# Homework 1

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## 1 Characterization of Convex function

**(1)** Let $f$ be the function of the consideration. First, we prove if the function is convex, its epigraph is a convex set.

$$\mathrm{epi}(f) = \{(x, y) : x \in \mathrm{dom}(f), y \geq f(x)\}$$

If $(x, y), (x', y') \in \mathrm{epi}(f)$, then $x, x' \in \mathrm{dom}(f)$ and $y \geq f(x), y' \geq f(x')$. For any $t \in (0, 1)$, let $z = tx + (1-t)x', w = ty + (1-t)y'$. As $\mathrm{dom}(f)$ is a convex set, by the definition of convex function, $z \in \mathrm{dom}(f)$.

$$f(z) \geq tf(x) + (1-t)f(x') \geq ty + (1-t)y' = w$$

Here, the first inequality comes from the definition of convex function, and the second inequality comes from $y \geq f(x), y' \geq f(x')$. Thus, $(z, w) = t(x, y) + (1-t)(x', y') \in \mathrm{epi}(f)$. This shows $\mathrm{epi}(f)$ is a convex set.

**(2)** Now for the converse direction, assume $\mathrm{epi}(f)$ is a convex set. First, we show the domain of $f$ is convex. For $x, x' \in \mathrm{dom}(f)$, let $y = f(x), y' = f(x')$. Then, $(x, y), (x', y') \in \mathrm{epi}(f)$. For any $t \in (0, 1)$

$$(tx + (1-t)x', ty + (1-t)y') = t(x, y) + (1-t)(x', y') \in \mathrm{epi}(f)$$

as $\mathrm{epi}(f)$ is a convex set. By the definition of $\mathrm{epi}(f)$, $tx + (1-t)x' \in \mathrm{dom}(f)$. Hence, $\mathrm{dom}(f)$ is convex. From above and the definition of $\mathrm{epi}(f)$, we also have $f(tx + (1-t)x') \leq ty + (1-t)f' = tf(x) + (1-t)f(x')$, which completes to show $f$ is convex.

## 2 Feasible Sets as a Convex set

**(1)** If $g$ is convex, then $\{g \leq 0\} := \{x \in \mathrm{dom}(g) : g(x) \leq 0\}$ is convex. This is because level set of a convex function is convex.

**(2)** If $h$ is affine, then $\{h = 0\} := \{x \in \mathrm{dom}(h) : h(x) = 0\}$ is convex. This is because $\{0\}$ as a subset of $\mathbb{R}$ is convex, and inverse affine image of convex set is convex.

**(3)** Given set $C$ is the intersection of $C_i := \{g_i \leq 0\}$ for $i = 1, \cdots, p$ and $D_j := \{h_j = 0\}$ for $j = 1, \cdots, q$. $C_i, D_j$ are convex by (1), (2). As arbitary intersection of convex sets is convex, $C$ is convex.

# 3 Examples of Convex and Concave Functions

**(a) Negative Entropy Function** For $g : \mathbb{R}_{++} \longrightarrow \mathbb{R}$ by $g(x) = x\log(x)$ is convex. This is because

$$g'(x) = 1 + \log(x), g''(x) = 1/x > 0$$

for all $x \in \mathrm{dom}(g)$ and by the second order characterization of convex function. For $x \in \mathbb{R}^d_{++}$, $h_i(x) = x_i$ is affine and thus

$$x \mapsto x_i \log(x_i) = g(h_i(x))$$

is convex by affine composition. $f$ is a sum of finite convex functions $g \circ h_i$ for $i = 1, \cdots, d$ and thus is convex.

**(b) Geometric Mean** is concave. Let $x, y \in \mathrm{dom}(f)$, $t \in (0, 1)$. Let $z = tx + (1-t)y$. Note that $f(z) \neq 0$.

$$\begin{aligned}
\frac{tf(x) + (1-t)f(y)}{f(tx + (1-t)y)} &= \frac{t\left(\Pi_{i\in[d]} x_i\right)^{1/d} + (1-t)\left(\Pi_{i\in[d]} y_i\right)^{1/d}}{\left(\Pi_{i\in[d]} z_i\right)^{1/d}} \\
&= t\left(\Pi_{i\in[d]} \frac{x_i}{z_i}\right)^{1/d} + (1-t)\left(\Pi_{i\in[d]} \frac{y_i}{z_i}\right)^{1/d} \\
&\leq \frac{t}{d}\sum_{i\in[d]} \frac{x_i}{z_i} + \frac{1-t}{d}\sum_{i\in[d]} \frac{y_i}{z_i} \\
&= \frac{1}{d}\sum_{i\in[d]} \frac{z_i}{z_i} \\
&= 1
\end{aligned}$$

This shows that $tf(x) + (1-t)f(y) \leq f(tx + (1-t)y)$ and so $f$ is concave.

