20200130 Yujun Kim MAS550 HW2

#2.1.1

$\mathbb{P}(X_1 \le x_1, \cdots, X_n \le x_n) = \int_A f(x)\,dx$, where $A = \prod_{i=1}^{n} (-\infty, x_i] \in \mathbb{R}^n$.

As $f = g_1(x_1) \cdots g_n(x_n) \ge 0$, we can apply Fubini:

$$\mathbb{P}(X_1 \le x_1, \cdots, X_n \le x_n) = \int_A f(x)\,dx$$

$$= \int_A g_1(x_1) \cdots g_n(x_n)\,dx$$

$$= \int_{A_1} \cdots \int_{A_n} g_1(x_1) \cdots g_n(x_n)\,dx_1 \cdots dx_n \quad (A_i = (-\infty, x_i])$$

$$= \left(\int_{A_1} g_1(x_1)\,dx_1\right) \cdots \left(\int_{A_n} g_n(x_n)\,dx_n\right)$$

$$= \mathbb{P}(X_1 \le x_1) \cdots \mathbb{P}(X_n \le x_n)$$

By theorem 2.1.8, $X_1, \cdots, X_n$ are independent.

#2.1.11

① We first solve "previous exercise":

If X, Y are independent, integer-valued R.V., $\mathbb{P}(X+Y=n) = \sum_m \mathbb{P}(X=m)(Y=n-m)$.

∵) This is a slight modification of pf of thm 2.1.15.

$$\mathbb{P}(X+Y=n) = \iint \mathbb{1}_{(x+y=n)}\,\mu(dx)\,\nu(dy), \text{ where } \mu, \nu: \text{prob. measure for } X, Y$$

$$= \iint \mathbb{1}_{(x+y=n)}\,\nu(dy)\,\mu(dx)$$

If $x = m$, $\int \mathbb{1}_{(x+y=n)}\,\nu(dy) = \sum_{l \in \mathbb{Z}} \mathbb{1}_{(x+y=n)}(m, l)\ \mathbb{P}(Y=l)$

$$= \mathbb{P}(Y = n-m)$$

Thus, $\mathbb{P}(X+Y=n) = \int \mathbb{P}(Y=n-m)\,\mu(dx) = \sum_{m \in \mathbb{Z}} \mathbb{P}(Y=n-m)\,\mathbb{P}(X=m)$

② If $X = \text{Poisson}(\lambda)$, $Y = \text{Poisson}(\mu)$, then $X+Y = \text{Poisson}(\lambda+\mu)$.

$\because$) By ①, $\mathbb{P}(X+Y=i) = \sum_j \mathbb{P}(Y=i-j)\mathbb{P}(X=j)$

$$= \sum_{j=0}^{i} \frac{e^{-\mu}\mu^{i-j}}{(i-j)!} \cdot \frac{e^{-\lambda}\lambda^j}{j!} \quad \left(\text{If } j<0 \text{ or } j>i, \text{ terms become } 0\right)$$

$$= \sum_{j=0}^{i} \frac{e^{-(\lambda+\mu)}}{i!} \frac{i!}{(i-j)!\,j!} \mu^{i-j}\lambda^j$$

$$= \frac{e^{-(\lambda+\mu)}}{i!} (\lambda+\mu)^i.$$

Thus, $X+Y = \text{Poisson}(\lambda+\mu)$.

#2.1.12

(i) By ① of #2.1.11,

$$\mathbb{P}(X+Y=i) = \sum_j \mathbb{P}(Y=i-j)\,\mathbb{P}(X=j)$$

$$= \sum_{j=0}^{i} \binom{m}{i-j} p^{i-j}(1-p)^{m-i+j} \binom{n}{j} p^j (1-p)^{n-j}$$

$$= \sum_{j=0}^{i} \binom{m}{i-j}\binom{n}{j} p^i (1-p)^{n+m-i}$$

$$= \binom{m+n}{i} p^i (1-p)^{n+m-i}$$

$\hookrightarrow \sum_{j=1}^{i} \binom{m}{i-j}\binom{n}{j} = \binom{m+n}{i}$ as

$$\binom{m+n}{i} = \text{Coeff. of } x^i \text{ on } (1+x)^{m+n}$$

$$= \sum_{j=0}^{i} (\text{Coeff. of } x^{i-j} \text{ on } (1+x)^m)(\text{Coeff. of } x^j \text{ on } (1+x)^n)$$

$$= \sum_{j=0}^{i} \binom{m}{i-j}\binom{n}{j}.$$

Thus, $X+Y = \text{Binomial}(n+m, p)$.

