#1 $f_X(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}$

$$\mu_X = \int_0^\infty \frac{x}{\beta} e^{-\frac{x}{\beta}} dx = \left[-x e^{-\frac{x}{\beta}} - \beta e^{-\frac{x}{\beta}} \right]_0^\infty = \beta.$$

$$\sigma_X^2 = \int_0^\infty (x-\beta)^2 \frac{1}{\beta} e^{-\frac{x}{\beta}} dx \qquad \frac{x^2}{\beta} - 2x + \beta.$$

$$= -x^2 e^{-\frac{x}{\beta}} \Big] + \int \beta e^{-\frac{x}{\beta}} dx$$

$$= \left[-x^2 e^{-\frac{x}{\beta}} - \beta^2 e^{-\frac{x}{\beta}} \right]_0^\infty = \beta^2. \Rightarrow \sigma_X = \beta.$$

Let $\mathbb{P}(|X-\mu_X| \geq k\sigma_X) = \mathbb{P}(|X-\beta| \geq k\beta) = \alpha$. As $k > 1$,

$\alpha = \mathbb{P}(X > (k+1)\beta)$.

$$= \int_{(k+1)\beta}^\infty \frac{1}{\beta} e^{-\frac{x}{\beta}} dx = \left[-e^{-\frac{x}{\beta}} \right]_{(k+1)\beta}^\infty = e^{-(k+1)}$$

By Chebyshev, $\mathbb{P}(|X-\mu_X| \geq k\sigma_X) \leq \frac{1}{k^2}$.

For large k, actual probability decay exponentially while bound by Chebyshev is inverse quadratic. Thus, bound not tight.

#2 $\mu_{\bar{X}_n} = p$, $\sigma^2_{\bar{X}_n} = \frac{1}{n} p(1-p)$.

$$\mathbb{P}(|\bar{X}_n - p| > \varepsilon) = \frac{p(1-p)}{n\varepsilon^2} \quad \text{(Chebyshev)} \quad \cdots \text{①}$$

$0 \le X_i \le 1$.

$$\mathbb{P}(|\bar{X}_n - p| > \varepsilon) = \mathbb{P}(\bar{X}_n - p > \varepsilon) + \mathbb{P}(\bar{X}_n - p < -\varepsilon)$$

Let $Y_i = X_i - p$. Then, $-p \le Y_i \le 1-p$, $\mathbb{E}(Y_i) = 0$. By Hoeffding,

$$\mathbb{P}(\bar{X}_n - p > \varepsilon) = \mathbb{P}(\sum Y_i \ge n\varepsilon) \le e^{-n\varepsilon t} \prod_{i=1}^{n} e^{t^2/8}$$

$$= e^{-n\varepsilon t + nt^2/8} \quad \forall t.$$

$$t = 4\varepsilon \Rightarrow \mathbb{P}(\bar{X}_n - p > \varepsilon) \le e^{-2n\varepsilon^2} \quad \cdots \text{②}$$

$$\mathbb{P}(\bar{X}_n - p < -\varepsilon) = \mathbb{P}(\sum -Y_i > n\varepsilon) \le e^{-n\varepsilon t} \prod_{i=1}^{n} e^{t^2/8}$$

$$= e^{-n\varepsilon t + nt^2/8}.$$

$$t = 4\varepsilon \Rightarrow \mathbb{P}(\bar{X}_n - p < -\varepsilon) \le e^{-2n\varepsilon^2} \quad \cdots \text{③}$$

②+③ $\Rightarrow \mathbb{P}(|\bar{X}_n - p| > \varepsilon) \le 2e^{-2n\varepsilon^2}$. (By Hoeffding)

$\lim_{n\to\infty} \dfrac{2e^{-2n\varepsilon^2}}{\frac{p(1-p)}{n\varepsilon^2}} = 0$. Thus, when n is large,

bound from Hoeffding is smaller than the bound from Chebyshev.

#3 ① Let $\mathbb{P}(X=0)=1$. We prove $X_n \xrightarrow{P} X$.

