Three-Body Project Progress Log

Branch Peng

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Project Initialization

We began the final project for PHYS5300 by investigating the classical three-body problem in two spatial dimensions, governed by Newtonian gravity. Each body was assumed to have the same mass $m = 1$, and the gravitational constant was set to $G = 1$ for simplicity.
A dedicated Python script, `three_body.py`, was created and placed in the `src/` directory. A virtual environment was initialized to manage dependencies, and the following key packages were installed:

- `numpy`
- `scipy`
- `matplotlib`
- `ipywidgets`
- `manim`

The project folder structure was organized as follows:

- `three_body_project/`
  - `src/` – Python scripts
  - `notebooks/` – Jupyter notebooks
  - `figures/` – Plots and PNGs
  - `animations/` – Manim animations and MP4s
  - `requirements.txt` – Dependencies
  - `README.md` – Project overview

Simulation: Verified Figure-Eight Orbit

We implemented a symplectic leapfrog (velocity Verlet) integrator to simulate the motion of three equal-mass bodies using canonical initial conditions corresponding to the periodic figure-eight orbit, originally identified numerically by Chenciner and Montgomery in 2000.

$$
\begin{aligned}
\vec{r}_1 &= (0.97000436,\ -0.24308753), & \vec{v}_1 &= (0.4662036850,\ 0.4323657300) \\
\vec{r}_2 &= (-0.97000436,\ 0.24308753), & \vec{v}_2 &= (0.4662036850,\ 0.4323657300) \\
\vec{r}_3 &= (0,\ 0), & \vec{v}_3 &= (-0.93240737,\ -0.86473146)
\end{aligned}
$$

These conditions were sourced from numerically verified implementations found in literature and simulation repositories, and are known to produce a symmetric, periodic orbit in which the three bodies follow the same path but are phase-shifted by one third of the period. The setup conserves both momentum and angular momentum. The simulation was run using a time step $\Delta t = 0.001$ over 20,000 steps to ensure smooth and accurate reproduction of the figure-eight trajectory.

Trajectory Visualization

We created a Python script, `plot_trajectories.py`, to visualize the 2D trajectories of the three masses. The plot below confirms the expected behavior: the bodies trace out a symmetric figure-eight path, remaining gravitationally bound with no drift or divergence.