(ii) Let $X_1 = \cdots = X_n = X = \text{Bernoulli}(p)$.

$$\left.\begin{aligned} &\mathbb{P}(X=1) = p = \binom{1}{1} p^1 (1-p)^{1-1} \\ &\mathbb{P}(X=0) = (1-p) = \binom{1}{0} p^0 (1-p)^{1-0} \\ &\mathbb{P}(X=m) = 0, \text{ if } m \neq 1 \text{ or } 0 \end{aligned}\right\} \Rightarrow X = \text{Binomial}(1, p).$$

Induction to prove $X_1 + \cdots + X_n =: S_n = \text{Binomial}(n,p)$.

(Base case) $n=1$. Holds by above argument.

(Induction) If $S_n = \text{Binomial}(n,p)$, then $S_{n+1} = S_n + X_{n+1}$, where

$S_n = \text{Binomial}(n,p)$, $X_{n+1} = \text{Binomial}(1,p)$.

By (i), $S_{n+1} = \text{Binomial}(n+1, p)$.

Thus, $S_n = \text{Binomial}(n,p)$ $\forall n \in \mathbb{N}$.

#2.1.14

$$P(XY \le z) = \iint \mathbb{1}_{(xy \le z)}\, \mu(dx)\, \nu(dy),$$ where μ, ν are prob. measure for with distribution F, G.

$$= \int_0^\infty \int_0^\infty \mathbb{1}_{(xy\le z)}\, \mu(dx)\, \nu(dy).$$

$$\int_0^\infty \mathbb{1}_{(xy\le z)}\, \mu(dx) = \int_0^{z/y} \mu(dx) = F(z/y). \quad (z/y = \infty \text{ if } y = 0)$$

Thus, $P(XY\le z) = \int_0^\infty F(z/y)\, \nu(dy)$

$$= \int_0^\infty F(z/y)\, dG(y).$$

#2.2.1

$$E\left[\left(\frac{S_n}{n} - \nu_n\right)^2\right] = E\left[\frac{1}{n^2}\left(\sum_{i=1}^n X_i - \mu_i\right)^2\right]$$

$= E\left[\frac{\sum\sum (X_i - EX_i)(X_j - EX_j)}{n^2}\right]$
$= E\left[\frac{1}{n^2}\sum (X_i - EX_i)^2\right] + E\left[\frac{1}{n^2}\sum [illegible]\right]$
$= \frac{1}{n^2}\sum_{i=1}^n E[(X_i - EX_i)^2] + 0$ (since $X_1, \cdots, X_n$ are [illegible])
$= \frac{1}{n^2}\sum Var(X_i)$

$$= \frac{1}{n^2} Var(S_n)$$

$$= \frac{1}{n^2}\sum_{i=1}^n Var(X_i) \quad (X_1, \cdots, X_n \text{ are uncorrelated})$$

$$= \frac{1}{n^2}\sum_{i=1}^n i \cdot \frac{Var(X_i)}{i}$$

(Lemma) Let $a_i \to 0$ as $i \to \infty$. Then $\frac{1}{n^2}\sum_{i=1}^n i a_i \to 0$ as $i \to \infty$.

$\because$) For $\varepsilon > 0$, $\exists N$ s.t. $0 \le a_i < \varepsilon$ $\forall n \ge N$.

$$\frac{1}{n^2}\sum_{i=1}^n i \cdot a_i = \frac{1}{n^2}\left(\sum_{i=1}^N i a_i + \sum_{i=N+1}^n i a_i\right)$$

$$\le \frac{1}{n^2}\left(C + \sum_{i=1}^n i \cdot \varepsilon\right), \text{ where } C = \sum_{i=1}^N i a_i.$$

$$= \frac{1}{n^2}\left(C + \frac{n(n+1)}{2}\varepsilon\right) \to \frac{\varepsilon}{2} \text{ as } n \to \infty.$$

Thus, $0 \le \liminf \frac{1}{n^2}\sum_{i=1}^{n} i a_i \le \limsup \frac{1}{n^2}\sum_{i=1}^{n} i a_i \le \frac{\varepsilon}{2}$.

