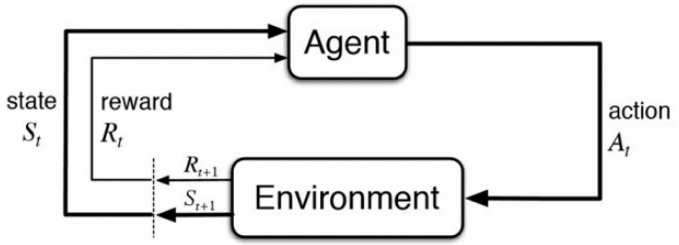
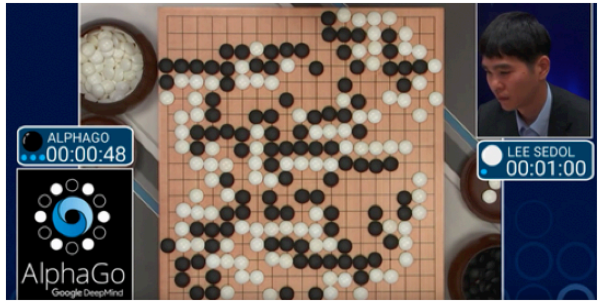



# Introduction on Optimal Control and RL

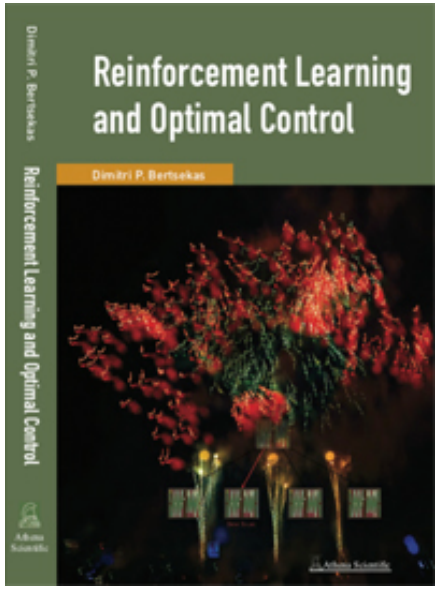
Bolei Zhou

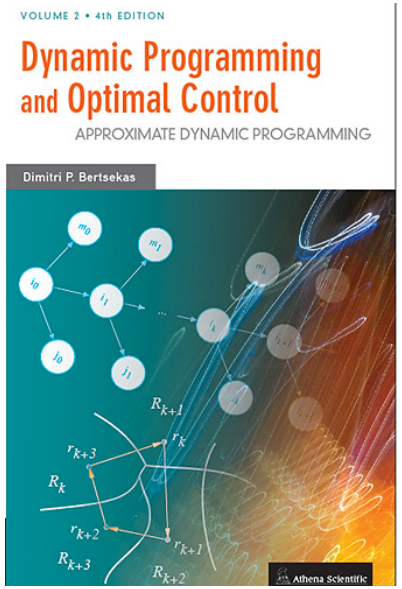
The Chinese University of Hong Kong

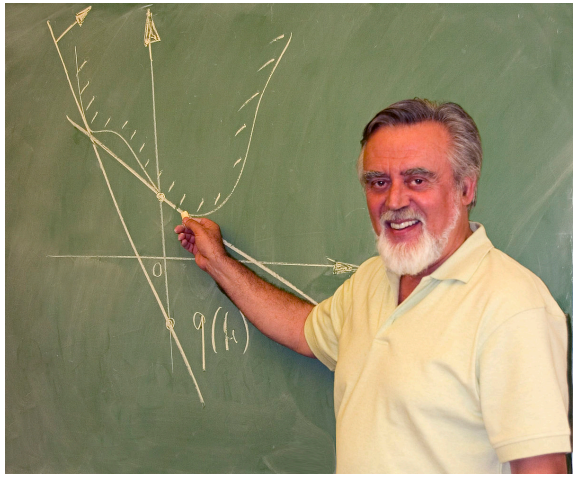
Some slides are from Bertsekas' and his book

# Prof. Dimitri P. Bertsekas from MIT

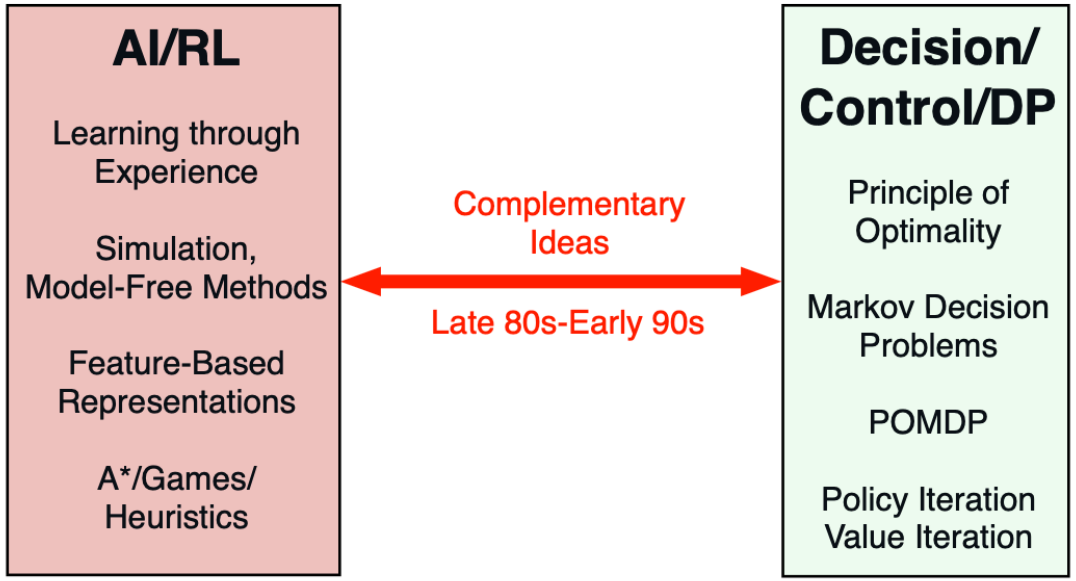
- Researcher on control and optimization
- dynamic programming, optimal control, and reinforcement learning
- https://web.mit.edu/dimitrib/www/RLbook.html
- https://github.com/cuhkrlcourse/bertsekas



# Reinforcement Learning and Optimal Control




## Historical highlights

- Exact DP, optimal control (Bellman, Shannon, 1950s ...)
- First major successes: Backgammon programs (Tesauro, 1992, 1996)
- Algorithmic progress, analysis, applications, first books (mid 90s ...)
- Machine Learning, BIG Data, Robotics, Deep Neural Networks (mid 2000s ...)
- AlphaGo and Alphazero (DeepMind, 2016, 2017)

# Reinforcement Learning and Optimal Control

## RL uses Max/Value, DP uses Min/Cost

- Reward of a stage = (Opposite of) Cost of a stage.
- State value = (Opposite of) State cost.
- Value (or state-value) function = (Opposite of) Cost function.

## Controlled system terminology

- Agent = Decision maker or controller.
- Action = Control.
- Environment = Dynamic system.

## Methods terminology

- Learning = Solving a DP-related problem using simulation.
- Self-learning (or self-play in the context of games) = Solving a DP problem using simulation-based policy iteration.
- Planning vs Learning distinction = Solving a DP problem with model-based vs model-free simulation.

# From the Perspective of Optimal Control

- Start with model-based implementation
- What is a system?

A deterministic DP problem involves a discrete-time dynamic system of the form

$$x_{k+1} = f_k(x_k, u_k), \qquad k = 0, 1, \ldots, N-1, \tag{1.1}$$

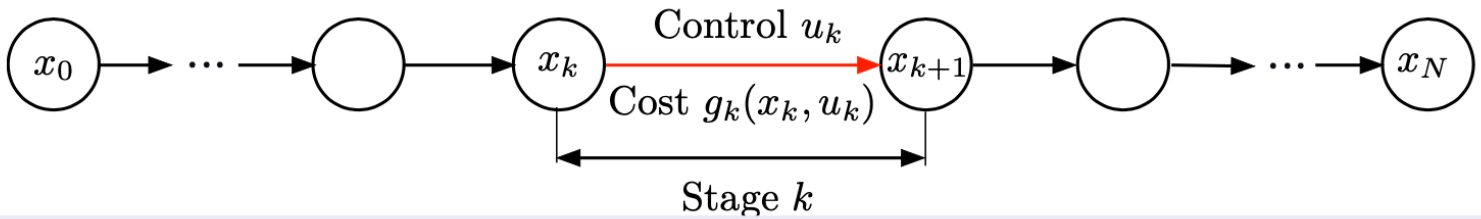
where

$k$ is the time index,

$x_k$ is the state of the system, an element of some space,

$u_k$ is the control or decision variable, to be selected at time $k$ from some given set $U_k(x_k)$ that depends on $x_k$,

# Deterministic Dynamic Programming




- Discrete-time system:

$$x_{k+1} = f_k(x_k, u_k), \qquad k = 0, 1, \ldots, N-1$$

where $x_k$: State, $u_k$: Control chosen from some constraint set $U_k(x_k)$

- Cost function:

