Exercise

1

Probability

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1. Probability

1.1 $B_n = A_n - \bigcup_{i<n} A_i \subseteq A_n$. Hence $\bigcup B_i \subset \bigcup A_i$.

For $x \in \bigcup A_i$, let n be the smallest integer s.t. $x \in A_n$.
Then, $x \in B_n$. Thus, $x \in \bigcup B_i$. Thus, $\bigcup A_i = \bigcup B_i$.

Suppose monotone decreasing case $A_1 \supset A_2 \supset \cdots$,

Let $C_n = A_n - \bigcup_{i>n} A_i \subseteq A_n$. $A = \lim_{n\to\infty} A_n = \bigcap_{i=1}^{\infty} A_i$

$$A_n = \left(\bigcup_{i \geq n} C_n\right) \cup A$$

$$\mathbb{P}(A_n) = \mathbb{P}\left(\left(\bigcup_{i \geq n} C_n\right) \cup A\right)$$

$$= \sum_{i \geq n} \mathbb{P}(C_n) + \mathbb{P}(A)$$

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \lim \sum_{i \geq n} \mathbb{P}(C_n) + \mathbb{P}(A) = \mathbb{P}(A)$$

1.3 $m > n$. $x \in B_m = \bigcup_{i \geq m} A_i \Rightarrow x \in A_i$ for $i \geq m$

$\Rightarrow x \in A_i$ for $i \geq n$

$\Rightarrow x \in \bigcup_{i \geq n} A_i = B_n$. i.e. $B_m \subseteq B_n$

$x \in C_n = \bigcap_{i \geq n} A_i \Rightarrow x \in A_i \ \forall i \geq n$

$\Rightarrow x \in A_i \ \forall i \geq m$

$\Rightarrow x \in \bigcap_{i \geq n} A_i = C_m$. i.e. $C_n \subseteq C_m$

(b) $\omega \in \bigcap B_n \Rightarrow \omega \in B_n = \bigcup_{i \geq n} A_i \quad \forall n.$

If ω belongs to finite A_i's, take N s.t. $\omega \notin A_i \ \forall i \geq N$

Then, $\omega \notin B_N = \bigcup_{i \geq N} A_i$. $\times$.

For any n, take $m > n$ s.t. $x \in A_m$. Then,

$x \in A_m \subseteq \bigcup_{i \geq n} A_i = B_n$. Thus, $x \in \bigcap B_n$.

(c). $\omega \in \bigcup C_n \Rightarrow \omega \in C_n$ for some n.

$\Rightarrow \omega \in \bigcap_{i \geq n} A_i$.

i.e. $\omega \in A_i \ \forall i \geq n$.

Hence, $\omega \in A_i$ except finite A_i's.

$\omega \in A_i$ except finite i's $\Rightarrow$ take N s.t. $\omega \in A_i \ \forall i \geq N$

Then $\omega \in \bigcap_{i \geq N} A_i = C_N$. Thus, $\omega \in \bigcup C_n$.

1.5 $\Omega = \{(a_1, \cdots, a_n) \mid a_i \in \{H, T\}, n \in \mathbb{Z}_+, a_i = H$ for exactly two a_i's, $a_n = H\}$

$\binom{k}{2} \times 2^{-k}$ ✗

$$\boxed{(k-1)2^{-k}}$$

1.6 Suppose $\mathbb{P}$ is a uniform distribution on Ω.

Let $\mathbb{P}(\{0\}) = \alpha$. Then, $\mathbb{P}(\{n\}) = \alpha \ \forall n$.

$\mathbb{P}(\Omega) = \mathbb{P}(\cup\{n\}) = \sum \mathbb{P}(\{n\}) = \sum \alpha = 1$

$\alpha = 0 \Rightarrow \mathbb{P}(\Omega) = 1$ ✗

$\alpha \neq 0 \Rightarrow \mathbb{P}(\Omega) = \infty$ ✗.

1.7 $B_n = A_n - \bigcup_{i<n} A_i$. B_n's are disjoint $\quad \subseteq A_n$

Clearly $\cup B_n \subseteq \cup A_n$.

If $w \in \cup A_n$, let n be the smallest s.t. $w \in A_n$.

Then, $w \in B_n$ so that $w \in \cup B_n$. Thus, $\cup B_n = \cup A_n$.

Now, $\mathbb{P}(\cup A_n) = \mathbb{P}(\cup B_n) = \sum \mathbb{P}(B_n) \leq \sum \mathbb{P}(A_n)$

Note that $\mathbb{P}(A_n) = \mathbb{P}(B_n) + \mathbb{P}(A_n \setminus B_n) \geq \mathbb{P}(B_n)$.

1.8 $\mathbb{P}((\cap A_i)^c) = \mathbb{P}(\cup A_i^c) \overset{\text{by \#1.7}}{\le} \sum \mathbb{P}(A_i^c) = 0.$

Thus, $\mathbb{P}(\cap A_i) = 1 - \mathbb{P}((\cap A_i)^c) = 1$

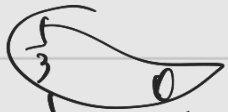
1.9 $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(AB)}{\mathbb{P}(B)}$

1. $\mathbb{P}(A|B) \ge 0 \ \forall A$ as $\mathbb{P}(AB), \mathbb{P}(B) \ge 0$.
2. $\mathbb{P}(\Omega|B) = \dfrac{\mathbb{P}(\Omega B)}{\mathbb{P}(B)} = \dfrac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$
3. Let A_i's be disjoint $\forall i \in \mathbb{Z}_+$

Then, $\mathbb{P}(\cup A_i | B) = \dfrac{\mathbb{P}((\cup A_i) B)}{\mathbb{P}(B)}$

$$= \frac{\mathbb{P}(\cup(A_i B))}{\mathbb{P}(B)} = \frac{\sum \mathbb{P}(A_i B)}{\mathbb{P}(B)}$$

$$= \sum \mathbb{P}(A_i | B)$$



$\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{3}$ $\frac{1}{3}$

1.10 12 13 23 32

$$\mathbb{P}(\omega_1 \omega_2) = \begin{cases} \frac{1}{6} & \text{if } 12 \text{ or } 13 \\ \frac{1}{3} & \text{if } 23 \text{ or } 32 \end{cases}$$

$\mathbb{P}($

1.11 $P(A^cB^c) = P((A\cup B)^c) = 1-P(A\cup B)$

$= 1-(P(A)+P(B)-P(AB))$

$= 1-P(A)-P(B)+P(A)P(B)$

$= (1-P(A))(1-P(B)) = P(A^c)P(B^c)$.

GG RR GR

1.12 A B C. D: see green

X: card I choose is A.

$P(D|A)=1,\ P(D|B)=0,\ P(D|C)=\frac{1}{2}$.

$P(A|D) = \frac{1}{1.5} = \frac{2}{3},\ P(B|D)=0,\ P(C|D)=\frac{1}{3}$.

1.13 (a) $\Omega = \{(a_1 \cdots a_n) \mid a_i \in \{H, T\},\ a_1 = \cdots = a_{n-1} \neq a_n\}$

(b) HHT, TTH $\to \frac{2}{8} = \frac{1}{4}$.

1.14 $P(A)=0 \Rightarrow P(AB)=0=P(A)P(B)\ \forall B$.

$\Rightarrow$ A is independent to B $\forall B$.

$P(A)=1 \Rightarrow P(A^c)=0 \Rightarrow A^c$ indep. to $B^c\ \forall B$

$\Rightarrow$ A indepe. to B (\#1.11)

$P(A) = P(AA) = P(A)P(A) \Rightarrow P(A) = 0$ or 1

1.15 (a) AO. AT.

$$\mathbb{P}(AT|AO) = \frac{\mathbb{P}(AO|AT)\mathbb{P}(AT)}{\mathbb{P}(AO)} = \frac{\mathbb{P}(AT)}{\mathbb{P}(AO)} = \frac{10}{37}$$

$$\mathbb{P}(AO) = 1-\left(\frac{3}{4}\right)^3 = \frac{37}{64}$$

$$\mathbb{P}(AT) = \left(\frac{1}{4}\right)^2 \times 3 \times \frac{3}{4} + \left(\frac{1}{4}\right)^3 = \frac{10}{64}$$

2 3

(b) $1-\left(\frac{3}{4}\right)^2 = \frac{7}{16}$

1.17 $\mathbb{P}(ABC) = \mathbb{P}(A(BC))$

$= \mathbb{P}(A|BC)\,\mathbb{P}(BC)$

$= \mathbb{P}(A|BC)\,\mathbb{P}(B|C)\,\mathbb{P}(C).$

1.18 $\mathbb{P}(A_1|B) = \dfrac{\mathbb{P}(A_1B)}{\mathbb{P}(B)} < \mathbb{P}(A_1)$

$\sum_i \mathbb{P}(A_i|B) = \sum_i \mathbb{P}(A_i) = 1.$

Thus, if $\mathbb{P}(A_1|B) < \mathbb{P}(A_1)$, then $\mathbb{P}(A_i|B) > \mathbb{P}(A_i)$ for some $i \in \{2, \dots, k\}$

1.19 $P(M)=0.3,\ P(W)=0.5,\ P(L)=0.2$

$P(V|M)=0.65,\quad P(V|W)=0.82,\quad P(V|L)=0.5.$

$$P(W|V)=\frac{P(WV)}{P(V)}=\frac{P(V|W)P(W)}{P(V|M)P(M)+P(V|W)P(W)+P(V|L)P(L)}$$

$$=\frac{0.41}{0.195+0.41+0.1}=\frac{0.41}{0.615}$$

1.20 (a) $P(H|C_i)=p_i.\quad P(C_i)=0.2$

$$P(C_i|H)=\frac{p_i}{\sum p_i}=\frac{p_i}{2.5}$$

$$\frac{1+4+9+16}{16}\times\frac{2}{5}$$

$$\frac{\cancel{30}\,6}{16}\times\frac{2}{\cancel{5}}$$

(b). $P(H)=0.2\sum p_i=0.5$

$$P(H_2|H_1)=P(H_2(\cup C_i)|H_1)$$

$$=\sum P(H_2C_i|H_1)\qquad \frac{P(H_1H_2C_i)}{P(H_1)}=$$

$$=\frac{\sum P(H_2|C_iH_1)P(H_1|C_i)P(C_i)}{P(H_1)}$$

$$=\frac{\sum P(H_2|C_i)P(H_1|C_i)P(C_i)}{P(H_1)}$$

$$=\frac{(0^2+0.25^2+0.5^2+0.75^2+1^2)\times 0.2}{0.5}=\frac{3}{4}$$

$$P(H_2|H_1) = \frac{P(H_1H_2)}{P(H_1)} = \frac{\sum P(H_1H_2C_i)}{P(H_1)}$$

$$=$$

(c) TTTH.