As $\varepsilon > 0$ is arbitary, $\lim \frac{1}{n^2}\sum_{i=1}^{n} i a_i = 0$.

Now, apply lemma for $a_i = \frac{Var(X_i)}{i}$. Then, $\frac{1}{n^2}\sum_{i=1}^{n} i\frac{Var(X_i)}{i} \to 0$.

i.e. $E\left[\left(\frac{S_n}{n} - \nu_n\right)^2\right] \to 0$ as $n \to \infty$.

Hence, we proved convergence in L^2.

By Markov's ineq, convergence in L^2 implies convergence in probability.

#2.2.2

Let $S_n = X_1 + \cdots + X_n$, $E X_n^2 = r(0) < \infty$.

$$E\left[\left(\frac{S_n}{n}\right)^2\right] = \frac{1}{n^2}E[S_n^2]$$
$$= \frac{1}{n^2}E\left[\sum_i\sum_j X_i X_j\right]$$
$$= \frac{1}{n^2}\left(\sum_i E[X_i^2] + 2\sum_{i<j} E[X_i X_j]\right)$$
$$= \frac{1}{n^2}\left(n r(0) + 2\sum_{k=1}^{n-1}\sum_{j-i=k} E[X_j X_i]\right)$$
$$= \frac{1}{n^2}\left(n r(0) + 2\sum_{k=1}^{n-1}(n-k) r(k)\right)$$

($\because (i,j) = (1,k+1), (2,k+2), \cdots (n-k,n)$
$\Rightarrow$ There are $n-k$ pairs s.t. $i<j$, $j-i=k$)

$$= \frac{r(0)}{n} + 2\sum_{k=1}^{n}\frac{n-k}{n^2} r(k).$$
$$= \frac{r(0)}{n} + 2\frac{\sum_{k=1}^{n} r(k)}{n} - 2\sum_{k=1}^{n}\frac{k}{n^2} r(k).$$

For $r(i) \to 0$, apply lemma proved in #2.2.1 (See above).

Then, $\frac{1}{n^2}\sum_{i=1}^{n} i\, r(i) \to 0$ as $n \to \infty$.

Hence, $\frac{r(0)}{n} \to 0$, $2\frac{\sum_{k=1}^{n} r(k)}{n} \to 0$ (thm from calculus)

and $\frac{1}{n^2}\sum_{i=1}^{n} i\, r(i) \to 0$.

i.e. $E\left[\left(\frac{S_n}{n}\right)^2\right] \to 0$ as $n \to \infty$.

Hence, we proved convergence in L^2. By Markov, $\frac{S_n}{n} \to 0$ in probability.

#2.2.3

(i) For $U \sim \mathrm{Unif}([0,1])$,

$\mathbb{E}[f(U)] = \int_0^1 f(x)\,dx$. (Change of variable formula)

$=: \mu$

$\mathbb{E}|f(U)| = \int_0^1 |f(x)|dx < \infty$. By thm 2.2.14, $I_n \to \mu = I$ in probability
$\hookrightarrow$ WLLN

(ii) $\mathbb{P}(|I_n - I| > \frac{a}{\sqrt{n}}) = \mathbb{P}(|I_n - I|^2 > \frac{a^2}{n})$

$\leq \frac{n}{a^2} Var(I_n)$ (Chebyshev)

$= \frac{n}{a^2} \times \frac{1}{n^2} \times \sum_{i=1}^{n} Var(f(U_i))$

$= \frac{1}{a^2} \int_0^1 (f(x) - \mu)^2 dx$, where $\mu = \int_0^1 f(x)dx$.

(a constant with respect to n).

2.2.7

$h(y) \geq 0$, $H(x) = \int_{(-\infty, x]} h(y)dy$.

$\int_{-\infty}^{\infty} h(y)\mathbb{P}(X \geq y)\,dy = \int_{-\infty}^{\infty} \int h(y) \mathbb{1}_{X \geq y}\, dP\, dy$

$= \int\int_{-\infty}^{\infty} h(y) \mathbb{1}_{X \geq y}\, dy\, dP$ (Fubini, as $h\mathbb{1}_{X\geq y} \geq 0$).

$= \int\int_{-\infty}^{X} h(y)\,dy\,dP = \int H(X)\,dP = \mathbb{E}[H(X)]$

2.3.8

Let $P(A_n) < 1$ $\forall n$ and $P(\cup A_n) = 1$, for independent events A_n.