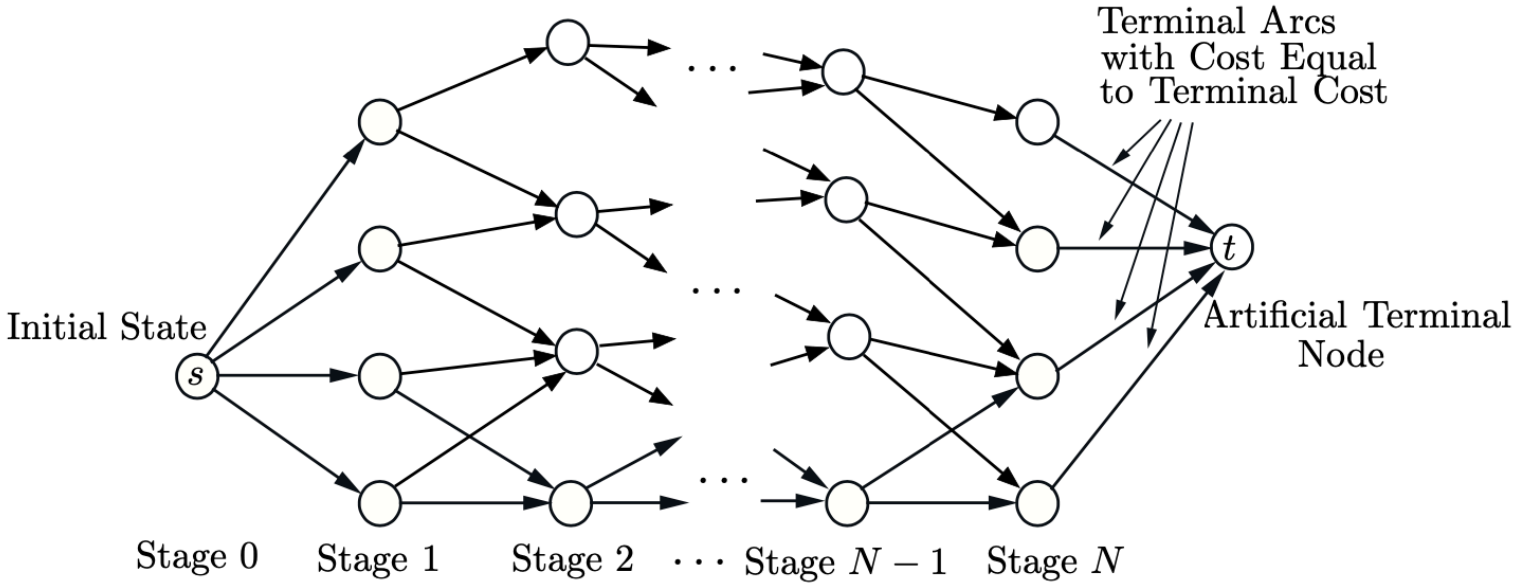
$$g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k)$$

- For given initial state $x_0$, minimize over control sequences $\{u_0, \ldots, u_{N-1}\}$

$$J(x_0; u_0, \ldots, u_{N-1}) = g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k)$$

- Control sequences correspond to paths from start node to end node in the graph
- Optimal cost function $J^*(x_0) = \min_{\substack{u_k \in U_k(x_k) \\ k=0,\ldots,N-1}} J(x_0; u_0, \ldots, u_{N-1})$

# Deterministic Dynamic Programming

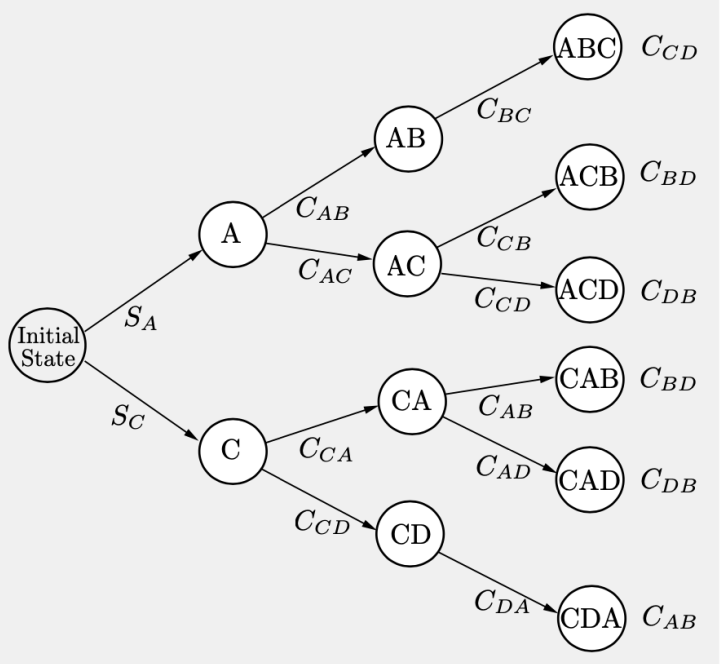



- Nodes correspond to states $x_k$
- Arcs correspond to state-control pairs $(x_k, u_k)$
- An arc $(x_k, u_k)$ has start node $x_k$ and end node $x_{k+1} = f_k(x_k, u_k)$
- A control sequence $\{u_0, \ldots, u_{N-1}\}$ corresponds to a sequence of arcs from the initial node $s$ to the terminal node $t$.
- An arc $(x_k, u_k)$ has a cost $g_k(x_k, u_k)$. The cost to optimize is the sum of the arc costs from $s$ to $t$.
- The problem is equivalent to finding a minimum cost/shortest path from $s$ to $t$.

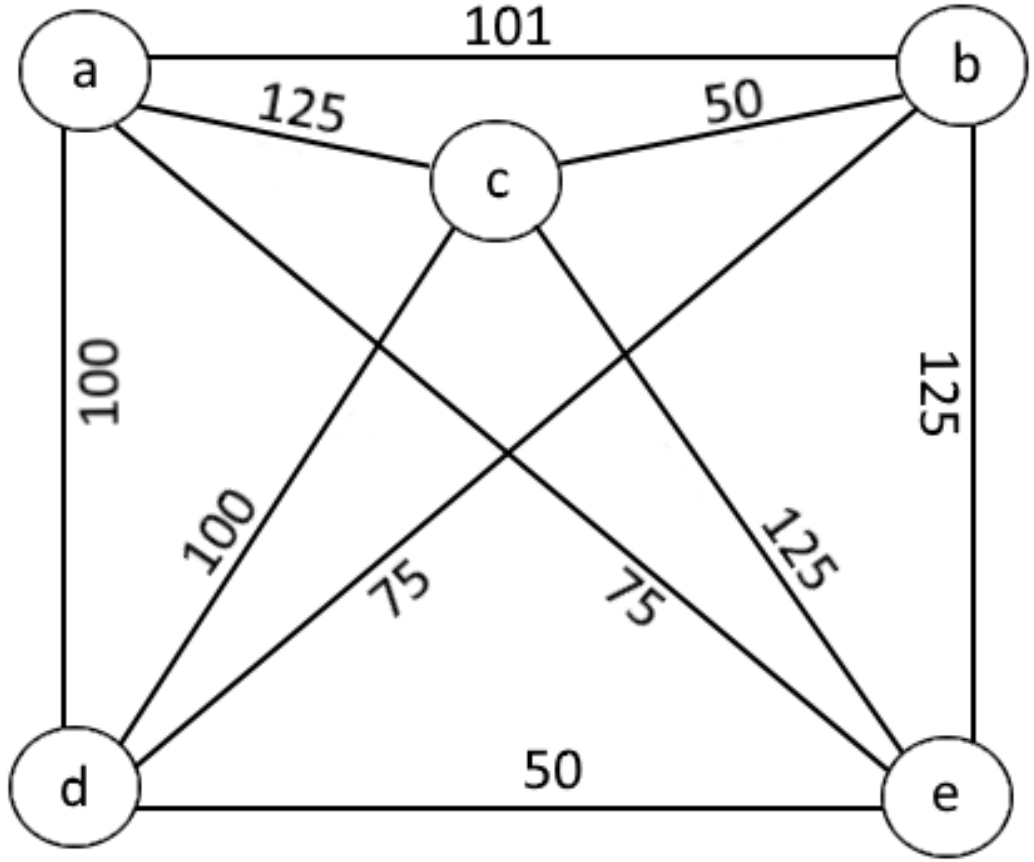
# Discrete Optimal Control: a deterministic scheduling problem

- Suppose that to produce a certain product, four operations (A, B, C, D) must be performed on a certain machine.
- We assume that operation B can be performed only after operation A has been performed (A -> B), and operation D can be performed only after operation C has been performed (C -> D)
- A sequence of three decisions
- For example, the cost for sequence ACDB is

$$S_A + C_{AC} + C_{CD} + C_{DB}.$$



# Discrete Optimal Control:
# Travelling salesman problem (TSP)



# Continuous-spaces Optimal Control: temperature control

**Example 1.1.2 (A Linear-Quadratic Problem)**

A certain material is passed through a sequence of $N$ ovens (see Fig. 1.1.4). Denote

$x_0$: initial temperature of the material,

$x_k$, $k = 1, \ldots, N$: temperature of the material at the exit of oven $k$,

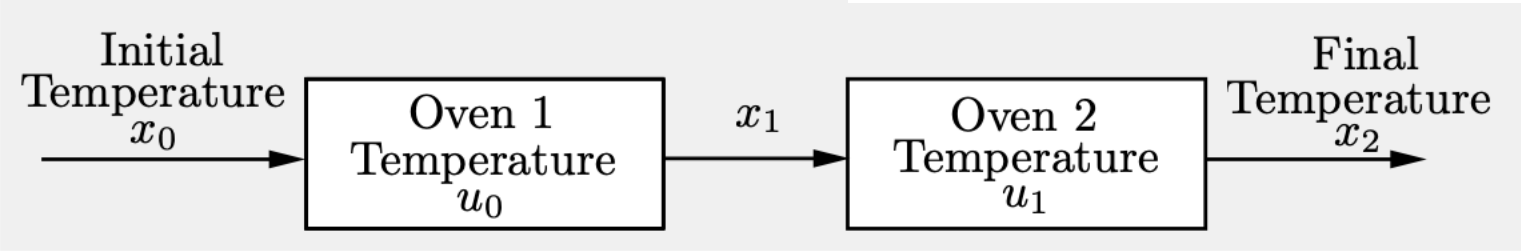
$u_{k-1}$, $k = 1, \ldots, N$: heat energy applied to the material in oven $k$. In practice there will be some constraints on $u_k$, such as nonnegativity. However, for analytical tractability one may also consider the case where $u_k$ is unconstrained, and check later if the solution satisfies some natural restrictions in the problem at hand.