$$\mathbb{P}(C_i|B_4) = \frac{\mathbb{P}(B_4|C_i)\,\mathbb{P}(C_i)}{\mathbb{P}(B_4)} = \frac{\mathbb{P}(B_4|C_i)}{\sum_j \mathbb{P}(B_4|C_j)} = \begin{cases} 0 & i=1 \\ \frac{3}{22} & 2 \\ \frac{16}{22} & 3 \\ \frac{3}{22} & 4 \\ 0 & 5. \end{cases}$$

$$\mathbb{P}(B_4|C_i) = (1-p_i)^3 p_i. = \begin{cases} 0 & i = 1 \\ \frac{3}{256} & 2 \\ \frac{16}{256} & 3 \\ \frac{3}{256} & 4 \\ 0 & 5 \end{cases}$$

2

Random Variable

2.1 Lemma & Prop.

$P(X=x) = F(x) - F(x^-)$ (Lemma)

$= F(x^+) - F(x^-)$ (F is right continuous.)

2.2

$F(x) = P(X \le x)$

$P(2 < X \le 4.8) = F(4.8) - F(2) = 1/10$

$P(2 \le X \le 4.8) = F(4.8) - F(2-) = 2/10$

2.3

1. Let $y_n < x \ \forall n$, $y_n \to x$.

$F(x) - F(x^-) = F(x) - \lim F(y_n)$

$= \lim F(x) - F(y_n)$

$= \lim P(y_n < X \le x)$

$X^{-1}((y_n, x]) \to X^{-1}(x)$ (Let $t \in \bigcap X^{-1}(y_n, x]$. $\Rightarrow X(t) \in (y_n, x] \ \forall n$

If $X(t) \ne x$, $\exists N$ s.t. $X(t) \notin (y_n, x]$ ※)

Thus, $P(y_n < X \le X) \to P(X^{-1}(x)) = P(X=x)$.

2. $\mathbb{P}(x<X\le y) = \mathbb{P}(X^{-1}((x,y]))$

$$= \mathbb{P}(X^{-1}((-\infty,y])) - \mathbb{P}(X^{-1}((-\infty,x]))$$

$$= \mathbb{P}(X\le y) - \mathbb{P}(X\le x)$$

$$= F(y) - F(x)$$

3. $X^{-1}((x,\infty)) \cap X^{-1}((-\infty,x]) = \emptyset$

$\qquad\qquad \cup \qquad\qquad = \Omega.$

Thus, $\mathbb{P}(X^{-1}(x,\infty)) + \mathbb{P}(X^{-1}(-\infty,x]) = \mathbb{P}(\qquad \cup \qquad) = 1.$

$\Rightarrow \mathbb{P}(X>x) = \mathbb{P}(X^{-1}(x,\infty)) = 1-\mathbb{P}(X^{-1}(-\infty,x])$

$$= 1-\mathbb{P}(X\le x) = 1-F(x)$$

4. $F(b)-F(a) = \mathbb{P}(a<X\le b)$. $\quad F(x) = \int_{-\infty}^{x} f(x)\,dx.$

$\mathbb{P}(X=x_0) = F(x_0) - F(x_0^-)$

$$= \lim_{y\to x_0^-} \int_{x_0^-}^{x_0} f(x)\,dx = 0 \quad \forall x_0.$$

Thus, $\mathbb{P}(a<x<b) = \mathbb{P}(a\le x\le b)$. Thus, $\mathbb{P}(a<X<b) = \cdots$

2.4 (a) $F(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{4}x & \text{if } x \in [0,1] \\ \frac{1}{4} & x \in (1,3] \\ \frac{1}{4} + \frac{3}{8}(x-3) & x \in (3,5] \\ 1 & x > 5 \end{cases}$

(b) $P(Y < \frac{1}{5}) = P(X > 5) = 0 \Rightarrow f_Y(y) = 0 \quad \forall y < \frac{1}{5}.$

Let $y \in [\frac{1}{5}, \frac{1}{3}]$. $P(Y \le y) = P(X \ge \frac{1}{y})$

$= 1 - \left(\frac{1}{4} + \frac{3}{8}\left(\frac{1}{y} - 3\right)\right) = \frac{15}{8} - \frac{3}{8y}$

$\Rightarrow f_Y(y) = F_Y'(y) = \frac{3}{8y^2}$

If $y \in [\frac{1}{3}, 1]$, $P(Y \le y) = P(X \ge \frac{1}{y})$

$= 1 - \frac{1}{4} = \frac{3}{4}.$

$\Rightarrow f_Y(y) = F_Y'(y) = 0.$

If $y \ge 1$, $P(Y \le y) = P(X \ge \frac{1}{y})$

$= 1 - \frac{1}{4y}$

$\Rightarrow f_Y(y) = F_Y'(y) = \frac{1}{16y^2}$

$y < \frac{1}{5} \Rightarrow P(Y \le y) = 1$.
$\Rightarrow f_Y(y) = 0$

Thus, $f_Y(y) = \begin{cases} 0 & \text{if } y < \frac{1}{5} \\ \frac{1}{8}(15 - 3y) & y \in [\frac{1}{5}, \frac{1}{3}] \\ 0 & y \in [\frac{1}{3}, 1] \\ \frac{1}{16y^2} & y > 1. \end{cases}$

2.5 X,Y indep $\overset{\text{def}}{\iff} \mathbb{P}(X\in A, Y\in B) = \mathbb{P}(X\in A)\mathbb{P}(Y\in B) \quad \forall A,B$.

Prove X,Y indep $\iff f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \forall x,y$.

$(\Rightarrow)$
$$\begin{aligned} f_{X,Y}(x,y) &= \mathbb{P}(X=x, Y=y) \\ &= \mathbb{P}(X=x)\,\mathbb{P}(Y=y) \\ &= f_X(x) f_X(y) \end{aligned}$$

$(\Leftarrow)$ Let $A = \{x_n\}$, $Y = \{y_m\}$ be given (countable.

$$\begin{aligned} \mathbb{P}(X\in A, Y\in B) &= \mathbb{P}(X\in \bigcup_n \{x_n\}, Y\in \bigcup_m \{y_m\}) \\ &= \sum_{n,m} \mathbb{P}(X=x_n, Y=y_m) \\ &= \sum_{n,m} f_{X,Y}(x_n, y_m) \\ &= \sum_{n,m} f_X(x_n) f_Y(y_m) \\ &= \Big(\sum_n f_X(x_n)\Big)\Big(\sum_m f_Y(y_m)\Big) \\ &= \mathbb{P}(X\in \bigcup_n \{x_n\})\,\mathbb{P}(Y\in \bigcup_m \{y_m\}) \\ &= \mathbb{P}(X\in A)\,\mathbb{P}(Y\in B) \end{aligned}$$

2.6 $f_Y(1) = \mathbb{P}(Y=1) = \mathbb{P}(I_A(X)=1) = \mathbb{P}(X\in A) = \int_A f(x)\,dx$

$f_Y(0) = \mathbb{P}(X\notin A) = 1 - \int_A f(x)\,dx$ similarly.

Let $\alpha := \int_A f(x)\,dx$. $F_Y(y) = \begin{cases} 0 & \text{if } y<0 \\ 1-\alpha & \text{if } y\in[0,1) \\ 1 & \text{if } y\in[1,\infty) \end{cases}$

2.7 If $z<0$, $\mathbb{P}(Z<z) = \mathbb{P}(\min\{X,Y\}<z)$
$= \mathbb{P}(X<z \text{ or } Y<z) = 0$

If $0\le z\le 1$ $\mathbb{P}(Z<z) = \mathbb{P}(\min\{X,Y\}<z\}$
$= \mathbb{P}(X<z \text{ or } Y<z)$
$= 1-\mathbb{P}(X>z \text{ and } Y>z)$
$= 1-(1-z)^2 = 2z-z^2$.

$\left| \text{or } \mathbb{P}(X<z \text{ or } Y<z) = \mathbb{P}(X<z)+\mathbb{P}(Y<z) - \mathbb{P}(X<z, Y<z) \right.$
$\left. = 2z-z^2 \right|$

If $z>1$, $\mathbb{P}(Z<z) = \mathbb{P}(\min\{X,Y\}<z)$
$= \mathbb{P}(X<z \text{ or } Y<z) = 1$

$$f_Z(z) = \begin{cases} 2-2z & \text{if } z\in[0,1] \\ 0 & \text{otherwise.} \end{cases}$$

2.8 $\mathbb{P}(X^+\le x) = \mathbb{P}(\max\{0,X\}\le x)$
$= \mathbb{P}(0\le x, X\le x) = \begin{cases} F(x) & \text{if } x\ge 0 \\ 0 & \text{otherwise.} \end{cases}$

Thus, $F_{X^+}(x) = \begin{cases} F(x) & \text{if } x\ge 0 \\ 0 & \text{otherwise.} \end{cases}$

2.9 $X \sim Exp(\beta) \Rightarrow f_X(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}$

$$F(x) = \int_{-\infty}^{x} f_X(t)\,dt = \left[-e^{-\frac{t}{\beta}}\right]_{t=0}^{x} = 1 - e^{-\frac{x}{\beta}}$$

$f_X(t) = 0 \quad \forall t < 0$

$$F^{-1}(q) := \inf\{x : F(x) > q\}$$
$$= \inf\{x : 1 - e^{-\frac{x}{\beta}} > q\}$$
$$= \inf\{x : x > -\beta \ln(1-q)\}$$
$$= \beta \ln(1-q)$$

$1 - q > e^{-\frac{x}{\beta}}$

$\ln(1-q) > -\frac{x}{\beta}$

$x > -\beta \ln(1-q)$

$X : \Omega \to \mathbb{R}$

2.10 Let $A, B \in \mathbb{R}$, $g, h : \mathbb{R} \to \mathbb{R}$.

$$\mathbb{P}(g(X) \in A, h(Y) \in B) = \mathbb{P}(X \in g^{-1}(A), Y \in h^{-1}(B))$$
$$= \mathbb{P}(X \in g^{-1}(A))\, \mathbb{P}(Y \in h^{-1}(B))$$
$$= \mathbb{P}(g(X) \in A)\, \mathbb{P}(h(Y) = B)$$

2.11 (a)

x	0	1
$f_X(x)$	$1-p$	p

y	0	1
$f_Y(y)$	p	$1-p$

$y \backslash x$	0	1
0	0	p
1	$1-p$	0

$\langle f_{X,Y}(x,y) \rangle$

$f_{X,Y}(0,0) = 0$

$f_X(0) f_Y(0) = p(1-p)$.

