MAS550 Homework 3   20200130 Yujun Kim

#3.4.2

(a) By CLT, $\dfrac{S_n}{\sigma\sqrt{n}} \Rightarrow Z$ (standard normal)

for $Var(X_i) = \sigma^2 < \infty$.

Thus, $\lim \mathbb{P}\left(\frac{S_n}{\sqrt{n}} > z\right) = \lim \mathbb{P}\left(\frac{S_n}{\sigma\sqrt{n}} > \frac{z}{\sigma}\right) = \mathbb{P}\left(Z > \frac{z}{\sigma}\right) > 0$.

As $\mathbb{P}\left(\limsup \frac{S_n}{\sqrt{n}} > z\right) \geq \limsup \mathbb{P}\left(\frac{S_n}{\sqrt{n}} > z\right) > 0$, using Kolmogorov on tail event $\left\{\limsup \frac{S_n}{\sqrt{n}} > z\right\}$, $\mathbb{P}\left(\limsup \frac{S_n}{\sqrt{n}} > z\right) = 1$.

Thus, by taking $z \to \infty$, $\mathbb{P}\left(\limsup \frac{S_n}{\sqrt{n}} = \infty\right) = 1$

(b) Suppose $\frac{S_n}{\sqrt{n}}$ converges to S in probability. $\sigma^2 = Var(X_i)$

As "convergence in prob." $\Rightarrow$ "conv. in distribution", $\sigma \mathcal{X}$, $\mathcal{X}$: standard normal, by CLT.

Then, $\left| \frac{S_{(m+1)!}}{\sqrt{(m+1)!}} - \frac{S_{m!}}{\sqrt{m!}} \right| \xrightarrow{P} 0$ as $m \to \infty$.

By independence, $\mathbb{P}\Big(\underbrace{0 < \frac{S_{m!}}{\sqrt{m!}} < 1, \ \frac{S_{(m+1)!} - S_{m!}}{\sqrt{(m+1)!}} < -2}_{(*)}\Big) = \underbrace{\mathbb{P}\left(0 < \frac{S_m}{\sqrt{m!}} < 1\right)}_{①} \underbrace{\mathbb{P}\left(\frac{S_{(m+1)!} - S_{m!}}{\sqrt{(m+1)!}} < -2\right)}_{②}$.

① $\longrightarrow \mathbb{P}(0 < \sigma\mathcal{X} < 1) > 0$ as $m \to \infty$

② $= \mathbb{P}\left(\frac{X_{m!+1} \cdots X_{(m+1)!}}{\sqrt{(m+1)! - m!}} < -2 \times \frac{\sqrt{(m+1)!}}{\sqrt{(m+1)! - m!}}\right) \longrightarrow \mathbb{P}(\sigma\mathcal{X} < -2) > 0$ as $m \to \infty$.

(*) implies $0 < \frac{S_m}{\sqrt{m!}}$, $\frac{S_{(m+1)!}}{\sqrt{(m+1)!}} < -1$. Thus,

① × ② $\leq \mathbb{P}\left(0 < \frac{S_m}{\sqrt{m!}}, \ \frac{S_{(m+1)!}}{\sqrt{(m+1)!}} < -1\right) \leq \mathbb{P}\left(\left|\frac{S_{m!}}{\sqrt{m!}} - \frac{S_{(m+1)!}}{\sqrt{(m+1)!}}\right| > 1\right)$.

Take liminf both side : $\liminf\limits_{m \to \infty} \mathbb{P}\left(\left|\frac{S_{m!}}{\sqrt{m!}} - \frac{S_{(m+1)!}}{\sqrt{(m+1)!}}\right| > 1\right) > 0$.

Now, we have $\frac{S_{m!}}{\sqrt{m!}}$ does not converge in probability. $\rightarrow\leftarrow$

Thus, $\frac{S_n}{\sqrt{n}}$ does not converge in probability.

#3.4.7

Let $X_i = Y_i - \mu$ so that $\mathbb{E}X_i = 0$. $Var(X_i) = Var(Y_i) < \infty$.

X_i : iid. $T_n := X_1 + \cdots + X_n$. $N_t = \sup\{m : S_m \le t\}$

By thm, $\frac{N_t}{t} \to \frac{1}{\mu}$ a.s. as $t \to \infty$.

Let t_n be any positive sequence that $t_n \to \infty$ as $n \to \infty$.

