MAS550 Probability Theory Homework 5

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#4.4.5

$\mathbb{E}(\mathbb{E}[Y|\mathcal{G}] - \mathbb{E}[Y|\mathcal{F}])^2 = \mathbb{E}(\mathbb{E}[Y|\mathcal{G}])^2 + \mathbb{E}(\mathbb{E}[Y|\mathcal{F}])^2$
$\qquad - 2\mathbb{E}(\mathbb{E}[Y|\mathcal{G}]\mathbb{E}[Y|\mathcal{F}])$.

Hence, it suffices to show $\mathbb{E}(\mathbb{E}[Y|\mathcal{G}]\mathbb{E}[Y|\mathcal{F}]) = \mathbb{E}(\mathbb{E}[Y|\mathcal{F}])^2 \cdots (*)$

$\mathbb{E}(\mathbb{E}[Y|\mathcal{G}]\mathbb{E}[Y|\mathcal{F}]) = \mathbb{E}(\mathbb{E}[\mathbb{E}[Y|\mathcal{F}]Y|\mathcal{G}])$,
as $\mathbb{E}[Y|\mathcal{F}] \in \mathcal{F} \subseteq \mathcal{G}$

$= \mathbb{E}(\mathbb{E}[Y|\mathcal{F}]Y)$

$\mathbb{E}(\mathbb{E}[Y|\mathcal{F}])^2 = \mathbb{E}(\mathbb{E}[Y|\mathcal{F}]Y|\mathcal{F})$, as $\mathbb{E}[Y|\mathcal{F}] \in \mathcal{F}$

Hence, we get $(*)$

#4.4.7

Square of Martingale is sub-Martingale by Jensen.
Thus, $Y_n = (X_n + c)^2$ is sub-Martingale, and nonnegative.
Let $\bar{X}_n = \max_{1 \le m \le n} X_m$, $\bar{Y}_n = \max_{1 \le m \le n} Y_m$

$\mathbb{P}(\bar{X}_n \ge \lambda) = \mathbb{P}(\bar{Y}_n \ge (\lambda + c)^2)$, for $c \ge 0$

By Doob, $(\lambda + c)^2 \mathbb{P}(\bar{Y}_n \ge (\lambda+c)^2) \le \mathbb{E}Y_n$

Thus, $\mathbb{P}(\bar{X}_n \ge \lambda) \le \frac{1}{(\lambda+c)^2}\mathbb{E}Y_n = \frac{1}{(\lambda+c)^2}\mathbb{E}(X_n + c)^2 = \varphi(c)$.

$\mathbb{E}(X_n) = \mathbb{E}X_0 = 0. \Rightarrow \varphi(c) = (\lambda+c)^{-2}(\mathbb{E}(X_n^2) + c^2)$

$\Rightarrow \varphi'(c) = (\lambda+c)^{-4}(2c(\lambda+c)^2 - 2(\lambda+c)(\mathbb{E}(X_n^2) + c^2))$

$\varphi'(c) = 0 \Leftrightarrow c(\lambda+c) = \mathbb{E}(X_n^2) + c^2$

$\Rightarrow c^* = \lambda^{-1}\mathbb{E}(X_n^2)$.

$$\varphi(c^*) = \frac{\mathbb{E}(X_n^2) + (\lambda^{-1}\mathbb{E}(X_n^2))^2}{(\lambda + \lambda^{-1}\mathbb{E}(X_n^2))^2} = \frac{\lambda^2\mathbb{E}(X_n^2) + \mathbb{E}(X_n^2)^2}{(\lambda^2 + \mathbb{E}(X_n^2)^2)^2}$$

$$= \frac{\mathbb{E}(X_n^2)}{\lambda^2 + \mathbb{E}(X_n^2)}$$

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#4.4.8

(i) $\mathbb{E}(\bar{X}_n \wedge M) = \int_0^\infty \mathbb{P}(\bar{X}_n \wedge M > \lambda)\, d\lambda$

$$\le \int_0^1 d\lambda + \int_1^\infty \mathbb{P}(\bar{X}_n \wedge M \ge \lambda)\, d\lambda$$

$$\le 1 + \int_1^\infty \left(\lambda^{-1} \int X_n^+ \mathbb{1}_{(\bar{X}_n \wedge M \ge \lambda)}\, dP\right) d\lambda$$

$$= 1 + \iint_1^{\bar{X}_n \wedge M} \lambda^{-1} X_n^+\, d\lambda\, dP$$

$$= 1 + \int X_n^+ \log(\bar{X}_n \wedge M)\, dP$$

(ii) ① $a\log a + b/e \le a\log^+ a + b/e$.