(b) $f_N(x) = e^{-\lambda}\frac{\lambda^x}{x!}$, $x \geq 0$

$$f_X(x) = \mathbb{P}(X=x) = \sum_n \mathbb{P}(X=x \mid N=n)\,\mathbb{P}(N=n)$$
$$= \sum_{n \geq x} \quad ''$$
$$= \sum_{n \geq x} \binom{n}{x} p^x (1-p)^{n-x} \cdot e^{-\lambda}\frac{\lambda^n}{n!} = \sum \frac{p^x(1-p)^{n-x}}{x!\,(n-x)!} e^{-\lambda}\lambda^n$$

$f_Y(y) = \sum_{n \geq y} \binom{n}{y} (1-p)^y p^{n-y} e^{-\lambda}\frac{\lambda^n}{n!}$ similarly.

$$f_{X,Y}(x,y) = \sum_n \mathbb{P}(X=x, Y=y \mid N=n)\,\mathbb{P}(N=n)$$
$$= \mathbb{P}(X=x, Y=y \mid N=x+y)\,\mathbb{P}(N=x+y)$$
$$= \binom{x+y}{x} p^x (1-p)^y e^{-\lambda}\frac{\lambda^{x+y}}{(x+y)!}$$

$$f_X(x) = \sum_{y \geq 0} \binom{x+y}{x} p^x (1-p)^y e^{-\lambda}\frac{\lambda^{x+y}}{(x+y)!} = \sum_{y \geq 0} e^{-\lambda}\frac{\lambda^x}{x!} p^x \frac{\lambda^y}{y!}(1-p)^y$$
$$= e^{-\lambda}\frac{(\lambda p)^x}{x!} e^{\lambda(1-p)}$$
$$= e^{-\lambda p}\frac{(\lambda p)^x}{x!}$$

$$f_Y(y) = \sum_{x \geq 0} \binom{x+y}{y} p^x (1-p)^y e^{-\lambda}\frac{\lambda^{x+y}}{(x+y)!}$$
$$= \sum_{x \geq 0} e^{-\lambda}\frac{(\lambda(1-p))^y}{y!} \times \frac{(\lambda p)^x}{x!}$$
$$= e^{-\lambda}\frac{(\lambda(1-p))^y}{y!} e^{\lambda p} = e^{-\lambda(1-p)}\frac{(\lambda(1-p))^y}{y!}.$$

$$f_X(x) f_Y(y) = e^{-\lambda}\frac{(\lambda p)^x (\lambda(1-p))^y}{x!\,y!}$$
$$= e^{-\lambda}\frac{\lambda^{x+y}}{x!\,y!} p^x (1-p)^y = \binom{x+y}{x} p^x (1-p)^y e^{-\lambda}\frac{\lambda^{x+y}}{(x+y)!}$$

2.12 Range$(X,Y) = R_X \times R_Y$.

Continuous R.V.? ← Assume

$$\mathbb{P}(X\in A, Y\in B) = \iint_{A\times B} f(x,y)\,dx\,dy$$

$$= \iint_{A\times B} g(x)h(y)\,dx\,dy$$

$$= \int_A\int_B g(x)h(y)\,dy\,dx = \int_A g(x)\int_B h(y)\,dy\,dx$$

$$= \int_A g(x)\,dx\int_B h(y)\,dy \quad \cdots\cdots ①$$

$$\mathbb{P}(X\in R_X, Y\in R_Y) = \int_{R_X} g(x)\,dx\int_{R_Y} h(y)\,dy = 1.$$

Let $\alpha := \int_{R_X} g(x)\,dx$, $\beta := \int_{R_Y} h(y)\,dy$. $\Rightarrow \alpha\beta = 1$

$$\mathbb{P}(X\in A) = \iint_{A\times R_Y} f\,dx\,dy = \beta\int_A g(x)\,dx$$

$$\mathbb{P}(Y\in B) = \iint_{R_X\times B} f\,dx\,dy = \alpha\int_B h(y)\,dy$$

$$\mathbb{P}(X\in A)\,\mathbb{P}(Y\in B) = \alpha\beta\int_A g(x)\,dx\int_B h(y)\,dy$$

$$= \int_A g(x)\,dx\int_B h(y)\,dy \quad \cdots\cdots ②$$

①, ② $\Rightarrow$ Done.

2.13 (a) $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

$P(Y \le y) = P(X \le \ln y)$

$= \Phi(\ln y)$ where $\Phi(x) = \int_{-\infty}^{x} f_X(t)dt$.

$= F_Y(y)$

$f_Y(y) = F_Y'(y) = \frac{d}{dy}(\Phi(\ln y)) = \frac{1}{y}\Phi'(\ln y) = \frac{1}{y} f_X(\ln y)$

$= \frac{1}{y\sqrt{2\pi}} e^{-\frac{1}{2}(\ln y)^2}$.

2.14 $F_R(r) = \begin{cases} 0 & \text{if } r < 0 \\ 1 & \text{if } r > 1 \end{cases}$

Let $r \in [0,1] \Rightarrow F_R(r) = P(R \le r) = P(X^2 + Y^2 \le r) = r^2$

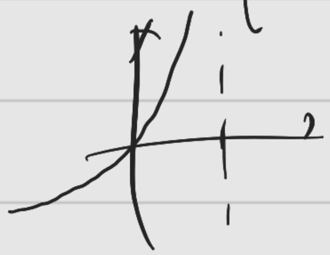
$f_R(r) = \begin{cases} 2r & \text{if } r \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$

2.15 $P(Y \le y) = P(F(X) \le y) = P(X \le F^{-1}(y)) = F(F^{-1}(y)) = y$.

$P(X \le x) = P(F^{-1}(U) \le x) = P(U \le F(x)) = F(x)$

Recall #2.9 $\Rightarrow F^{-1}(q) = \beta \ln(1-q)$ for $Exp(\beta)$

Take $U \sim \text{Uniform}(0,1) \to E = -\beta \ln(1-U) \sim Exp(\beta)$



2.16 $f_X(x) = e^{-\lambda}\dfrac{\lambda^x}{x!}$, $f_Y(y) = e^{-\mu}\dfrac{\mu^y}{y!}$, $x, y \geq 0$

$Z \sim \text{Binomial}(n, p)$ if $f_Z(z) = \begin{cases} \binom{n}{z} p^z (1-p)^{n-z} & z = 0, \cdots, n \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{P}(X = x \mid X+Y = n) = 0$ for $x \neq 0, \cdots, n$

For $x = 0, \cdots, n$, $\mathbb{P}(X = x \mid X+Y = n) = \dfrac{\mathbb{P}(X+Y = n \mid X = x)\,\mathbb{P}(X = x)}{\mathbb{P}(X+Y = n)}$

$$= \frac{\mathbb{P}(Y = n-x \mid X = x)\,\mathbb{P}(X = x)}{\mathbb{P}(X+Y = n)}$$

$$= \frac{\mathbb{P}(Y = n-x)\,\mathbb{P}(X = x)}{\mathbb{P}(X+Y = n)} \quad \cdots (*)$$

$$\mathbb{P}(X+Y = n) = \sum_y \mathbb{P}(X+Y = n \mid Y = y)\,\mathbb{P}(Y = y)$$

$$= \sum_{y=0}^{n} \mathbb{P}(X = n-y \mid Y = y)\,\mathbb{P}(Y = y)$$

$$= \sum_{y=0}^{n} \mathbb{P}(X = n-y)\,\mathbb{P}(Y = y) = \sum_{y=0}^{n} e^{-\lambda}\frac{\lambda^{n-y}}{(n-y)!} e^{-\mu}\frac{\mu^y}{y!}$$

$$= \sum_{y=0}^{n} e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^n}{n!}\,\frac{n!}{(n-y)!\,y!}\left(\frac{\lambda}{\lambda+\mu}\right)^{n-y}\left(\frac{\mu}{\lambda+\mu}\right)^y$$

$$= e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^n}{n!}$$

$$(*) = \frac{e^{-\lambda}\frac{\lambda^x}{x!}\, e^{-\mu}\frac{\mu^{n-x}}{(n-x)!}}{e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^n}{n!}} = \frac{n!}{x!\,(n-x)!}\left(\frac{\lambda}{\lambda+\mu}\right)^x\left(\frac{\mu}{\lambda+\mu}\right)^{n-x}$$

$$= \binom{n}{x}\pi^x(1-\pi)^{n-x}.$$

2.17 $\iint_{(-\infty,\frac{1}{2})\times\{\frac{1}{2}\}} f\,dx\,dy = \int_{-\infty}^{\frac{1}{2}} \int_{\frac{1}{2}}^{\frac{1}{2}} f\,dy\,dx = 0$

2.19 $f_Y(y) = f_X(s(y)) \left|\frac{ds(y)}{dy}\right|$ $\quad Y = r(X)$

$s = r^{-1}$ (r is strictly monotone) Assumption

Note that r is inc $\Leftrightarrow$ s is inc / r is dec $\Leftrightarrow$ s is dec.