Suppose $\sum \mathbb{P}(A_n) < \infty$. Then by Borel Cantelli, $\mathbb{P}(A_n \text{ i.o.}) = 0$.

$\mathbb{P}((\bigcup_{n \leq m} A_n)^c) = P(\bigcap_{n \leq m} A_n^c) = \prod_{n \leq m} \mathbb{P}(A_n^c) = \prod_{n \leq m} (1 - \mathbb{P}(A_n))$
$\hookrightarrow$ By independence.

Take $m \to \infty$. $\mathbb{P}((\bigcup_n A_n)^c) = \prod_n (1-\mathbb{P}(A_n)) = 0$, with $1-\mathbb{P}(A_n) > 0 \ \forall n$.

Note that $e^{-x} \le 1 - \frac{1}{2}x$ for $\forall x \in [0, \frac{1}{2})$,

↳ see lemma below

As we assumed $\sum \mathbb{P}(A_n) < \infty$, if we let $I := \{n \in \mathbb{N} \mid \mathbb{P}(A_n) \ge \frac{1}{4}\}$,

then $|I| < \infty$.

Thus, $\prod_{n \le m} (1-\mathbb{P}(A_n)) = \prod_{\substack{n \le m \\ n \in I}} (1-\mathbb{P}(A_n)) \prod_{\substack{n \le m \\ n \notin I}} (1 - \frac{1}{2} \times 2\mathbb{P}(A_n))$

$$\ge \left[\prod_{n \in I} (1-\mathbb{P}(A_n))\right] \times e^{-2\sum_{n \le m, n \notin I} \mathbb{P}(A_n)} \quad \text{(By lemma)}$$

$$\ge \left[\prod_{n \in I} (1-\mathbb{P}(A_n))\right] \cdot e^{-2\sum_{n \le m} \mathbb{P}(A_n)}.$$

Take $m \to \infty$ on both side, $0 = \prod_n (1-\mathbb{P}(A_n)) \ge \left[\prod_{n \in I} (1-\mathbb{P}(A_n))\right] e^{-2\sum_n \mathbb{P}(A_n)}$.

As $\prod_{n \in I} (1-\mathbb{P}(A_n)) > 0$, $e^{-2\sum_n \mathbb{P}(A_n)} > 0$, we have a contradiction.

↳ Because $|I| < \infty$ ↳ Because $\sum_n \mathbb{P}(A_n) < \infty$.

Hence, $\sum \mathbb{P}(A_n) = \infty$.

(Lemma) $e^{-x} \le 1 - \frac{1}{2}x \ \forall x \in [0, \frac{1}{2})$.

∵) Let $\varphi(x) = e^{-x}$, $\psi(x) = 1 - \frac{1}{2}x$.

$\varphi(0) = \psi(0) = 1$.

$\varphi'(x) = -e^{-x} \le -e^{-\frac{1}{2}} \le -\frac{1}{2} = \psi'(x)$ for $\forall x \in [0, \frac{1}{2})$

$-e^{-\frac{1}{2}} \le -\frac{1}{2} \Leftrightarrow e^{-\frac{1}{2}} \ge \frac{1}{2} \Leftrightarrow \sqrt{e} \le 2 \Leftrightarrow e \le 4$

2.3.11 $\mathbb{P}(X_n = 1) = p_n$, $\mathbb{P}(X_n = 0) = 1 - p_n$.

(i) $X_n \to 0$ in probability $\Leftrightarrow p_n \to 0$.

∵) For $\varepsilon \ge 1$, $\mathbb{P}(|X_n| > \varepsilon) = 0$.

For $0 < \varepsilon < 1$, $\mathbb{P}(|X_n| > \varepsilon) = p_n$

Hence, $[\forall \varepsilon, \ \mathbb{P}(|X_n| > \varepsilon) \to 0 \text{ as } n \to \infty]$

$\Leftrightarrow [\forall 0 < \varepsilon < 1, \ \mathbb{P}(|X_n| > \varepsilon) \to 0 \text{ as } n \to \infty]$

$\Leftrightarrow [\, p_n \to 0 \text{ as } n \to \infty \,]$.

Thus, we proved the statement.

(ii) $X_n \to 0$ a.s. $\Leftrightarrow \sum p_n < \infty$.

