MAS550 Probability Theory HW4, 20200130 Yujun Kim

#4.1.2

$a^2 \mathbb{1}_{|X|\geq a} \leq X^2 \Rightarrow \mathbb{1}_{|X|\geq a} \leq a^{-2}X^2$.

By thm 4.1.9, $\mathbb{P}(|X|\geq a \mid \mathcal{F}) = \mathbb{E}(\mathbb{1}_{|X|\geq a} \mid \mathcal{F})$

$\leq \mathbb{E}(a^{-2}X^2 \mid \mathcal{F})$ (by (b))

$= a^{-2}\mathbb{E}(X^2 \mid \mathcal{F})$. (by (a))

#4.1.3

$(X+\theta Y)^2 \geq 0$. Thus, for $\forall \theta$,

$0 = \mathbb{E}[0 \mid \mathcal{G}] \leq \mathbb{E}[(X+\theta Y)^2 \mid \mathcal{G}]$

$= \underbrace{\mathbb{E}(X^2 \mid \mathcal{G})}_{c} + 2\theta \underbrace{\mathbb{E}(XY \mid \mathcal{G})}_{b} + \theta^2 \underbrace{\mathbb{E}(Y^2 \mid \mathcal{G})}_{a}$ (By linearity)

$\Rightarrow b^2 - ac \leq 0$

Thus, $\mathbb{E}(XY \mid \mathcal{G})^2 \leq \mathbb{E}(X^2 \mid \mathcal{G})\mathbb{E}(Y^2 \mid \mathcal{G})$

#4.1.6

$\mathbb{E}(\{X - \mathbb{E}(X\mid\mathcal{G})\}^2) = \mathbb{E}(\{X - \mathbb{E}(X\mid\mathcal{F}) + \mathbb{E}(X\mid\mathcal{F}) - \mathbb{E}(X\mid\mathcal{G})\}^2)$

$= \mathbb{E}(\{X - \mathbb{E}(X\mid\mathcal{F})\}^2) + \mathbb{E}(\{\mathbb{E}(X\mid\mathcal{F}) - \mathbb{E}(X\mid\mathcal{G})\}^2)$

by linearity

$+ 2\underbrace{\mathbb{E}((X - \mathbb{E}(X\mid\mathcal{F}))(\mathbb{E}(X\mid\mathcal{F}) - \mathbb{E}(X\mid\mathcal{G})))}_{(*)}$

and so it suffices to show $(*) = 0$.

As $\mathcal{G} \subseteq \mathcal{F}$, $\mathbb{E}(X\mid\mathcal{G}) = \mathbb{E}(\mathbb{E}(X\mid\mathcal{G})\mid\mathcal{F}) = \mathbb{E}(\mathbb{E}(X\mid\mathcal{F})\mid\mathcal{G})$.

and $\mathbb{E}(X\mid\mathcal{G}), \mathbb{E}(X\mid\mathcal{F}) \in \mathcal{F}$.

Using $Z = \mathbb{E}(X\mid\mathcal{F}) - \mathbb{E}(X\mid\mathcal{G}) \in \mathcal{F}$ and thm 4.1.14,

$Z\mathbb{E}(X\mid\mathcal{F}) = \mathbb{E}(ZX\mid\mathcal{F})$

$\Rightarrow \mathbb{E}(Z\mathbb{E}(X\mid\mathcal{F})) = \mathbb{E}(\mathbb{E}(ZX\mid\mathcal{F})) = \mathbb{E}(ZX)$.

$\Rightarrow \mathbb{E}(Z(X - \mathbb{E}(X\mid\mathcal{F}))) = 0 \Leftrightarrow (*)$

Hence, we are done.

#4.1.7

Note that $\mathcal{G} = \{\phi, \Omega\}$ is independent to any R.V.
Thus, $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$.

From #4.1.6, $\mathbb{E}(\{X - \mathbb{E}(X|\mathcal{G})\}^2) = \mathbb{E}(\{X - \mathbb{E}(X)\}^2)$
$= Var(X)$. ... ①

$\mathbb{E}(\{X - \mathbb{E}(X|\mathcal{F}_1)\}^2) = \mathbb{E}(X^2) - 2\mathbb{E}(X\mathbb{E}(X|\mathcal{F}_1))$
$+ \mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)^2)$. ... ②

$\mathbb{E}(X\mathbb{E}(X|\mathcal{F}_1)) = \mathbb{E}(\mathbb{E}(X\mathbb{E}(X|\mathcal{F}_1))|\mathcal{F}_1)$
$= \mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)\mathbb{E}(X|\mathcal{F}_1))$
$= \mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)^2)$ ... ③

Using ③ to ①, $\mathbb{E}(\{X - \mathbb{E}(X|\mathcal{F}_1)\}^2) = \mathbb{E}(X^2) - \mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)^2)$
$= \mathbb{E}(\mathbb{E}(X^2|\mathcal{F}_1)) - \mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)^2)$
$= \mathbb{E}(\mathbb{E}(X^2|\mathcal{F}_1) - \mathbb{E}(X|\mathcal{F}_1)^2)$
$= \mathbb{E}(Var(X|\mathcal{F}_1))$. ... ④

