MAS550 Probability Theory Homework 1. 20200130 Yujun Kim

#1.1.1

We have to check several things.

① $\mathcal{F}$ is a σ-field.

i) By definition, $\mathcal{F}$ is closed under complement.

ii) Let $A_i \in \mathcal{F}$ $\forall i \in \mathbb{N}$. If A_i are countable $\forall i$, so does $\bigcup_i A_i$.
Otherwise, A_j^c should be countable for some j.
Thus, $(\bigcup_i A_i)^c \subset A_j^c$ is countable and thus $\bigcup A_i \in \mathcal{F}$.

iii) If closed under countable union and complement, it is closed under countable intersection as $\bigcap A_i = (\bigcup A_i^c)^c$.

② P is a probability measure.

i) $P(\Omega)=1$, as $\Omega^c = \phi$ is countable.

ii) $P(A) \geq 0$ $\forall A \in \mathcal{F}$ by def.

iii) To show for countable disjoint A_i, $P(\bigcup A_i) = \sum P(A_i)$, we divide into two cases.

If A_i are countable $\forall i$, then so does $\bigcup A_i$
$\Rightarrow P(\bigcup A_i) = \sum P(A_i) = 0$.

Otherwise, A_j^c is countable for some j. As A_i are disjoint, $\forall i \neq j$, A_i should be countable. Thus, $P(A_i)=0$ $\forall i \neq j$.
$(\bigcup A_i)^c \subset A_j^c$ is countable. Thus, $P(\bigcup A_i) = 1$, $\sum P(A_i) = P(A_j) = 1$.

#1.1.3

From topology, $\mathbb{R}^d$ with it's standard topology is second countable with the basis of the form $\mathcal{B} = \{\prod_{i=1}^{d}(a_i, b_i) : a_i, b_i \in \mathbb{Q}\}$.
Topology generated by a basis is collection of arbitary union of $\mathcal{B}$.
As $\mathcal{B}$ is countable all unions are countable union.
Hence, all open sets are in $\sigma(\mathcal{B})$. Thus, $\mathcal{R}^d \subseteq \sigma(\mathcal{B})$.
$\mathcal{B} \subseteq \mathcal{R}^d$. Thus, $\mathcal{R}^d = \sigma(\mathcal{B})$.

#1.2.4

As $F(x)$ is non-decreasing continuous, $\lim_{x\to\infty} F(x)=1$, $\lim_{x\to-\infty} F(x)=0$,
$F^{-1}(y) := \sup\{x : F(x)=y\}$ is well-def. $\forall y\in(0,1)$. (By IVT).
The supremum is attained by $x_y \in \{x : F(x)=y\}$ as the set is closed.
Then, $F(F^{-1}(y)) = F(x_y) = y$. $\forall y\in(0,1)$.
Now, $\mathbb{P}(Y\le y) = \mathbb{P}(F\circ X \le y) = \mathbb{P}(F\circ X \le F(F^{-1}(y)))$
$= \mathbb{P}(X \le F^{-1}(y))$ ($\because F$ is monotone (non-dec.))
$= F(F^{-1}(y)) = y$.

#1.2.5

$\mathbb{P}(g(X)\le y) = \mathbb{P}(X\le g^{-1}(y)) = F(g^{-1}(y)) = F(g^{-1}(y)) - F(\alpha)$. $(F(\alpha)=0)$.
From $\mathbb{P}(\alpha\le X\le\beta)=1$, we have $\mathbb{P}(X>\beta \text{ or } X<\alpha)=0$.
Thus, $f\equiv 0$ a.e. outside $[\alpha,\beta]$.

$$\mathbb{P}(g(X)\le y) = F(g^{-1}(y)) - F(g^{-1}(g(\alpha)))$$
$$= \int_{g^{-1}(g(\alpha))}^{g^{-1}(y)} f(t)\,dt$$
$$\xrightarrow{t=g^{-1}(u)} = \int_{g(\alpha)}^{y} \frac{f(g^{-1}(u))}{g'(g^{-1}(u))}\,du.$$
$$dt = \frac{1}{g'(g^{-1}(u))}du.$$

Hence, $G(y) := \mathbb{P}(g(X)\le y)$ has $G'(y) = \dfrac{f(g^{-1}(y))}{g'(g^{-1}(y))}$.

Thus, for $y > g(\alpha)$, $g(X)$ has density $\dfrac{f(g^{-1}(y))}{g'(g^{-1}(y))} = h(y)$

$f(t)=0$ $\forall t>\beta$. Thus, $h(y)=0$ $\forall y>g(\beta)$.