We assume a system equation of the form

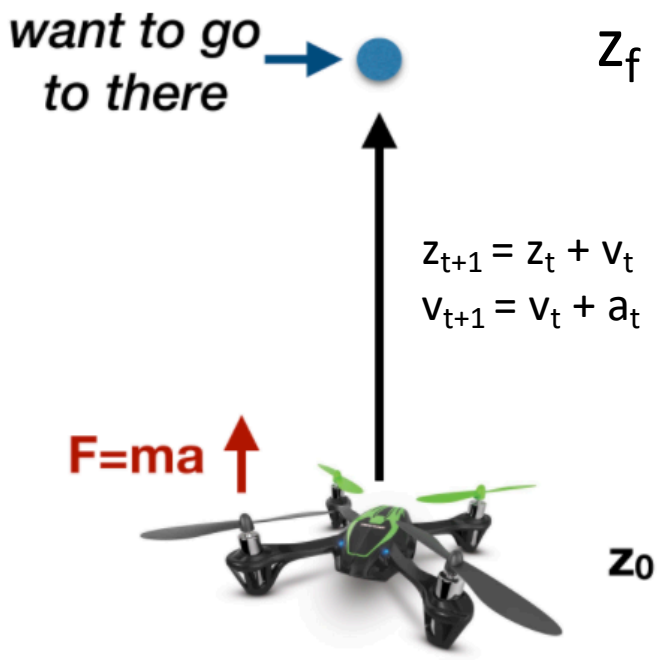
$$x_{k+1} = (1 - a)x_k + au_k, \qquad k = 0, 1, \ldots, N - 1,$$

where $a$ is a known scalar from the interval $(0, 1)$. The objective is to get the final temperature $x_N$ close to a given target $T$, while expending relatively little energy. We express this with a cost function of the form

$$r(x_N - T)^2 + \sum_{k=0}^{N-1} u_k^2,$$



# Continuous-spaces Optimal Control: quadcopter control




**F** is force

**a** is acceleration

**v** is velocity

**z** is vertical position

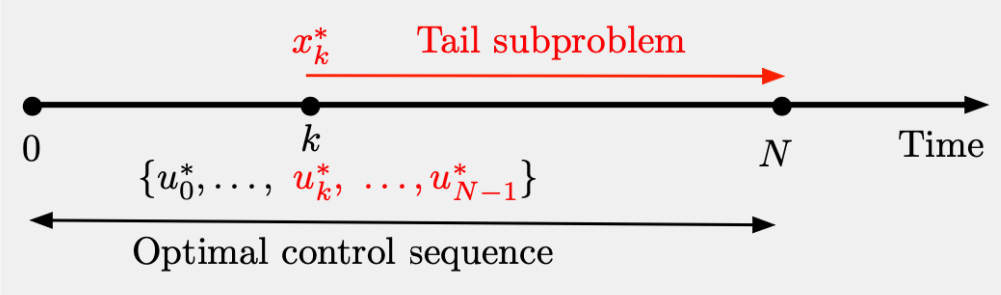
# Dynamic Programming: the principle of optimality

**Principle of Optimality**

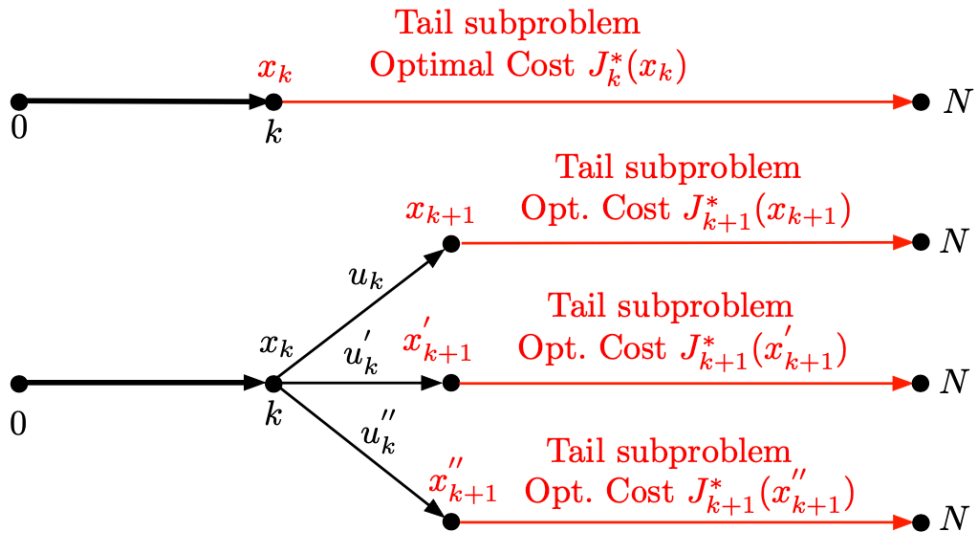
Let $\{u_0^*, \ldots, u_{N-1}^*\}$ be an optimal control sequence, which together with $x_0$ determines the corresponding state sequence $\{x_1^*, \ldots, x_N^*\}$ via the system equation (1.1). Consider the subproblem whereby we start at $x_k^*$ at time $k$ and wish to minimize the "cost-to-go" from time $k$ to time $N$,

$$g_k(x_k^*, u_k) + \sum_{m=k+1}^{N-1} g_m(x_m, u_m) + g_N(x_N),$$

over $\{u_k, \ldots, u_{N-1}\}$ with $u_m \in U_m(x_m)$, $m = k, \ldots, N-1$. Then the truncated optimal control sequence $\{u_k^*, \ldots, u_{N-1}^*\}$ is optimal for this subproblem.



# Dynamic Programming




Consider solution of the tail subproblem that starts at $x_k$ at time $k$ and minimizes over $\{u_k, \ldots, u_{N-1}\}$ the "cost-to-go" from $k$ to $N$,

$$g_k(x_k, u_k) + \sum_{m=k+1}^{N-1} g_m(x_m, u_m) + g_N(x_N), \qquad \text{Optimal Cost-to-Go: } J_k^*(x_k)$$

To solve it, choose $u_k$ to min (1st stage cost + Optimal tail problem cost) or

$$\min_{u_k \in U_k(x_k)} \left[ g_k(x_k, u_k) + J_{k+1}^*\left(f_k(x_k, u_k)\right) \right]$$

# Dynamic Programming: 2 steps

## Going backward

**DP Algorithm for Deterministic Finite Horizon Problems**

Start with

$$J_N^*(x_N) = g_N(x_N), \qquad \text{for all } x_N, \tag{1.3}$$

and for $k = 0, \ldots, N-1$, let

$$J_k^*(x_k) = \min_{u_k \in U_k(x_k)} \Big[ g_k(x_k, u_k) + J_{k+1}^*\big(f_k(x_k, u_k)\big) \Big], \qquad \text{for all } x_k. \tag{1.4}$$

Compute optimal cost-to-go function $J^*_K$

## Going forward

**Construction of Optimal Control Sequence $\{u_0^*, \ldots, u_{N-1}^*\}$**

Set

$$u_0^* \in \arg \min_{u_0 \in U_0(x_0)} \Big[ g_0(x_0, u_0) + J_1^*\big(f_0(x_0, u_0)\big) \Big],$$

and

$$x_1^* = f_0(x_0, u_0^*).$$

Sequentially, going forward, for $k = 1, 2, \ldots, N-1$, set

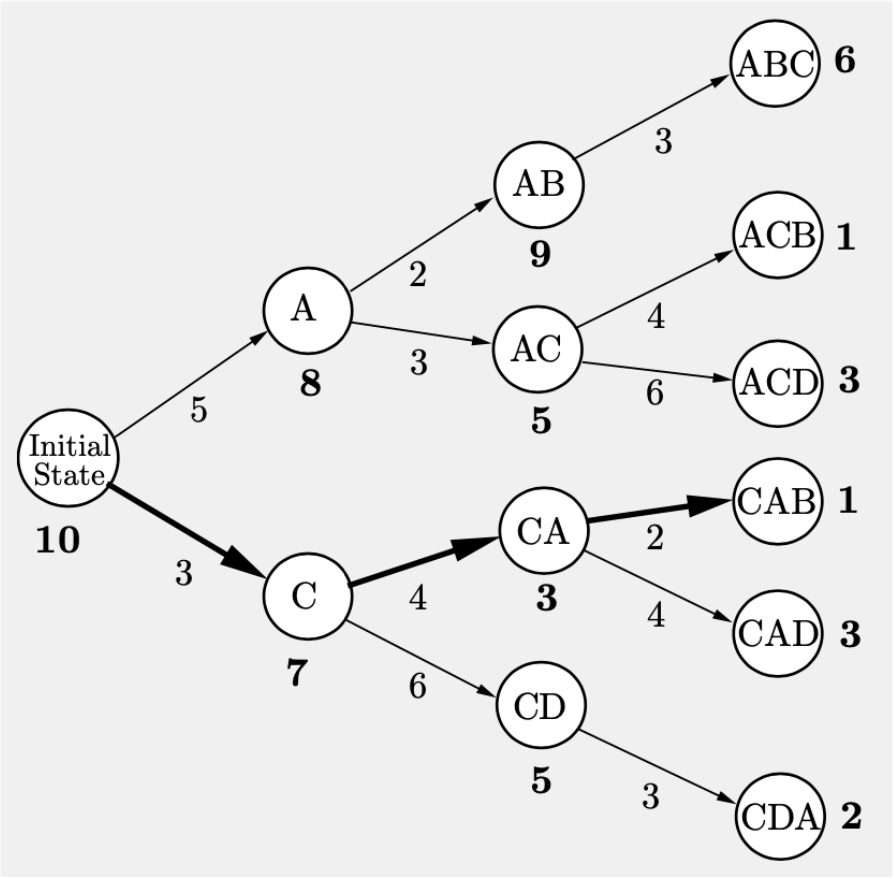
$$u_k^* \in \arg \min_{u_k \in U_k(x_k^*)} \Big[ g_k(x_k^*, u_k) + J_{k+1}^*\big(f_k(x_k^*, u_k)\big) \Big], \tag{1.7}$$

