Homework 2

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IE539 Convex Optimization

October 2023

1 Closed Form Optimal Solution

(a) Let f be the objective function. Then, $\nabla^2 f = (1+2\lambda)I \succ 0$. By first order optimality condition,

$$\nabla f(x^*) = x^* - z + 2\lambda x^* = 0$$

Thus, $x^* = z/(1+2\lambda)$. The optimal value of the objective function is $f(x^*)$.

(b) Let f be the objective function. Then for $x_i \neq 0$,

$$\frac{\partial f}{\partial x_i} = x_i - z_i + \lambda \cdot \operatorname{sgn}(x_i)$$

where $\operatorname{sgn}(t)$ is the sign of t. We divide cases into three: $z_i > \lambda, z_i < -\lambda, -\lambda \leq z_i \leq \lambda$. For each case, we analyze by considering both $x_i > 0$ and $x_i < 0$. Note that f being not differentiable at $x_i = 0$ does not matter by the continuity of f.

- If $z_i > \lambda$. By considering both $x_i > 0, x_i < 0$, for $x_i \neq 0$

  $$\frac{\partial f}{\partial x_i} \begin{cases} > 0 \text{ if } x_i > z_i - \lambda \\ < 0 \text{ if } x_i < z_i - \lambda \end{cases}$$

  Hence, x_i minimize f when $x_i = z_i - \lambda$.

- If $z_i < -\lambda$. By considering both $x_i > 0, x_i < 0$, for $x_i \neq 0$

  $$\frac{\partial f}{\partial x_i} \begin{cases} > 0 \text{ if } x_i > z_i + \lambda \\ < 0 \text{ if } x_i < z_i + \lambda \end{cases}$$

  Hence, x_i minimize f when $x_i = z_i + \lambda$

- If $-\lambda \leq z_i \leq \lambda$. By considering both $x_i > 0, x_i < 0$, for $x_i \neq 0$

  $$\frac{\partial f}{\partial x_i} \begin{cases} > 0 \text{ if } x_i > 0 \\ < 0 \text{ if } x_i < 0 \end{cases}$$

  Hence, x_i minimize f when $x_i = 0$.

Thus, we have closed for for x^*.

$$x_i^* = \begin{cases} z_i - \lambda & \text{if } z_i > \lambda \\ 0 & \text{if } -\lambda \leq z_i \leq \lambda \\ z_i + \lambda & \text{otherwise} \end{cases}$$
$$= max\{z_i - \lambda, 0\} + min\{z_i + \lambda, 0\}$$

The optimal value of the objective function is $f(x^*)$.

2 SOCP is CP

(Step 0) Basic Settings
Let (SOCP) be given as following.

$$\begin{aligned} &\text{minimize } f^T x \\ &\text{subject to } ||A_i x + b_i||_2 \leq c_i^T x + d_i \text{ for } i = 1, \cdots, m, \\ &\qquad\qquad Ex = g \end{aligned}$$

(CP) is the program of the following form

$$\begin{aligned} &\text{minimize } u^T z \\ &\text{subject to } Px + q \in K \end{aligned}$$

for some regular cone K. By a cone to be a regular means it is closed, pointed and has a nonempty interior. We will represent the equivalent problem of (SOCP) in the form of (CP). By introducing dummy rows, WLOG, we may assume A_i and b_i are of the same size respectively. Let $A_i \in \mathcal{M}_{l\times n}, b_i \in \mathbb{R}^l$ for all $i = 1, \cdots, m$.

(Step 1) We first deal with the constraint $Ex = g$. Let x_0 be any vector such that $Ex_0 = g$. Then, any other vector satisfying $Ex = g$ is in form $x = x_0 + h$ where $h \in Null(E)$, the null space of E. Let r be the rank of E. Let N be the matrix of size $n \times (n-r)$ with columns formed by the basis of $Null(E)$. Then, any vector in $Null(E)$ is of the form Nz, where $z \in \mathbb{R}^{n-r}$. Hence, $Ex = g$ if and only if