$\mathbb{P}(g(X)\le g(\alpha)) = \mathbb{P}(X\le\alpha)=0$. Thus, $h(y)\equiv 0$ a.e. for $y\le g(\alpha)$.

Thus, $$\boxed{h(y) = \begin{cases} \dfrac{f(g^{-1}(y))}{g'(g^{-1}(y))} & \text{if } y\in(g(\alpha),g(\beta)) \\ 0 & \text{otherwise.} \end{cases}}$$

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#1.2.7 (i) $F(X^2 \le x) = \mathbb{P}(X^2 \le x)$

$= \mathbb{P}(-\sqrt{x} \le X \le \sqrt{x})$ for $x \ge 0$.

$= \mathbb{P}(-\sqrt{x} < X \le \sqrt{x})$

$= F(\sqrt{x}) - F(-\sqrt{x})$.

$=: G(x)$.

$G'(x) = f(\sqrt{x})\frac{1}{2\sqrt{x}} + f(-\sqrt{x})\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}}(f(\sqrt{x})+f(-\sqrt{x}))$, for $x>0$.

$G(x)=0 \ \forall x \le 0$. Thus, $G'(x)=0 \ \forall x\le 0$.

Thus, density g of X^2 is $g(x) = \begin{cases} \frac{1}{2\sqrt{x}}(f(\sqrt{x})+f(-\sqrt{x})) & \text{for } x>0 \\ 0 & \text{otherwise} \end{cases}$.

(ii) If $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, then $g(x) = \begin{cases} \frac{1}{\sqrt{2\pi x}} e^{-\frac{1}{2}x} & \text{for } x>0 \\ 0 & \text{otherwise} \end{cases}$.

#1.3.4 (i) Let τ be the collection of all open sets in $\mathbb{R}$. Then, $\mathcal{R} = \sigma(\tau)$.

By definition of continuous fn, $f^{-1}(U)$ is open in $\mathbb{R}^d$ $\forall U \in \tau$.

Thus, $f^{-1}(U) \in \mathcal{R}^d$ $\forall U\in\tau$. By theorem 1.3.1, f is measurable.

#1.3.5 ① Claim: f is l.s.c. $\Leftrightarrow$ $\{f\le a\}$ closed $\forall a\in\mathbb{R}$.

($\Rightarrow$) We show complement of $\{f\le a\}$ is open.

If $x \notin \{f\le a\}$, then $f(x)>a$. Thus, $\liminf_{y\to x} f(y) \ge f(x) > a$.

$\liminf_{y\to x} f(y) = \lim_{\varepsilon\to 0} \inf_{|y-x|<\varepsilon} f(y) > a$. Thus, $\exists \varepsilon>0$ s.t.

$\inf_{|y-x|<\varepsilon} f(y) > a$. Then, $B_\varepsilon(x) \subset \{f>a\}$. Hence, $\{f>a\}$ is open

and so $\{f\le a\}$ is closed.

($\Leftarrow$) For any x, let $\varepsilon>0$ be given, and let $a=f(x)-\varepsilon<f(x)$.
As $x\in\{f>a\}$ and $\{f>a\}$ is open, $\exists\,\delta>0$ s.t.
$B_\delta(x)\subset\{f>a\}$. Then, $\inf_{|y-x|<\delta} f(y)\ge a$. $\Rightarrow \inf_{|y-x|<\delta'} f(y)\ge a \ \forall\, 0<\delta'\le\delta$

Thus, $\liminf_{y\to x} f(y)\ge a=f(x)-\varepsilon$. As $\varepsilon>0$ is arbitrary,
$\liminf_{y\to x} f(y)\ge f(x)$.

② Semicontinuous ftns are msb.
By ①, l.s.c ftns are msb.
If f is u.s.c, then $-f$ is l.s.c. $\Rightarrow -f$ is msb $\Rightarrow f$ is msb.

#1.4.1

Let $E_n=\{f>\frac{1}{n}\}$. Then, $f>\frac{1}{n}\mathbb{1}_{E_n}$ $\forall n$ so that
$0=\int f\,d\mu \ge \int \frac{1}{n}\mathbb{1}_{E_n}\,d\mu=\frac{1}{n}\mu(E_n)\ge 0$.