and

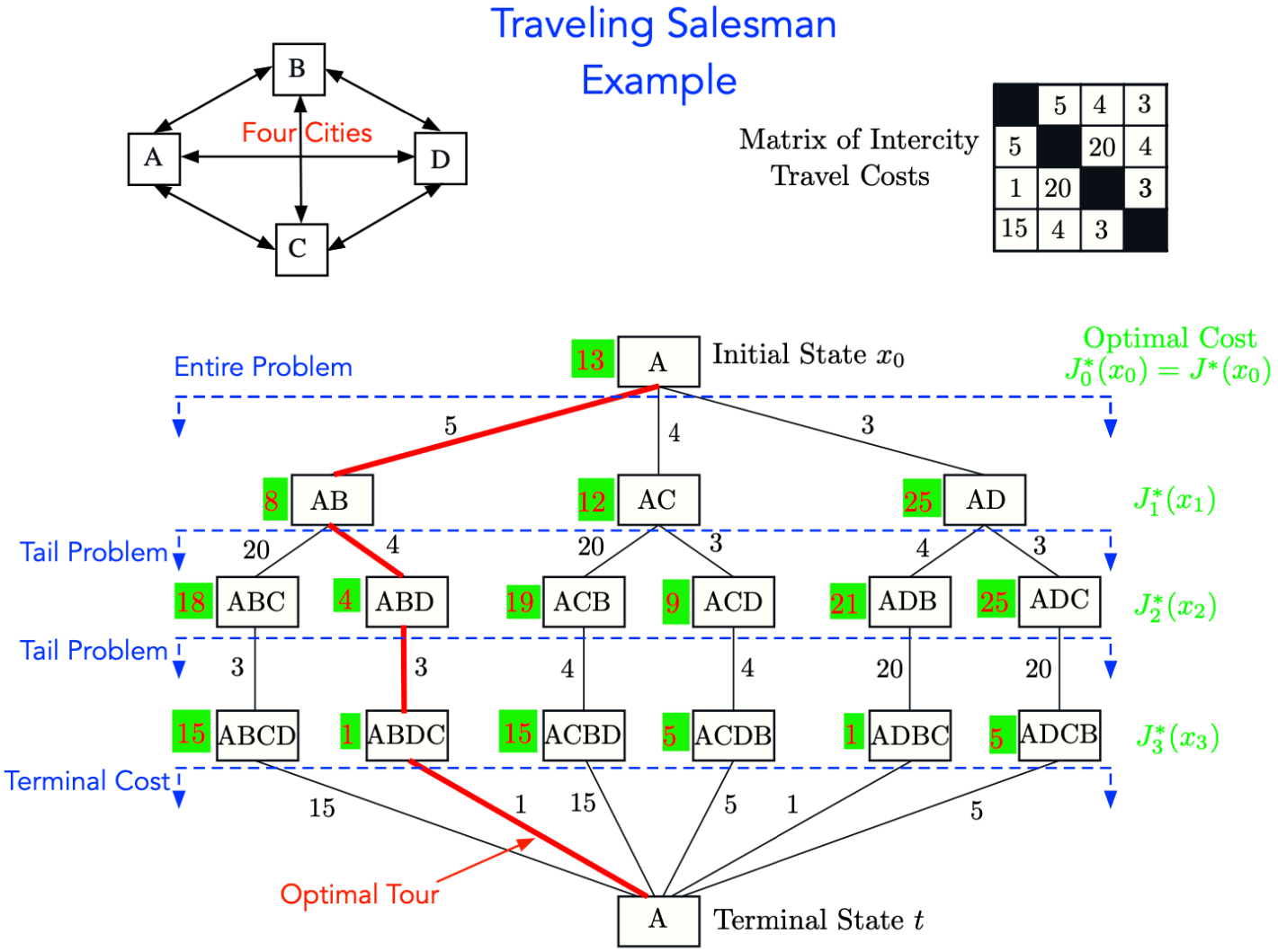
$$x_{k+1}^* = f_k(x_k^*, u_k^*). \tag{1.8}$$

# Solving the scheduling problem with DP

- The DP algorithm is based on this idea: it proceeds sequentially, by solving all the tail subproblems of a given time length, using the solution of the tail subproblems of shorter time length.



# Solving Traveling Salesman problem with DP



# Solving the Linear Quadratic Optimal Control

- Analytical solution for the temperature control

As defined in Example 1.1.2, the terminal cost is

$$g_2(x_2) = r(x_2 - T)^2.$$

Thus the DP algorithm starts with

$$J_2^*(x_2) = g_2(x_2) = r(x_2 - T)^2,$$

For the next-to-last stage, we have [cf. Eq. (1.4)]

$$J_1^*(x_1) = \min_{u_1}\big[u_1^2 + J_2^*(x_2)\big] = \min_{u_1}\Big[u_1^2 + J_2^*\big((1-a)x_1 + au_1\big)\Big].$$

Substituting the previous form of $J_2^*$, we obtain

$$J_1^*(x_1) = \min_{u_1}\Big[u_1^2 + r\big((1-a)x_1 + au_1 - T\big)^2\Big]. \qquad (1.21)$$

This minimization will be done by setting to zero the derivative with respect to $u_1$. This yields

$$0 = 2u_1 + 2ra\big((1-a)x_1 + au_1 - T\big),$$

and by collecting terms and solving for $u_1$, we obtain the optimal temperature for the last oven as a function of $x_1$:

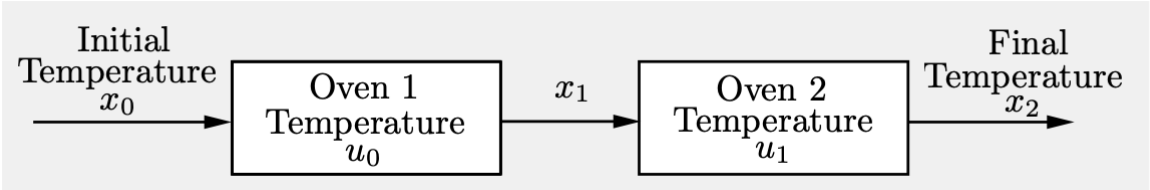
$$\mu_1^*(x_1) = \frac{ra\big(T - (1-a)x_1\big)}{1 + ra^2}. \qquad (1.22)$$

By substituting the optimal $u_1$ in the expression (1.21) for $J_1^*$, we obtain

$$\begin{aligned} J_1^*(x_1) &= \frac{r^2a^2\big((1-a)x_1 - T\big)^2}{(1+ra^2)^2} + r\left((1-a)x_1 + \frac{ra^2\big(T-(1-a)x_1\big)}{1+ra^2} - T\right)^2 \\ &= \frac{r^2a^2\big((1-a)x_1 - T\big)^2}{(1+ra^2)^2} + r\left(\frac{ra^2}{1+ra^2} - 1\right)^2\big((1-a)x_1 - T\big)^2 \\ &= \frac{r\big((1-a)x_1 - T\big)^2}{1+ra^2}. \end{aligned}$$

We now go back one stage. We have [cf. Eq. (1.4)]

$$J_0^*(x_0) = \min_{u_0}\big[u_0^2 + J_1^*(x_1)\big] = \min_{u_0}\Big[u_0^2 + J_1^*\big((1-a)x_0 + au_0\big)\Big],$$



# Solving the Linear Quadratic Optimal Control

- Analytical solution for the temperature control

and by substituting the expression already obtained for $J_1^*$, we have

$$J_0^*(x_0) = \min_{u_0} \left[ u_0^2 + \frac{r\big((1-a)^2 x_0 + (1-a)a u_0 - T\big)^2}{1 + ra^2} \right].$$