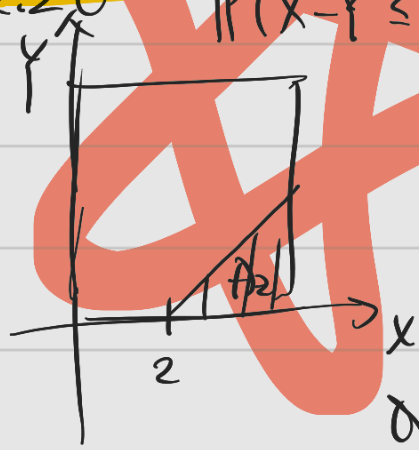
① r, s inc

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(r(X) \le y)$$
$$= \mathbb{P}(X \le s(y)) = F_X(s(y))$$
$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}(F_X(s(y)) = f_X(s(y))\,\frac{ds(y)}{dy}$$
$$= f_X(s(y)) \left|\frac{ds(y)}{dy}\right|.$$

② r, s dec

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(r(X) \le y)$$
$$= \mathbb{P}(X \ge s(y)) = 1 - F_X(s(y))$$
$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}(1 - F_X(s(y)) = -f_X(s(y))\,\frac{ds(y)}{dy}$$
$$= f_X(s(y)) \left|\frac{ds(y)}{dy}\right|.$$

2.20 $\mathbb{P}(X-Y\le z) = \begin{cases} 0 & \text{if } z<-1 \\ 1 & \text{if } z>1 \end{cases}$ $\quad -1\le X-Y\le 1$



$$\mathbb{P}(X-Y\le z) = |A_z| = \begin{cases} \frac{1}{2}(1-z)^2 & \text{for } z\in[0,1] \\ \frac{1}{2}(1+z)^2 & \text{for } z\in[-1,0] \end{cases}$$

Or, $z\in[0,1] \Rightarrow \mathbb{P}(X-Y\le z) = \mathbb{P}(X\le Y+z)$

$=$

Define $f(x,y) := f_X(x)f_Y(y)$. Then, we can show f is a joint PDF. i.e. $\mathbb{P}((X,Y)\in A) = \iint_A f \, dx\,dy$.

2.21 $P(Y \le y) = P(X_1 \le y, \dots, X_n \le y)$

$$= \prod_{i=1}^{n} P(X_i \le y)$$

$$= \left(\int_0^y \frac{1}{\beta} e^{-\frac{x}{\beta}} dx \right)^n$$

$$= \left(\left[-e^{-\frac{x}{\beta}} \right]_0^y \right)^n$$

$$= \left(1 - e^{-\frac{y}{\beta}}\right)^n = F_Y(y)$$

$$\frac{dF_Y(y)}{dy} = n\left(1 - e^{-\frac{y}{\beta}}\right)^{n-1} \frac{1}{\beta} e^{-\frac{n}{\beta}}$$

$$= \frac{n}{\beta}\left(1 - e^{-\frac{y}{\beta}}\right)^{n-1} e^{-\frac{n}{\beta}}$$

$X(a,b) \le X(m,n)$ $= m-n+1$

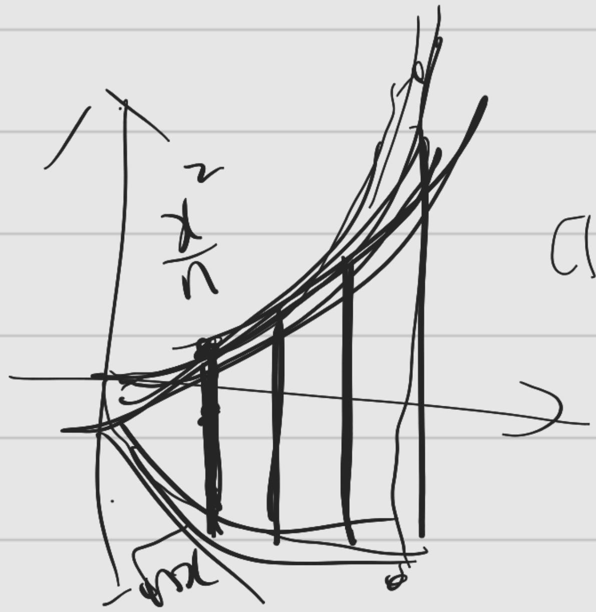
n $= \cancel{b-a+1}$

$b+1$ $b-a+1+2$

m $a-1$ a b

$m \ge a-1$

$n \le b+1$



$\frac{x^2}{n} + \sqrt{nx} + 1 \quad +2$

$f \quad \sum f(n) \quad \int f(x)$

$$\sum_{x=0}^{n} \left\lfloor \frac{x^2}{n} \right\rfloor + \left\lceil \sqrt{nx} \right\rceil + 1$$

$$n - 2n \le a_n \le \int_0^n \frac{x^2}{n} + \sqrt{nx} + 1 \, dx + 2n$$

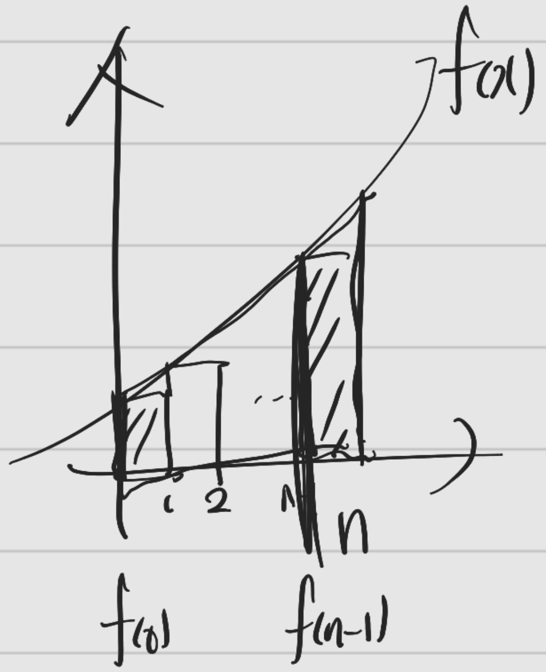
$$= \left[\frac{x^3}{3n} + \frac{2\sqrt{n}}{3} x^{\frac{3}{2}} + x \right]_0^n + 2n$$

$$= n^2 + 3n$$

$$c_n \le a_n \le b_n$$

$$c_n \le 80\,\frac{a_n}{4n^2+\cdots} \le 80\,\frac{\frac{(n+1)^3}{3n}+\frac{2}{3}\sqrt{n}(n+1)^{\frac{3}{2}}+3(n+1)}{4n^2+\cdots}$$

$\to 20$



$$\int_0^n f(x)\,dx \ge f(0)+\cdots+f(n-1)$$

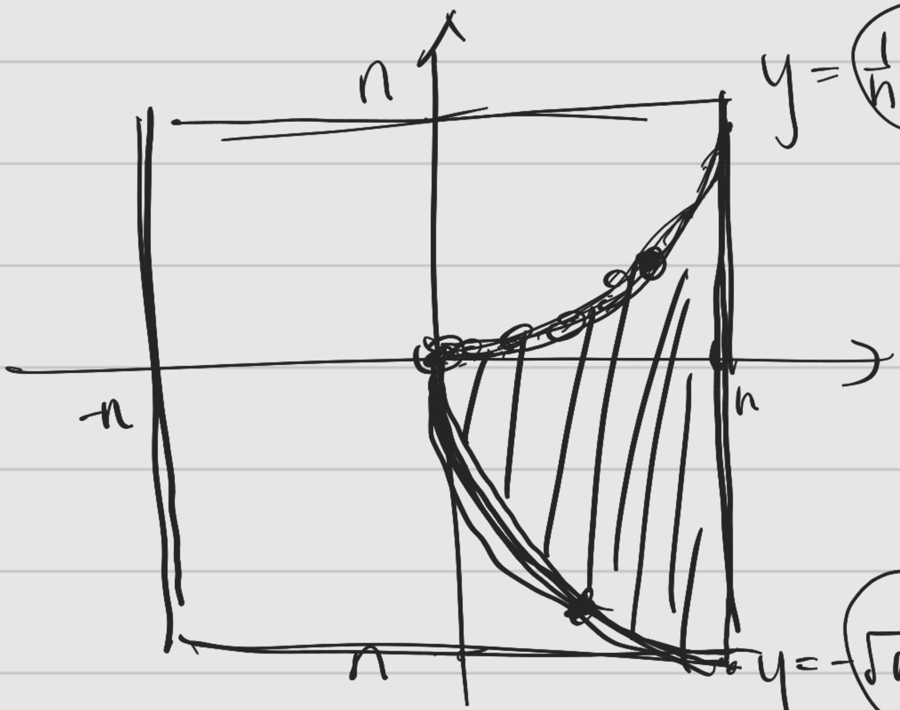
$$f(x) = \frac{1}{n}x^2+\sqrt{nx}+1+2$$

$$a_n \le f(0)+\cdots+f(n) \le \int_0^{n+1} f(x)\,dx$$

$$= \int_0^{n+1} \frac{1}{n}x^2+\sqrt{nx}+3\,dx$$

$$= \left[\frac{1}{3n}x^3+\frac{2}{3}\sqrt{n}\,x^{\frac{3}{2}}+3x\right]_0^{n+1}$$

$$= \frac{(n+1)^3}{3n}+\frac{2}{3}\sqrt{n}(n+1)^{\frac{3}{2}}+3(n+1) = b_n$$



$y = \frac{1}{n}x^2$

$(x, \frac{x^2}{n})$

$\hookrightarrow (\frac{x^2}{n}, -x)$

$\frac{(2n+1)^2-1}{4}$

$(a,b)=(b,-a)$

$y=-\sqrt{nx}$

$-x = -\sqrt{n\frac{x^2}{n}} = -x$

$\frac{(2n+1)^2}{4} \le a_n \le \frac{(2n+1)^2}{4} + n+1$

8o $4n^2+$

3 Expectation

3.1

1. Suppose we play a game where we start with c dollars. On each play of the game you either double or halve your money, with equal probability. What is your expected fortune after n trials?