($\Rightarrow$) If $\sum p_n = \infty$, then by 2^{nd} Borel Cantelli,

$\mathbb{P}(\{X_n = 1\} \text{ i.o}) = 1$. i.e. $X_n = 1$ infinitely often almost surely.

If $\{X_n = 1\}$ i.o, then X_n cannot converge to 0.

Thus, $[X_n \to 0 \text{ a.s.}]$ is false.

($\Leftarrow$) Let $\sum p_n < \infty$. By Borel Cantelli,

$\mathbb{P}(\{X_n = 1\} \text{ i.o.}) = 0$.

If not $[\{X_n = 1\} \text{ i.o.}]$, then $[X_n \to 0]$

Thus, $X_n \to 0$ a.s.

2.3.14 $\sup X_n < \infty$ a.s. $\Leftrightarrow \sum_n \mathbb{P}(X_n > A) < \infty$ for some A.

($\Rightarrow$) Let $Y = \sup X_n$. Then, $X_n \le \sup_n X_n = Y \ \forall n$

If $\sum_n \mathbb{P}(X_n > N) = \infty$, by second Borel Cantelli, for all N,

$\mathbb{P}(X_n > N \text{ i.o.}) = 1$.

$[X_n > N \text{ i.o.}]$ implies $[\sup X_n > N]$. Thus, $\mathbb{P}(\sup X_n > N) = 1$.

Take $N \to \infty$. $\mathbb{P}(\sup X_n = \infty) = 1$. (By thm 1.1.1 (iv))

Thus, $\mathbb{P}(\sup X_n < \infty) = 1 - \mathbb{P}(\sup X_n = \infty) = 1 - 1 = 0 \neq 1$.

($\Leftarrow$) Suppose $\mathbb{P}(\sup X_n = \infty) = \alpha > 0$.

Then, $\mathbb{P}(X_n > A) \ge \mathbb{P}(\sup X_n = \infty) = \alpha > 0 \ \forall n.\ \forall A$.

Thus, $\sum_n \mathbb{P}(X_n > A) \ge \sum_n \alpha = \infty \ \forall A$.

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2.3.15 (i) Let $A_n = \{X_n > a \log n\}$ for some constant a.

Then, $\mathbb{P}(A_n) = e^{-a\log n} = \frac{1}{n^a}$.

If $a>1$, then $\sum \mathbb{P}(A_n) < \infty$ so that $\mathbb{P}(A_n \text{ i.o.}) = 0$ by B-C.

$\Rightarrow \limsup \frac{X_n}{\log n} \le a$ a.s. $\forall a > 1$

If $a \le 1$, then $\sum \mathbb{P}(A_n) = \infty$ so that $\mathbb{P}(A_n \text{ i.o.}) = 1$ by 2nd B-C.

$\Rightarrow \limsup \frac{X_n}{\log n} \ge a$ a.s. $\forall a \le 1$.

Hence, $\limsup \frac{X_n}{\log n} = 1$ a.s.

(ii) $M_n \ge X_n$ so that

$\limsup \frac{M_n}{\log n} \ge \limsup \frac{X_n}{\log n} = 1$ a.s. (By (i))

Let $B_n = \{M_n \le \log n\}$

$\mathbb{P}(B_n) = \mathbb{P}(\max_{1\le m\le n} X_m \le \log n) = \mathbb{P}((e^{-\log n})^n) = \frac{1}{n^n}$.

$\sum \mathbb{P}(B_n) < \infty$, so that by B-C, $\mathbb{P}(B_n \text{ i.o.}) = 0$.

Thus, $\liminf \frac{M_n}{\log n} = 1$ a.s.

Hence, $\lim \frac{M_n}{\log n} = 1$ a.s.

2.3.18 Let $A_n = \{|\frac{X_n}{n^\alpha} - a| > \varepsilon\}$.

$\mathbb{P}(A_n) = \mathbb{P}(\{|X_n - \mathbb{E}X_n| > \varepsilon n^\alpha\})$

$\le \mathbb{P}(\{|X_n - \mathbb{E}X_n|^2 > \frac{\varepsilon^2}{B} n^{2\alpha-\beta} Var(X_n)\})$

$\le \frac{B}{\varepsilon^2} n^{\beta - 2\alpha}$ by Markov.