We minimize with respect to $u_0$ by setting the corresponding derivative to zero. We obtain

$$0 = 2u_0 + \frac{2r(1-a)a\big((1-a)^2 x_0 + (1-a)a u_0 - T\big)}{1 + ra^2}.$$

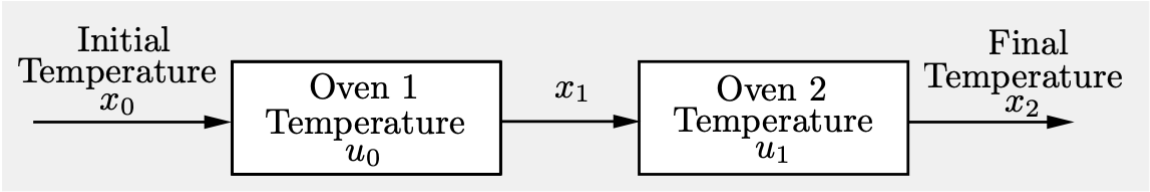
This yields, after some calculation, the optimal temperature of the first oven:

$$\mu_0^*(x_0) = \frac{r(1-a)a\big(T - (1-a)^2 x_0\big)}{1 + ra^2\big(1 + (1-a)^2\big)}. \tag{1.23}$$

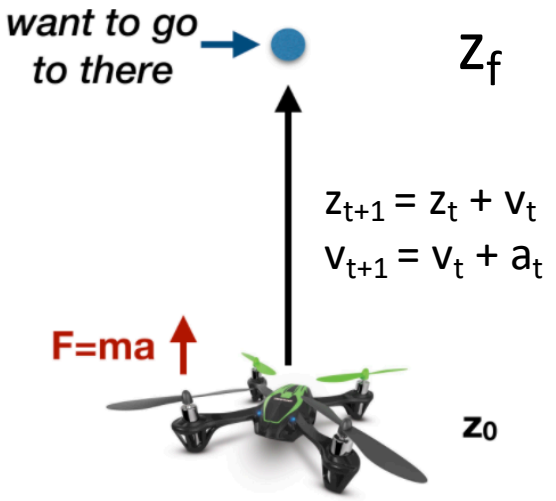
The optimal cost is obtained by substituting this expression in the formula for $J_0^*$. This leads to a straightforward but lengthy calculation, which in the end yields the rather simple formula

$$J_0^*(x_0) = \frac{r\big((1-a)^2 x_0 - T\big)^2}{1 + ra^2\big(1 + (1-a)^2\big)}.$$

**what simplifies the solution is the quadratic nature of the cost and the linearity of the system equation.**



# Quadrotor control problem with analytical solution




**F** is force
**a** is acceleration
**v** is velocity
**z** is vertical position

Note that the quadrotor dynamics we derived from Newton's Laws are linear. The state of the quadrotor is its vertical position and velocity $x_t = [z_t; v_t]$. The input $u_t$ is the propeller force. Newton's Laws written in matrix form are thus

$$x_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} (u_t - g)$$

Thus linear dynamic system: $x_{t+1} = Ax_t + Bu_t,$

http://www.argmin.net/2018/02/08/lqr/

control objective. If we want the quadrotor to fly to location $z_f$, we could specify

$$R_t[x_t, u_t] = |z_t - z_f|$$

or

$$R_t[x_t, u_t] = (z_t - z_f)^2$$

or

$$R_t[x_t, u_t] = \begin{cases} 1 & z_t = z_f \\ 0 & \text{otherwise} \end{cases}$$

# Quadrotor control problem with analytical solution

$$\begin{aligned} \textbf{maximize}_{u_t} \quad & \mathbb{E}_{e_t}\left[\sum_{t=0}^{N} R_t[x_t, u_t]\right] \\ \textbf{subject to} \quad & x_{t+1} = f(x_t, u_t, e_t) \\ & (x_0 \text{ given}). \end{aligned}$$

$$\begin{aligned} \textbf{minimize}_{u_t, x_t} \quad & \tfrac{1}{2}\sum_{t=0}^{N}\left\{x_t^T Q x_t + u_t^T R u_t\right\} \\ & \quad + \tfrac{1}{2} x_{N+1}^T S x_{N+1}, \\ \textbf{subject to} \quad & x_{t+1} = A x_t + B u_t, \\ & \quad \text{for } t = 0, 1, \ldots, N, \\ & (x_0 \text{ given}). \end{aligned}$$

The Lagrangian for the LQR problem has the form:

$$\begin{aligned} \mathcal{L}(x, u, p) := \sum_{t=0}^{N} \Big[ & \frac{1}{2} x_t^T Q x_t + \frac{1}{2} u_t^T R u_t \\ & - p_{t+1}^T (x_{t+1} - A x_t - B u_t) \Big] \\ & + \frac{1}{2} x_{N+1}^T S x_{N+1}. \end{aligned}$$

After some derivation we can compute u_t as:

$$u_t = -(R + B^T M_{t+1} B)^{-1} B^T M_{t+1} A x_t \,.$$

http://www.argmin.net/2018/02/08/lqr/

# DP Algorithm: formal statement

**Go backward to compute the optimal costs $J_k^*(x_k)$ of the $x_k$-tail subproblems**

Start with

$$J_N^*(x_N) = g_N(x_N), \qquad \text{for all } x_N,$$

and for $k =$ $N$-1, ..., 0 , let

$$J_k^*(x_k) = \min_{u_k \in U_k(x_k)} \Big[ g_k(x_k, u_k) + J_{k+1}^*\big(f_k(x_k, u_k)\big) \Big], \qquad \text{for all } x_k.$$

The optimal cost $J^*(x_0)$ is obtained at the last step: $J_0^*(x_0) = J^*(x_0)$.

**Go forward to construct optimal control sequence $\{u_0^*, \ldots, u_{N-1}^*\}$**

Start with

$$u_0^* \in \arg\min_{u_0 \in U_0(x_0)} \Big[ g_0(x_0, u_0) + J_1^*\big(f_0(x_0, u_0)\big) \Big], \qquad x_1^* = f_0(x_0, u_0^*).$$

Sequentially, going forward, for $k = 1, 2, \ldots, N-1$, set

$$u_k^* \in \arg\min_{u_k \in U_k(x_k^*)} \Big[ g_k(x_k^*, u_k) + J_{k+1}^*\big(f_k(x_k^*, u_k)\big) \Big], \qquad x_{k+1}^* = f_k(x_k^*, u_k^*).$$

# Problem with exact DP

**Go backward to compute the optimal costs $J_k^*(x_k)$ of the $x_k$-tail subproblems**

Start with

$$J_N^*(x_N) = g_N(x_N), \qquad \text{for all } x_N,$$

and for $k =$ $N$-1, ..., 0 , let

$$J_k^*(x_k) = \min_{u_k \in U_k(x_k)} \Big[ g_k(x_k, u_k) + J_{k+1}^*\big(f_k(x_k, u_k)\big) \Big], \qquad \text{for all } x_k.$$

The optimal cost $J^*(x_0)$ is obtained at the last step: $J_0^*(x_0) = J^*(x_0)$.

**Go forward to construct optimal control sequence $\{u_0^*, \ldots, u_{N-1}^*\}$**

Start with

$$u_0^* \in \arg \min_{u_0 \in U_0(x_0)} \Big[ g_0(x_0, u_0) + J_1^*\big(f_0(x_0, u_0)\big) \Big], \qquad x_1^* = f_0(x_0, u_0^*).$$

Sequentially, going forward, for $k = 1, 2, \ldots, N-1$, set

$$u_k^* \in \arg \min_{u_k \in U_k(x_k^*)} \Big[ g_k(x_k^*, u_k) + J_{k+1}^*\big(f_k(x_k^*, u_k)\big) \Big], \qquad x_{k+1}^* = f_k(x_k^*, u_k^*).$$

**Approximation idea**: Replace $J_k^*$ with an approximation $\tilde{J}_k$ (possibly a neural net).

# Approximation in Value Space

- The construction of suitable approximate cost-to-go functions $\tilde{J}_k$

**Approximation in Value Space - Use of $\tilde{J}_k$ in Place of $J_k^*$**

Start with

$$\tilde{u}_0 \in \arg \min_{u_0 \in U_0(x_0)} \Big[ g_0(x_0, u_0) + \tilde{J}_1\big(f_0(x_0, u_0)\big) \Big],$$

and set

$$\tilde{x}_1 = f_0(x_0, \tilde{u}_0).$$

Sequentially, going forward, for $k = 1, 2, \ldots, N-1$, set

$$\tilde{u}_k \in \arg \min_{u_k \in U_k(\tilde{x}_k)} \Big[ g_k(\tilde{x}_k, u_k) + \tilde{J}_{k+1}\big(f_k(\tilde{x}_k, u_k)\big) \Big], \qquad (1.9)$$

and

$$\tilde{x}_{k+1} = f_k(\tilde{x}_k, \tilde{u}_k). \qquad (1.10)$$

# Q-Factors and Q-Learning

**Approximation in Value Space - Use of $\tilde{J}_k$ in Place of $J_k^*$**

Start with

$$\tilde{u}_0 \in \arg \min_{u_0 \in U_0(x_0)} \Big[ g_0(x_0, u_0) + \tilde{J}_1\big(f_0(x_0, u_0)\big) \Big],$$

and set

$$\tilde{x}_1 = f_0(x_0, \tilde{u}_0).$$

Sequentially, going forward, for $k = 1, 2, \ldots, N-1$, set

$$\tilde{u}_k \in \arg \min_{u_k \in U_k(\tilde{x}_k)} \Big[ g_k(\tilde{x}_k, u_k) + \tilde{J}_{k+1}\big(f_k(\tilde{x}_k, u_k)\big) \Big], \qquad (1.9)$$

and

$$\tilde{x}_{k+1} = f_k(\tilde{x}_k, \tilde{u}_k). \qquad (1.10)$$

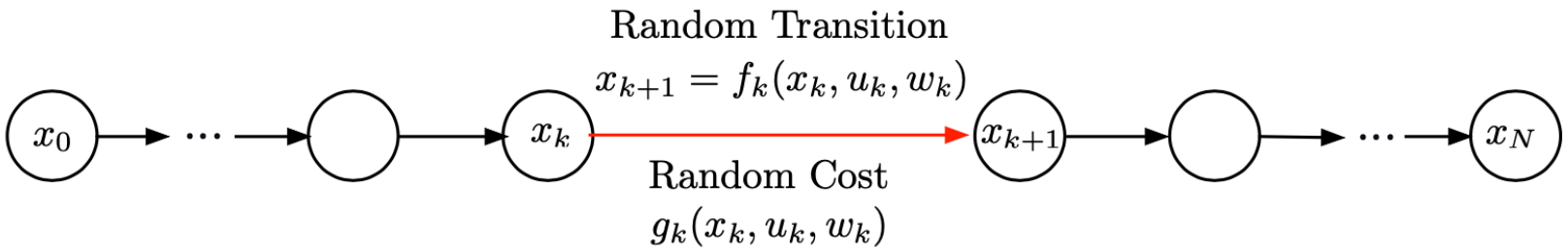
Q-factor of ($x_k$, $u_k$):

$$\tilde{Q}_k(x_k, u_k) = g_k(x_k, u_k) + \tilde{J}_{k+1}\big(f_k(x_k, u_k)\big),$$