**(c) Conjugate** of a function is convex. For each $y \in \mathbb{R}^d$,

$$f_y(x) := < y, x > - f(y)$$

is an affine function in $x$ and thus convex.

$$f^*(x) = \sup_{y\in\mathbb{R}^d}\{< y, x > - f(y)\} = \sup_{y\in\mathbb{R}^d} f_y(x)$$

is a point-wise supremum of convex functions $f_y$ and thus is convex.

**(d) Sum of k largest component** is convex. Let $S_d$ be the set of all permutation $\tau : [d] \longrightarrow [d]$. Let $k \leq d$ be given. For each fixed $\tau$, define function $f_\tau : \mathbb{R}^d \longrightarrow \mathbb{R}$ by

$$x \mapsto x_{\tau(1)} + \cdots + x_{\tau(k)}$$

which is a affine function of $x$. Hence, $f_\tau$ is convex function. Let

$$g(x) = \max\{f_\tau(x) : \tau \in S_d\}$$

$g$ is a arbitrary point-wise maximum of convex function and thus convex. I claim that the given $f$ in the question is indeed equal to $g$. For each $x \in \mathbb{R}^d$, $f(x) = x_{\sigma(1)} + \cdots + x_{\sigma(k)} = f_\sigma(x)$ for some $\sigma \in S_d$. By definition of $g$, $f(x) \leq g(x)$. Sum of largest distinct $k$ elements in the (multi)set is greater or equal to sum of any distinct $k$ elements in the (multi)set. Hence for any $\tau \in S_d$,

$$f(x) = x_{\sigma(1)} + \cdots + x_{\sigma(k)} \geq x_{\tau(1)} + \cdots + x_{\tau(k)} = f_\tau(x)$$

Take maximum over $\tau \in S_d$ to obtain $f(x) \geq g(x)$. Thus, $f = g$ is convex.

# 4 Uncertainty Qualification Problem is Convex

**(a) Nuclear Norm is a norm**. To prove the given norm is a norm, we have to check three things. Let $||\cdot||$ denote the 2-norm.

- $||S||_{nuc} \geq 0$ and the equality holds if and only if $S = 0$.
  If $(\lambda, v)$ is an eigenpair of $S^TS$, then

  $$||Sv||^2 = v^TS^TSv = \lambda v^Tv = \lambda||v||^2 \geq 0$$

  As $||v||^2 > 0$, we have $\lambda \geq 0$. Thus all eigenvalues of $S^TS$ is nonnegative. Thus, $||S||_{nuc}$ is a real value and $||S||_{nuc} \geq 0$. If $||S||_{nuc} = 0$, then all the eigenvalues of $S^TS$ should be 0. As $S^TS$ is symmetric, it has eigendecomposition

  $$S^TS = V\Lambda V^T$$

  for some diagonal $\Lambda$ and orthonormal $V$. If all eigenvalues of $S^TS$ are 0, then the diagonal matrix $\Lambda$ which has eigenvalues as its diagonal elements should be the zero matrix. This shows $S^TS$ is a zero matrix. For any $x$,

  $$||Sx||^2 = x^TS^TSx = x^T0x = 0$$

  and so $Sx$ is a zero vector. Taking $x = e_i$, the i-th element of the standard basis, $Se_i = 0$ shows i-th column of $S$ is a zero vector. Thus, $S$ is a zero matrix.