Let X_n be the expected fortune after n trials.

$X_0 = C$. $X_1 = \frac{1}{2} \times (2C) + \frac{1}{2} \times \frac{C}{2} = \frac{5}{4} C$.

$X_2 = \frac{1}{4} \times 4C + \frac{1}{2} \times C + \frac{1}{4} \times \frac{C}{4} = C + \frac{C}{2} + \frac{C}{16} = \frac{25}{16} C$.

$$X_n = \sum_{k=0}^{n} 2^{-n} \binom{n}{k} 2^{2k-n} = \sum_{k=0}^{n} \binom{n}{k} 2^{2(k-n)} = \sum_{k=0}^{n} \binom{n}{k} 4^{k-n} \cdot 1^{n-k}$$

$$= 4^{-n} \sum_{k=0}^{n} \binom{n}{k} 4^{k} 1^{n-k} = 4^{-n} (4+1)^n = \left(\frac{5}{4}\right)^n.$$

3.2

2. Show that $\mathbb{V}(X) = 0$ if and only if there is a constant c such that $P(X = c) = 1$.

$(\Rightarrow)$ $\mathbb{V}(X) = \mathbb{E}(X - \mathbb{E}[X])^2 \geq 0$. If $\mathbb{V}(X) = 0$, $\mathbb{E}[X] = \mu < \infty$.

If $\mathbb{P}(X = \mu) < 1$, then $X \neq \mu$ on some positive measure set. Let $A_n = \{x \mid (x-\mu)^2 > \frac{1}{n}\}$, $A = \bigcup_{n=1}^{\infty} A_n$.

Then, $\mathbb{P}(A) = \mathbb{P}(\bigcup_{i=1}^{\infty} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n) > 0$.

$\because A_1 \subseteq A_2 \subseteq \cdots$, follows from theorem in Lebesgue integration theory

Thus, $\exists N$ such that $\mathbb{P}(A_N) > 0$. Then,

$\mathbb{V}(X) = \mathbb{E}(X - \mathbb{E}[X])^2 \geq \frac{1}{N} \mathbb{P}(A_N) > 0$. $\times$

Thus, $\mathbb{P}(X = \mu) = 1$ and so $\mu = c$ works.

$(\Leftarrow)$ $\mathbb{E}[X] = c \times \mathbb{P}(X=c) = c \times 1 = c$.

$\mathbb{V}(X) = \mathbb{E}(X-\mathbb{E}[X])^2 = \mathbb{E}(X-c)^2 = (c-c)^2 \times \mathbb{P}(X=c) = 0$.

3.3

3. Let $X_1, \ldots, X_n \sim \text{Uniform}(0,1)$ and let $Y_n = \max\{X_1, \ldots, X_n\}$. Find $\mathbb{E}(Y_n)$.

$\mathbb{P}(Y \le y) = \prod_{i=1}^{n} \mathbb{P}(X_i \le y) = y^n$ for $\forall y \in (0,1)$.

$f_Y(y) = \frac{dF_Y}{dy} = ny^{n-1}$. $\mathbb{E}[Y] = \int_0^1 y f_Y(y) dy = \int_0^1 ny^n dy = \frac{n}{n+1}$.

3.4

4. A particle starts at the origin of the real line and moves along the line in jumps of one unit. For each jump the probability is p that the particle will jump one unit to the left and the probability is $1-p$ that the particle will jump one unit to the right. Let X_n be the position of the particle after n units. Find $\mathbb{E}(X_n)$ and $\mathbb{V}(X_n)$. (This is known as a **random walk**.)

Let $\mathbb{P}(Y_i = -1) = p$, $\mathbb{P}(Y_i = 1) = 1-p$. Then, $X_n = \sum_{i=1}^{n} Y_i$.

$\mathbb{E}[Y_i] = (-1) \times p + 1 \times (1-p) = 1-2p$,

$\mathbb{V}(Y_i) = \mathbb{E}(Y_i - \mathbb{E}[Y_i])^2 = (-2+2p)^2 \times p + (2p)^2 (1-p)$

$= 4p(1-p)(1-p + p) = 4p(1-p)$.

3.5

5. A fair coin is tossed until a head is obtained. What is the expected number of tosses that will be required?

n toss required with probability 2^{-n}.

$P(X=n) = 2^{-n}$.

$\Rightarrow \mathbb{E}(X) = \sum n 2^{-n} = 2^{-1} + 2\cdot 2^{-2} + 3\cdot 2^{-3} + \cdots$

$2\mathbb{E}(X) = 1 + 2\cdot 2^{-1} + 3\cdot 2^{-2} + \cdots$

$\mathbb{E}(X) = 1 + 2^{-1} + 2^{-2} + \cdots = 2$.

3.6

6. Prove Theorem 3.6 for discrete random variables.

3.6 Theorem (The Rule of the Lazy Statistician). *Let* $Y = r(X)$. *Then*

$$\mathbb{E}(Y) = \mathbb{E}(r(X)) = \int r(x)dF_X(x). \tag{3.3}$$

Let $\mathbb{P}(X=x_i) = p_i \ \forall i \in \mathbb{N}$.

$\mathbb{E}(Y) = \mathbb{E}(r(X)) = \sum_{i=1}^{\infty} p_i r(x_i) = \int r(x) dF_X(x)$.

(Actually, for $Q = \{r(x_i)\}$

$\mathbb{P}(Y = r(x')) = \sum_{r(x)=r(x')} \mathbb{P}(X=x)$

So $\mathbb{E}[r(X)] = \sum_{q\in Q} q\, \mathbb{P}(Y=q)$

$= \sum_{q\in Q} q \sum_{r(x_i)=q} \mathbb{P}(X=x_i)$

$= \sum r(x_i)\,\mathbb{P}(X=x_i)$.

3.7

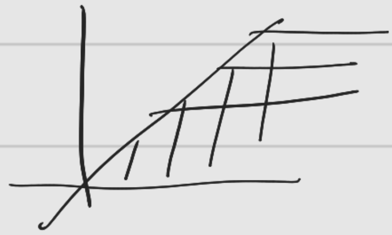
7. Let X be a continuous random variable with CDF F. Suppose that $P(X>0)=1$ and that $\mathbb{E}(X)$ exists. Show that $\mathbb{E}(X)=\int_0^\infty \mathbb{P}(X>x)dx$.

Hint: Consider integrating by parts. The following fact is helpful: if $\mathbb{E}(X)$ exists then $\lim_{x\to\infty} x[1-F(x)]=0$.

$$\mathbb{E}(X) = \int_0^\infty x f_X(x)\,dx$$

$$= \left[x(F(x)-1)\right]_0^\infty - \int_0^\infty F(x)-1\,dx$$

$$= 0 + \int_0^\infty \mathbb{P}(X>x)\,dx$$



X

$$\lim_{x\to\infty} x[1-F(x)] = \lim_{x\to\infty} \frac{1-F(x)}{\frac{1}{x}} = \lim_{x\to\infty} x^2 f_X(x) = \alpha$$

If $\alpha \neq 0$, $\lim_{x\to\infty} \frac{f_X(x)}{\frac{\alpha}{x^2}} = 1$.

$$\Rightarrow \mathbb{E}(X) \geq \int_N^\infty x\,(1-\varepsilon)\frac{\alpha}{x^2}\,dx = \infty. \quad \cancel{\times}$$

$$\int_u^\infty x f_X(x)\,dx \geq u\int_u^\infty f_X(x)\,dx = u[1-F(u)].$$

$$\Rightarrow \lim_{u\to\infty} u[1-F(u)] = 0.$$

3.8

8. Prove Theorem 3.17.

3.17 Theorem. *Let $X_1, \ldots, X_n$ be IID and let $\mu = \mathbb{E}(X_i)$, $\sigma^2 = \mathbb{V}(X_i)$. Then*

$$\mathbb{E}(\overline{X}_n) = \mu, \quad \mathbb{V}(\overline{X}_n) = \frac{\sigma^2}{n} \quad \text{and} \quad \mathbb{E}(S_n^2) = \sigma^2.$$

(pf) $\mathbb{E}(\overline{X}_n) = \mathbb{E}(n^{-1}\sum X_i) = n^{-1}\mathbb{E}(\sum X_i) = n^{-1}\sum \mathbb{E}(X_i)$

$= n^{-1} \cdot n\mu = \mu$

$\mathbb{V}(\overline{X}_n) = \mathbb{V}(n^{-1}\sum X_i) = n^{-2}\mathbb{V}(\sum X_i) = n^{-2}\sum \mathbb{V}(X_i)$

$= n^{-2} \cdot n\sigma^2 = \frac{\sigma^2}{n}$.

$\mathbb{E}(S_n^2) = \mathbb{E}\left(\frac{1}{n-1}\sum (X_i - \overline{X}_n)^2\right) = \frac{1}{n-1}\sum \mathbb{E}(X_i - \overline{X}_n)^2$.

$X_i - \overline{X}_n = X_i - \frac{1}{n}\sum X_j = \frac{n-1}{n}X_i - \frac{1}{n}\sum_{j\neq i} X_j$. $\mathbb{E}(X_i - \overline{X}_n) = 0$.

$\Rightarrow \mathbb{E}(X_i - \overline{X}_n)^2 = \mathbb{V}\left(\frac{n-1}{n}X_i - \frac{1}{n}\sum_{j\neq i} X_j\right)$

$= \left(\frac{n-1}{n}\right)^2 \sigma^2 + \frac{1}{n^2} \times (n-1)\sigma^2$. (By independence)

$= \frac{(n-1)}{n}\sigma^2$.

$\Rightarrow \mathbb{E}(S_n^2) = \frac{1}{n-1}\sum \mathbb{E}(X_i - \overline{X}_n)^2 = \sigma^2$.