$\because$) It suff. to prove $a\log a \le a\log^+ a$, for $a>0$

If $a \ge 1$, $a\log a = a\log^+ a$, as $\log a \ge 0$.

If $0 < a \le 1$, $a\log a < 0 = a\log^+ a$.

② $a\log b \le a\log a + b/e$, for $\forall b, \forall a>0$.

Fix $a>0$.

$a\log b - a\log a - b/e =: \varphi(b)$.

$\varphi(b)$ is concave. $\varphi'(b) = 0 \Rightarrow \frac{a}{b} = \frac{1}{e}$

$\Rightarrow b = ae$.

$\varphi(ae) = a\log ae - a\log a - ae/e$

$= a\log e - a = 0$.

Thus, $\varphi(b) \le 0\ \forall b$. i.e. $a\log b \le a\log a + b/e\ \forall b$.

#4.4.9

$\mathbb{E}(X_m - X_{m-1})(Y_m - Y_{m-1}) = \mathbb{E}X_mY_m - \mathbb{E}X_mY_{m-1} - \mathbb{E}X_{m-1}Y_m + \mathbb{E}X_{m-1}Y_{m-1}$

$$\begin{aligned}\mathbb{E}X_mY_{m-1} &= \mathbb{E}[\mathbb{E}[X_mY_{m-1}|\mathcal{F}_{m-1}]] \\ &= \mathbb{E}[\mathbb{E}[X_m|\mathcal{F}_{m-1}]Y_{m-1}] \\ &= \mathbb{E}[X_{m-1}Y_{m-1}]\end{aligned}$$

Similarly, $\mathbb{E}[X_{m-1}Y_m] = \mathbb{E}[X_{m-1}Y_{m-1}]$.

Thus, $\mathbb{E}(X_m - X_{m-1})(Y_m - Y_{m-1}) = \mathbb{E}X_mY_m - \mathbb{E}X_{m-1}Y_{m-1}$.

$\Rightarrow \sum_{m=1}^{n} \mathbb{E}(X_m - X_{m-1})(Y_m - Y_{m-1}) = \mathbb{E}X_nY_n - \mathbb{E}X_0Y_0$.

#4.4.10

① $\mathbb{E}X_n^2 \le M < \infty$ for some constant M.

$X_n = X_0 + \sum_{i=1}^{n} \xi_i$.

$\mathbb{E}X_n^2 = \mathbb{E}X_0^2 + \sum_{i=1}^{n}\mathbb{E}\xi_i^2 + 2\sum_{i=1}^{n}\mathbb{E}X_0\xi_i + 2\sum_{\substack{i\neq j \\ i,j\in[n]}}\mathbb{E}\xi_i\xi_j$

$X_0 \in \mathcal{F}_0 \subseteq \mathcal{F}_{i-1}$, $\Rightarrow \mathbb{E}(X_0\xi_i) = \mathbb{E}X_0(X_i - X_{i-1}) = 0$, by thm 4.7.7

If $i<j$, then $\xi_i \in \mathcal{F}_i \subseteq \mathcal{F}_{j-1} \Rightarrow \mathbb{E}(\xi_i\xi_j) = \mathbb{E}\xi_i(X_j - X_{j-1}) = 0$, "

Thus, $\mathbb{E}X_n^2 = \mathbb{E}X_0^2 + \sum_{i=1}^{n}\mathbb{E}\xi_i^2 \le \mathbb{E}X_0^2 + \sum\mathbb{E}\xi_i^2 =: M < \infty$

② By thm 4.4.6, $X_n \to X_\infty$ a.s. in L^2.