Define $a_n = \frac{t_n}{\mu}$, $N_n := N_{t_n}$. Then, $\frac{N_n}{a_n} = \frac{\mu N_{t_n}}{t_n} \to 1$ a.s. as $n \to \infty$.

By #3.4.6, $\frac{T_{N_n}}{\sigma\sqrt{a_n}} = \frac{S_{N_n} - \mu N_n}{\sigma\sqrt{a_n}} \Rightarrow Z$ (standard normal).

By definition of N_t, $S_{N_t} \le t < S_{N_t+1} \Rightarrow S_{N_n} \le t_n \le S_{N_n+1}$.

Thus, ① $\frac{S_{N_n} - \mu N_n}{\sigma\sqrt{a_n}} \le \frac{t_n - \mu N_n}{\sigma\sqrt{a_n}} \le \frac{S_{N_n+1} - \mu N_n}{\sigma\sqrt{a_n}} = \frac{Y_{N_n+1}}{\sigma\sqrt{a_n}} + \frac{S_{N_n} - \mu N_n}{\sigma\sqrt{a_n}}$

$$\mathbb{P}\left(\frac{S_{N_n+1} - \mu N_n}{\sigma\sqrt{a_n}} \le z + \varepsilon\right) \le \mathbb{P}\left(\frac{Y_{N_n+1}}{\sigma\sqrt{a_n}} \le \varepsilon \text{ and } \frac{S_{N_n} - \mu N_n}{\sigma\sqrt{a_n}} \le z\right)$$
$$= \mathbb{P}\left(\frac{Y_{N_n+1}}{\sigma\sqrt{a_n}} \le \varepsilon\right)\mathbb{P}\left(\frac{S_{N_n} - \mu N_n}{\sigma\sqrt{a_n}} \le z\right).$$

By Markov, $\mathbb{P}\left(\frac{Y_{N_n+1}}{\sigma\sqrt{a_n}} > \varepsilon\right) \to 0$ as $n \to \infty$. Thus,

$\mathbb{P}\left(\frac{S_{N_n+1} - \mu N_n}{\sigma\sqrt{a_n}} \le z + \varepsilon\right) \to \mathbb{P}\left(\frac{S_{N_n} - \mu N_n}{\sigma\sqrt{a_n}} \le z\right)$.

Take $\varepsilon \to 0$ so that ② $\mathbb{P}\left(\frac{S_{N_n+1} - \mu N_n}{\sigma\sqrt{a_n}} \le z\right) \to \mathbb{P}\left(\frac{S_{N_n} - \mu N_n}{\sigma\sqrt{a_n}} \le z\right)$.

From ①, ② we conclude $\mathbb{P}\left(\frac{t_n - \mu N_n}{\sigma\sqrt{a_n}} \le z\right) \to \mathbb{P}\left(\frac{S_{N_n} - \mu N_n}{\sigma\sqrt{a_n}} \le z\right)$.

Thus, $\frac{t_n - \mu N_n}{\sigma\sqrt{a_n}} \Rightarrow Z$ as $n \to \infty$. Hence, $\frac{\mu N_n - t_n}{\sigma\sqrt{a_n}} = \frac{\mu N_{t_n} - t_n}{\sqrt{\frac{\sigma^2 t_n}{\mu}}} \Rightarrow Z$. as $n \to \infty$

This holds for any $t_n \to \infty$. Thus, $\frac{\mu N_t - t}{\sqrt{\frac{\sigma^2 t}{\mu}}} \Rightarrow Z$ as $t \to \infty$

#3.4.10

$X_1, X_2, \cdots$ independent.

$\mathrm{Var}(S_n) = \sum_{i=1}^{n} \mathrm{Var}(X_i) \le \sum_{m=1}^{n} \mathbb{E}X_i^2 \le nM^2 < \infty$.

Let $X_{n,m} = \dfrac{X_m - \mathbb{E}X_m}{\sqrt{\mathrm{Var}(S_n)}}$ for $m = 1, \cdots, n$. $n \in \mathbb{N}$.

① $\mathbb{E}X_{n,m} = 0$

② $\sum_{m=1}^{n} \mathbb{E}X_{n,m}^2 = \dfrac{1}{\mathrm{Var}(S_n)} \sum_{m=1}^{n} \mathrm{Var}(X_m) = 1 \quad \forall n$

③ $\forall_{\varepsilon>0}$, $\exists n_0$ s.t. $\sqrt{\mathrm{Var}(S_n)} > \dfrac{2M}{\varepsilon}$, for $\forall n \ge n_0$.