$\frac{1}{\sqrt{2\pi}}$

3.10 By Lazy statistician's rule,

$\mathbb{E}(Y) = \int e^x f(x)\,dx = \int e^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2} \cdot e^{\frac{1}{2}}\,dx$

$= e^{\frac{1}{2}} \int \cdot \sim dx = e^{\frac{1}{2}}$.

$\mathbb{V}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$

$$= \int e^{2x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx - e$$

$$= \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2)^2} \cdot e^2 dx - e = e^2 - e = e(e-1).$$

3.12

12. Prove the formulas given in the table at the beginning of Section 3.4 for the Bernoulli, Poisson, Uniform, Exponential, Gamma, and Beta. Here are some hints. For the mean of the Poisson, use the fact that $e^a = \sum_{x=0}^{\infty} a^x/x!$. To compute the variance, first compute $\mathbb{E}(X(X-1))$. For the mean of the Gamma, it will help to multiply and divide by $\Gamma(\alpha+1)/\beta^{\alpha+1}$ and use the fact that a Gamma density integrates to 1. For the Beta, multiply and divide by $\Gamma(\alpha+1)\Gamma(\beta)/\Gamma(\alpha+\beta+1)$.

Bernoulli $X \sim \text{Bernoulli}(p)$. $\mathbb{E}(X) = p \times 1 + (1-p) \times 0 = p$.

$\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = (p \times 1^2 + (1-p) \times 0^2) - p^2 = p - p^2 = p(1-p)$

Exponential $X \sim \text{Exp}(\beta)$. $f_X(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}$, $x \geq 0$.

$$\mathbb{E}(X) = \int_0^\infty \frac{x}{\beta} e^{-\frac{x}{\beta}} dx = \left[-x e^{-\frac{x}{\beta}} - \beta e^{-\frac{x}{\beta}}\right]_0^\infty = \beta.$$

$\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

$$= \int_0^\infty \frac{x^2}{\beta} e^{-\frac{x}{\beta}} dx - \beta^2 = \left[-x^2 e^{-\frac{x}{\beta}}\right]_0^\infty + \int_0^\infty 2x e^{-\frac{x}{\beta}} dx - \beta^2$$

$$= 0 + 2\beta \mathbb{E}(X) - \beta^2 = 2\beta^2 - \beta^2 = \beta^2$$

Poisson $f_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x \ge 0, \quad x \in \mathbb{Z}$

$$\mathbb{E}(X) = \sum_{x=0}^{\infty} x e^{-\lambda}\frac{\lambda^x}{x!}dx = \lambda\sum_{x=1}^{\infty} e^{-\lambda}\frac{\lambda^{x-1}}{(x-1)!}dx$$

$$= \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda$$

$$\mathbb{E}(X(X-1)) = \sum_{x=0}^{\infty} x(x-1)e^{-\lambda}\frac{\lambda^x}{x!}dx$$

$$= \sum_{x=2}^{\infty} \quad \text{''}$$

$$= \sum_{x=2}^{\infty} \lambda^2 e^{-\lambda} \cdot \frac{\lambda^{x-2}}{(x-2)!}dx = \lambda^2.$$

$$\mathbb{E}(X^2) = \lambda^2 + \lambda.$$

$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda.$$

Uniform $X \sim \text{Uniform}(a,b)$

$$\mathbb{E}(X) = \int_a^b x\frac{1}{b-a}dx = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}.$$

$$\mathbb{E}(X^2) = \int_a^b x^2\frac{1}{b-a}dx = \frac{b^3-a^3}{3(b-a)} = \frac{1}{3}(b^2+ba+a^2).$$

$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{1}{3}(b^2+ba+a^2) - \left(\frac{b+a}{2}\right)^2 \qquad \frac{1}{3} - \frac{1}{\cancel{8}}$$

$$= \frac{1}{12}(b^2 - 2ab + a^2) = \frac{1}{12}(b-a)^2$$

Gamma $X \sim Gamma(\alpha, \beta)$ $f_X(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, x>0$

$$\mathbb{E}(X) = \int_0^\infty \frac{1}{\beta^\alpha \Gamma(\alpha)} x^\alpha e^{-x/\beta} dx = \frac{\beta \Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha\beta$$

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy = \frac{1}{\alpha} y e^{-y}\Big]_0^\infty + \int_0^\infty \frac{y^\alpha}{\alpha} e^{-y} dy$$

$$= \frac{1}{\alpha} \Gamma(\alpha+1).$$

$$\mathbb{E}(X^2) = \int_0^\infty \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha+1} e^{-\frac{x}{\beta}} dx = \frac{\beta^2 \Gamma(\alpha+2)}{\Gamma(\alpha)} = \beta^2 \alpha(\alpha+1)$$

$$\mathbb{V}(X) = \beta^2 \alpha(\alpha+1) - (\alpha\beta)^2$$

$$= \beta^2 \alpha.$$

Beta $X \sim Beta(\alpha, \beta)$. $f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}, 0<x<1$

$$\mathbb{E}(X) = \int_0^1 x f(x) dx = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^\alpha (1-x)^{\beta-1} dx$$

$$= \int_0^1 \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)} \times \left(\frac{\alpha}{\alpha+\beta}\right) x^{\alpha}(1-x)^{\beta-1}\,dx$$

$$= \frac{\alpha}{\alpha+\beta}.$$

$$\mathbb{E}(X^2) = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha+1}(1-x)^{\beta-1}\,dx$$

$$= \int_0^1 \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta)} x^{\alpha+1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$V(X) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2$$

$$= \frac{\alpha}{\alpha+\beta} \cdot \frac{(\alpha+1)(\alpha+\beta) - \alpha(\alpha+\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$= \frac{\alpha}{\alpha+\beta} \frac{\beta}{(\alpha+\beta)(\alpha+\beta+1)} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

3.13

13. Suppose we generate a random variable X in the following way. First we flip a fair coin. If the coin is heads, take X to have a Unif(0,1) distribution. If the coin is tails, take X to have a Unif(3,4) distribution.

(a) Find the mean of X.

(b) Find the standard deviation of X.

(a). $f_X(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0,1] \cup [3,4] \\ 0 & \text{otherwise} \end{cases}$.

$$\mathbb{E}(X) = \int x f_X(x) dx = \int_0^1 \frac{1}{2} x dx + \int_3^4 \frac{1}{2} x dx$$
$$= \frac{1}{4} + \frac{7}{4} = 2.$$

$$\mathbb{E}(X^2) = \int x^2 f_X(x) dx = \int_0^1 \frac{1}{2} x^2 dx + \int_3^4 \frac{1}{2} x^2 dx$$
$$= \frac{1}{6} + \frac{64-27}{6} = \frac{38}{6} = \frac{19}{3}.$$

$$\mathbb{V}(X) = \frac{19}{3} - 2^2 = \frac{7}{3}.$$

3.14 Let $\mathbb{E}(X_i) = \mu_i$. Then, $\mathbb{E}(\sum a_i X_i) = \sum a_i \mu_i$.

$\mathbb{E}(Y_j) = \mu_j'$ $\qquad \mathbb{E}(\sum_j b_j Y_j) = \sum b_j \mu_j'$.

$$\mathrm{Cov}(\sum a_i X_i, \sum b_j Y_j) = \mathbb{E}((\sum a_i(X_i - \mu_i))(\sum b_j (Y_j - \mu_j')))$$
$$= \mathbb{E}(\sum_{i,j} a_i b_j (X_i - \mu_i)(Y_j - \mu_j'))$$
$$= \sum_{i,j} a_i b_j \mathbb{E}(\qquad)$$
$$= \sum_{i,j} a_i b_j \mathrm{Cov}(X_i, Y_j).$$

3.15 $\mathbb{E}(2X-3Y+8) = \int_0^1\int_0^2 (2x-3y+8)\,\frac{1}{3}(x+y)\,dy\,dx$.

$\frac{1}{3}\int_0^1\int_0^2 2(x^2+xy)\,dy\,dx = \frac{1}{3}\int_0^1 4x^2+4x\,dx = \frac{1}{3}\left(\frac{4}{3}+2\right) = \frac{10}{9}$

$\int_0^2\int_0^1 -xy-y^2\,dx\,dy = \int_0^2 -\frac{1}{2}y-y^2\,dy = -\frac{1}{4}y^2-\frac{1}{3}y^3\Big]_0^2 = -1-\frac{8}{3}$

$= -\frac{11}{3}$.

$f_X(x) = \int_0^2 \frac{1}{3}(x+y)\,dy = \frac{2}{3}x+\frac{2}{3} = \frac{2}{3}(x+1)$ $\quad \frac{1}{3}(x+1)^2$.

$f_Y(y) = \int_0^1 \frac{1}{3}(x+y)\,dx = \frac{1}{6}+\frac{y}{3}$.

$\mathbb{E}(X) = \int_0^1 \frac{2}{3}(x^2+x)\,dx = \frac{2}{3}\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{2}{3}\times\frac{5}{6} = \frac{5}{9}$.

$\mathbb{E}(Y) = \int_0^2 \frac{1}{6}(2y^2+y)\,dy = \frac{1}{6}\left(\frac{2}{3}\times 2^3+\frac{1}{2}\times 2^2\right) = \frac{1}{6}\left(\frac{16}{3}+2\right) = \frac{1}{6}\times\frac{22}{3} = \frac{11}{9}$.

$\mathbb{E}(2X-3Y+8) = \frac{10}{9}-\frac{33}{9}+8 = \frac{72-23}{9} = \frac{49}{9}$.

$\mathbb{E}(2X-3Y+8)^2 = \mathbb{E}(4X^2-6XY+9Y^2+16(2X-3Y)+64)$.

$\mathbb{E}(X^2) = \int_0^1 \frac{2}{3}(x^3+x^2)\,dx = \frac{2}{3}\left(\frac{1}{4}+\frac{1}{3}\right) = \frac{7}{18}$.

