

#3.4.2

(a) By CLT, $\frac{S_n}{\sqrt{n}} \Rightarrow Z$ (standard normal)for $\text{Var}(X_i) = \sigma^2 < \infty$ Thus, $\lim P\left(\frac{S_n}{\sqrt{n}} > z\right) = \lim P\left(\frac{S_n}{\sqrt{n}} > \frac{z}{\sigma}\right) = P(Z > \frac{z}{\sigma}) > 0$ As $P(\limsup \frac{S_n}{\sqrt{n}} > z) \geq \limsup P\left(\frac{S_n}{\sqrt{n}} > z\right) > 0$, using Kolmogorov on tail event $\{\limsup \frac{S_n}{\sqrt{n}} > z\}$, $P(\limsup \frac{S_n}{\sqrt{n}} > z) = 1$.Thus, by taking $z \rightarrow \infty$, $P(\limsup \frac{S_n}{\sqrt{n}} = \infty) = 1$ (b) Suppose $\frac{S_n}{\sqrt{n}}$ converges to S in probability. $\sigma^2 = \text{Var}(X_i)$ As "convergence in prob." \Rightarrow "conv. in distribution"; δX , X : standard normal, by CLT.Then, $\left| \frac{S_{(m+1)}!}{\sqrt{(m+1)!}} - \frac{S_m!}{\sqrt{m!}} \right| \xrightarrow{P} 0 \text{ as } m \rightarrow \infty$.By independence, $P(0 < \frac{S_m}{\sqrt{m!}} < 1, \frac{S_{(m+1)}! - S_m!}{\sqrt{(m+1)!}} < -2) = P(0 < \frac{S_m}{\sqrt{m!}} < 1) \underbrace{P(\frac{S_{(m+1)}! - S_m!}{\sqrt{(m+1)!}} < -2)}$ (1) $\rightarrow P(0 < \sigma X < 1) > 0 \text{ as } m \rightarrow \infty$ (2) $= P\left(\frac{X_{m+1} \cdots X_{(m+1)}}{\sqrt{(m+1)!} - m!} < -2 \times \frac{\sqrt{(m+1)!}}{\sqrt{(m+1)!} - m!}\right) \rightarrow P(\sigma X < -2) > 0 \text{ as } m \rightarrow \infty$ (*) implies $0 < \frac{S_m}{\sqrt{m!}}, \frac{S_{(m+1)}!}{\sqrt{(m+1)!}} < -1$. Thus,(1) \times (2) $\leq P(0 < \frac{S_m}{\sqrt{m!}}, \frac{S_{(m+1)}!}{\sqrt{(m+1)!}} < -1) \leq P\left(|\frac{S_m}{\sqrt{m!}} - \frac{S_{(m+1)}!}{\sqrt{(m+1)!}}| > 1\right)$ Take \liminf both side: $\liminf_{m \rightarrow \infty} P\left(|\frac{S_m}{\sqrt{m!}} - \frac{S_{(m+1)}!}{\sqrt{(m+1)!}}| > 1\right) > 0$.Now, we have $\frac{S_m}{\sqrt{m!}}$ does not converge in probability. $\cancel{\rightarrow}$
Thus, $\frac{S_n}{\sqrt{n}}$ does not converge in probability.

#3.4.7

Let $X_i = Y_i - \mu$ so that $\mathbb{E}X_i = 0$. $\text{Var}(X_i) = \text{Var}(Y_i) < \infty$.

X_i : iid. $T_n := X_1 + \dots + X_n$. $N_t = \sup\{m : S_m \leq t\}$

By thm, $\frac{N_t}{t} \rightarrow \frac{1}{\mu}$ a.s. as $t \rightarrow \infty$.

Let t_n be any positive sequence that $t_n \rightarrow \infty$ as $n \rightarrow \infty$.

Define $a_n = \frac{t_n}{\mu}$, $N_n := N_{t_n}$. Then, $\frac{N_n}{a_n} = \frac{\mu N_{t_n}}{t_n} \rightarrow 1$ a.s. as $n \rightarrow \infty$.

By #3.4.6, $\frac{T_{N_n}}{\sigma \sqrt{a_n}} = \frac{S_{N_n} - \mu N_n}{\sigma \sqrt{a_n}} \Rightarrow \mathcal{Z}$ (standard normal).

By definition of N_t , $S_{N_t} \leq t < S_{N_t+1} \Rightarrow S_{N_n} \leq t_n \leq S_{N_n+1}$.

Thus, ① $\frac{S_{N_n} - \mu N_n}{\sigma \sqrt{a_n}} \leq \frac{t_n - \mu N_n}{\sigma \sqrt{a_n}} \leq \frac{S_{N_n+1} - \mu N_n}{\sigma \sqrt{a_n}} = \frac{Y_{N_n+1}}{\sigma \sqrt{a_n}} + \frac{S_{N_n} - \mu N_n}{\sigma \sqrt{a_n}}$

② $\frac{Y_{N_n+1}}{\sigma \sqrt{a_n}} \xrightarrow{P} 0$ by Markov, $\frac{S_{N_n} - \mu N_n}{\sigma \sqrt{a_n}} \Rightarrow \mathcal{Z}$

weak convergence is preserved under addition.