$$x = x_0 + Nz, \text{ for some } z \in \mathbb{R}^{n-r}$$

(Step 2) Now we translate constraints $||A_i x + b_i||_2 \leq c_i^T x + d_i$ by temrs of z. By inserting $x = x_0 + Nz$, the constraint is equivalent to

$$||P_i z + q_i||_2 \leq c_i'^T z + d_i'$$

where $P_i = A_i N, q_i = A_i x_0 + b_i, c_i' = N^T c_i, d_i' = c_i^T x_0 + d_i$. We introduce new variable $s_i \in \mathbb{R}$ and translate above constraint into

$$||P_i z + q_i||_2 \leq s_i \text{ and } s_i \leq c_i'^T z + d_i'$$

(Step 4) Making the objective function into (CP) form.
$f^T x = f^T(x_0 + Nz) = f^T x_0 + (N^T f)^T z$. Minimizing this is equivalent to minimizing $u^T z$ where $u = N^T f$. Let $s = [s_1 \cdots s_m]^T$. Then, we can extend u to $\bar{u}$ with last m elements equal to 0 so that $u^T z = \bar{u}^T \begin{bmatrix} z \\ s \end{bmatrix}$

(Step 5) Making the constraint into (CP) form.
Define a cone $K_1 = \{(y,t) \in \mathbb{R}^l \times \mathbb{R} : ||y||_2 \leq t\}$. We prove it is a regular cone. It is closed for being a inverse image of closed set $[0, \infty)$ under a continuous map $\phi(y,t) = t - ||y||_2$. Clearly it has a nonempty interior $\{(y,t) \in \mathbb{R}^l \times \mathbb{R} : ||y||_2 < t\}$. It is pointed. If $(y,t), (-y,-t) \in K_1$, then $||y||_2 \leq t, -t$ so that $t = 0$ and $y = 0$. Thus K_1 is a regular cone.

Now, $||P_i z + q_i||_2 \leq s_i$ is equivalent to

$$\begin{bmatrix} P_i & 0 \\ 0 & e_i^T \end{bmatrix} \begin{bmatrix} z \\ s \end{bmatrix} + \begin{bmatrix} q_i \\ 0 \end{bmatrix} = \begin{bmatrix} P_i z + q_i \\ s_i \end{bmatrix} \in K_1$$

and $s_i \leq c_i'^T z + d_i'$ is equivalent to $c_i'^T z + d_i' - s_i = \begin{bmatrix} c_i'^T & -e_i^T \end{bmatrix} \begin{bmatrix} z \\ s \end{bmatrix} + d_i' \in \mathbb{R}_+$, where e_i is the i-th standard basis of $\mathbb{R}^m$. Let

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & e_1^T \\ \vdots & \vdots \\ P_m & 0 \\ 0 & e_m^T \\ c_1'^T & -e_1^T \\ \vdots & \vdots \\ c_m'^T & -e_m^T \end{bmatrix}, q = \begin{bmatrix} q_1 \\ 0 \\ \vdots \\ q_m \\ 0 \\ d_1 \\ \vdots \\ d_m \end{bmatrix}$$

which is obtained by vertically stacking all matrices we found above. Let $K = (K_1)^m \times (\mathbb{R}_+)^m$. Then, the overall constraint is equivalent to

$$P \begin{bmatrix} z \\ s \end{bmatrix} + q \in K$$

K is a product of regular cones $K_1, \mathbb{R}_+$. Using

- finite product of closed sets is closed
- finite product of open sets is open
- finite product of cone is a cone
- finite product of pointed cone is a pointed cone

K is regular cone and we proved the statement.

(Conclusion) We obtained equivalent formulation in (CP):

$$\text{minimize } \bar{u}^T \begin{bmatrix} z \\ s \end{bmatrix}$$
$$\text{subject to } Px + q \in K$$

If we obtain $\begin{bmatrix} z \\ s \end{bmatrix}$ minimizing (CP), $x = x_0 + Nz$ minimize (SOCP). Note that another way to deal with this problem is to prove (SOCP) is (SDP) and prove (SDP) is (CP).

3 Convergence of GD with Decaying Step Size

For this question, further assume f is convex and has global optimum x^*. First, we derive some useful inequalities as four **lemma**.