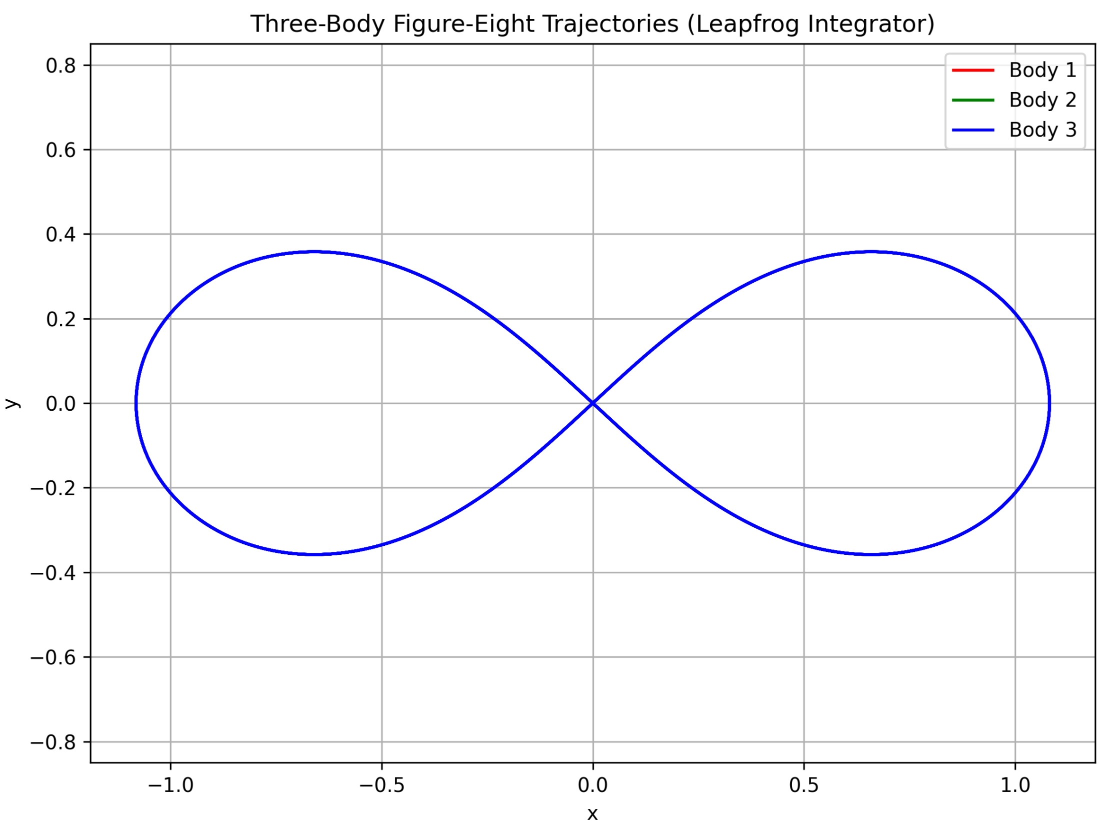

Figure 1: Trajectories of the three bodies using the leapfrog symplectic integrator.

Conservation Diagnostics

To evaluate the stability and physical fidelity of the numerical integrator, we tracked the total energy and angular momentum throughout the simulation.
Figure 2 shows the evolution of these quantities over time. The total energy remains stable and oscillates around a constant value of $E \approx -1.287142$ with only very small fluctuations on the order of 10^{-7}, a typical behavior for symplectic integrators which trade perfect energy conservation for long-term bounded drift. The total angular momentum remains effectively zero, with variations on the order of 10^{-14}, indicating excellent conservation. This confirms that the leapfrog integrator preserves the key physical symmetries of the system and is highly suited for long-term integration of gravitational orbits.

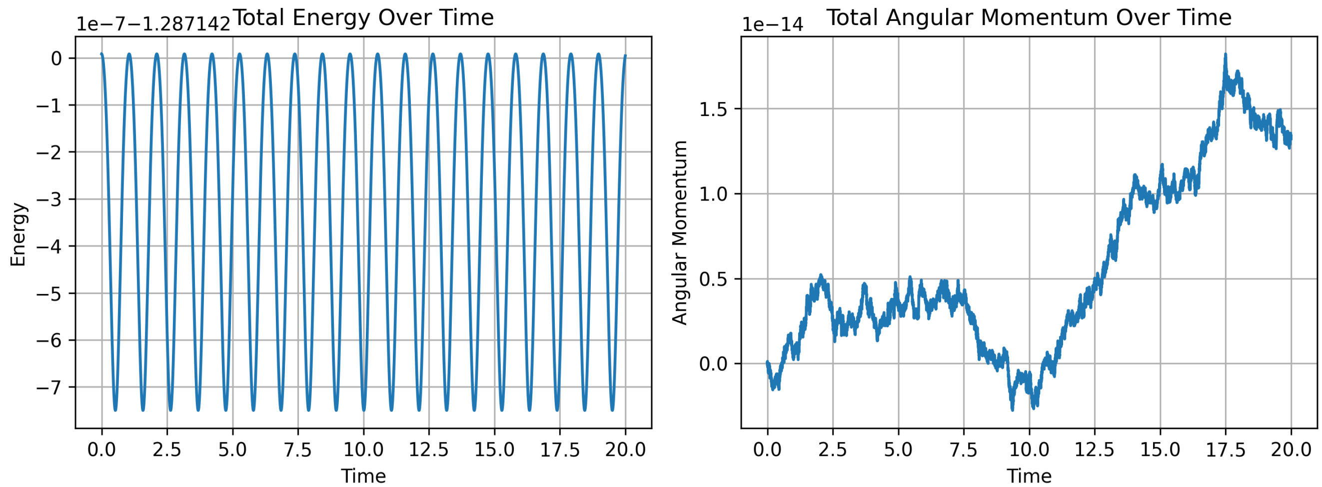

Figure 2: Total energy (left) and total angular momentum (right) as functions of time during the leapfrog simulation.