For $n \ge n_0$, $|X_{n,m}| < \dfrac{\varepsilon}{2M} |X_m - \mathbb{E}X_m| \le \varepsilon$

$\Rightarrow \sum_{m=1}^{n} \mathbb{E}(|X_{n,m}|^2 ; |X_{n,m}| > \varepsilon) = \mathbb{E} \sum_{m=1}^{n} |X_{n,m}|^2 \mathbb{1}_{|X_{n,m}| > \varepsilon} = 0$

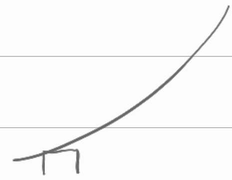
$\Rightarrow \lim \sum_{m=1}^{n} \mathbb{E}(|X_{n,m}|^2 ; |X_{n,m}| > \varepsilon) = 0$

By Lindeberg-Feller, $X_{n,1} + \cdots + X_{n,n} = \dfrac{S_n - \mathbb{E}S_n}{\sqrt{\mathrm{Var}(X_n)}} \Rightarrow \chi$.

#3.4.13

(i) If $\beta > 1$, $\sum_{j=1}^{\infty} \mathbb{P}(X_j \neq 0) = \sum \frac{1}{j^\beta} < \infty$. By 1st B-C,
$\mathbb{P}(X_j \neq 0 \text{ i.o.}) = 0$.
Thus, $\mathbb{P}(X_j = 0 \text{ eventually}) = 1$. If $X_j = 0$ eventually,
$\sum X_j$ converges so that $S_n \to S_\infty$.
Thus, $S_n \to S_\infty$ a.s.

(ii) $\beta < 1$. $\mathbb{E}X_j = 0$. $Var(X_j) = \mathbb{E}X_j^2 = \frac{1}{2j^\beta} j^2 \times 2 = j^{2-\beta}$.
$Var(S_n) = \sum_{j=1}^{n} Var(X_j) = \sum_{j=1}^{n} j^{2-\beta}$.
Thus, $\int_0^n t^{2-\beta}\,dt \le Var(S_n) \le \int_1^{n+1} t^{2-\beta}\,dt$



$$\Rightarrow \quad \frac{1}{3-\beta} n^{3-\beta} \le Var(S_n) \le \frac{1}{3-\beta}\left((n+1)^{3-\beta} - 1\right). \quad \cdots (*)$$

Let $X_{n,m} = \frac{X_m}{n^{(3-\beta)/2}}$.

① $\mathbb{E}X_{n,m} = 0$

② $\sum_{m=1}^{n} \mathbb{E}X_{n,m}^2 = \frac{1}{n^{(3-\beta)}} Var(S_n) = \frac{1}{n^{3-\beta}} \sum_{j=1}^{n} j^{2-\beta} \xrightarrow{\text{by } (*)} c^2$ for some c

③ For $\varepsilon > 0$, $\exists n_0$ s.t. $\varepsilon \cdot n^{\frac{3-\beta}{2}} > n$ $\forall n \ge n_0$. $(\beta < 1)$.
Thus, $|X_{n,m}| = \frac{|X_m|}{n^{(3-\beta)/2}} \le \frac{m}{n^{(3-\beta)/2}} \le \frac{n}{n^{(3-\beta)/2}} < \varepsilon$ $\forall n \ge n_0$, $m \le n$.

Thus, $\sum_{m=1}^{n} \mathbb{E}(|X_{n,m}|^2 ; |X_{n,m}|^2 > \varepsilon) = 0$ $\forall n \ge n_0$ so that its limit by $n \to \infty$ is 0.
By Lindeberg Feller, $X_{n,1} + \cdots + X_{n,n} = \frac{S_n}{n^{(3-\beta)/2}} \Rightarrow c\chi$.

(iii) $\beta = 1$. $\varphi_j(t) := \mathbb{E}\, e^{itX_j} = 1 - \frac{1}{j} + \frac{1}{2j}(e^{ijt} + e^{-ijt})$
$= 1 + \frac{1}{j}(\cos(jt) - 1)$

$\varphi(t) := \mathbb{E}\, e^{itS_n/n} = \prod_{j=1}^{n} \varphi_j\left(\frac{t}{n}\right) = \prod_{j=1}^{n}\left[1 + \frac{1}{j}\left(\cos\left(\frac{jt}{n}\right) - 1\right)\right] \approx e^{\sum_{j=1}^{n} \frac{1}{j}\left(\cos\left(\frac{jt}{n}\right) - 1\right)}$

$\sum_{j=1}^{n} \frac{1}{j}\left(\cos\left(\frac{jt}{n}\right) - 1\right) \to \int_0^1 x^{-1}(\cos(xt) - 1)\,dx$ so the result follows.
↳ by Riemann sum.