Optimal Q-factor of ($x_k$, $u_k$):

$$Q_k^*(x_k, u_k) = g_k(x_k, u_k) + J_{k+1}^*\big(f_k(x_k, u_k)\big).$$

# Stochastic DP Problems




- Stochasticity in the form of a random "disturbance" $w_k$ (e.g., physical noise, market uncertainties, demand for inventory, unpredictable breakdowns, etc)
- Cost function:

$$E\left\{g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k)\right\}$$

- Policies $\pi = \{\mu_0, \ldots, \mu_{N-1}\}$, where $\mu_k$ is a "closed-loop control law" or "feedback policy"/a function of $x_k$. Specifies control $u_k = \mu_k(x_k)$ to apply when at $x_k$.
- For given initial state $x_0$, minimize over all $\pi = \{\mu_0, \ldots, \mu_{N-1}\}$ the cost

$$J_\pi(x_0) = E\left\{g_N(x_N) + \sum_{k=0}^{N-1} g_k\left(x_k, \mu_k(x_k), w_k\right)\right\}$$

- Optimal cost function $J^*(x_0) = \min_\pi J_\pi(x_0)$

# Stochastic DP Algorithm

**Go backward to compute the optimal costs $J_k^*(x_k)$ of the $x_k$-tail subproblems**

- Start with $J_N^*(x_N) = g_N(x_N)$, and for $k = 0, \ldots, N-1$, let

$$J_k^*(x_k) = \min_{u_k \in U_k(x_k)} E\Big\{ g_k(x_k, u_k, w_k) + J_{k+1}^*\big(f_k(x_k, u_k, w_k)\big) \Big\}, \qquad \text{for all } x_k.$$

- The optimal cost $J^*(x_0)$ is obtained at the last step: $J_0^*(x_0) = J^*(x_0)$.
- The optimal control function $\mu_k^*$ can be constructed simultaneously with $J_k^*$, and consists of the minimizing $u_k^* = \mu_k^*(x_k)$ above.

**Go forward to implement the optimal policy, given $J_1^*, \ldots, J_{N-1}^*$**

Sequentially, starting with $x_0$, observe $x_k$ and apply

$$u_k^* \in \arg \min_{u_k \in U_k(x_k)} E\Big\{ g_k(x_k, u_k, w_k) + J_{k+1}^*\big(f_k(x_k, u_k, w_k)\big) \Big\}.$$

Issues: Need to compute $J_{k+1}^*$ (possibly off-line), compute expectation for each $u_k$, minimize over all $u_k$

Approximation idea: Use $\tilde{J}_k$ in place of $J_k^*$, and approximate $E\{\cdot\}$ and $\min_{u_k}$.

# DP and Approximation in Value Space

**Go backwards, $k = N-1, \ldots, 0$, using**

$$J_N(x_N) = g_N(x_N)$$

$$J_k(x_k) = \min_{u_k} E_{w_k}\Big\{ g_k(x_k, u_k, w_k) + J_{k+1}\big(f_k(x_k, u_k, w_k)\big)\Big\}$$

$J_k(x_k)$: Optimal cost-to-go starting from state $x_k$

Approximate DP is motivated by the ENORMOUS computational demands of exact DP

**Approximation in value space: Use an approximate cost-to-go function $\tilde{J}_{k+1}$**

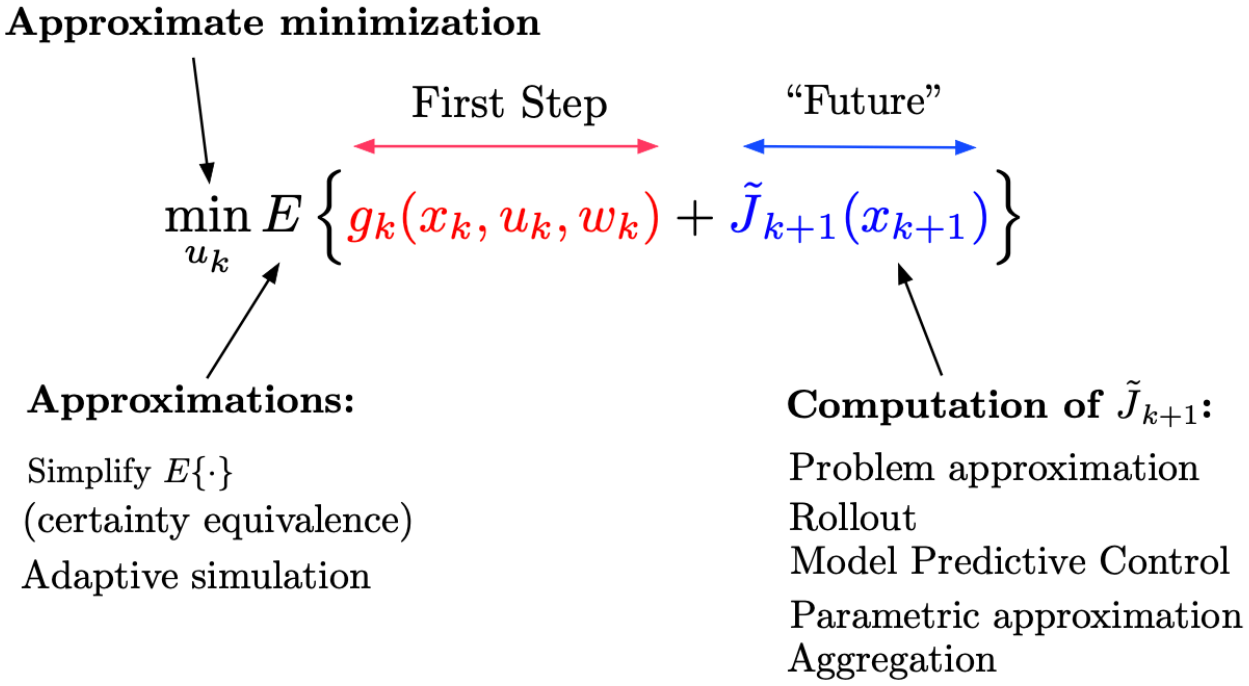
$$\tilde{\mu}_k(x_k) \in \arg\min_{u_k} E_{w_k}\Big\{ g_k(x_k, u_k, w_k) + \tilde{J}_{k+1}\big(f_k(x_k, u_k, w_k)\big)\Big\}$$

**There is also a multistep lookahead version**

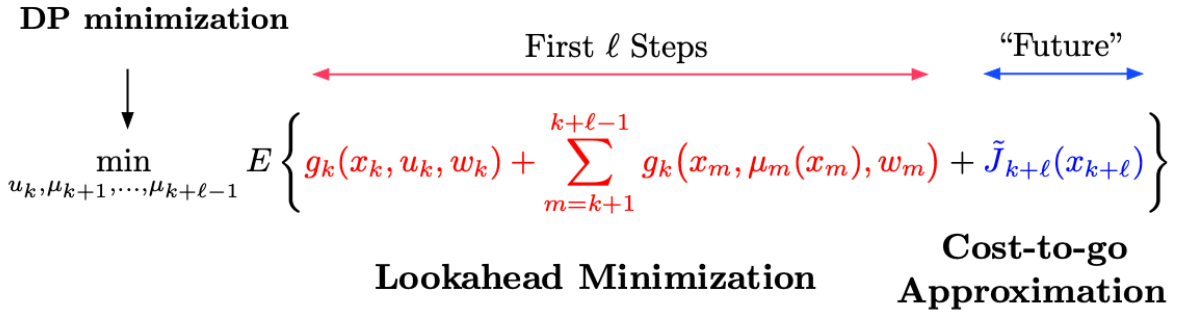
At state $x_k$ solve an $\ell$-step DP problem with terminal cost function approximation $\tilde{J}_{k+\ell}$.
Use the first control in the optimal $\ell$-step sequence.