Integrator Comparison: Leapfrog vs RKF45

To evaluate the effectiveness of the leapfrog integrator, we compared its performance with that of a high-accuracy, non-symplectic Runge–Kutta–Fehlberg 4(5) method (RKF45), implemented via `scipy`'s `solve_ivp`.

Both integrators were initialized with identical conditions and run for the same total simulation time. Figure 3 demonstrates the difference in long-term behavior between the two methods.

The symplectic leapfrog integrator exhibits excellent conservation of energy and angular momentum, with bounded oscillations that do not grow over time. In contrast, RKF45 shows significant drift in both quantities, despite its higher local precision. This behavior confirms that symplectic integrators are superior for long-term evolution of Hamiltonian systems like the three-body problem.

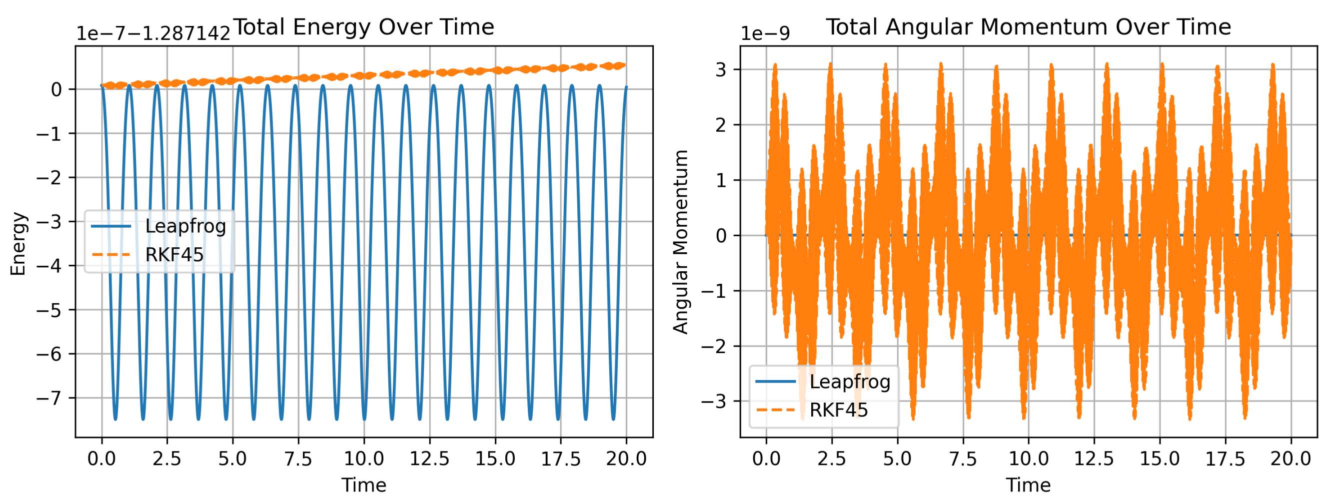

Figure 3: Comparison of total energy (left) and angular momentum (right) over time for Leapfrog (symplectic) and RKF45 (non-symplectic) integrators.

Trajectory Comparison

In addition to monitoring conserved quantities, we directly compared the orbital trajectories generated by both integrators. Figure 4 shows the overlaid figure-eight paths traced by the three bodies using both Leapfrog and RKF45.

As expected, both methods initially agree very closely, reproducing the iconic symmetric orbit. However, as shown previously, the RKF45 solution begins to drift in total energy and angular momentum, meaning that its long-term accuracy would degrade — even if the orbit initially appears visually correct.

