# Week 4: Off-policy Learning and Connection between Optimal Control and RL

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# This Week

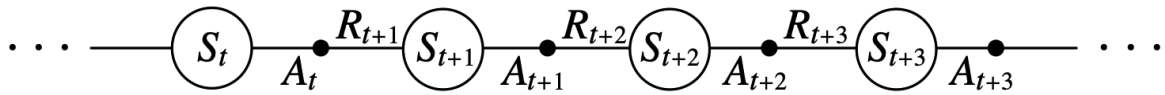
1. On-policy learning and off-policy learning
2. Importance sampling
3. Introduction on the connection between optimal control and RL

# On-policy Learning vs. Off-policy Learning

1. On-policy learning: Learn about policy $\pi$ from the experience collected from $\pi$
    1. Behave non-optimally in order to explore all actions, then reduce the exploration. e.g., $\epsilon$-greedy
2. Another important approach is **off-policy learning** which essentially uses **two different polices**:
    1. the one which is being learned about and becomes the optimal policy
    2. the other one which is more exploratory and is used to generate trajectories
3. Off-policy learning: Learn about policy $\pi$ from the experience sampled from another policy $\mu$
    1. $\pi$: target policy
    2. $\mu$: behavior policy

# On-policy Control: SARSA

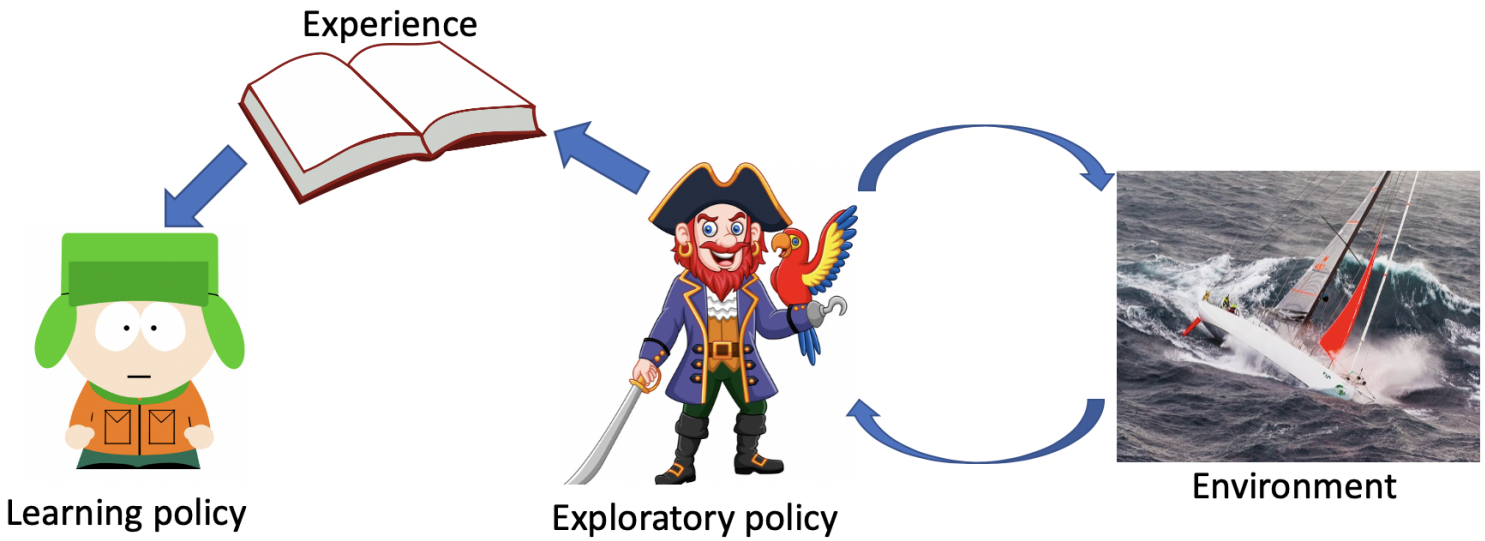
1. Trajectory is collected by the $\epsilon$-greedy policy that depends on $Q(S_t, A_t)$,




