to prove the existence of orthogonal system for any surface using the result we proved. In fact, more can be said.

Let A,B,C D be (locally defined) smooth real-valued functions in (u, v). Write:

$$\vec{P} = A\vec{r}_u + B\vec{r}_v$$
$$\vec{Q} = C\vec{r}_u + D\vec{r}_v$$

Suppose $\vec{P}$ and $\vec{Q}$ are linearly independent as (u, v) varies.

We can prove the following theorem:

Theorem 2. *Under the above conditions, for each (u, v), there is a local diffeomorphism from a neighbourhood of (u, v) to (x, y) such that, $\vec{P} \parallel \vec{r}_x$, and $\vec{Q} \parallel \vec{r}_y$. Here $\parallel$ means parallel as vectors when evaluated at each point.*

Proof. By existence of integrating factor, we have μ and λ,both non-vanishing in a neighbourhood, such that,

$$\mu(Ddu - Cdv) = dx \tag{2.3.1}$$
$$\lambda(-Bdu + Adv) = dy. \tag{2.3.2}$$

The notations are suggestive.

Note we have,

$$dx \wedge dy = (\mu\lambda)(DA - CB)du \wedge dv$$

Since $\vec{P}$,$\vec{Q}$ are linearly independent , the determinant of the matrix formed by having them as column, which is $AD - BC = -(DA - CB)$, is non-zero. Therefore, $dx \wedge dy$ is non-vanishing in a neighbourhood. So by the inverse function theorem(after going back and forth between tangent space and cotangent space), we have local diffeomorphism $(u, v) \to (x, y)$.

Now, by explicitly comparing (2.3.1) and (2.3.2) with

$$dx = x_u du + x_v dv \tag{2.3.3}$$
$$dy = y_u du + y_v dv \tag{2.3.4}$$

Applying inverse function theorem again:

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\mu(DA-CB)}A & \frac{1}{\lambda(DA-CB)}C \\ \frac{1}{\mu(DA-CB)}B & \frac{1}{\lambda(DA-CB)}D \end{bmatrix} \tag{2.3.5}$$

We used Cramer's rule for the matrix equality on the right.
Explicit computation using chain rule then shows:

$$\vec{r}_x = \vec{r}_u u_x + \vec{r}_v v_x = \frac{1}{\mu(DA - CB)}(A\vec{r}_u + B\vec{r}_v) \parallel \vec{P}$$
$$\vec{r}_y = \vec{r}_u u_y + \vec{r}_v v_y = \frac{1}{\lambda(DA - CB)}(C\vec{r}_u + D\vec{r}_v) \parallel \vec{Q}$$

□

Remark. *The truth is, I started off using Cramer's rule, and tried several possible configurations until I found the right pair of differential forms.*

What this result means is: if we have a function θ defined on surface, with $0 < \theta \leq \frac{\pi}{2}$, varying smoothly(in coordinates), then we can find a different coordinate system so that the basis vectors of the tangent plane determined by the new coordinates have angle precisely θ at each point. From this point of view, the existence of orthogonal system is immediate.

Corollary. *For any surface, orthogonal coordinates always exists locally.*

Proof. $\theta \equiv \frac{\pi}{2}$ is as smooth as a function can be. More explicitly, pick arbitrary coordinates (u, v) and apply above theorem using $\vec{P} = \vec{r}_u$ and $\vec{Q} = \frac{F}{E}\vec{r}_u - \vec{r}_v$.

□

3 Elliptic System and Isothermal Coordinates

We say a coordinate system is isothermal if the first fundamental form in the coordinates is of the form:

$$I = \rho^2(du^2 + dv^2) \tag{3.0.1}$$

This would mean(see Do Carmo [3, p229,section4.2]), the surface is locally conformal to the plane. It is a surprising fact that any surface is locally conformal to the plane. One possible proof is to transform the first fundamental form into (3.0.1).

The idea is to use integrating factor again. But this time, we will get a system of equations rather than just one equation. The existence of a non-trivial solution to this system will be rather delicate. We will analyse everything carefully, but refer the reader to a paper which is closer to our approach in spirit for the proof of existence.

3.1 The Basic Setting

By the results in section 2, we may assume without loss of generality :

$$I = Edu^2 + Gdv^2 \tag{3.1.1}$$

By definition of E and G, we have, $E > 0$ and $G > 0$.