This further emphasizes the importance of using structure-preserving integrators like Leapfrog for gravitational simulations over long timescales.

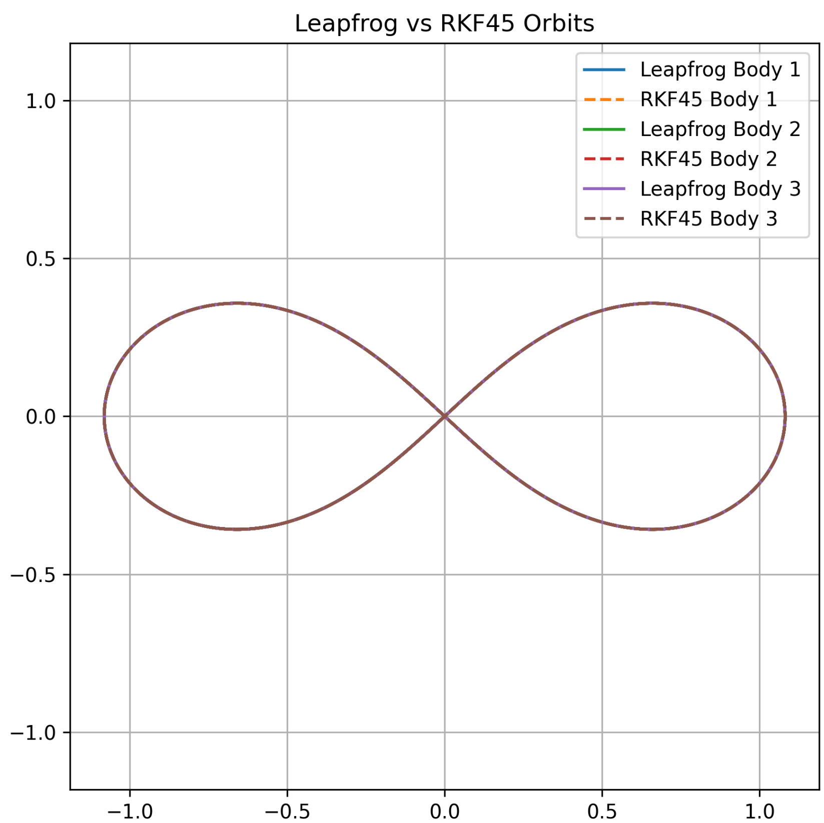

Figure 4: Overlaid figure-eight trajectories for all three bodies using Leapfrog (solid) and RKF45 (dashed). The paths remain nearly identical over a single orbit period.

Orbit Animation

To further illustrate the dynamics of the three-body system, we created an animation of the figure-eight trajectory using `matplotlib.animation`. Each body leaves behind a fading trail as it moves along the orbit, allowing viewers to track the motion of all three masses in time.

The animation demonstrates the remarkable periodicity and symmetry of the figure-eight solution, highlighting how each body follows the same path but offset in phase by one third of the orbital period.

A sample frame is shown below, and the full animation is available in the `animations/` folder as an MP4 file.