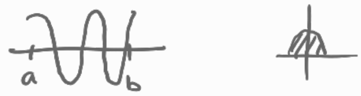
Thus, $\mu(E_n)=0$ $\forall n$. As $E_n\uparrow E:=\{f>0\}$, $\underline{\mu(E_n)\uparrow\mu(E)}$
↳ monotone conv. thm (MCT)

Hence, $\mu(E)=0$. i.e. $f=0$ a.e.

#1.4.4

① If g is a simple function, $\lim_{n\to\infty}\int g(x)\cos nx\,dx=0$.
By linearity of integration, it suffices to consider $g=\mathbb{1}_E$ for msb E.
Given $\varepsilon>0$, take open $U\supseteq E$ s.t. $\mu(U-E)<\varepsilon$. Then, $\mu(U)<\underline{\mu(E)+\varepsilon<\infty}$.
↳ as $\mathbb{1}_E$ is integrable

$$\begin{aligned}\left|\int \mathbb{1}_E\cos nx\,dx\right| &\le \left|\int(\mathbb{1}_U-\mathbb{1}_{U\setminus E})\cos nx\,dx\right|\\ &\le \left|\int \mathbb{1}_U\cos nx\,dx\right|+\left|\int \mathbb{1}_{U\setminus E}\cos nx\,dx\right|\\ &\le \left|\int \mathbb{1}_U\cos nx\,dx\right|+\int\left|\mathbb{1}_{U\setminus E}\cos nx\right|dx\\ &\le \left|\int \mathbb{1}_U\cos nx\,dx\right|+\mu(U\setminus E)\,dx. < \left|\int \mathbb{1}_U\cos nx\,dx\right|+\varepsilon \quad \cdots ①\end{aligned}$$



For $I=(a,b)$, $\left|\int_I \cos nx\,dx\right| \le \int_{-\frac{\pi}{2n}}^{\frac{\pi}{2n}} \cos nx\,dx$

$\le \int_{-\frac{\pi}{2n}}^{\frac{\pi}{2n}} 1\,dx = \frac{\pi}{n}$. $\cdots (*)$ } case for step ftn.

Let $U=\bigcup_{i=1}^{\infty} I_i$ be countable disjoint unions of open intervals.

$\exists N$ s.t. $U_N = \bigcup_{i=1}^{N} I_i$, has $\mu(U_N) > \mu(U) - \varepsilon$.

Then, $\left|\int_U \cos nx\,dx\right| = \left|\int_{U_N} \cos nx\,dx + \sum_{i=N+1}^{\infty} \int_{I_i} \cos nx\,dx\right|$

$\le \left|\int_{U_N} \cos nx\,dx\right| + \sum_{i=N+1}^{\infty} \mu(I_i)$

$< \left|\int_{U_N} \cos nx\,dx\right| + \varepsilon$.

From $(*)$, $\left|\int_{U_N} \cos nx\,dx\right| \le \sum_{i=1}^{N} \left|\int_{I_i} \cos nx\,dx\right| \le \sum_{i=1}^{N} \frac{\pi}{n} = \frac{N\pi}{n}$.

Thus, $\left|\int_U \cos nx\,dx\right| < \frac{N\pi}{n} + \varepsilon. \longrightarrow \varepsilon$ as $n\to\infty$. $\cdots$ ②

From ①,②, $\lim_{n\to\infty} \left|\int_E \cos nx\,dx\right| \le 2\varepsilon$.

As $\varepsilon>0$ is arbitrary, $\lim_{n\to\infty} \left|\int_E \cos nx\,dx\right| = 0$.

② Now we consider general integrable g.

For $\varepsilon>0$,

Take simple ftn φ s.t. $\int |g-\varphi|\,dx < \varepsilon$

$\left|\int \varphi \cos nx\,dx\right| \to 0$ as $n\to\infty$, by ①

Then, $\left|\int g\cos nx\,dx\right| \le \left|\int (g-\varphi)\cos nx\,dx\right| + \left|\int \varphi\cos nx\,dx\right|$

$\le \int |(g-\varphi)\cos nx|\,dx + \left|\int \varphi\cos nx\,dx\right|$

$\le \int |g-\varphi|\,dx + \left|\int \varphi\cos nx\,dx\right|$

$\le \varepsilon + \left|\int \varphi\cos nx\,dx\right| \longrightarrow \varepsilon$ as $n\to\infty$

i.e. $\lim_{n\to\infty} \left|\int g\cos nx\,dx\right| \le \varepsilon$. As $\varepsilon>0$ is arbitrary, done.