- $\sum_{t=1}^{T} \frac{1}{t} \leq log(T) + 1$
  $p(x) = \frac{1}{x}$ is decreasing. From a theorem from calculus,

  $$\sum_{t=2}^{T} \frac{1}{t} = \sum_{t=2}^{T} p(t) \leq \int_{x=1}^{T} p(x)dx = log(T)$$

  Add 1 on both side to obtain the inequality.

- $\sum_{t=1}^{T} \frac{1}{\sqrt{t}} \geq 2(\sqrt{T+1} - 1)$
  $q(x) = \frac{1}{\sqrt{x}}$ is decreasing. From a theorem from calculus,

  $$\sum_{t=1}^{T} \frac{1}{\sqrt{t}} = \sum_{t=1}^{T} q(t) \geq \int_{x=1}^{T+1} q(x)dx = 2(\sqrt{T+1} - 1)$$

- $\sum_{t=1}^{T} \frac{1}{t^2} \leq 2$
  $r(x) = \frac{1}{x^2}$ is decreasing. From a theorem from calculus,

  $$\sum_{t=2}^{T} \frac{1}{t^2} = \sum_{t=2}^{T} r(t) \leq \int_{t=1}^{T} \frac{1}{x^2}dx = 1 - \frac{1}{T} \leq 1$$

  Add 1 on both side to obtain the inequality

- $\sum_{t=1}^{T} \frac{1}{t} \geq log(T+1)$
  $s(x) = \frac{1}{x}$ is decreasing. From a theorem from calculus,

  $$\sum_{t=1}^{T} \frac{1}{t} = \sum_{t=1}^{T} s(t) \geq \int_{x=1}^{T+1} s(x)dx = log(T+1)$$

We next derive a common iteration-wise recursion for both (a), (b).

$$\begin{aligned}
||x_{t+1} - x^*||^2 &= ||x_t - \eta_t \nabla f(x_t) - x^*||^2 \\
&= ||x_t - x^*||^2 - 2\eta_t \nabla f(x_t)^T (x_t - x^*) + \eta_t^2 ||\nabla f(x_t)||^2 \\
&\leq ||x_t - x^*||^2 - 2\eta_t (f(x_t) - f(x^*)) + \eta_t^2 L^2
\end{aligned}$$

where in the inequality, we used f is convex and L-Lipschitz. Then,

$$2\eta_t (f(x_t) - f(x^*)) \leq ||x_t - x^*||^2 - ||x_{t+1} - x^*||^2 + \eta_t^2 L^2$$

Sum the equation for $t = 1, \cdots, T$.

$$2\sum_{t=1}^{T} \eta_t (f(x_t) - f(x^*)) \leq ||x_1 - x^*||^2 - ||x_{T+1} - x^*||^2 + L^2 \sum_{t=1}^{T} \eta_t^2 \tag{1}$$

$$\leq ||x_1 - x^*||^2 + L^2 \sum_{t=1}^{T} \eta_t^2 \tag{2}$$

(a) By choosing $\eta_t = \frac{1}{\sqrt{t}}$ in (2) and using the first lemma,

$$2\sum_{t=1}^{T} \eta_t (f(x_t) - f(x^*)) \leq ||x_1 - x^*||^2 + L^2(log(T) + 1)$$

Divide each side by $2\sum_{t=1}^{T} \eta_t$ and using the second lemma,

$$\begin{aligned}
(\sum_{t=1}^{T} \eta_t)^{-1} \left(\sum_{t=1}^{T} \eta_t (f(x_t) - f(x^*)) \right) &\leq (2\sum_{t=1}^{T} \eta_t)^{-1} \left(||x_1 - x^*||^2 + L^2(log(T) + 1) \right) \\
&\leq \frac{||x_1 - x^*||^2 + L^2(log(T) + 1)}{4(\sqrt{T+1} - 1)} \\
&= \mathcal{O}\left(\frac{log(T)}{\sqrt{T}} \right)
\end{aligned}$$