Following the remark about I in section 2, we may write:

$$I = (\sqrt{E}du + i\sqrt{G}dv)(\sqrt{E}du - i\sqrt{G}dv) \tag{3.1.2}$$

If there were a complex integrating factor(non-vanishing by definition) $\mu = p + qi$(here p and q are real) satisfying,

$$dU = \mu(\sqrt{E}du + i\sqrt{G}dv) \tag{3.1.3}$$

We would have,taking complex conjugate,

$$d\bar{U} = \bar{\mu}(\sqrt{E}du - i\sqrt{G}dv). \tag{3.1.4}$$

Hence,

$$I = \frac{1}{|\mu|^2}dUd\bar{U}. \tag{3.1.5}$$

Now, write,

$$\begin{aligned} dU &= (dx + idy) \\ &= (p + iq)(\sqrt{E}du + i\sqrt{G}dv) \\ &= (p\sqrt{E}du - q\sqrt{G}dv) + i(q\sqrt{E}du + p\sqrt{G}dv) \end{aligned}$$

Similar to section 2, taking,

$$\det \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \sqrt{EG}(p^2 + q^2) \tag{3.1.6}$$

Which is non-vanishing by hypothesis, so the inverse function theorem applies. In (x, y), the fundamental form is:

$$I = \frac{1}{|\mu|^2}(dx^2 + dy^2) \tag{3.1.7}$$

This will prove the claim. But recall, in the very beginning, we assumed such pair p and q existed. This is not so clear.

We try to use the Poincaré's lemma again. Applying d to real and imaginary part separately, by a similar calculation as in section 2, we arrive at the following system:

$$\sqrt{E}p_v + p\sqrt{E}_v + \sqrt{G}q_u + q\sqrt{G}_u = 0 \tag{3.1.8}$$
$$-\sqrt{E}q_v - q\sqrt{E}_v + \sqrt{G}p_u + p\sqrt{G}_u = 0 \tag{3.1.9}$$

The only requirement is the existence of a pair of p and q which do not vanish simultaneously.Note we are taking partials of square-roots, not square-rooting the partials.

As we remarked, E and G are non-vanishing, so we may divide 3.1.8 and 3.1.9 by $\sqrt{E}$ and $-\sqrt{E}$(the choice for doing this is will be apparent later). This allows us to write the system more compactly as

$$\vec{X}_v + A\vec{X}_u + B\vec{X} = 0 \tag{3.1.10}$$

Where $\vec{X} = \begin{bmatrix} p \\ q \end{bmatrix}$, A and B are some real-valued matrices whose entries are C^1 in (u, v). This is very much like 2.2.1: they are both linear, first order, and homogeneous. The big difference, and this is what makes this case difficult, is that we have a coupled system.

3.2 Analysis of the System, Existence Proof for Analytic Metric

We now look more closely at the system 3.1.10. As it turns out, being a system is not the only thing which makes it difficult. We analyse the matrix A more carefully.

$$A = \begin{bmatrix} 0 & \sqrt{\frac{G}{E}} \\ -\sqrt{\frac{G}{E}} & 0 \end{bmatrix} \tag{3.2.1}$$

This matrix has no real eigen-values! In some literature, this is called an elliptic system [4]. Unlike hyperbolic systems, where the method of characteristic can be generalize [5, section7.3], an elliptic system cannot be solved this way,precisely because of the above observation. The failure of method of characteristics is a big difference between the two equations. In fact, it played a major role in the proof of existence of non-trivial solution of equation 2.2.1 .

We can nevertheless deduce local existence of non-trivial solution when everything is analytic in (u, v). This can be done by imposing non-vanishing analytic initial condition. The existence then follows as a result of the (proof of) Cauchy–Kowalevski theorem [5](This is also why we wrote the equation in the form 3.1.10).

The analytic result was known to Gauss [8, p466].

3.3 A Guide to Different Approaches and Relevant Papers

In a way, the exposition could stop in the last subsection. Since it is beginning to become less self-contained when a strong result like Cauchy–Kowalevski theorem is invoked. Knowing that it is a difficult problem, it may be even more inspiring to have a look at what lies ahead. We will briefly talk about other approaches to reduce the first fundamental form into (3.1.7), and point to established results for interested readers. We will also give a guide to the paper which resolves the technical problem concerning existence of non-trivial solution for the system (3.1.10).