#1.5.2

$\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p}$, $\|f\|_\infty = \inf\{M : \mu(|f|>M)=0\}$.

① $\|f\|_p \le \|f\|_\infty \quad \forall p>1$.

$|f| \le \|f\|_\infty + \varepsilon$ a.e. $\forall \varepsilon>0$. Let $E_n = \{f < \|f\|_\infty + \frac{1}{n}\} \downarrow \{f \le \|f\|_\infty\}$, $\mu(E_n)<\infty$.

This shows $|f| \le \|f\|_\infty$ a.e. Hence, $\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p}$

$$\le \left(\int \|f\|_\infty^p d\mu\right)^{1/p}$$

$$= \left(\|f\|_\infty^p\right)^{1/p} = \|f\|_\infty.$$

② $\limsup \|f\|_p \ge \|f\|_\infty$

Let $\varepsilon>0$ be given. Then, $\exists M > \|f\|_\infty - \varepsilon$ s.t. $\mu(|f|>M)>0$.

Hence, $\int |f|^p d\mu \ge \int M^p \mathbb{1}_{\{|f|>M\}} d\mu$

$$= M^p \mu(|f|>M).$$

$\Rightarrow \|f\|_p \ge \left(M^p \mu(|f|>M)\right)^{1/p} = M \left(\mu(|f|>M)\right)^{1/p}$

Take limsup both side $\Rightarrow \limsup_{p\to\infty} \|f\|^p \ge M$ ($\because \mu(|f|>M)>0$).

Thus, $\|f\|_\infty - \varepsilon \le M \le \limsup_{p\to\infty} \|f\|^p$.

As $\varepsilon>0$ is arbitrary, done.

From ①, ②, $\|f\|_\infty = \limsup_{p\to\infty} \|f\|^p$.

#1.5.7

(i) $f\wedge n \ge 0$ and $f\wedge n \uparrow f$ as $n\to\infty$.

By MCT, $\int f\wedge n \, d\mu \uparrow \int f d\mu$ as $n\to\infty$

(ii) Given $\varepsilon>0$, $\exists n$ s.t. $\int |g|\wedge n \, d\mu > \int |g| d\mu - \frac{\varepsilon}{2}$ (by (i)).

Let $\delta = \frac{\varepsilon}{2n}$. Then for $\mu(A)<\delta$,

$$\int_A |g|\wedge n \, d\mu \le \int_A n \, d\mu = n\mu(A) < n\delta = \frac{\varepsilon}{2}.$$

Hence, $\int_A |g| - (|g|\wedge n) d\mu \le \int |g| - (|g|\wedge n) d\mu < \frac{\varepsilon}{2}$

$\Rightarrow \int_A |g| d\mu < \int_A |g|\wedge n \, d\mu + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

#1.5.10

First, we show $\int \Sigma |f_n| d\mu = \Sigma \int |f_n| d\mu$ to deduce $\int \Sigma |f_n| d\mu < \infty$.
Let $g_m = \sum_{n=1}^{m} |f_n|$, and $g = \Sigma |f_n|$.
Then, $g_m \uparrow g$ and g is integrable. By MCT, $\int g_m d\mu \uparrow \int g d\mu$.
Thus, $\int \Sigma |f_n| d\mu = \int g d\mu = \lim_{m\to\infty} \int g_m d\mu = \lim_{m\to\infty} \int \sum_{n=1}^{m} |f_n| d\mu$

$$= \lim_{m\to\infty} \sum_{n=1}^{m} \int |f_n| d\mu = \Sigma \int |f_n| d\mu.$$

For $h_m = \sum_{n=1}^{m} f_n$, and $h = \Sigma f_n$, we have $|h_m| \le g$ $\forall m$. and $h_m \to h$.
By DCT, $\int h_m d\mu \to \int h d\mu$ as $m \to \infty$. (From above, g is integrable)
i.e. $\int \Sigma f_n d\mu = \int h d\mu = \lim_{m\to\infty} \int h_m d\mu = \lim_{m\to\infty} \int \sum_{n=1}^{m} f_n d\mu = \lim_{m\to\infty} \sum_{n=1}^{m} \int f_n d\mu$

$$= \Sigma \int f_n d\mu.$$

#1.6.6

By C-S ineq, $\left(\int |Y \mathbb{1}_{(Y>0)}| d\mu\right)^2 = \left(\int Y \mathbb{1}_{Y>0} d\mu\right)^2$

$$\le \left(\int Y^2 d\mu\right)\left(\int \mathbb{1}_{Y>0}^2 d\mu\right)$$
$$= (\mathbb{E} Y^2)\left(\int \mathbb{1}_{Y>0} d\mu\right)$$
$$= (\mathbb{E} Y^2)(\mathbb{P}(Y>0)).$$