$\mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)) = \mathbb{E}(X) = \mathbb{E}(X|\mathcal{G})$ so that
$Var(\mathbb{E}(X|\mathcal{F}_1)) = \mathbb{E}(\{\mathbb{E}(X|\mathcal{F}_1) - \mathbb{E}(X|\mathcal{G})\}^2)$. ... ⑤

①, ④, ⑤ on #4.1.6 gives desired equality:
$Var(X) = \mathbb{E}(Var(X|\mathcal{F}_1)) + Var(\mathbb{E}(X|\mathcal{F}_1))$.

#4.1.8

$Var(X) = \mathbb{E}(Var(X|N)) + var(\mathbb{E}(X|N))$.

i) $\mathbb{E}(X|N) = \mu N$

$\int_{N=n} X\,d\mathbb{P} = \int_{N=n} Y_1 + \cdots + Y_n \,d\mathbb{P}$

$= n\mu \int 1_{N=n} d\mathbb{P} = \mu \int_{N=n} N\,d\mathbb{P}$

$\{N=n\}$ generate $\sigma(N)$. Thus, $E(X|N) = \mu N$.

ii) $Var(X|N) = \sigma^2 N$.

$\int_{N=n} X^2 d\mathbb{P} = \int_{N=n} (Y_1 + \cdots + Y_n)^2 d\mathbb{P}$

$= \sum_i \int_{N=n} Y_i^2 d\mathbb{P} + \sum_{i \neq j} \int_{N=n} Y_i Y_j \,d\mathbb{P}$

$= \{n(\sigma^2 + \mu^2) + n(n-1)\mu^2\} \int 1_{N=n} d\mathbb{P}$

$= \int_{N=n} N\sigma^2 + N^2\mu^2 \,d\mathbb{P}$.

$\Rightarrow \mathbb{E}[X^2 - N^2\mu^2 | N] = \mathbb{E}[X^2|N] - \mathbb{E}[X|N]^2 = N\sigma^2$.

The formula follows from #4.1.7

#4.1.9

$\mathbb{E}(XY) = \mathbb{E}(\mathbb{E}(XY|\mathcal{G})) \overset{\text{as } X = \mathbb{E}(Y|\mathcal{G}) \in \mathcal{G}}{=} \mathbb{E}(X\,\mathbb{E}(Y|\mathcal{G})) = \mathbb{E}(X^2)$.

Thus, $\mathbb{E}((X-Y)^2) = \mathbb{E}X^2 + \mathbb{E}Y^2 - 2\mathbb{E}XY$

$= \mathbb{E}X^2 + \mathbb{E}Y^2 - 2\mathbb{E}X^2 = 0$.

#4.2.4

$\sup X_n < \infty$, $\xi_n = X_n - X_{n-1}$. $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$

Let $N = \inf\{n \mid X_n^+ \geq M\}$. $X_{N \wedge n}$ is subMG, $X_{N\wedge n}^+ \leq M + \sup \xi_n^+$.

Then, $E[X_{N\wedge n}^+] \leq M + E[\sup \xi_n^+] < \infty$.

Hence, $X_{N\wedge n}$ conv. a.s. $\{N=\infty\} \uparrow \Omega$ as $M \to \infty$.

Thus, $N=\infty$ a.s. i.e. X_n converges a.s. as well.

#4.2.5

$\xi_i = \begin{cases} -1 & \text{with probability } \frac{i^2}{i^2+1} \\ i^2 & \text{with probability } \frac{1}{i^2+1}. \end{cases}$ $\xi_1, \xi_2, \cdots$ indep.

Then, $\mathbb{E}\xi_i = 0$. Let $X_n = \xi_1 + \cdots + \xi_n$, $\{\mathcal{F}_n\}$: canonical filtration

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = \mathbb{E}(X_n + \xi_{n+1}|\mathcal{F}_n)$$
$$= X_n + \mathbb{E}(\xi_{n+1}|\mathcal{F}_n)$$
$$= X_n + \mathbb{E}(\xi_{n+1}) = X_n.$$

Hence, X_n is $\mathcal{F}_n$-MG.

$\sum P(\xi_i \neq -1) = \sum \frac{1}{i^2+1} < \infty.$

By 1st B-C, $P(\xi_i \neq -1 \text{ i.o.}) = 0$. i.e. $\xi_i = -1$ eventually a.s.

Hence, $X_n = -\infty$ a.s.

#4.2.6

(i) X_n is MG. $\exists X$ s.t. $X_n \to X$ a.s.

$\mathbb{P}(|X_{n+1} - X_n| > \varepsilon^2) \geq \mathbb{P}(X_n > \varepsilon)\mathbb{P}(|Y_{n+1} - 1| > \varepsilon) \to 0$ as $n \to \infty$.