$\mathbb{E}(Y^2) = \int_0^2 \frac{1}{6}(2y^3+y^2)\,dy = \frac{1}{6}\left(8+\frac{8}{3}\right) = \frac{16}{9}$.

$$\mathbb{E}(XY) = \int_0^1\int_0^2 \frac{xy}{3}(x+y)\,dy\,dx = \int_0^1\int_0^2 \frac{1}{3}(x^2y + xy^2)\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{6}x^2y^2 + \frac{1}{9}xy^3\right]_0^2 dx$$

$$= \int_0^1 \frac{2}{3}x^2 + \frac{8}{9}x\,dx = \frac{2}{9} + \frac{8}{9} = \frac{10}{9}.$$

$\rightarrow$ Calculate . .

3.11 Second follows from first with $s(Y) = 1$.

$$\mathbb{E}(r(X)s(Y) \mid X = x) = \int r(x)s(y)\, f_{Y|X}(y|x)\,dy$$

$$= r(x)\int s(y) f_{Y|X}(y|x)\,dy$$

$$= r(x)\,\mathbb{E}(s(Y) \mid X = x)$$

i.e. $\mathbb{E}(r(X)s(Y) \mid X) = r(X)\,\mathbb{E}(s(Y) \mid X)$.

3.12 $\mathbb{E}(Y \mid X = x) = \int y\, f_{Y|X}(y|x)\,dy = \mu_Y(x)$, $\mathbb{E}(\mu_Y(X)) = \mu_Y$

$$\mathbb{V}(Y \mid X = x) = \int (y - \mu_Y(x))^2 f_{Y|X}(y|x)\,dy.$$

$$\mathbb{E}(\mathbb{V}(Y|X)) = \int (y-\mu_Y(x))^2 f_{Y|X}(y|x)\,dy\, f_X(x)\,dx$$

$$= \iint (y-\mu_Y(x))^2 f_{X,Y}(x,y)\,dy\,dx$$

$$\mathbb{V}(\mathbb{E}(Y|X)) = \int \left(\int y f_{Y|X}(y|x)\,dy - \mu_Y\right)^2 f_X(x)\,dx$$

$$= \int \left[\int y\,(f_{Y|X}(y|x) - f_Y(y))\,dy\right]^2 f_X(x)\,dx$$

$$\mathbb{E}(\mathbb{V}(Y|X)) = \mathbb{E}(Y-\mu_Y(X))^2 .$$

$$\mathbb{V}(\mathbb{E}(Y|X)) = \mathbb{V}(\mu_Y(X))$$

$$= \mathbb{E}(\mu_Y(X) - \mu_Y)^2 .$$

$$\mathbb{E}(Y-\mu_Y)^2 = \mathbb{E}(Y-\mu_Y(X) + \mu_Y(X) - \mu_Y)^2$$

$$= \mathbb{E}(\mathbb{V}(Y|X)) + \mathbb{V}(\mathbb{E}(Y|X))$$

$$+ 2\,\mathbb{E}(Y-\mu_Y(X))(\mu_Y(X)-\mu_Y) .$$

$$\mathbb{E}((Y-\mu_Y(X))\mu_Y(X)) =$$

$$\mu_Y(x) = \int y f_{Y|X}(y|x)\,dy$$

$$\mathbb{E}((Y-\mu_X(X))\mu_Y) = 0 .$$

$$\mathbb{E}(Y\mu_Y(x)) = \iint y \int y f_{Y|X}(y|x)dy \; f_{X,Y}(x,y)\,dx\,dy$$

$$= \iint y \int y f_{Y|X}(t|x)dx \; f_{Y|X}(y|x) f_X(x)\,dy\,dx$$

$$= \int \left(\int y f_{Y|X}(y|x)\,dy\right)^2 f_X(x)\,dx$$

$$= \int \mu_X(x)^2 f_X(x)\,dx = \mathbb{E}(\mu_X(x)^2).$$

3.18 $\rho(X,Y) = 0 \quad \mathbb{E}(X|Y=y) = \int x f_{X|Y}(x|y)\,dx = c.$

$\frac{\partial}{\partial y} \Rightarrow \quad x \frac{\partial}{\partial y} f_{X|Y}(x|y) = 0.$

$\Rightarrow \quad f_{X|Y}(x|y) = f(x)$ for some f

$\Rightarrow \quad f_{XY} = f(x) f_Y(y)$

$\int f_{XY}\,dy = f(x) \int f_Y(y)dy = f_X(x) \quad \Rightarrow \quad f_X(x) = f(x).$

i.e. $f_{X|Y}(x) = f_X(x). \Rightarrow f_{XY} = f_X f_Y.$

$$\mathrm{Cov}(X,Y) = \mathbb{E}(X-\mu_X)(Y-\mu_Y)$$

$$= \iint (X-\mu_X)(Y-\mu_Y)\, f_X f_Y\,dx\,dy$$

$$= \int (Y-\mu_Y) \int x-\mu_x f_X dx \, f_Y dy$$

$$= \int (Y-\mu_Y) \cdot 0 f_Y dy = 0 .$$

3.20

$$\mathbb{E}(AX) = \int A x f_X(x) dx = A \int x f_X(x) dx = A\mathbb{E}(X)$$

$$\mathbb{V}(X) = \left[Cov(X_i, X_j) \right] = \left[\mathbb{E}(X_i-\mu_i)(X_j-\mu_j) \right]$$

$$= \mathbb{E}\left[(X_i-\mu_i)(X_j-\mu_j) \right] = \mathbb{E}\left[(X-\mu)(X-\mu)^T \right].)$$

$$= \mathbb{E}\left[X_i X_j - \mu_i \mu_j \right] = \mathbb{E}\left[XX^T \right] - \mu\mu^T .$$

$$\mathbb{V}(AX) = \mathbb{E}(AXX^TA^T) - A\mu\mu^TA^T$$

$$= A(\mathbb{E}(XX^T) - \mu\mu^T)A^T = A\Sigma A^T .$$

3.21 $\mathbb{E}(Y|X) = X . \Rightarrow \int y f_{Y|X}(y|x) dy = x .$

$\mathbb{E}(Y) = \mathbb{E}[\mathbb{E}(Y|X)] = \mathbb{E}(X) .$

$\mathbb{E}(XY) = \mathbb{E}(\mathbb{E}(XY|X))$

$$= \int \int xy\, f_{Y|X}(y|x)\,dy\, f_X(x)\,dx.$$

$$= \int x \int y\, f_{Y|X}(y|x)\,dy\, f_X(x)\,dx$$

$$= \int x^2 f_X(x)\,dx = \mathbb{E}(X^2).$$

$$\mathrm{Cov}(X,Y) = E(XY) - \mu_X\mu_Y = E(X^2) - \mu_X^2 = V(X).$$

3.22 (a) $\mathbb{P}(Y=1) = b \quad \mathbb{P}(Y=0) = 1-b$

$\mathbb{P}(Z=1) = 1-a, \quad \mathbb{P}(Z=0) = a.$

$\mathbb{P}(Y=1, Z=0) = \mathbb{P}(X<a) = a$

$\mathbb{P}(Y=1, Z=0) = \mathbb{P}(Y=1)\mathbb{P}(Z=0) \Leftrightarrow ab = a$

$\Leftrightarrow a=0 \text{ or } b=1. \quad \cancel{\times}.$

Thus dependent.

$a<x<1. \quad 0<x<a$

(b) $\mathbb{E}(Y|Z=1) = 1 \times \frac{b-a}{1-a} + 0 \times \frac{1-b}{1-a} = \frac{b-a}{1-a}.$

$\mathbb{E}(Y|Z=0) = 1 \times \frac{a}{a} + 0 \times \frac{0}{a} = 1$

$\mathbb{E}(Y|Z) = \left(\frac{b-a}{1-a} - 1\right)Z + 1. = \frac{b-1}{1-a} \cdot Z + 1.$

3.23 Poisson $f_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x \geq 0, x \in \mathbb{Z}.$

$X \sim \mathrm{Poisson}(\lambda)$

$$\varphi_X(t) = \mathbb{E}(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} \sum \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}.$$

Normal $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

$x^2 - 2\mu x + \mu^2 - 2t\sigma^2 x.$

$$\varphi_X(t) = \int e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

$$= \int \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu-\sigma^2 t)^2} e^{t\mu + \frac{1}{2}\sigma^2 t^2} dx$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$(x-\mu)^2 + t(x-\mu)$

Gamma $f_X(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, \ x > 0.$

$$\varphi_X(t) = \int_0^\infty e^{tx} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$$

$$= \int_0^\infty \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\left(\frac{1-\beta t}{\beta}\right)x}$$

$$= \int_0^\infty \frac{\left(\frac{\beta}{1-\beta t}\right)^\alpha}{\beta^\alpha} \frac{1}{\left(\frac{\beta}{1-\beta t}\right)^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{1}{\frac{\beta}{1-\beta t}} x}.$$

$$= \left(\frac{1}{1-\beta t}\right)^\alpha$$

3.24 $f_{X_i}(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}$.

$$\varphi_{X_i}(t) = \int_0^\infty e^{tx} \frac{1}{\beta} e^{-\frac{x}{\beta}} dx = \int_0^\infty \frac{1}{\beta} e^{-\frac{1-\beta t}{\beta} x} dx$$

$$= \left[-\frac{1}{1-\beta t} e^{-\frac{1-\beta t}{\beta} x} \right]_0^\infty = \frac{1}{1-\beta t}.$$

$$\varphi_{\Sigma X_i}(t) = \prod \varphi_{X_i}(t) = \left(\frac{1}{1-\beta t}\right)^n = \varphi_Y(t), \quad Y \sim Gamma(n, \beta).$$

Thus, $\Sigma X_i = Y$ by theorem 3.33.

4

Inequalities

4.1 Refer HW2.

$e^{-(k+1)}$ vs $\frac{1}{k^2}$.