#3.7.1

① $\mathbb{P}(T>t)=e^{-\lambda t}$ for $t=2^{-n}$.

$\mathbb{P}(T>1)=e^{-\lambda}$ by setting $\lambda=-\log(\mathbb{P}(T>1))$.

Use induction on n:

i) $n=1 \Rightarrow$ given above

ii) $$\mathbb{P}(T>2^{-n})=\mathbb{P}(T>2^{-n}\mid T>2^{-(n+1)})\,\mathbb{P}(T>2^{-(n+1)})$$
$$+\,\cancelto{0}{\mathbb{P}(T>2^{-n}\mid T\le 2^{-(n+1)})}\,\mathbb{P}(T\le 2^{-(n+1)})$$
$$=\mathbb{P}(T>2^{-(n+1)})^2$$

Thus, $\mathbb{P}(T>2^{-(n+1)})^2=e^{-\lambda 2^{-n}} \Rightarrow \mathbb{P}(T>2^{-(n+1)})=e^{-\lambda 2^{-n}/2}=e^{-\lambda 2^{-(n+1)}}$.

② $\mathbb{P}(T>t)=e^{-\lambda t}\ \forall t=m2^{-n}$.

Use induction on m.

i) $m=1 \Rightarrow$ Holds by ①

ii) $$\mathbb{P}(T>(m+1)2^{-n})=\mathbb{P}(T>(m+1)2^{-n}\mid T>m2^{-n})\,\mathbb{P}(T>m2^{-n})$$
$$+\,\cancelto{0}{\mathbb{P}(T>(m+1)2^{-n}\mid T\le m2^{-n})}\,\mathbb{P}(T\le m2^{-n})$$
$$=\mathbb{P}(T>2^{-n})\,\mathbb{P}(T>m2^{-n})$$
$$=e^{-\lambda 2^{-n}}\,e^{-\lambda m 2^{-n}}=e^{-\lambda(m+1)2^{-n}}.$$

③ $\mathbb{P}(T>t)=e^{-\lambda t}$ for $\forall t\ge 0$.

$\mathbb{P}(T>t)=:h(t)$ is non-increasing, and $h(t)=e^{-\lambda t}$ for $\forall t=m2^{-n}$.

$\mathbb{P}(T\le t)=1-h(t)$ is right continuous. $\Rightarrow$ $h(t)=1-\mathbb{P}(T\le t)$ is also right cont.

For $t\ge 0$, $\exists\, t_k\in\{m2^{-n}\mid m,n\in\mathbb{N}\}$ s.t. $t_k\ge t$ and $t_k\to t$.

Thus, $h(t)=\lim_{k\to\infty}h(t_k)=\lim_{k\to\infty}e^{-\lambda t_k}=e^{-\lambda t}$ (By continuity of $t\mapsto e^{-\lambda t}$)

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#3.7.3

pdf of $\exp(\lambda)$ is $\lambda e^{-\lambda t}$ for $t \geq 0$.

cdf of $\exp(\lambda)$ is $1-e^{-\lambda t}$ for $t \geq 0$

① Thus, $\mathbb{P}(V>t) = \mathbb{P}(T_i > t \quad \forall i=1,\cdots,n)$

$$= \prod_{i=1}^{n} \mathbb{P}(T_i > t) \qquad \text{(By independence).}$$

$$= \prod_{i=1}^{n} (1-\mathbb{P}(T_i \leq t))$$

$$= \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-(\lambda_1+\cdots+\lambda_n)t} \quad \text{for } t \geq 0,$$