# DP and Approximation in Value Space

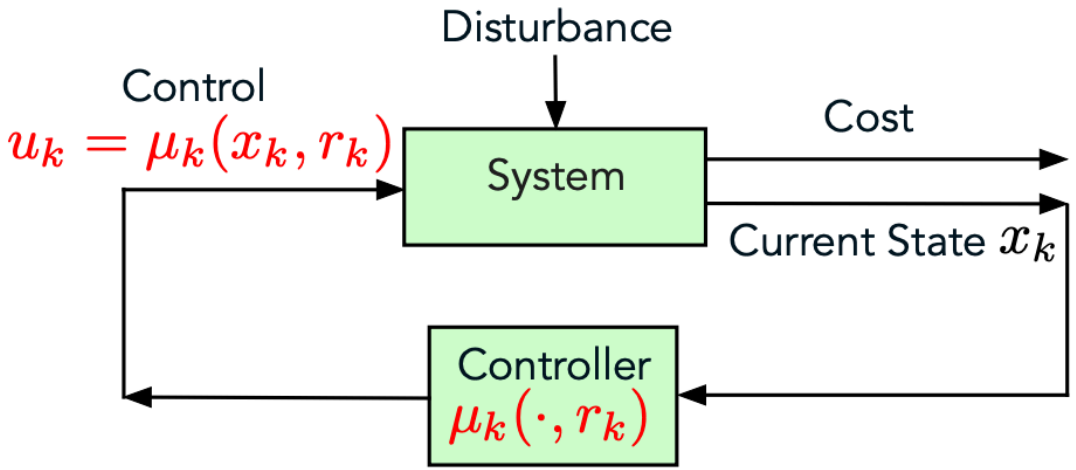
One-step case at state $x_k$:




Multistep case at state $x_k$:



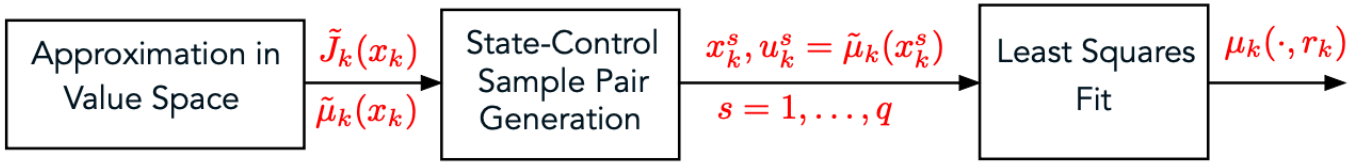
# Approximation in Policy Space: The Major Alternative to Approximation in Value Space




- Idea: Select the policy by optimization over a suitably restricted class of policies.
- The restricted class is usually a parametric family of policies $\mu_k(x_k, r_k)$, $k = 0, \ldots, N-1$, of some form, where $r_k$ is a parameter (e.g., a neural net).
- Important advantage once the parameters $r_k$ are computed: The computation of controls during on-line operation of the system is often much easier; i.e.,

At state $x_k$ apply $u_k = \mu_k(x_k, r_k)$

# Approximation in Policy Space: The Major Alternative to Approximation in Value Space




- Compute approximate cost-to-go functions $\tilde{J}_{k+1}$, $k = 0, \ldots, N-1$.
- This defines a suboptimal policy $\tilde{\mu}_k$, $k = 0, \ldots, N-1$, through

$$\tilde{\mu}_k(x_k) \in \arg \min_{u_k \in U_k(x_k)} E\Big\{ g_k(x_k, u_k, w_k) + \tilde{J}_{k+1}\big(f_k(x_k, u_k, w_k)\big) \Big\}$$

- Generate a training set consisting of a large number $q$ of sample pairs $(x_k^s, u_k^s)$, $s = 1, \ldots, q$, where $u_k^s = \tilde{\mu}_k(x_k^s)$. Approximate $\tilde{\mu}_k$ using some form of parametric least squares fit.
- Example: Introduce a parametric family of policies $\mu_k(x_k, r_k)$, $k = 0, \ldots, N-1$, of some form, where $r_k$ is a parameter vector. Then obtain $r_k$ by

$$r_k \in \arg \min_r \sum_{s=1}^{q} \big\| u_k^s - \mu_k(x_k^s, r) \big\|^2$$

- Some other possibilities: Nearest neighbor optimization (use the control $u_k^s$ of the sampled state $x_k^s$ nearest to $x_k$), or interpolation.

# Model-Based Versus Model-Free Implementation for Stochastic Problems

Our use of the term "model-free": A method is called model-free if it involves calculations of expected values using Monte Carlo simulation.

## Principal example

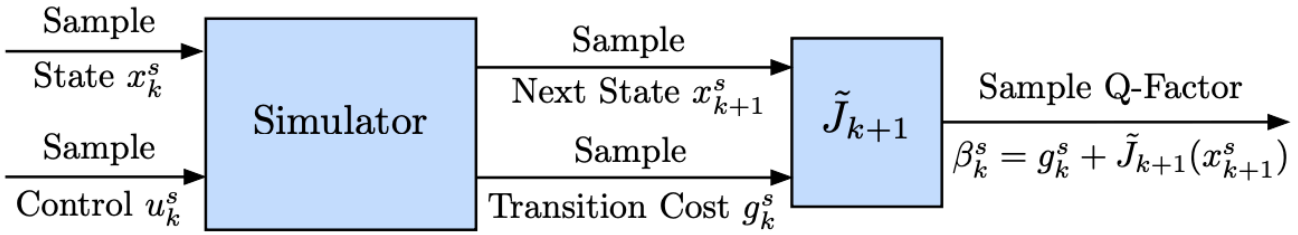
- Calculate $\tilde{J}_k$ in model-free fashion (rollout is a possibility to be discussed later).
- Implement model-free control minimization with one of three possible methods:
  - (1) On-line, at state $x_k$ calculate by simulation the needed Q-factors for lookahead minimization

$$\tilde{Q}_k(x_k, u_k) = E\Big\{g_k(x_k, u_k, w_k) + \tilde{J}_{k+1}\big(f_k(x_k, u_k, w_k)\big)\Big\}$$

    and choose $u_k$ that minimizes $\tilde{Q}_k(x_k, u_k)$
  - (2) Off-line "train" a parametric class of policies $\tilde{\mu}_k(x_k, r_k)$ by approximation in policy space on top of approximation in value space (see previous slide)
  - (3) Off-line "train" a parametric family of Q-factors $\tilde{Q}_k(x_k, u_k, r_k)$ and choose $u_k$ that minimizes $\tilde{Q}_k(x_k, u_k, r_k)$ (see next slide)

# Model-Free Q-Factor Parametric Approximation

- Heavily involve Monte-Carlo simulation




- Use the simulator to collect a large number of "representative" samples of state-control-successor states-stage cost quadruplets $(x_k^s, u_k^s, x_{k+1}^s, g_k^s)$, and corresponding sample Q-factors

$$\beta_k^s = g_k^s + \tilde{J}_{k+1}(x_{k+1}^s), \qquad s = 1, \ldots, q$$

- Introduce a parametric family of Q-factors $\tilde{Q}_k(x_k, u_k, r_k)$.
- Determine the parameter vector $\bar{r}_k$ by the least-squares fit

$$\bar{r}_k \in \arg\min_{r_k} \sum_{s=1}^{q} \left(\tilde{Q}_k(x_k^s, u_k^s, r_k) - \beta_k^s\right)^2$$

- Use the policy

$$\tilde{\mu}_k(x_k) \in \arg\min_{u_k \in U_k(x_k)} \tilde{Q}_k(x_k, u_k, \bar{r}_k)$$

# Rollout: start with a base policy, get a better policy

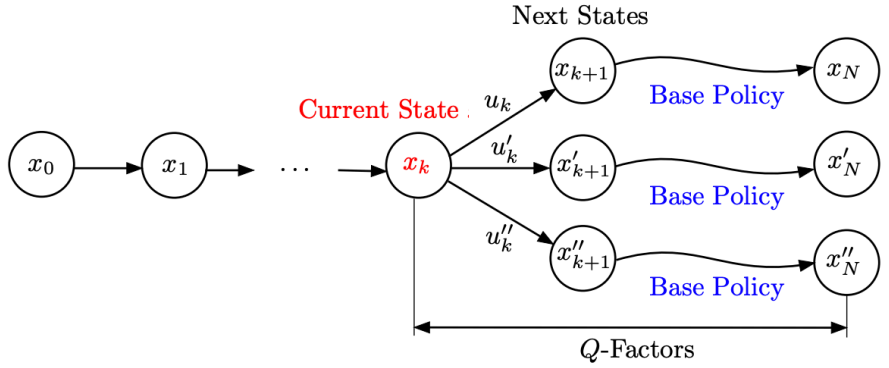
First $\ell$ Steps "Future"

$$\text{At } x_k \longrightarrow \min_{u_k,\mu_{k+1},\cdots,\mu_{k+\ell-1}} E\Big\{g_k(x_k,u_k,w_k) + \sum_{m=k+1}^{k+\ell-1} g_m\big(x_m,\mu_m(x_m),w_m\big) + \tilde{J}_{k+\ell}(x_{k+\ell})\Big\}$$

Rollout Control | Lookahead Minimization | Cost of Base Policy

**Use the cost of the base/suboptimal policy at the end of $\ell$-step lookahead**

**Deterministic Rollout ($\ell = 1$) - Avoids Expensive Simulation**

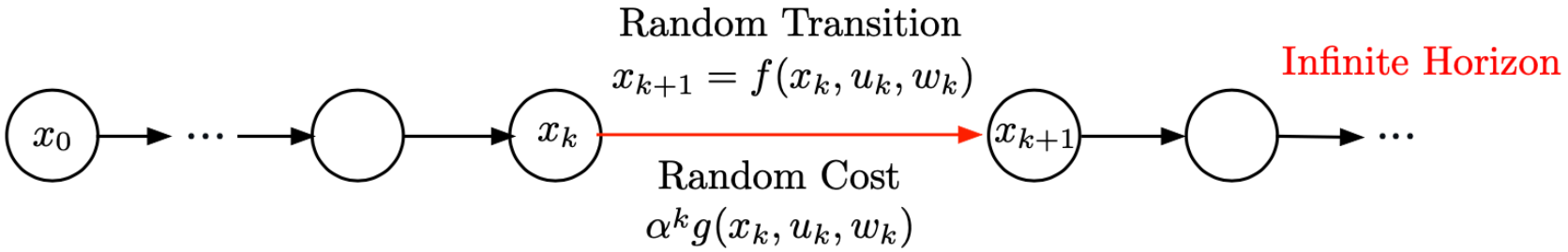



- At state $x_k$, for every pair $(x_k, u_k)$, $u_k \in U_k(x_k)$, we generate a Q-factor

$$\tilde{Q}_k(x_k,u_k) = g_k(x_k,u_k) + H_{k+1}\big(f_k(x_k,u_k)\big)$$

  using the base policy [$H_{k+1}(x_{k+1})$ is the base policy cost starting from $x_{k+1}$].
- We select the control $u_k$ with minimal Q-factor.
- We move to the next state $x_{k+1}$, and continue.