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#4.6.4

If $\limsup X_n < \infty$, then for $\varepsilon > 0$, $\exists N$ s.t. $X_n < \limsup X_n + \varepsilon$ $\forall n \geq N$

Consider $\mathcal{F}_n = \sigma(X_1, \cdots X_n) \uparrow \mathcal{F}_\infty$, $D \in \mathcal{F}_\infty$.

Thus, $\mathbb{P}(D | X_1, \cdots, X_n) = \mathbb{E}[\mathbb{1}_D | \mathcal{F}_n] \to \mathbb{1}_D$ a.s.

$\mathbb{P}(D | X_1, \cdots, X_n) \geq \delta(x) > 0$ a.s. on $\{X_n \leq x\}$

Take $x = \limsup X_n + \varepsilon$. Then $\mathbb{P}(D | X_1, \cdots, X_n) \geq \delta(x) > 0$ a.s. $\forall n \geq N$

By taking $n \to \infty$, $\mathbb{1}_D > \delta(x) > 0$ a.s.

i.e. $\limsup X_n < \infty$ implies D a.s.

As $\mathbb{P}(\limsup X_n < \infty$ or $\limsup X_n = \infty) = 1$, done.

#4.6.6

① X_n is Martingale.

$\mathbb{E}[X_{n+1} | \mathcal{F}_n] \overset{!}{=} X_n(\alpha + \beta X_n) + (1 - X_n)\beta X_n$

$= (\alpha + \beta) X_n = X_n$.

As X_n are uniformly integrable, $X_n \to X_\infty$ in L_1 and a.s.

↳ For being bdd on $[0, 1]$.

If $X_n = x \in (0, 1)$, then

$$X_n - X_{n+1} = \begin{cases} x - (\alpha + \beta x) = (1-\beta)x - \alpha = \alpha(x-1) > 0 \\ x - \beta x = (1-\beta)x = \alpha x > 0. \end{cases}$$

Thus, If $X_n \to x \in (0,1)$, X_n is not Cauchy. ⚡

Hence, $\mathbb{P}(X_\infty \in \{0, 1\}) = 1$.

$\theta = \mathbb{E}X_0 = \mathbb{E}X_n \to \mathbb{E}X_\infty = \mathbb{P}(X_\infty = 1)$, as $X_\infty \in \{0, 1\}$.

Thus, $\mathbb{P}(X_\infty = 1) = \theta$.

#4.6.7

Thm 4.6.8 deals case where Y_n is fixed.

$$\mathbb{E}|\mathbb{E}[Y_n|\mathcal{F}_n]-\mathbb{E}[Y|\mathcal{F}_\infty]| \le \mathbb{E}|\mathbb{E}[Y_n|\mathcal{F}_n]-\mathbb{E}[Y|\mathcal{F}_n]| + \mathbb{E}|\mathbb{E}[Y|\mathcal{F}_n]-\mathbb{E}[Y|\mathcal{F}_\infty]|$$

$$= \mathbb{E}|\mathbb{E}[Y_n-Y|\mathcal{F}_n]| + \mathbb{E}|\mathbb{E}[Y|\mathcal{F}_n]-\mathbb{E}[Y|\mathcal{F}_\infty]|.$$

$$\le \mathbb{E}(\mathbb{E}[|Y_n-Y|\,|\mathcal{F}_n]) + \mathbb{E}|\mathbb{E}[Y|\mathcal{F}_n]-\mathbb{E}[Y|\mathcal{F}_\infty]|$$

$$= \mathbb{E}|Y_n-Y| + \mathbb{E}|\mathbb{E}[Y|\mathcal{F}_n]-\mathbb{E}[Y|\mathcal{F}_\infty]|$$

As $Y_n \to Y_n$ in L_1, $\mathbb{E}|Y_n-Y| \to 0$ as $n\to\infty$

By thm 4.6.8, $\mathbb{E}[Y|\mathcal{F}_n] \to \mathbb{E}[Y|\mathcal{F}_\infty]$ in L_1.