3.3.1 Alternative: Beltrami's Equation

Many works were done to extend Gauss's result. Investigations on the existence of isothermal coordinates under weaker hypotheses on regularity were carried out extensively. All the approaches to this problem essentially come down to a existence problem of some system of PDEs. And all of these equations are "elliptic".

In addition to the integrating factor approach, there is one other approach. This approach is to solve the Beltrami's equation. It is in fact more mainstream.

We derive the Beltrami's equation(following closely the presentation of Imayoshi and Taniguchi [7, page20-21]) from our problem (3.1.7), and see how the existence of a solution will solve the problem. We then give the definition in general, and provide some resources for additional reading.

We first agree on standard notations:

$$dz = dx + idy, \quad d\bar{z} = dx - idy, \quad \partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$$

Let $f(x, y) = u + iv$ be a complex valued function, identifying (x, y) with $z = x + iy$. Standard computation(found in any book on complex variable) shows:

$$df = f_z dz + f_{\bar{z}} d\bar{z}. \tag{3.3.1}$$

A fundamental for I expressed in terms of (x, y) in its utmost general form is:

$$I = Edx^2 + 2Fdxdy + Gdy^2 \tag{3.3.2}$$

Note $EG - F^2 > 0$ by definition of E,F,G and inner product:by Cauchy-Schwarz $EG = F^2$ iff $\vec{r}_x \parallel \vec{r}_y$, which is impossible when (x, y) are coordinates. so $\sqrt{EG - F^2}$ is real and strictly positive. Now, we want to express I in the form(guided by isothermal coordinate):

$$I = \lambda |dz + \mu d\bar{z}|^2 \tag{3.3.3}$$

Here λ and μ are to be determined. Of course, we require $\lambda > 0$ as constrained by I. But μ can be imaginary.The choice of multiplying $d\bar{z}$ by μ is arbitrary in the sense that we could have normalized its coefficient instead. After large amount of trials(much was done by Gauss), the standard choices are the following:

$$\lambda = \frac{1}{4}(E + G + 2\sqrt{EG - F^2}), \quad \mu = \frac{E - G + 2iF}{4\lambda} \tag{3.3.4}$$

The exact expressions are almost irrelevant for our purpose. What is important is that $\mu < 1$. This follows from the expression

$$|\mu|^2 = \frac{E + G - 2\sqrt{EG - F^2}}{E + G + 2\sqrt{EG - F^2}} \tag{3.3.5}$$

This is easily seen by writing the numerator of $|\mu|^2$ as a difference of squares.

We now try to see what condition we should put on local diffeomorphism $f = u + iv$ in order for it to be isothermal. We require

$$I = \rho^2(du^2 + dv^2) \tag{3.3.6}$$
$$= \rho^2|f_z dz + f_{\bar{z}} d\bar{z}|^2 \tag{3.3.7}$$
$$= \rho^2|f_z|^2|dz + \frac{f_{\bar{z}}}{f_z} d\bar{z}|^2 \tag{3.3.8}$$

Here ρ is just a (at least C^1) function to be specified. Compare with equation 3.3.3, we need,

$$f_{\bar{z}} = \mu f_z \tag{3.3.9}$$

In general, equation of the form in 3.3.9 with $|\mu| < 1$ is called the Beltrami's equation. The $|\cdot|$ may be replaced by $\|\cdot\|_\infty$ .This is typical when the conditions on μ's regularity are relaxed. So the existence of isothermal coordinates can be reduced to finding diffeomorhism which is also a solution to the Beltrami's equation. This is because, once 3.3.9 is satisfied, taking appropriate quotient gives the right ρ.

It is worth comparing equation 3.3.9 with equation 3.1.2. Equation 3.1.2 has very simple higher order term, but it has zeroth order term. Equation 3.3.9 may have complicated coefficients, but it is homogeneous in degree.

