IE541 Homework 1

20200130 Yujun Kim

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1 Problem 1

The sample space is

$\Omega = \{(a_1, \cdots, a_n) | n \in \mathbb{N}, a_i \in \{H, T\}, a_n = H$ and exactly one of $a_1, \cdots a_{n-1}$ is H

Let $\Omega_k = \{(a_1, \cdots, a_n) \in \Omega | n = k\}$ be the event that exactly k toss is required. Then,

$$P(\Omega_k) = (k-1) \times 2^{-k}$$

2 Problem 2

Let A, B, C each denote the event that the card I choose has color $(green/green), (red/red), (green/red)$ on each side. Let D be the event that the color of the side of the card I have seen is green. Then, $P(D|A) = 1, P(D|B) = 0, P(D|C) = 1/2$. Thus,

$$P(A|D) = \frac{P(AD)}{P(D)} = \frac{P(AD)P(A)}{P(A)P(D)} = P(D|A)\frac{P(A)}{P(D)} = \frac{1/3}{1/2} = \frac{2}{3}$$

The event that the other side is also green is equal to the event A. Hence, the probability the question is asking is 2/3.

3 Problem 3

Let sample space be the set of all computer owners. Let M be the set of Macintosh user, W be the set of Windows user, and L be the set of Linux users. Assume M, W, L are disjoint. Let V be the set of infected users with virus. Then, $P(M) = 0.3, P(W) = 0.5, P(L) = 0.2, P(V|M) = 0.65, P(V|W) = 0.82, P(V|L) = 0.5$. Then,

$$P(W|V) = \frac{P(WV)}{P(V)} = \frac{P(V|W)P(W)}{P(V|M)P(M) + P|V|W)P(W) + P(V|L)P(L)} = 0.41/0.615 = 2/3$$

This is the desired probability.

4 Problem 4

We show x, y are independent if and only if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.

($\Rightarrow$) If $\int_E f = 0$ on arbitary measurable set E, then $f = 0$ almost everywhere. Let rectangle denote the set of the form $A \times B$. Every measurable set in $\Omega_X \times \Omega_Y$ differs to countable union of almost disjoint rectangles by measure zero set. Thus, if $\int_{A\times B} f = 0$ for every rectangles, then $\int_E f = 0$ for any measurable set E in the domain and thus $f = 0$ almost everywhere. Now, for any A, B,

$$\begin{aligned} P(X \in A, Y \in B) - P(X \in A)P(Y \in B) &= \iint_{A\times B} f_{X,Y}(x,y)dxdy - \int_A f_X(x)dx \int_B f_Y(y)dy \\ &= \iint_{A\times B} f_{X,Y}(x,y) - f_X(x)f_Y(y)dxdy = 0 \end{aligned}$$

Thus, the above argument, we have $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ almost everywhere.

($\Leftarrow$) We have

$$\begin{aligned} P(X \in A, Y \in B) &= \iint_{A\times B} f_{X,Y}(x,y)dxdy \\ &= \iint_{A\times B} f_X(x)f_Y(y)dxdy \\ &= \int_A f_X(x)dx \int_B f_Y(y)dy \\ &= P(X \in A)P(Y \in B) \end{aligned}$$

5 Problem 5

The CDF of X is

$$F(x) = \int_{-\infty}^{x} f_x(t)dt = \begin{cases} 0 & \text{if } x \le 0 \\ x/4 & \text{if } x \in (0,1] \\ 1/4 & \text{if } x \in (1,3] \\ 1/4 + 3(x-3)/8 & \text{if } x \in (3,5] \\ 1 & \text{if } x > 5 \end{cases}$$

6 Problem 6

(a) Fix x_2. $\int f_{1|2}(x_1|x_2)dx_1 = \int_0^{x_2} c_1 x_1/x_2^2 dx_1 = c_1/2 = 1$. Thus, $c_1 = 2$. $\int f_2(x_2)dx_2 = \int_0^1 c_2 x_2^4 dx_2 = c_2/5 = 1$. Thus, $c_2 = 5$.