Thus, $\mathbb{E}|\mathbb{E}[Y|\mathcal{F}_n]-\mathbb{E}[Y|\mathcal{F}_\infty]| \to 0$ as $n\to\infty$

i.e. $\mathbb{E}[Y_n|\mathcal{F}_n] \to \mathbb{E}[Y|\mathcal{F}_\infty]$ in L_1.

#4.7.2

$$|\mathbb{E}(Y_n|\mathcal{F}_n)-\mathbb{E}(Y_{-\infty}|\mathcal{F}_{-\infty})|$$

$$\le \underbrace{|\mathbb{E}(Y_n|\mathcal{F}_n)-\mathbb{E}(Y_{-\infty}|\mathcal{F}_n)|}_{①} + \underbrace{|\mathbb{E}(Y_{-\infty}|\mathcal{F}_n)-\mathbb{E}(Y_{-\infty}|\mathcal{F}_{-\infty})|}_{②}$$

$$① = |\mathbb{E}(Y_n-Y_{-\infty}|\mathcal{F}_n)|$$

$$\le \mathbb{E}[|Y_n-Y_{-\infty}|\,|\mathcal{F}_n] = ③$$

$$\lim_{n\to\infty} ③ \le \lim \mathbb{E}[\sup_{m,n\le N} |Y_m-Y_n|\,|\mathcal{F}_n]$$

$$= \mathbb{E}[\sup_{m,n\le N} |Y_m-Y_n|\,|\mathcal{F}_{-\infty}].$$

Take $N\to-\infty$. Then $\sup_{m,n\le N}|Y_m-Y_n| \to 0$ a.s. as $N\to-\infty$.

Thus, $\lim_{n\to\infty} ③ = 0$. $\Rightarrow \lim_{n\to\infty} ① = 0$ $\quad\Rightarrow \mathbb{E}[Y_n|\mathcal{F}_n] \to \mathbb{E}[Y_{-\infty}|\mathcal{F}_{-\infty}]$ in L_1.

As $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, $\lim_{n\to\infty} ② = 0$.

#4.7.4

As X_i are exchangable,

$$\mathbb{E}(X_1+\cdots+X_n)^2 = \mathbb{E}\left[\sum_i X_i^2 + 2\sum_{i\neq j} X_i X_j\right]$$
$$= n\mathbb{E}X_1^2 + n(n-1)\mathbb{E}X_1X_2 \geq 0.$$

Thus, $\mathbb{E}X_1X_2 \geq -\frac{1}{n-1}\mathbb{E}X_1^2$. Take $n\to\infty$ to conclude $\mathbb{E}X_1X_2 \geq 0$.

#4.8.3

Let $X_n = S_n^2 - n\sigma^2$

Then, $\mathbb{E}|X_T| \leq \mathbb{E}|S_T^2|$
$$\leq \mathbb{E}[S_T^2 \mathbb{1}_{T=\infty}] + \mathbb{E}[S_T^2 \mathbb{1}_{T<\infty}]$$
$$\leq a^2 + \mathbb{E}(a+\xi_T)^2$$
$$= 2a^2 + \sigma^2 < \infty.$$

$X_n \mathbb{1}_{T>n}$ is uniformly integrable:

$\because$) $|X_n \mathbb{1}_{T>n}| \leq a^2 \ \forall n. \Rightarrow \lim_{M\to\infty} \sup_n \mathbb{E}(|X_n \mathbb{1}_{T>n}|; |X_n \mathbb{1}_{T>n}|) = 0$

By thm 4.8.2, $X_{T\wedge n}$ is uniformly integrable.

Thus, $0 = \mathbb{E}X_{T\wedge n} \to \mathbb{E}X_T$.

i.e. $\mathbb{E}T \geq \mathbb{E}S_T^2/\sigma^2 \geq a^2/\sigma^2$, by def. of X_T.

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#4.8.4

Take X_n as #4.8.3.

$\mathbb{E} S_{T\wedge n}^2 = \sigma^2 \mathbb{E}[T\wedge n] \to \sigma^2 \mathbb{E}T$ by MCT, ... ①

for $\sigma^2 (T\wedge n) \uparrow \sigma^2 T$ as $n \uparrow \infty$.

As $\mathbb{E} S_{T\wedge n}^2 < \infty$, $S_{T\wedge n}$ converges in L^2.