The study of Beltrami's equation was very active at some point(roughly up to the end of the 60s). Any books that specialize on "Elliptic PDE" will discuss it. In geometry, it often appears in the context of quasi-conformal mapping. A very readable introduction of the theory is the first two chapters of Wendland's book [9]. There is a whole book [6] dedicated to this equation. Beltrami equation is now a mature topic, very much like the Laplace equation. Local existence is proved using the continuity method [6, page50]. Many results under weaker regularity assumptions were also proved.

3.3.2 Chern's Proof:Existence of Integrating Factor By Integro-differential Equation

There is a different way to prove the existence of integrating factor satisfying equation 3.1.3. We present the approach of Chern [2], because his steps can be broken down more easily. Actually, we will present his technical lemmas, but modify the proof more suitable for our choice of coordinates.

Chern's method is very common in the study of PDEs: He introduced some functions defined by integral operator. He then showed, by estimating the integral, the regularity of the function defined by integral. He then showed one can choose the input function appropriately so that the output is a solution to the PDE. We will state the integrals he considered, and explain the connection between the steps. We will omit the estimates, which can be found in Chern's paper. It should be stressed it was really the estimates which made everything work.

Notations are adjusted to fit our proof. Start off with the complex differential form in equation 3.1.3, call it θ(following Chern). We want to rewrite it in complex notation:

$$\begin{aligned}\theta &= \sqrt{E} + i\sqrt{G} \\ &= \alpha dz + \beta d\bar{z} \\ &= (A + iB)(dx + idy) + (C + iD)(dx - idy) \\ &= \{A + C + (B + D)i\}dx + \{(A - D)i + (D - B)\}dy\end{aligned}$$

Here, A,B,C,D are real valued functions to be determined, α and β complex functions determined by A,B,C,D. We require them to be chosen so that all the above equalities hold. This is equivalent to, by comparison:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} \sqrt{E} \\ 0 \\ 0 \\ \sqrt{G} \end{bmatrix} \tag{3.3.10}$$

The matrix has determinant 2, so it is invertible: A,B,C,D can be specified by the right-hand side. By translating the coordinates, we may assume the point of interest has coordinates $(0, 0)$. After scaling the coordinates, we can assume A,B,C,D are chosen so that $\alpha = 0$ and $\beta = 1$ at $(0, 0)$. Now,everything is set-up nicely for Chern's technical lemmas to work.

We need to recall some standard definitions.

Definition 3.1 (Hölder Continuity)**.** We say a complex valued function is Hölder-continuous with Hölder-exponent $\lambda > 0$ in an open connected region

$R \subset \mathbb{R}^d$, if there is a fixed constant $C > 0$ such that $|f(x)-f(y)| \leq C\|x-y\|^\lambda$, $\forall x, y \in R$. Denote the set of all these functions by $C^{0,\lambda}(R)$.

The norms in the definition are the standard ones. We required R to be open just so that we can draw open balls inside R. It is then clear that f is constant if $\lambda > 1$. $C^{0,\lambda}(R) \subset C^{0,\delta}(R)$ if $\lambda > \delta > 0$. $\lambda = 1$ corresponds to the Lipschitz functions. By the mean-value theorem(inequality), if R is bounded and f is $C^1(\bar{R})$, then f is Lipschitz in R, so Hölder for any lower exponent. It is useful to remember that $C^{0,\lambda}(R)$ is a vector space over $\mathbb{C}$.

After some lengthy but straightforward computations, Chern prove the following technical lemma.

Lemma 3.1 (Chern's Main Lemma)**.** Let D be a ball centered at the origin, let $f \in C^{0,\lambda}(D)$, with $0 < \lambda < 1$. Define F(z) by the equation

$$-2\pi i F(z) = 2i \iint_D \frac{f(w)}{w-z} dxdy \tag{3.3.11}$$

Here the surface integral is computed by separating real and imaginary parts, and $w = x + yi$. It looks like the Cauchy's integral formula, but that is a line-integral. Among many of the estimates for F, there are existences of F_z and $F_{\bar{z}}$,both Hölder continuous with exponent λ.And the equality $F_{\bar{z}} = f$. Further more, F is also in $C^{0,\lambda}(D)$.

Using this lemma, he was able to prove the theorem about the existence and uniqueness of what he called an "integro-differential equation".