$\mathbb{P}(|X-X_n| = \frac{1}{n}) = 1-\frac{1}{n^2}$, $\mathbb{P}(|X-X_n| = n) = \frac{1}{n^2}$.

Let $\varepsilon>0$ be given. Take N with $\frac{1}{N}<\varepsilon$

Then $\mathbb{P}(|X-X_n|>\varepsilon) \le 1-\mathbb{P}(|X-X_n|=\frac{1}{n}) = 1-(1-\frac{1}{n^2}) = \frac{1}{n^2}$ $\forall n \ge N$,

i.e. $\mathbb{P}(|X-X_n|>\varepsilon) \to 0$ as $n\to\infty$. i.e. $X_n \xrightarrow{P} X$.

② Let X be as above. We prove X_n does not converge in qm.

If $X_n \xrightarrow{qm} Y$ for some Y, then $X_n \xrightarrow{P} Y$. Thus, $Y=X$. i.e. $X_n \xrightarrow{qm} X$ (X from ①)

However, $\mathbb{E}(X_n-X)^2 = \frac{1}{n^2}\times(1-\frac{1}{n^2}) + n^2 \times \frac{1}{n^2} \to 1$ as $n\to\infty$.

Thus, X_n does not converges to X in qm. Hence, X_n does not converge in qm to any distribution.

#4 Let Y be a distribution $\mathbb{P}(Y=\mu_1/\mu_2)=1$.

$Y_n \rightsquigarrow Y$. i.e. $Y_n \rightsquigarrow \mu_1/\mu_2$. (If $\mu_2 \neq 0$).

$\because$) WLLN : $\bar{X}_1 \xrightarrow{P} \mu_1$, $\bar{X}_2 \xrightarrow{P} \mu_2$. Suppose $\mu_2 \neq 0$ and $0<\delta<|\mu_2|$.

If $|\bar{X}_1-\mu_1|<\delta$, $|\bar{X}_2-\mu_2|<\delta$, then $\frac{\mu_1-\delta}{\mu_2+\delta} < Y_n < \frac{\mu_1+\delta}{\mu_2-\delta}$

Let $\varepsilon=\varepsilon(\delta)=\max\{\frac{\mu_1+\delta}{\mu_2-\delta}-\frac{\mu_1}{\mu_2}, \frac{\mu_1}{\mu_2}-\frac{\mu_1-\delta}{\mu_2+\delta}\}$. Then, $|\bar{X}_i-\mu_i|<\delta$, $i=1,2 \Rightarrow |Y_n-\mu_1/\mu_2|<\varepsilon$

$\Rightarrow \mathbb{P}(|Y_n-\mu_1/\mu_2|\le\varepsilon) \ge \mathbb{P}(|\bar{X}_1-\mu_1|<\delta)\,\mathbb{P}(|\bar{X}_2-\mu_2|<\delta)$ by above

So that $\mathbb{P}(|Y_n-\mu_1/\mu_2|>\varepsilon) \le 1-\mathbb{P}(|\bar{X}_1-\mu_1|<\delta)\mathbb{P}(|\bar{X}_2-\mu_2|<\delta)$.

Given $\varepsilon' > 0$, take $0 < \delta < |\mu_2|$ so that $\varepsilon = \varepsilon(\delta) < \varepsilon'$. ($\because \lim_{\delta \to 0} \varepsilon(\delta) = 0$)

Then, $\mathbb{P}(|Y_n - \mu_1/\mu_2| > \varepsilon') \leq \mathbb{P}(|Y_n - \mu_1/\mu_2| > \varepsilon) \leq 1 - \mathbb{P}(|\bar{X}_1 - \mu_1| < \delta)\, \mathbb{P}(|\bar{X}_2 - \mu_2| < \delta)$

As $\bar{X}_1 \xrightarrow{P} \mu_1$, $\bar{X}_2 \xrightarrow{P} \mu_2$, we have $\mathbb{P}(|\bar{X}_i - \mu_i| < \delta) = 1 - \mathbb{P}(|\bar{X}_i - \mu_i| > \delta)$

$\longrightarrow 1$ as $n \to \infty$.