As $S_{T\wedge n} \to S_T$ a.s., we have $S_{T\wedge n} \to S_T$ in L^2.

Thus, $\mathbb{E}(S_{T\wedge n}^2) \to \mathbb{E} S_T^2$. ... ②

By using ①, ②, we have $\mathbb{E} S_T^2 = \sigma^2 \mathbb{E} T$.

#4.8.5

(a) $\mathbb{E}\xi_i = p - q$. Thus, $\mathbb{E}Y_i = 0$, $Var(Y_i) = Var(X_i)$

$= \mathbb{E}X_i u^2 - (\mathbb{E}X_i)^2$.

$Z_n = (S_n - (p-q)n)^2 - n(1-(p-q)^2)$ is Martingale.

$\Rightarrow Z_{n\wedge V_0}$ is Martingale.

$\mathbb{E} Z_{0\wedge V_0} = 0$. Thus, $\mathbb{E} Z_{n\wedge V_0} = 0 \Rightarrow$

$(1-(p-q)^2)\mathbb{E}V_0 = \mathbb{E}(S_{V_0} - (p-q)V_0)^2$.

Using $\mathbb{E}S_{V_0} = 0$ by def. of V_0, we have

$(1-(p-q)^2)\mathbb{E}V_0 = (p-q)^2 \mathbb{E}V_0^2$.

Thus, $\mathbb{E}V_0^2 = \dfrac{1-(p-q)^2}{(p-q)^2} \mathbb{E}V_0 = \dfrac{1-(p-q)^2}{(q-p)^3} x$,

as $\mathbb{E}_x V_0 = \dfrac{x}{q-p}$.

(b) As it is proportional to $E_x V_0 \sim x$.

#4.8.6

(a) $X_n = \frac{\exp(\theta S_n)}{(\phi(\theta))^n}$ is exp. Martingale, $\phi(\theta) = \mathbb{E} e^{\xi\theta}$.

$X_0 = e^{\theta x}$. Thus,

$\mathbb{E}(X_{n\wedge V_0}) = \mathbb{E}(X_0) = e^{\theta x}$.

$\Rightarrow e^{\theta x} = \mathbb{E}_x\left(\frac{\exp(\theta S_{n\wedge V_0})}{\phi(\theta)^{n\wedge V_0}}\right)$

By BCT, RHS $\longrightarrow \mathbb{E}_x\left(\frac{\exp(\theta S_{V_0})}{\phi(\theta)^{V_0}}\right) = \mathbb{E}(\phi(\theta)^{-V_0})$

(b) $\phi(\theta) = 1/s. = pe^{\theta} + qe^{-\theta}$

$\Rightarrow pe^{2\theta} - e^{\theta}/s + q = 0$.

$\Rightarrow pse^{2\theta} - e^{\theta} + qs$

$\Rightarrow e^{\theta} = \frac{1 - \sqrt{1 - 4pqs^2}}{2ps}$

$\Rightarrow e^{\theta x} = \left(\frac{1 - \sqrt{1 - 4pqs^2}}{2ps}\right)^x = \mathbb{E}_x(\phi(\theta)^{-V_0}) = \mathbb{E}_x(s^{V_0})$

#4.8.8

① $X_{\tau\wedge n}$ is uniformly integrable.

$S_{\tau\wedge n}$ is bdd $\Rightarrow X_{\tau\wedge n}$ also. Thus, is uniformly integrable.

② By optional stopping thm, $\mathbb{E}[X_{n\wedge\tau}] = \mathbb{E}[X_0] = 1$

By DCT, $\mathbb{E}[X_\tau] = \mathbb{E}[\lim X_{n\wedge\tau}]$

$= \lim \mathbb{E}[X_{n\wedge\tau}] = 1$.

③ By Chebyshev,

$1 = \mathbb{E}X_\tau \geq e^{a\theta_0}\mathbb{P}(X_\tau \geq e^{a\theta_0}) = e^{a\theta_0}\mathbb{P}(S_\tau \leq a)$

Thus, $\mathbb{P}(S_\tau \le a) \le e^{-a\theta_0}$.