- $||tS||_{nuc} = |t| \cdot ||S||_{nuc}$ for scala $t$
  Note that by linearity of matrix multiplication, eigenvalues of $tX$ is $t$ times the eigenvalue of $X$. i.e. if $(\lambda, v)$ is an eigenpair of $X$,

  $$(tX)v = t(Xv) = t(\lambda v) = (t\lambda)v$$

  and so $(t\lambda, v)$ is an eigenpair of $tX$. $||tS||_{nuc}$ is square root of the sum of the square of eigenvalues of $(tS)^T(tS) = t^2S^TS$. By the note above, this is

  $$||tS||_{nuc} = \sum_{i=1}^{d} \sqrt{t^2\lambda_i(S^TS)} = |t| \sum_{i=1}^{d} \sqrt{\lambda_i(S^TS)} = |t| \cdot ||S||_{nuc}$$

- $||S+T||_{nuc} \le ||S||_{nuc} + ||T||_{nuc}$
  To prove this, we first need to prove

  $$||S||_{nuc} = \sup_{\sigma_1(X)\le 1} < X, S >$$

  where $\sigma_1(X)$ is the largest singular value of $X$ and $< A, B >:= tr(A^T B)$.
  ($\le$)
  Let $S = U\Sigma V^T$ be the SVD of $S$. Then, $||S||_{nuc} = tr(\Sigma)$. This is straightforward from $S^T S = V\Sigma^2 V^T$. Take $Q = UV^T$. $Q$ has $\sigma_1(Q) = 1$ as all its singular values are 1. $Q^T S = VU^T U\Sigma V^T = V\Sigma V^T$. Thus, $< Q, S >= tr(Q^T S) = tr(V\Sigma V^T) = tr(\Sigma) = ||S||_{nuc}$. This shows

  $$||S||_{nuc} \le \sup_{\sigma_1(X)\le 1} < X, S >$$

  ($\ge$)
  Now, to prove converse, let $Q$ be a matrix with $\sigma_1(Q) \le 1$. Then,

  $$tr(Q^T S) = tr(Q^T U\Sigma V^T) = tr(V^T Q^T U\Sigma V^T V) = tr(V^T Q^T U\Sigma)$$

  Thus, $< Q, S >= tr(Q^T S) = tr(V^T Q^T U\Sigma) =< U^T QV, \Sigma >$.

  $$\begin{aligned} < U^T QV, \Sigma > &= \sum_{i=1}^{n} (U^T QV)_{ii}\Sigma_{ii} \\ &= \sum_{i=1}^{n} u_i^T Q v_i \Sigma_{ii} \\ &\le \sum_{i=1}^{n} ||u_i|| \cdot ||Qv_i||\Sigma_{ii} \\ &\le \sum_{i=1}^{n} \sigma_1(Q)||u_i|| \cdot ||v_i||\Sigma_{ii} \\ &= \sum_{i=1}^{n} \sigma_1(Q)\Sigma_{ii} \\ &\le \sum_{i=1}^{n} \Sigma_{ii} = tr(\Sigma) = ||S||_{nuc} \end{aligned}$$

  Hence, $< Q, S >\le ||S||_{nuc}$. Take supremum over all $Q$ with $\sigma_1(Q) \le 1$. Then, the converse inequality is proved.

  Now, using above fact,

  $$\begin{aligned} ||S+T||_{nuc} &= \sup_{\sigma_1(X)\le 1} < X, S+T > \\ &= \sup_{\sigma_1(X)\le 1} < X, S > + < X, T > \\ &\le \sup_{\sigma_1(X)\le 1} < X, S > + \sup_{\sigma_1(X)\le 1} < X, T > \\ &= ||S||_{nuc} + ||T||_{nuc} \end{aligned}$$

By above three argument, nuclear norm is a norm.

**(b)** $\mathcal{C} := \{S \in \mathbb{R}^{d\times d} : \bar{\Sigma} + S \succeq 0\}$ is convex. Suppose $S, T \in \mathcal{C}$. For any $t \in (0, 1)$,

$$\bar{\Sigma} + (tS + (1-t)T) = t(\bar{\Sigma} + S) + (1-t)(\bar{\Sigma} + T)$$

Positive scala multiplication of positive semidefinite(PSD) matrix is PSD. Sum of two PSD matrices is PSD. These two facts are by the linearity of matrix multiplication and the definition of PSD matrix. We conclude $\bar{\Sigma}+(tS+(1-t)T)$ is PSD and therefore $tS + (1-t)T \in \mathcal{C}$. Hence, $\mathcal{C}$ is a convex set.

**(c)** From (a), $||\cdot||_{nuc}$ is a norm. The feasible set is intersection of $\mathcal{C}$(defined in (b)) and a norm ball. As intersection of two convex set is convex, the feasible set is convex.

By the linearity of matrix multiplication, the objective function is linear with respect to our variable $S$ and so is convex. Thus, the formulation is a convex optimization.