② $\mathbb{P}(I=i) = \mathbb{P}(V=T_i) = \mathbb{P}(T_j \geq T_i \quad \forall j=1,\cdots,n)$

$$= \mathbb{P}(T_j \geq T_i \quad \forall j \neq i)$$

$$= \int_0^\infty \left(\int_{[t_i,\infty)^{n-1}} \prod_{j\neq i} \lambda_j e^{-\lambda_j t_j} dt_1 \cdots dt_{i-1} dt_{i+1} \cdots dt_n \right) \lambda_i e^{-\lambda_i t} dt_i$$

$$= \int_0^\infty \left\{ \prod_{j\neq i} \left(\int_{t_i}^\infty \lambda_j e^{-\lambda_j t_j} dt_j \right) \right\} \lambda_i e^{-\lambda_i t_i} dt_i$$

$$= \int_0^\infty \left\{ \prod_{j\neq i} \left(e^{-\lambda_j t_i} \right) \right\} \lambda_i e^{-\lambda_i t_i} dt$$

$$= \lambda_i \int_0^\infty e^{-(\lambda_1+\cdots+\lambda_n)t} dt$$

$$= \frac{\lambda_i}{\lambda_1+\cdots+\lambda_n}.$$

③ V and I are independent.

$$\mathbb{P}(V>t, I=i) = \mathbb{P}(T_i > t, \; T_j \geq T_i \quad \forall j \neq i)$$

$$= \int_t^\infty \left(\prod_{j\neq i} \int_{t_i}^\infty \lambda_j e^{-\lambda_j t_j} dt_j \right) \lambda_i e^{-\lambda_i t_i} dt_i$$

$$= \int_t^\infty \left(\prod_{j\neq i} e^{-\lambda_j t_i} \right) \lambda_i e^{-\lambda_i t_i} dt_i$$

$$= \frac{\lambda_i}{\lambda_1+\cdots+\lambda_n} e^{-(\lambda_1+\cdots+\lambda_n)t}$$

$$= \mathbb{P}(I=i)\,\mathbb{P}(V>t).$$

By thm, V, I are independent.

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#3.8.4

(i) X: symmetric stable with index α, $p<\alpha$.

$$\mathbb{P}(|X|>\tfrac{2}{u}) \le u^{-1}\int_{-u}^{u}(1-\varphi(t))\,dt.$$

$$\varphi(t) = e^{-b|t|^\alpha} \approx 1-b|t|^\alpha \quad \text{so that}$$

$$\mathbb{P}(|X|>x) \underset{u=\frac{2}{x}}{\le} \frac{x}{2}\int_{-\frac{2}{x}}^{\frac{2}{x}} b|t|^\alpha dt = C|x|^{-\alpha} \text{ for some } C$$

$$\mathbb{E}|X|^p = \int_0^\infty p x^{p-1}\,\mathbb{P}(|X|>x)\,dx$$

$$\le \int_0^1 p x^{p-1}dx + \int_1^\infty p x^{p-\alpha-1}dx < \infty, \text{ as } p-\alpha-1<-1.$$

(ii) $\mathbb{P}(|X|\ge x) \ge Cx^{-\alpha}$.

$$\mathbb{E}|X|^\alpha = \int_0^\infty \alpha x^{\alpha-1}\,\mathbb{P}(|X|>x)\,dx$$

$$\ge \int_0^\infty \alpha x^{\alpha-1} C x^{-\alpha}dx = \alpha C\int_0^\infty \frac{1}{x}dx = \infty.$$

Thus, $\mathbb{E}|X|^\alpha = \infty$.

$\mathbb{E}e^{itX} \sim 1-b|t|^\alpha$.

#3.8.5

(i) By thm 3.8.8, $\exists$ iid $X_1, X_2, \cdots$ s.t. $\dfrac{X_1+\cdots+X_k-b_k}{a_k} \Rightarrow Y$

$X_{i1}, X_{i2}, \cdots$ s.t. $\dfrac{X_{i1}+\cdots+X_{ik}-b_{ik}}{a_{ik}} \Rightarrow Y_i$

$a_n = \inf\{x \mid \mathbb{P}(|X_1|>x) \le \frac{1}{n}\}$

$\mathbb{P}(|X_1|>a_n) = a_n^{-\alpha} L(a_n) = \frac{1}{n} \Rightarrow a_n \sim n^{1/\alpha}$

$a_{nk} \sim (nk)^{1/\alpha}$

Thus, $\dfrac{a_{nk}}{a_n} \longrightarrow k^{1/\alpha}$ as $n \to \infty$.

By pf of thm 3.8.8, $\alpha_k = \dfrac{a_{nk}}{a_n} \longrightarrow k^{1/\alpha}$.

(ii) If $\alpha<1$, by #3.8.1, we can let $b_n=0$.

$-\dfrac{kb_n-b_{nk}}{a_n}=0$ so that $\beta_k=0$.