# Stochastic DP Algorithm: Infinite Horizon Extension




## Infinite number of stages, and stationary system and cost

- System $x_{k+1} = f(x_k, u_k, w_k)$ with state, control, and random disturbance.
- Policies $\pi = \{\mu_0, \mu_1, \ldots\}$ with $\mu_k(x) \in U(x)$ for all $x$ and $k$.
- Optimal cost function $J^*(x_0) = \min_\pi J_\pi(x_0)$ satisfies Bellman's equation

$$J^*(x) = \min_{u \in U(x)} E\Big\{ g(x,u,w) + \alpha J^*\big(f(x,u,w)\big) \Big\}$$

- Optimal policy: Applies at $x$ the minimizing $u$ above, regardless of stage $k$.
- When there are finitely many states, $i = 1, \ldots, n$, Bellman's equation is written in terms of the $i \to j$ transition probabilities $p_{ij}(u)$ as

$$J^*(i) = \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha J^*(j)\big)$$

- Approximation possibility: Use $\tilde{J}$ in place of $J^*$, and approximate $E\{\cdot\}$ and $\min_u$

# Value Iteration

$$J_{k+1}(x) = \min_{u \in U(x)} \underset{w}{E} \Big\{ g(x,u,w) + J_k\big(f(x,u,w)\big) \Big\}, \qquad k = 0, 1, \ldots, \quad (4.1)$$

The optimal infinite horizon cost is the limit of the corresponding $N$-stage optimal costs as $N \to \infty$; i.e.,

$$J^*(x) = \lim_{N \to \infty} J_N(x) \qquad (4.2)$$

for all states $x$.

The following equation should hold for all states $x$,

$$J^*(x) = \min_{u \in U(x)} \underset{w}{E} \Big\{ g(x,u,w) + J^*\big(f(x,u,w)\big) \Big\}. \qquad (4.3)$$

# Value Iteration for Discounted Problems

**Bellman Equation and Value Iteration for Discounted Problems:**

For all $i = 1, \ldots, n$, we have

$$J^*(i) = \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha J^*(j)\big).$$

For all $i = 1, \ldots, n$, and any initial conditions $J_0(1), \ldots, J_0(n)$, the VI algorithm generates the sequence $\{J_k\}$ according to

$$J_{k+1}(i) = \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha J_k(j)\big).$$

**Proposition 4.3.1: (Convergence of VI)** Given any initial conditions $J_0(1), \ldots, J_0(n)$, the sequence $\{J_k(i)\}$ generated by the VI algorithm

$$J_{k+1}(i) = \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha J_k(j)\big), \qquad i = 1, \ldots, n,$$

converges to the optimal cost $J^*(i)$ for each $i$.

# DP (Bellman) operators that map the vector J into another vectors

$$(TJ)(i) = \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha J(j)\big), \qquad i = 1, \ldots, n, \quad (4.13)$$

and

$$(T_\mu J)(i) = \sum_{j=1}^{n} p_{ij}(\mu(i))\Big(g\big(i,\mu(i),j\big) + \alpha J(j)\Big), \qquad i = 1, \ldots, n, \quad (4.14)$$

Fixed point property of the DP mapping:

$$\|J_k - J^*\| \leq \rho^k \|J_0 - J^*\|.$$

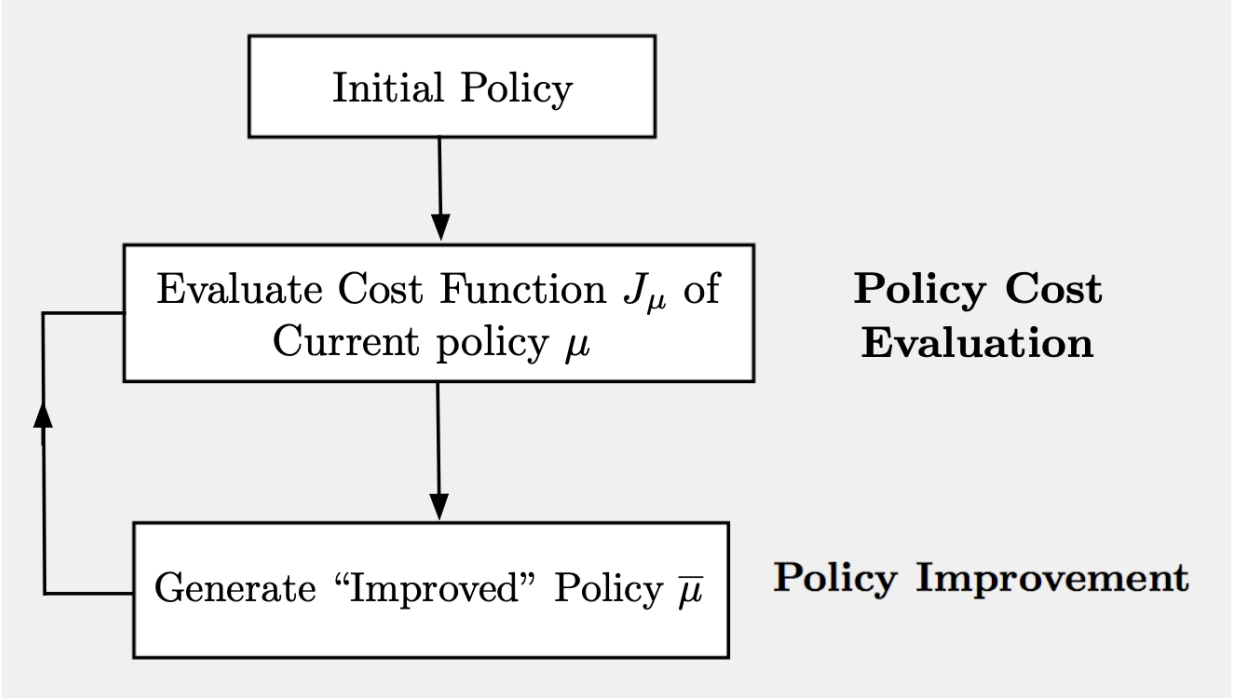
**Proposition 4.3.5: (Contraction Property of the DP Operator)** The DP operators $T$ and $T_\mu$ of Eqs. (4.13) and (4.14) are contraction mappings of modulus $\alpha$ with respect to the maximum norm

$$\|J\| = \max_{i=1,\ldots,n} \big|J(i)\big|. \quad (4.15)$$

In particular, for any two $n$-dimensional vectors $J$ and $J'$, we have

$$\|TJ - TJ'\| \leq \alpha \|J - J'\|, \qquad \|T_\mu J - T_\mu J'\| \leq \alpha \|J - J'\|.$$

# Policy Iteration




**Exact Policy Iteration: Discounted Problems**

Given the typical policy $\mu^k$:

**Policy evaluation** computes $J_{\mu^k}(i)$, $i = 1, \ldots, n$, as the solution of the (linear) system of Bellman equations

$$J_{\mu^k}(i) = \sum_{j=1}^{n} p_{ij}\big(\mu^k(i)\big)\Big(g\big(i, \mu^k(i), j\big) + \alpha J_{\mu^k}(j)\Big), \quad i = 1, \ldots, n, \tag{4.25}$$

(cf. Prop. 4.2.3).

**Policy improvement** then computes a new policy $\mu^{k+1}$ as

$$\mu^{k+1}(i) \in \arg\min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i, u, j) + \alpha J_{\mu^k}(j)\big), \quad i = 1, \ldots, n. \tag{4.26}$$

The process is repeated with $\mu^{k+1}$ used in place of $\mu^k$, unless we have $J_{\mu^{k+1}}(i) = J_{\mu^k}(i)$ for all $i$, in which case the algorithm terminates with the policy $\mu^k$.

Convergence of exact PI: $J_{\mu^{k+1}}(i) \leq J_{\mu^k}(i), \qquad \text{for all } i \text{ and } k,$

# Resources

- Chapter 1,2,4 of the Berteskas' textbook (more theory driven)
  - https://github.com/cuhkrlcourse/bertsekas
- Reinforcement learning theory monograph:
  - By Alekh Agarwal, Nan Jiang, Sham Kakade: https://rltheorybook.github.io/rl_monograph_AJK.pdf
- Ben Recht (Berkeley Professor in control theory): An outsider's tour of reinforcement learning: http://www.argmin.net/2018/06/25/outsider-rl/