Let $\hat{x}_T = (\sum_{t=1}^{T} \eta_t)^{-1} (\sum_{t=1}^{T} \eta_t x_t)$ which is a convex combination of $x_1, \cdots, x_T$. By convexity of f,

$$f(\hat{x}_T) \leq (\sum_{t=1}^{T} \eta_t)^{-1} \sum_{t=1}^{T} \eta_t f(x_t)$$

Put this in above inequality and obtain

$$f(\hat{x}_T) - f(x^*) = \mathcal{O}\left(\frac{log(T)}{\sqrt{T}} \right)$$

(b) By choosing $\eta_t = \frac{1}{t}$ in (2) and using the third lemma,

$$2\sum_{t=1}^{T} \eta_t (f(x_t) - f(x^*)) \leq ||x_1 - x^*||^2 + 2L^2$$

Divide each side by $2\sum_{t=1}^T \eta_t$ and using the fourth lemma,

$$\begin{aligned}
(\sum_{t=1}^T \eta_t)^{-1}\left(\sum_{t=1}^T \eta_t(f(x_t)-f(x^*))\right) &\leq (2\sum_{t=1}^T \eta_t)^{-1}\left(||x_1-x^*||^2+2L^2\right)\\
&\leq \frac{||x_1-x^*||^2+2L^2}{2log(T+1)}\\
&= \mathcal{O}\left(\frac{1}{log(T)}\right)
\end{aligned}$$

Let $\hat{x}_T = (\sum_{t=1}^T \eta_t)^{-1}(\sum_{t=1}^T \eta_t x_t)$ which is a convex combination of $x_1, \cdots, x_T$. By convexity of f,

$$f(\hat{x}_T) \leq (\sum_{t=1}^T \eta_t)^{-1}\sum_{t=1}^T \eta_t f(x_t)$$

Put this in above inequality and obtain

$$f(\hat{x}_T) - f(x^*) = \mathcal{O}\left(\frac{1}{log(T)}\right)$$

4 Convergence of Projected GD

For this question, further assume f is convex and has a constrained optimum x^* over C. Also, let C be convex. Let $||\cdot||$ denote the l_2-norm.

(a) By definition, $x_{t+1} = Proj_C(x_t - \eta_t \nabla f(x_t)), x^* = Proj_C(x^*)$. From the lecture, projection is a contraction mapping.

$$||x_{t+1}-x^*||^2 = ||Proj_C(x_t-\eta_t\nabla f(x_t)) - Proj_C(x^*)||^2 \leq ||x_t-\eta_t\nabla f(x_t)-x^*||^2$$

(b) Let $\eta_t = \eta = \frac{||x_1-x^*||}{L\sqrt{T}}$. By (a), convexity of f and Lipschitzness of f,

$$\begin{aligned}
||x_{t+1}-x^*||^2 &\leq ||x_t-\eta\nabla f(x_t)-x^*||^2\\
&= ||x_t-x^*||^2 - 2\eta\nabla f(x_t)^T(x_t-x^*) + \eta^2||\nabla f(x_t)||^2\\
&\leq ||x_t-x^*||^2 - 2\eta(f(x_t)-f(x^*)) + \eta^2 L^2
\end{aligned}$$

Thus, $2\eta(f(x_t)-f(x^*)) \leq ||x_t-x^*||^2 - ||x_{t+1}-x^*||^2 + \eta^2L^2$. Sum this equation over $t = 1, \cdots, T$ and divide each side by $2\eta T$ to obtain

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^T(f(x_t)-f(x^*)) &\leq \frac{1}{2\eta T}(||x_1-x^*||^2 - ||x_{T+1}-x^*||^2 + T\eta^2L^2)\\
&\leq \frac{||x_1-x^*||^2}{2\eta T} + \eta L^2/2 = \frac{L||x_1-x^*||}{\sqrt{T}}
\end{aligned}$$

where the last equality is due to the definition of η. Let $\bar{x}_T = \frac{1}{T}\sum_{t=1}^T x_t$, which is a convex combination of $x_1, \cdots, x_T$. By convexity of f, $f(\bar{x}_T) \leq \frac{1}{T}\sum_{t=1}^T f(x_t)$.