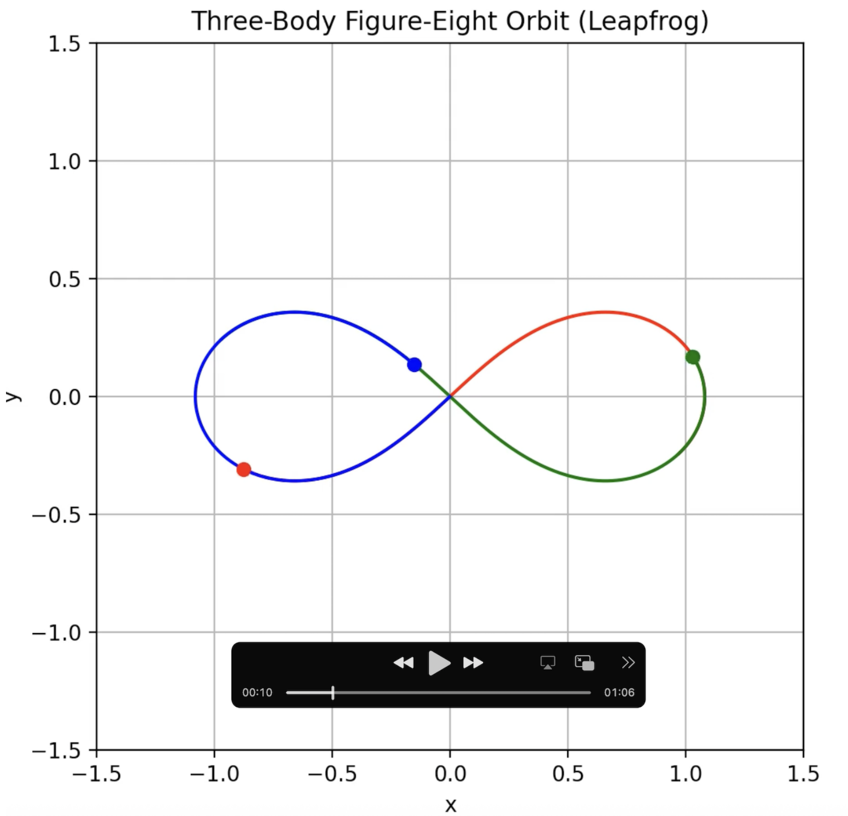

Figure 5: Sample frame from the animated three-body figure-eight orbit.

Project Summary and Reflection

This project explored the classic three-body problem in Newtonian gravity, focusing on the special case of the periodic, symmetric figure-eight orbit. The objective was to simulate this system numerically, analyze its conservation properties, and compare the behavior of different numerical integrators.

We began by setting up the figure-eight initial conditions, which are known to produce a stable, choreographed orbit in which all three bodies trace the same path offset by a phase shift. These conditions are particularly sensitive to numerical precision and conservation, making them a compelling test case for long-term simulation.

We implemented the `leapfrog` (velocity Verlet) integrator, a symplectic method known for preserving energy and angular momentum over long timescales. Using a step size of $\Delta t = 0.001$ and 20,000 steps, we verified that the leapfrog integrator accurately reproduced the expected figure-eight orbit and maintained nearly constant energy and angular momentum throughout the simulation. This was demonstrated both through trajectory visualization and diagnostic plots.

To evaluate the reliability of the leapfrog method, we compared it to the Runge–Kutta–Fehlberg 4(5) (RKF45) method using `scipy`'s `solve_ivp`. While both integrators produced visually identical orbits over one cycle, RKF45 exhibited significant drift in both energy and angular momentum, illustrating the advantage of symplectic methods for Hamiltonian systems. Side-by-side diagnostic plots and overlaid trajectories reinforced this finding.

To enhance visualization, we generated a smooth animation of the orbit using `matplotlib.animation`, and exported a representative frame to include in the report. This provided a more intuitive understanding of the system's symmetry and periodic motion.

Overall, the project synthesized knowledge from classical mechanics, numerical methods, and scientific computing. The final implementation not only reproduced a famous solution to the three-body prob-

lem, but also highlighted important principles in computational physics such as structure preservation, numerical drift, and long-term integration fidelity.
Through this project, we demonstrated:

- Proficiency in implementing and evaluating different numerical integration schemes
- The practical importance of symplectic integrators for orbital dynamics
- Visualization and diagnostic techniques for dynamic systems
- Critical thinking in interpreting physical vs. numerical behavior

This project fulfilled trajectory visualization, conserved quantity analysis, integrator comparison, animation, and comprehensive documentation.

References

[1] Alain Chenciner and Richard Montgomery. “A Remarkable Periodic Solution of the Three-Body Problem in the Case of Equal Masses.” *Annals of Mathematics*, vol. 152, no. 3, 2000, pp. 881–901.