Thus, $\mathbb{P}(A_{n^c}) = \frac{B}{\varepsilon^2} n^{c(\beta-2\alpha)}$ so that $\sum \mathbb{P}(A_{n^c}) < \infty$ for $c > \frac{1}{2\alpha-\beta}$ $(c \in \mathbb{N})$.

By B-C, $\mathbb{P}(A_{n^c} \text{ i.o.}) = 0$. i.e. $\frac{X_{n^c}}{(n^c)^\alpha} \to a$ a.s.

As $0 \le X_1 \le X_2 \le \cdots$, for any m with $n^c \le m < (n+1)^c$,

we have $\frac{X_{n^c}}{(n+1)^{c\alpha}} \le \frac{X_m}{m^\alpha} \le \frac{X_{(n+1)^c}}{n^\alpha}$.

As $\frac{X_{n^c}}{(n+1)^{c\alpha}} = \left(\frac{n}{n+1}\right)^{c\alpha} \frac{X_{n^c}}{n^{c\alpha}} \to a$ as $n\to\infty$

$\frac{X_{n^c}}{n^{c\alpha}} = \left(\frac{n+1}{n}\right)^{c\alpha} \frac{X_{(n+1)^c}}{(n+1)^{c\alpha}} \to a$ as $n\to\infty$,

we conclude by Sandwitch thm, $\frac{X_m}{m^\alpha} \to a$ as $m\to\infty$

2.4.1 $X_1, X_2, \cdots$ are iid, $X_i \sim X$ with distribution F, $X\geq 0$, $EX<\infty$

$Y_1, Y_2, \cdots$ are iid, $Y_i\sim Y$ with distribution G, $Y\geq 0$, $EY<\infty$

Let $Z_n = \sum_{i=1}^{n}(X_i+Y_i)$, $N_t = \sup\{n : Z_n \leq t\}$, $W_n = \sum_{i=1}^{n} X_i$.

Then, $W_{N_t} \leq R_t \leq W_{N_t+1}$, by definition of N_t.

$Z_{N_t} \leq t < Z_{N_t+1}$, by definition of N_t.

Thus, $\frac{W_{N_t}}{Z_{N_t+1}} \leq \frac{R_t}{t} \leq \frac{W_{N_t+1}}{Z_{N_t}}$

By thm 2.4.7, $\frac{N_t}{t} \to \frac{1}{EX+EY} > 0$ a.s. Thus, $N_t\to\infty$ a.s. as $t\to\infty$

$\frac{W_{N_t+1}}{Z_{N_t}} = \frac{W_{N_t+1}}{N_t+1} \times \frac{N_t}{Z_{N_t}} \times \frac{N_t+1}{N_t} \to \frac{EX}{EX+EY}$ a.s. as $N_t\to\infty$ by SLLN.

$\frac{W_{N_t}}{Z_{N_t+1}} = \frac{W_{N_t}}{N_t} \times \frac{N_t+1}{Z_{N_t+1}} \times \frac{N_t}{N_t+1} \to \frac{EX}{EX+EY}$ a.s. as $N_t\to\infty$ by SLLN

Hence, as $t\to\infty$, $N_t\to\infty$ a.s. and using Sandwitch thm,

$\frac{R_t}{t} \to \frac{EX}{EX+EY}$ as $t\to\infty$

2.4.2 $\frac{|X_{n+1}|}{|X_n|}$ are iid with density $f(r) = \frac{2\pi r}{\pi \cdot 1^2} = 2r$ for $r\in[0,1]$.

Let $R_n = \frac{|X_n|}{|X_{n-1}|}$. Then, $R_1, R_2, \cdots$ are i.i.d, $|X_n| = R_n \cdots R_1 |X_0|$

$n^{-1}\log|X_n| = n^{-1}\log(R_n\cdots R_1 |X_0|)$

$$= \frac{(\log R_1) + \cdots + (\log R_n)}{n}$$

$$\mathbb{E} \log R_n = \int_0^1 \frac{2\pi r}{\pi \cdot 1^2} \log r \, dr$$ (change of variable formula)

$$= \int_0^1 2r \log r \, dr$$

$$= \left[r^2 \log r \right]_0^1 - \int_0^1 \frac{r^2}{r} dr$$

$$= -\frac{1}{2}.$$

By SLLN, $n^{-1} \log |X_n| \longrightarrow -\frac{1}{2}$ a.s.