2. $\epsilon$-greedy policy for one step, then bootstrap the action value function:

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \Big[ R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t) \Big]$$

# Off-policy Learning




1. Following behaviour policy $\mu(a|s)$ to collect data

$$S_1, A_1, R_2, ..., S_T \sim \mu$$
$$\text{Update } \pi \text{ using } S_1, A_1, R_2, ..., S_T$$

2. It leads to many benefits:
   1. Learn about optimal policy while following exploratory policy
   2. Learn from observing humans or other agents
   3. Re-use experience generated from old policies $\pi_1, \pi_2, ..., \pi_{t-1}$

# Off-Policy Control with Q-Learning

1. The target learning policy $\pi$ is greedy on $Q(s, a)$

$$\pi(s) = \arg\max_{a'} Q(s, a')$$

2. The behavior policy could be totally random, but we let it improve, thus the behavior policy $\mu$ is $\epsilon$-greedy on $Q(s, a)$
3. Thus Q-learning target:

$$\begin{aligned} R_{t+1} + \gamma Q(S_{t+1}, A') &= R_{t+1} + \gamma Q(S_{t+1}, \arg\max Q(S_{t+1}, a')) \\ &= R_{t+1} + \gamma \max_{a'} Q(S_{t+1}, a') \end{aligned}$$

4. Thus the Q-Learning update becomes

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \Big[ R_{t+1} + \gamma \max_{a} Q(S_{t+1}, a) - Q(S_t, A_t) \Big]$$

# Q-learning algorithm

**Q-learning (off-policy TD control) for estimating $\pi \approx \pi_*$**

Initialize $Q(s,a)$, for all $s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
  Initialize $S$
  Repeat (for each step of episode):
    Choose $A$ from $S$ using policy derived from $Q$ (e.g., $\epsilon$-greedy)
    Take action $A$, observe $R$, $S'$
    $Q(S,A) \leftarrow Q(S,A) + \alpha\big[R + \gamma \max_a Q(S',a) - Q(S,A)\big]$
    $S \leftarrow S'$
  until $S$ is terminal

# Comparison of Sarsa and Q-Learning

1. Sarsa: On-Policy TD control

   Choose action $A_t$ from $S_t$ using policy derived from $Q$ with $\epsilon$-greedy

   Take action $A_t$, observe $R_{t+1}$ and $S_{t+1}$

   Choose action $A_{t+1}$ from $S_{t+1}$ using policy derived from $Q$ with $\epsilon$-greedy

   $$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)\Big]$$

2. Q-Learning: Off-Policy TD control

   Choose action $A_t$ from $S_t$ using policy derived from $Q$ with $\epsilon$-greedy

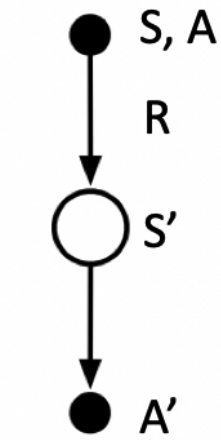
   Take action $A_t$, observe $R_{t+1}$ and $S_{t+1}$

   Then 'imagine' $A_{t+1}$ as $\arg\max Q(S_{t+1}, a')$ in the update target

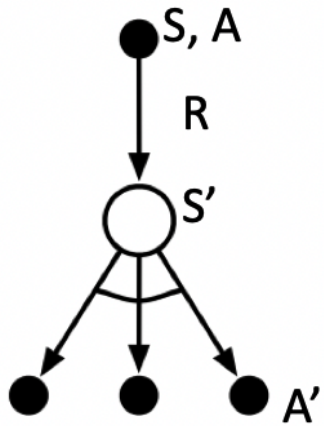
   $$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)\Big]$$

# Comparison of Sarsa and Q-Learning

1. Backup diagram for Sarsa and Q-learning




Sarsa




Q-learning

2. In Sarsa, A and A' are sampled from the same policy so it is on-policy
3. In Q Learning, A and A' are from different policies, with A being more exploratory and A' determined directly by the max operator

# Off-policy Learning with Importance Sampling

What is importance sampling?