4.2 $\sigma_X^2 = \lambda,\ \mu_X = \lambda.$

$$P(X \geq 2\lambda) = P(|X-\mu_X| \geq \sqrt{\lambda}\,\sigma_X) \leq \frac{1}{(\sqrt{\lambda})^2} = \frac{1}{\lambda}.$$

4.3 HW2 #2

4.4 (a). $\mathbb{P}(C_n \text{ contains } p) = \mathbb{P}(\hat{p}_n - \varepsilon_n < p < \hat{p}_n + \varepsilon_n)$

$$= \mathbb{P}(|\hat{p}_n - p| < \varepsilon_n)$$

$$= 1 - \mathbb{P}(|\hat{p}_n - p| > \varepsilon_n)$$

$$\geq 1 - 2e^{-2n\varepsilon_n^2} = 1-\alpha.$$

$$\varepsilon_n = \sqrt{\frac{1}{2n}\log\left(\frac{2}{\alpha}\right)} \Rightarrow 2e^{-2n\varepsilon_n^2} = 2e^{-\log\frac{2}{\alpha}} = \alpha.$$

4.5 $Z \sim N(0,1).$ $\qquad f_Z'(t) = -t f_Z(t) \Rightarrow f_Z(t) = -\frac{1}{t} f_Z'(t).$

$$\mathbb{P}(Z > t) = \int_t^\infty f_Z(u)\,du = \int_t^\infty -\frac{f_Z'(u)}{u}\,du \leq \int_t^\infty \frac{-f_Z'(u)}{t}\,dt = \frac{1}{t} f_Z(t).$$

$$\left(-f_Z'(u) > 0,\ \frac{1}{u} < \frac{1}{t}\right).$$

$$= \frac{1}{t\sqrt{2\pi}} e^{-\frac{1}{2}t^2}.$$

$\mathbb{P}(Z > t) = \mathbb{P}(Z < -t)$

Thus, $\mathbb{P}(|Z| > t) = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{1}{2}t^2}}{t}$.

4.1 $\bar{X}_n \sim N(0, \frac{1}{n})$. $\Rightarrow \sqrt{n}\,\bar{X}_n \sim N(0,1)$

By Mill, $\mathbb{P}(|\bar{X}_n| > t) = \mathbb{P}(|\sqrt{n}\,\bar{X}_n| > \sqrt{n}t) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{t^2}{2n}}}{\sqrt{n}t} = \sqrt{\frac{2}{n\pi}} \frac{e^{-\frac{t^2}{2n}}}{t}$.

By Chebyshev, $\mathbb{P}(|\bar{X}_n| > t) = \mathbb{P}(|\bar{X}_n| > \sqrt{\frac{1}{n}} \cdot \sqrt{n}t) \leq \frac{1}{(\sqrt{n}t)^2} = \frac{1}{nt^2}$.

5

Convergence of Random Variables

§ Quick Review

$X_n \xrightarrow{qm} X \Rightarrow X_n \xrightarrow{P} X$.

$\because$) $\mathbb{E}(X_n - X)^2 \to 0$ as $n \to \infty$

$$\mathbb{P}(|X_n - X| > \varepsilon) = \mathbb{P}(|X_n - X|^2 > \varepsilon^2) \le \frac{\mathbb{E}(X_n - X)^2}{\varepsilon^2} \to 0 \text{ as } n \to \infty$$

$X_n \xrightarrow{P} X \Rightarrow X_n \rightsquigarrow X$.

$\because$) Let $X \sim F$ and F be conti at x.

$$F_n(x) = \mathbb{P}(X_n \le x) = \mathbb{P}(X_n \le x, X \le x+\varepsilon), \mathbb{P}(X_n \le x, X > x+\varepsilon)$$
$$\le F(x+\varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon)$$

$$F(x-\varepsilon) = \mathbb{P}(X \le x-\varepsilon) = \mathbb{P}(X \le x-\varepsilon, X_n \le x) + \mathbb{P}(X \le x-\varepsilon, X_n > x)$$
$$\le F_n(x) + \mathbb{P}(|X_n - X| > \varepsilon)$$

If $X = c$, $X_n \rightsquigarrow X \Rightarrow X_n \xrightarrow{P} X$.

$$P(|X_n - c| > \varepsilon) = \mathbb{P}(X_n < c-\varepsilon) + \mathbb{P}(X_n > c+\varepsilon)$$
$$\le F_n(c-\varepsilon) + 1 - F_n(c+\varepsilon)$$
$$\to F(c-\varepsilon) + 1 - F(c+\varepsilon) \quad (n \to \infty)$$
$$= 0 + 1 - 1 = 0$$

$$
\begin{array}{ccccccccc}
 & & & & & & as & & \\
 & & & & & & \downarrow & & \\
qm & = & L_2 & \longrightarrow & L_1 & \longrightarrow & p & \longrightarrow & dis
\end{array}
$$

$(\mathbb{E}|X_n - X|)^2 \le \mathbb{E}(X_n - X)^2$. (Jensen).

$\frac{\mathbb{E}|X_n - X|}{\varepsilon}$

$\mathbb{P}(|X_n - X| > \varepsilon) \longrightarrow 0$? Given $\varepsilon > 0$,

Let $A_n = \{s \mid |X_n(s) - X(s)| < \varepsilon\}$

$B_m = \bigcap_{n \ge m} A_n$. $B_1 \subseteq B_2 \subseteq \cdots$

$C = \bigcup B_m$.

$X_n \xrightarrow{as} X$. $\Rightarrow \mathbb{P}(C) = 1$

$\Rightarrow \lim_{m \to \infty} \mathbb{P}(B_m) = 1 \Rightarrow \lim \mathbb{P}(A_n) = 1$

$\mathbb{P}(|X_n - X| > \varepsilon) \le 1 - \mathbb{P}(A_n) \longrightarrow 0$

5.1

(a) $S_n^2 = \frac{1}{n-1}\sum_i (X_i - \overline{X}_n)^2$.

$\mathbb{E}(X_i - \overline{X}_n) = 0$.

$$\mathbb{V}(X_i - \overline{X}_n) = \mathbb{V}\left(\frac{n-1}{n} X_i - \frac{1}{n}\sum_{j\neq i} X_j\right)$$
$$= \mathbb{V}\left(\frac{n-1}{n} X_i\right) + \frac{1}{n^2}\sum_{j\neq i} \mathbb{V}(X_j)$$
$$= \left(\frac{n-1}{n}\right)^2 \sigma^2 + \frac{n-1}{n^2}\sigma^2 = \frac{n-1}{n^2}\cdot n\sigma^2 = \frac{n-1}{n}\sigma^2.$$

$$\mathbb{E}(S_n^2) = \frac{1}{n-1}\sum \mathbb{E}(X_i - \overline{X}_n)^2 = \frac{1}{n-1}\sum \mathbb{V}(X_i - \overline{X})$$
$$= \frac{n}{n-1}\cdot\frac{n-1}{n}\sigma^2 = \sigma^2.$$

$g(\mathbb{E}(X)) \leq \mathbb{E}(g(X))$

5.2

(⇒) $\mathbb{E}(X_n-b)^2 \to 0$.

$$(\mathbb{E}(X_n)-b)^2 = (\mathbb{E}(X_n-b))^2 \le \mathbb{E}(X_n-b)^2 \to 0.$$

Thus, $\lim \mathbb{E}(X_n) \to b$.

$$\mathbb{V}(X_n) = \mathbb{E}(X_n - \mathbb{E}(X_n))^2$$
$$= \mathbb{E}(X_n - b + b - \mathbb{E}(X_n))^2$$
$$= \mathbb{E}(X_n-b)^2 + 2\mathbb{E}(X_n-b)(b-\mathbb{E}(X_n)) + \mathbb{E}(b-\mathbb{E}(X_n))^2.$$
$$= \cancelto{0}{\mathbb{E}(X_n-b)^2} + 2\cancelto{0}{(b-\mathbb{E}(X_n))}\,\mathbb{E}\cancelto{0}{(X_n-b)} + \cancelto{0}{(b-\mathbb{E}(X_n))^2}.$$

$L_2 \to L_1$ $\quad \mu_n \to b$.

(⇐) $\mathbb{E}(X_n-b)^2 = \mathbb{E}(X_n-\mu_n+\mu_n-b)^2$ $\quad (\mu_n = \mathbb{E}(X_n))$

$$= \mathbb{E}(X_n-\mu_n)^2 + 2\mathbb{E}(X_n-\mu_n)(\mu_n-b) + \mathbb{E}(\mu_n-b)^2$$
$$= \cancelto{0}{\mathbb{V}(X_n)} + 2\cancelto{0}{(\mu_n-b)}\,\underbrace{\mathbb{E}(X_n-\mu_n)}_{=0} + \cancelto{0}{(\mu_n-b)^2}.$$

5.3

Let $\sigma^2 = \mathbb{V}(X_1) < \infty$. $\quad \mathbb{E}(\bar{X}_n) = \mu$.

$$\mathbb{E}(\bar{X}_n-\mu)^2 = \mathbb{V}(\bar{X}_n) = \frac{1}{n^2}\sum \mathbb{V}(X_i) = \frac{1}{n}\mathbb{V}(X_1) = \frac{\sigma^2}{n} \to 0 \text{ as } n \to \infty$$

5.4

Does not converge in qm:

$$\mathbb{E}(X-X_n)^2 = \left(\frac{1}{n}\right)^2\left(1-\frac{1}{n^2}\right) + n^2 \cdot \frac{1}{n^2}$$

$$= \frac{1}{n^2}\left(1-\frac{1}{n^2}\right)+1 \to 1 \text{ as } n\to\infty.$$

Converge in Probability:

$$\mathbb{P}\left(|X-X_n| = \frac{1}{n}\right) = 1-\frac{1}{n^2}, \quad \mathbb{P}(|X-X_n| = n) = \frac{1}{n^2}.$$

$\Rightarrow$ Given $\varepsilon>0$, take $n > \frac{1}{\varepsilon}$.

$$\mathbb{P}(|X-X_n| > \varepsilon) \leq 1-\mathbb{P}\left(|X-X_n| = \frac{1}{n}\right) = 1-\left(1-\frac{1}{n^2}\right) = \frac{1}{n^2} \to 0.$$