(b) $f_{1|2}(x_1|x_2) = \frac{f_{12}(x_1,x_2)}{f_2(x_2)}$. Thus, $f_{12}(x_1,x_2) = f_{1|2}(x_1|x_2)f_2(x_2) = \begin{cases} 10x_1x_2^2 & \text{for } 0 < x_1 < x_2 \\ 0 & \text{otherwise} \end{cases}$

(c)

$$P(\frac{1}{4} < X_1 < \frac{1}{2}|X_2 = \frac{5}{8}) = \int_{\frac{1}{4}}^{\frac{1}{2}} f_{1|2}(x_1|\frac{5}{8})dx_1$$
$$= \int_{\frac{1}{4}}^{\frac{1}{2}} 2\left(\frac{8}{5}\right)^2 x_1 dx_1$$
$$= \left[\frac{64}{25}x_1^2\right]_{\frac{1}{4}}^{\frac{1}{2}} = \frac{12}{25}$$

(d)

$$f_1(x_1) = \int f_{12}(x_1, x_2)dx_2 = \int_{x_1}^{1} 10x_1x_2^2dx_2 = \frac{10}{3}x_1(1 - x_1^3)$$

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$$P(\frac{1}{4} < X_1 < \frac{1}{2}) = \int_{\frac{1}{4}}^{\frac{1}{2}} f_1(x_1)dx_1 = 149/512 \approx 0.291$$

7 Problem 7

$X_n = Y_1 + \cdots + Y_n$, where $P(Y_i = 1) = 1 - p, P(Y_i = -1) = p$. Then, $E(Y_i) = 1 \times (1 - p) + (-1) \times p = 1 - 2p$. Thus,

$$\mathbb{E}(X_n) = \mathbb{E}(Y_1 + \cdots Y_n) = \mathbb{E}(Y_1) + \cdots + \mathbb{E}(Y_n) = n(1 - 2p)$$

Now, $\mathbb{V}(Y_i) = \mathbb{E}(Y_i^2) - \mathbb{E}(Y_i)^2 = 1^2 \times (1 - p) + (-1)^2 \times p - (1 - 2p)^2 = 4p(1 - p)$. Thus,

$$\mathbb{V}(X_n) = \mathbb{V}(Y_1 + \cdots + Y_n) = \mathbb{V}(Y_1) + \cdots \mathbb{V}(Y_n) = 4np(1 - p)$$

8 Problem 8

$$\mathbb{E}(Y) = \mathbb{E}(e^X) = \int e^x \frac{1}{\sqrt{2\pi}}e^{-x^2}dx$$
$$= \int \frac{1}{\sqrt{2\pi}}e^{-(x^2-x)}dx$$
$$= \int \frac{1}{\sqrt{2\pi}}e^{-(x-1/2)^2}e^{1/4}dx$$
$$= e^{1/4}\int \frac{1}{\sqrt{2\pi}}e^{-x^2}dx$$
$$= e^{1/4} \times 1$$

The last equality holds as $\int \frac{1}{\sqrt{2\pi}}e^{-x^2}dx = 1$ for $f_X(x)$ being a PDF.

$$
\begin{aligned}
\mathbb{V}(Y) &= \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 \\
\mathbb{E}(Y^2) = \mathbb{E}(e^{2X}) &= \int e^{2x} \frac{1}{\sqrt{2\pi}} e^{-x^2} dx \\
&= \int \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2} e dx = e
\end{aligned}
$$

Thus, $\mathbb{V}(Y) = e - (e^{\frac{1}{4}})^2 = e - e^{\frac{1}{2}}$

9 Problem 9

(a) Y, Z are dependent. Note that $P(Y = 1) = b, P(Y = 0) = 1 - b, P(Z = 1) = 1 - a, P(Z = 0) = a$. Then,

$$P(Y = 1, Z = 1) = b - a \neq P(Y = 1)P(Z = 1) = b(1 - a) = b - ab$$

unless $b - a = b - ab$ or equivalently, $a(1 - b) = 0$. This is not possible as $a \neq 0, b \neq 1$. Thus, $P(Y = 1, Z = 1) \neq P(Y = 1)P(Z = 1)$.

(b)

$$
\begin{aligned}
\mathbb{E}(Y|Z = 1) &= 1 \times \frac{b - a}{1 - a} + 0 \times \frac{1 - b}{1 - a} = \frac{b - a}{1 - a} \\
\mathbb{E}(Y|Z = 0) &= 1 \times \frac{a}{a} + 0 \times \frac{0}{a} = 1 \\
\mathbb{E}(Y|Z) &= \frac{b - a}{1 - a} z + 1 \times (1 - z) = \frac{b - 1}{1 - a} z + 1
\end{aligned}
$$