Let W be a complex valued function on a ball D centred at the origin. Let $A, B \in C^{0,\lambda}(D)$, both vanish at the origin. Define $Z(W) = AW_z + BW_{\bar{z}}$.Let C be another function in $C^{0,\lambda}(D)$. Let σ be a complex differentiable function on D which vanishes at the origin.

Theorem 3. *Under the above conditions, the equation*

$$2\pi i W(z) + 2i \iint_D \frac{Z(W) + CW}{w-z} dxdy = \sigma(z) \tag{3.3.12}$$

has a unique solution W in a (possibly small) neighbourhood of $(0,0)$.

The proof is essentially successive approximation by a series of functions. The uniform convergence is proved using estimates from the main lemma.

One crucial observation Chern made was, if $C \equiv 0$, and $\sigma = z$, then the solution W satisfies $W_z(0,0) \neq 0$.

Now we can prove the theorem. Looking back at $\theta = \alpha dz + \beta d\bar{z}$. We proved that it can be assumed that $\alpha = 0$ and $\beta = 1$ at the origin. We try and see how a local diffeomorphism should behave in order to go to a system with isothermal coordinates. We required:

$$dW = W_z dz + W_{\bar{z}} d\bar{z} = \frac{1}{\rho}(\alpha dz + \beta d\bar{z}) \tag{3.3.13}$$

As remarked, we can take $\alpha = A$ and $\beta = 1 - B$, A and B defined by these two equations. These are not the same as the functions defining α and β. Construct the operator Z in the integral according to this choice of A and B. Choose $\sigma = z$. Then by Chern's observation, $W_z \neq 0$ at $(0,0)$. Moreover,taking $\partial_{\bar{z}}$ of equation 3.3.12, by main lemma and Cauchy-Riemann, we get,after dividing out $2\pi i$,

$$(1 - B)W_{\bar{z}} - AW_z = 0 \tag{3.3.14}$$

This shows that $\rho = \frac{\beta}{W_z} = \frac{\alpha}{W_{\bar{z}}} \neq 0$ near $(0,0)$.
There is a slight problem with this identity, which was not mentioned in Chern's original paper. Recall we demanded $\alpha = 0$ at $(0,0)$. So one has to interpret $\frac{\alpha}{W_{\bar{z}}}$ as a limit at $(0,0)$. Even then, there is a chance that A is identically 0, so the two equations are not quite equivalent. One fix is to notice, as long as A is not identically 0, there is no problem, by an argument using continuity. If A is identically 0, then notice this means $\theta = \beta d\bar{z}$,so the original coordinates already give the desired form.
It is therefore enough to assume the identity about ρ holds. This also shows W is a local diffeomorphism from $z = x + yi$ to its image. Putting everything together proves the existence of complex integrating factor of our differential form, and by analysis of section 3.1, gives local isothermal coordinates. This also indirectly proves the existence of non-trivial solution to our seemingly more difficult elliptic system. The difference between our approach and other approaches using integrating factor(like Chern's, with different regularity assumptions) is that our solution to the equation is the integrating factor, while their solution is the actual coordinates.

Acknowledgements

Materials in this note are representatives of my mathematical knowledge related to geometry by the end of my second year. Some of the steps taken in this note continue to be my motivation to learn geometry and PDE, and the intersection of the two fields. This note is intended to pass on this motivation to the younger students. I learned all of the geometric results in Professor Nabutovsky's Classical Differential Geometry class(MAT363). I was made aware of the connection between integrating factor and PDE by Professor Panchenko, who also told me about the Cauchy–Kowalevski theorem. I learned the Poincaré's lemma twice, first time in Professor Bar-Natan's MAT257(Analysis II), and its variant in Professor Panchenko's MAT267(ODE). The specific approach using integrating factor comes from Chern's book [1], whose notations and factorization I adopted in equation 3.1.2 . Chern only stated in his book the existence of complex integrating factor when the metric is analytic, without proof. Upon completion of this note, I found the notations and factorization are the same as those in Gauss's work [8]. I suppose that's where Chern's inspiration came from. The only thing original in this exposition that I am aware of is the use of Poincaré's lemma, so the PDEs I got were all different from those in the literature. This partly because this was the only tool I could think of back then. I could only find out the exact details of the relevant PDEs after completing Professor Burchard's MAT351. I recommend all these courses and books to math students at UofT.

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