Thus, $\mathbb{P}(|Y_n - \mu_1/\mu_2| > \varepsilon') \longrightarrow 0$ as $n \to \infty$. i.e. $Y_n \xrightarrow{P} \mu_1/\mu_2$

and so $Y_n \rightsquigarrow \mu_1/\mu_2$.

In case $\mu_2 = 0$, no need to consider !

#5 $P(\hat{\theta} \le t) = P(X_1, \cdots, X_n \le t) = \left(\frac{t}{\theta}\right)^n \Rightarrow f_{\hat{\theta}}(t) = \frac{n}{\theta^n} t^{n-1} \quad \forall t \in [0, \theta]$

① $\text{bias} = E_\theta(\hat{\theta}) - \theta$

$$\Rightarrow E_\theta(\hat{\theta}) = \int_0^\theta t \frac{n}{\theta^n} t^{n-1} dt = \frac{n}{n+1}\theta$$

$$\Rightarrow \text{bias} = \frac{n}{n+1}\theta - \theta = \frac{\theta}{n+1}$$

② $se = se(\hat{\theta}) = \sqrt{\mathbb{V}(\hat{\theta})}$, where $\mathbb{V}(\hat{\theta}) = E[\hat{\theta}^2] - E[\hat{\theta}]^2$

$$\mathbb{V}(\hat{\theta})^2 = \int_0^\theta \frac{n}{\theta^n} t^{n+1} dt - \left(\frac{n}{n+1}\theta\right)^2 = \frac{n}{n+2}\theta^2 \quad \left(\frac{n}{n+1}\right)^2 \theta^2 = \frac{n}{(n+2)(n+1)^2}\theta^2$$

$$\Rightarrow se = \sqrt{\frac{n}{(n+2)(n+1)^2}}\,\theta$$

MSE $= \text{bias}^2 + \mathbb{V}(\hat{\theta})$ (By theorem)

$$= \left(\frac{\theta}{n+1}\right)^2 + \frac{n}{(n+2)(n+1)^2}\theta^2 = \left(1 + \frac{n}{n+2}\right)\frac{\theta^2}{(n+1)^2}$$

$$= \frac{2(n+1)\theta^2}{(n+2)(n+1)^2} = \frac{2\theta^2}{(n+2)(n+1)}$$

#6. $\mathbb{E}_\theta(\hat{\theta}) = \mathbb{E}[2\bar{X}_n] = \mathbb{E}\left[\frac{2\sum X_i}{n}\right] = n\cdot\left(\frac{2}{n}\times\frac{\theta}{2}\right) = \theta.$

$\text{bias} = \mathbb{E}_\theta(\hat{\theta}) - \theta = \theta - \theta = 0.$

$\mathbb{V}(\hat{\theta}) = \mathbb{V}(2\bar{X}_n) = \mathbb{V}\left(\frac{2}{n}\sum X_i\right) = \frac{4}{n^2}\sum \mathbb{V}(X_i).$

$$\mathbb{V}(X_i) = \mathbb{E}\left(X_i - \frac{\theta}{2}\right)^2 = \int_0^\theta \left(t-\frac{\theta}{2}\right)^2 \frac{1}{\theta}\,dt$$

$$= \left[\frac{1}{3\theta}\left(t-\frac{\theta}{2}\right)^3\right]_0^\theta = \frac{\theta^2}{24} - \left(-\frac{\theta^2}{24}\right) = \frac{\theta^2}{12}.$$

$$\Rightarrow \mathbb{V}(\hat{\theta}) = \frac{4}{n^2}\times n \times \frac{\theta^2}{12} = \frac{\theta^2}{3n}$$

$$\Rightarrow se = \frac{\theta}{\sqrt{3n}}.$$

$$MSE = \text{bias}^2 + \mathbb{V}(\hat{\theta}) = \frac{\theta^2}{3n}.$$