As Y_n is iid, $\mathbb{P}(Y_n = 1) < 1$, $\mathbb{P}(|Y_{n+1} - 1| > \varepsilon) = \delta_\varepsilon > 0$.

Thus, $P(X_n > \varepsilon) \to 0$ as $n \to \infty$. i.e. $X_n \xrightarrow{P} 0$. Hence, $X_n \to 0$ a.s.

ineq. strict, as $\mathbb{P}(Y > 1) > 0$, $\mathbb{P}(Y < 1) > 0$.

(ii) $\mathbb{E}\log Y < \log \mathbb{E}Y = \log 1 = 0.$

By strong law of large nbrs,

$$\frac{1}{n}\log X_n = \frac{1}{n}\sum_{m=1}^{n}\log Y_m \to \mathbb{E}\log Y < 0.$$

#4.2.8

Let $Z_n = \prod_{i=1}^{n}(1+Y_n) \in \mathcal{F}_n \Rightarrow Z_n \geq 0$

Then, $\mathbb{E}\left(\frac{X_{n+1}}{Z_n}\Big|\mathcal{F}_n\right) = \frac{1}{Z_n}\mathbb{E}(X_{n+1}|\mathcal{F}_n)$

$$\leq \frac{1}{Z_n}(1+Y_n)X_n = \frac{X_n}{Z_{n-1}}.$$

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Thus, $\frac{X_{n+1}}{Z_n}$ is $\mathcal{F}_n$-super MG, $\frac{X_{n+1}}{Z_n} \geq 0$.

By thm 4.2.12, $\frac{X_{n+1}}{Z_n} \to W$ a.s.

$Z_n \leq e^{\sum_{k=1}^{n} Y_i} \leq e^{\sum Y_n} < \infty$ a.s.

Z_n is monotone increasing bounded above a.s. $\Rightarrow Z_n \to Z$ a.s.

Hence, $X_{n+1} = \frac{X_{n+1}}{Z_n} Z_n \to WZ$ a.s.

#4.3.3

Let $N = \inf\{k \mid \sum_{m=1}^{k} Y_m > M\}$, which is a stopping time.
Let $Z_n = X_n - \sum_{k=1}^{n-1} Y_k \in \mathcal{F}_n$.

$$\begin{aligned}\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[X_{n+1} - \sum_{k=1}^{n} Y_k \mid \mathcal{F}_n] \\ &= \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] - \sum_{k=1}^{n} Y_k \\ &\leq X_n + Y_n - \sum_{k=1}^{n} Y_k = X_n - \sum_{k=1}^{n-1} Y_k = Z_n\end{aligned}$$

Hence, Z_{n+1} is $\mathcal{F}_n$-super MG

By thm. $Z_{n\wedge N}$ is superMG.

$Z_{n\wedge N} + M = X_{n\wedge N} - \sum_{k=1}^{n\wedge N-1} Y_k + M \geq 0$ is nonnegative superMG.

By MG convergence thm, $Z_{n\wedge N} + M$ converges a.s.

$\Rightarrow Z_{n\wedge N}$ converges a.s.

$\{\sum Y_m \leq M\} \uparrow \Omega$ as $M \uparrow$. Thus, $\{N = \infty\}$ a.s. $\Rightarrow Z_{n\wedge N} = Z_n$ a.s.

Thus, $Z_n = X_n - \sum_{k=1}^{n-1} Y_k$ converges a.s. to Z_∞

$\Rightarrow X_n = Z_n + \sum_{k=1}^{n-1} Y_k \to Z_\infty + \sum Y_k$ a.s.

#4.3.12

$p_k = \mathbb{P}(\xi_i^m = k)$, $\varphi(\rho) = \rho$ for $\rho < 1$.

① ρ^{Z_n} is MG.

$$\begin{aligned}
\mathbb{E}(\rho^{Z_{n+1}} \mid \mathcal{F}_n) &= \mathbb{E}(\rho^{\xi_1^{n+1} + \cdots + \xi_{Z_n}^{n+1}} \mid \mathcal{F}_n) \\
&= \prod_{i=1}^{Z_n} (\mathbb{E}(\rho^{\xi_i^{n+1}} \mid \mathcal{F}_n)) \\
&= \prod_{i=1}^{Z_n} \varphi(\rho) \\
&= \rho^{Z_n}
\end{aligned}$$

② If $Z_0 = x$, $Z_n = Z_n^1 + \cdots + Z_n^x$, a sum of x indep.
Galton Watson process. $\mathbb{P}(Z_n^i = 0$ for some $n) = \rho$ $\forall 1 \le i \le x$.
$Z_n = 0$ for some $n \Leftrightarrow Z_n^i = 0$ for some n_i $\forall 1 \le i \le x$.
By independence, $\mathbb{P}(Z_n = 0$ for some $n) = \prod_{i=1}^{x} (Z_n^i = 0$ for some $n_i)$
$= \rho^x$ (thm 4.3.12)