$\int Y \mathbb{1}_{Y>0} d\mu = \int Y \mathbb{1}_{Y>0} + Y \mathbb{1}_{Y=0} d\mu$

$$= \int Y \cdot 1 d\mu = \mathbb{E} Y.$$

Thus, $(\mathbb{E} Y)^2 \le (\mathbb{E} Y^2)\, \mathbb{P}(Y>0)$.

$\Rightarrow \mathbb{P}(Y>0) \ge (\mathbb{E} Y)^2 / (\mathbb{E} Y^2)$.

#1.6.11

If $0<j<k$, then $\varphi(x) = x^{k/j}$ is convex on $x \ge 0$.
First, $\mathbb{E}|X|^k = \mathbb{E}(\varphi(|X|^j)) < \infty$. If $\mathbb{E}|X|^j < \infty$, then by Jensen,
$\mathbb{E}|X|^k = \mathbb{E}(\varphi(|X|^j)) \ge \varphi(\mathbb{E}|X|^j) = (\mathbb{E}|X|^j)^{k/j}$
Thus, $(\mathbb{E}|X|^k)^{j/k} \ge \mathbb{E}|X|^j$.

Now it suffices to prove $\mathbb{E}|X|^j < \infty$.
As $|X|^j \le |X|^k$ for $\forall |X| \ge 1$,

$$\mathbb{E}|X|^j = \int |X|^j d\mu = \int |X|^j \mathbb{1}_{(|X|\ge 1)} + |X|^j \mathbb{1}_{(|X|<1)} d\mu$$

$$\le \int |X|^k \mathbb{1}_{(|X|\ge 1)} d\mu + \int \mathbb{1}_{(|X|<1)} d\mu$$

$$\le \int |X|^k d\mu + 1 = (\mathbb{E}|X|^k) + 1 < \infty.$$

Thus, $\mathbb{E}|X|^j < \infty$.

#1.7.2

Let $E = \{(x,y) : 0 \le y < g(x)\}$. λ : Lebesgue measure on $\mathbb{R}$

Then, $(\mu\times\lambda)(E) = \int_{X\times Y} \mathbb{1}_E \, d(\mu\times\lambda)$

$$= \int_X \int_Y \mathbb{1}_E \, d\lambda \, d\mu$$

$$= \int_X \lambda(E_x) \, d\mu$$

$$= \int_X g(x) \, d\mu \qquad (E_x = [0, g(x)))$$

Fubini holds as $\mathbb{1}_E \ge 0$.

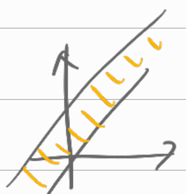
$$(\mu\times\lambda)(E) = \int_{X\times Y} \mathbb{1}_E \, d(\mu\times\lambda)$$

$$= \int_Y \int_X \mathbb{1}_E \, d\mu \, d\lambda$$

$$= \int_Y \mu(E_y) \, d\lambda, \quad (E_y = \{x : 0 \le y < g(x)\})$$

$$= \int_0^\infty \mu(E_y) \, d\lambda = \int_0^\infty \mu(\{x : g(x) > y\}) \, d\lambda$$

#1.7.4



$$F(x) = \mu(-\infty, x] = \int \mathbb{1}_{(-\infty, x]}(y) \, d\mu(y).$$

$$F(x+c) - F(x) = \int \mathbb{1}_{(-\infty, x+c]} - \mathbb{1}_{(-\infty, x]} \, d\mu = \int \mathbb{1}_{(x, x+c]} \, d\mu.$$

$$\int F(x+c) - F(x) \, dx = \iint \mathbb{1}_{(x, x+c]}(y) \, d\mu \, dx$$

$$= \iint \mathbb{1}_{(x < y \le x+c)} \, d(\mu\times\lambda)$$

$$= \int\int \mathbb{1}_{[y-c,y)}(x)\,dx\,d\mu$$
$$= \int y-(y-c)\,d\mu = \int c\,d\mu$$
$$= c\,\mu(\mathbb{R}).$$

Here again, Fubini holds as $\mathbb{1}_{(x<y\leq x+c)}(x,y)\geq 0$.