1. Estimate the expectation of a function

$$E_{x\sim P}[f(x)] = \int f(x)P(x)dx \approx \frac{1}{n}\sum_i f(x_i)$$

2. But sometimes it is difficult to sample $x$ from $P(x)$, then we can sample x from another distribution $Q(x)$, then correct the weight

$$\begin{aligned}\mathbb{E}_{x\sim P}[f(x)] &= \int P(x)f(x)dx \\ &= \int Q(x)\frac{P(x)}{Q(x)}f(x)dx \\ &= \mathbb{E}_{x\sim Q}\left[\frac{P(x)}{Q(x)}f(x)\right] \approx \frac{1}{n}\sum_i \frac{P(x_i)}{Q(x_i)}f(x_i)\end{aligned}$$

# Off-policy Learning with Importance Sampling

1. Expected return: $\mathbb{E}_{\tau\sim\pi}[r(\tau)]$ where $r(.)$ is the reward function and $\pi$ is the policy
2. Estimate the expectation of return using trajectories $\tau_i$ sampled from another policy (behavior policy $\mu$)

$$\begin{aligned}\mathbb{E}_{\tau\sim\pi}[g(\tau)] &= \int P(\tau)r(\tau)d\tau \\ &= \int Q(\tau)\frac{P(\tau)}{Q(\tau)}r(\tau)d\tau \\ &= \mathbb{E}_{\tau\sim\mu}\left[\frac{P(\tau)}{Q(\tau)}r(\tau)\right] \\ &\approx \frac{1}{n}\sum_i \frac{P(\tau_i)}{Q(\tau_i)}r(\tau_i)\end{aligned}$$

# Off-Policy Monte Carlo with Importance Sampling

1. Generate episode from behavior policy $\mu$ and compute the generated return $G_t$

$$S_1, A_1, R_2, ..., S_T \sim \mu$$

2. Weight return $G_t$ according to similarity between policies
   1. Multiply importance sampling corrections along whole episode

$$G_t^{\pi/\mu} = \frac{\pi(A_t|S_t)}{\mu(A_t|S_t)} \frac{\pi(A_{t+1}|S_{t+1})}{\mu(A_{t+1}|S_{t+1})} ... \frac{\pi(A_T|S_T)}{\mu(A_T|S_T)} G_t$$

3. Update value towards correct return

$$V(S_t) \leftarrow V(S_t) + \alpha(G_t^{\pi/\mu} - V(S_t))$$

# Off-Policy TD with Importance Sampling

1. Use TD targets generated from $\mu$ to evaluate $\pi$
2. Weight TD target $R + \gamma V(S')$ by importance sampling
3. Only need a single importance sampling correction

$$V(S_t) \leftarrow V(S_t) + \alpha\Big(\frac{\pi(A_t|S_t)}{\mu(A_t|S_t)}\big(R_{t+1} + \gamma V(S_{t+1})\big) - V(S_t)\Big)$$

4. Policies only need to be similar over a single step

# Why don't use importance sampling on Q-Learning?

1. Off-policy TD

$$V(S_t) \leftarrow V(S_t) + \alpha\Big(\frac{\pi(A_t|S_t)}{\mu(A_t|S_t)}\big(R_{t+1} + \gamma V(S_{t+1})\big) - V(S_t)\Big) \quad (1)$$

2. Short answer: 1,Q-learning is a deterministic policy, no action probability. 2,Q-learning does not make expected value estimates over the policy distribution. For the full answer click here
3. Remember bellman optimality backup from value iteration

$$Q(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) \max_{a'} Q(s',a') \quad (2)$$

   1. Q-learning can be considered as sample update of value iteration, except instead of using the expected value over the transition dynamics, we use the sample collected from the environment

$$Q(s,a) \leftarrow r + \gamma \max_{a'} Q(s',a') \quad (3)$$

      Q-learning is over the transition distribution, not over policy distribution thus no need to correct different policy distributions

# Reinforcement Learning and Optimal Control

1. Dimitri P. Bertsekas on reinforcement learning and optimal control
2. https://web.mit.edu/dimitrib/www/RLbook.html

# Next Week

1. Value function approximation
2. Deep Q Learning