Put this in above inequality and obtain

$$f(\bar{x}_T) - f(x^*) \le \frac{1}{T}\sum_{t=1}^{T}(f(x_t) - f(x^*)) \le \frac{L||x_1 - x^*||}{\sqrt{T}}$$

(c) By (a), convexity of f and Lipschitzness of f,

$$\begin{aligned} ||x_{t+1} - x^*||^2 &\le ||x_t - \eta_t \nabla f(x_t) - x^*||^2 \\ &= ||x_t - x^*||^2 - 2\eta_t \nabla f(x_t)^T(x_t - x^*) + \eta_t^2 ||\nabla f(x_t)||^2 \\ &\le ||x_t - x^*||^2 - 2\eta_t(f(x_t) - f(x^*)) + \eta_t^2 L^2 \end{aligned}$$

By rearranging the terms,

$$f(x_t) - f(x^*) \le \frac{1}{2\eta_t}(||x_t - x^*||^2 - ||x_{t+1} - x^*||^2) + \frac{\eta_t L^2}{2}$$

Sum this inequality over $t = 1, \cdots, T$ and divide both side by T to obtain

$$\frac{1}{T}\sum_{t=1}^{T}(f(x_t) - f(x^*)) \le \frac{1}{T}\sum_{t=1}^{T}\frac{1}{2\eta_t}(||x_t - x^*||^2 - ||x_{t+1} - x^*||^2) + \frac{1}{T}\sum_{t=1}^{T}\frac{\eta_t L^2}{2} \tag{3}$$

To bound the second term on the RHS of (3), note that

$$\sum_{t=1}^{T}\eta_t = \sum_{t=1}^{T}\frac{1}{\sqrt{t}} \le \int_{u=0}^{T}\frac{1}{\sqrt{u}}du = 2\sqrt{T}$$

Hence, $\frac{1}{T}\sum_{t=1}^{T}\frac{\eta_t L^2}{2} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$. To bound the first term on the RHS of (3),

$$\begin{aligned} \sum_{t=1}^{T}\frac{1}{\eta_t}(||x_t - x^*||^2 - ||x_{t+1} - x^*||^2) &= \sum_{t=1}^{T}\frac{1}{\eta_t}||x_t - x^*||^2 - \sum_{t=1}^{T}\frac{1}{\eta_{t+1}}||x_{t+1} - x^*||^2 \\ &\quad + \sum_{t=1}^{T}(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t})||x_{t+1} - x^*||^2 \\ &\le ||x_1 - x^*||^2 + \sum_{t=1}^{T}(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t})||x_{t+1} - x^*||^2 \end{aligned}$$

As $\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} = \sqrt{t+1} - \sqrt{t} = \frac{1}{\sqrt{t+1}+\sqrt{t}} \le \frac{1}{\sqrt{t}}$ and $||x_{t+1} - x^*||^2 \le R^2$ for all t, we have

$$\sum_{t=1}^{T}\frac{1}{\eta_t}(||x_t - x^*||^2 - ||x_{t+1} - x^*||^2) \le ||x_1 - x^*||^2 + \sum_{t=1}^{T}\frac{R^2}{\sqrt{t}} \le ||x_1 - x^*||^2 + 2R^2\sqrt{T} = \mathcal{O}\left(\sqrt{T}\right)$$

Hence, $\frac{1}{T}\sum_{t=1}^{T}\frac{1}{2\eta_t}(||x_t - x^*||^2 - ||x_{t+1} - x^*||^2) = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$. Now we have upper bound for RHS of (3) so that

$$\frac{1}{T}\sum_{t=1}^{T}(f(x_t) - f(x^*)) = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$$

Let $\bar{x}_T = \frac{1}{T}\sum_{t=1}^{T} x_t$, which is a convex combination of $x_1, \cdots, x_T$. By convexity of f, $f(\bar{x}_T) \le \frac{1}{T}\sum_{t=1}^{T} f(x_t)$. Put this in above inequality and obtain

$$f(\bar{x}_T) - f(x^*) \le \frac{1}{T}\sum_{t=1}^{T}(f(x_t) - f(x^*)) = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$$