By ②, $\frac{S_{N_n+1} - \mu N_n}{\sigma \sqrt{a_n}} \Rightarrow \mathcal{Z}$ as well. \Rightarrow By ①, $\frac{t_n - \mu N_n}{\sigma \sqrt{a_n}} \Rightarrow \mathcal{Z}$

Thus, $\frac{t_n - \mu N_n}{\sigma \sqrt{a_n}} \Rightarrow \mathcal{Z}$. Hence, $\frac{a_n t_n - t_n}{\sigma \sqrt{a_n}} = \frac{\mu N_{t_n} - t_n}{\sqrt{\frac{\sigma^2 t_n}{n}}} \Rightarrow \mathcal{Z}$ as $n \rightarrow \infty$

This holds for any $t_n \rightarrow \infty$. Thus, $\frac{\mu N_t - t}{\sqrt{\frac{\sigma^2 t}{n}}} \Rightarrow \mathcal{Z}$ as $t \rightarrow \infty$

#3.4.10

X_1, X_2, \dots independent.

$$\text{Var}(S_n) = \sum_{i=1}^n \text{Var}(X_i) \leq \sum_{m=1}^n E|X_i|^2 \leq nM^2 < \infty.$$

Let $X_{n,m} = \frac{X_m - EX_m}{\sqrt{\text{Var}(S_n)}}$ for $m=1, \dots, n$, $n \in \mathbb{N}$.

① $E X_{n,m} = 0$

② $\sum_{m=1}^n E X_{n,m}^2 = \frac{1}{\text{Var}(S_n)} \sum_{m=1}^n \text{Var}(X_m) = 1 \quad \forall n$

③ $\forall \varepsilon > 0$, $\exists n_0$ s.t. $\sqrt{\text{Var}(S_n)} > \frac{2M}{\varepsilon}$, for $\forall n \geq n_0$.

For $n \geq n_0$, $|X_{n,m}| \leq \frac{\varepsilon}{2M} |X_m - EX_m| \leq \varepsilon$

$$\Rightarrow \sum_{m=1}^n E(|X_{n,m}|^2; |X_{n,m}| > \varepsilon) = E \sum_{m=1}^n |X_{n,m}|^2 \mathbb{1}_{|X_{n,m}| > \varepsilon} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sum_{m=1}^n E(|X_{n,m}|^2; |X_{n,m}| > \varepsilon) = 0$$

By Lindeberg-Feller, $X_{n,1} + \dots + X_{n,n} = \frac{S_n - ES_n}{\sqrt{\text{Var}(S_n)}} \Rightarrow X$.

#3.4.13

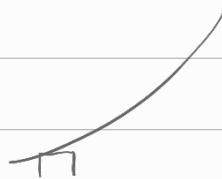
(i) If $\beta > 1$, $\sum_{j=1}^{\infty} P(X_j \neq 0) = \sum \frac{1}{j^\beta} < \infty$. By 1st B-C,
 $P(X_j \neq 0 \text{ i.o.}) = 0$.

Thus, $P(X_j = 0 \text{ eventually}) = 1$. If $X_j = 0$ eventually,
 $\sum X_j$ converges so that $S_n \rightarrow S_\infty$.
 Thus, $S_n \rightarrow S_\infty$ a.s.

(ii) $\beta < 1$. $E[X_j] = 0$. $Var(X_j) = E[X_j^2] = \frac{1}{2} j^{-2\beta} \times 2 = j^{-2\beta}$.

$$Var(S_n) = \sum_{j=1}^n Var(X_j) = \sum_{j=1}^n j^{-2\beta}$$

$$\text{Thus, } \int_0^n t^{2-\beta} dt \leq Var(S_n) \leq \int_1^{n+1} t^{2-\beta} dt$$



$$\Rightarrow \frac{1}{3-\beta} n^{3-\beta} \leq Var(S_n) \leq \frac{1}{3-\beta} ((n+1)^{3-\beta} - 1). \quad \cdots (*)$$

$$\text{Let } X_{n,m} = \frac{X_m}{n^{(3-\beta)/2}}.$$

$$\textcircled{1} \quad E[X_{n,m}] = 0$$

$$\textcircled{2} \quad \sum_{m=1}^n E[X_{n,m}]^2 = \frac{1}{n^{(3-\beta)}} Var(S_n) = \frac{1}{n^{3-\beta}} \sum_{j=N}^n j^{-2\beta} \xrightarrow{\text{by } (*)} C^2 \text{ for some } C$$

$$\textcircled{3} \quad \text{For } \varepsilon > 0, \exists n_0 \text{ s.t. } \varepsilon \cdot n^{\frac{3-\beta}{2}} > n \quad \forall n \geq n_0. \quad (\beta < 1).$$

$$\text{Thus, } |X_{n,m}| = \frac{|X_m|}{n^{(3-\beta)/2}} \leq \frac{m}{n^{(3-\beta)/2}} \leq \frac{n}{n^{(3-\beta)/2}} < \varepsilon \quad \forall n \geq n_0, m \leq n.$$

Thus, $\sum_{m=1}^n E(|X_{n,m}|^2 ; |X_{n,m}|^2 > \varepsilon) = 0 \quad \forall n \geq n_0$ so that its limit by $n \rightarrow \infty$ is 0.

By Lindeberg Feller, $X_{n,1} + \dots + X_{n,n} = \frac{S_n}{n^{(3-\beta)/2}} \Rightarrow cX$.

$$\textcircled{iii} \quad \beta = 1. \quad \varphi_j(t) := E e^{itX_j} = 1 - \frac{1}{j} + \frac{1}{2j} (e^{itj} + e^{-itj}) \\ = 1 + \frac{1}{j} (\cos(jt) - 1)$$

$$\varphi(t) := E e^{itS_n/n} = \prod_{j=1}^n \varphi_j\left(\frac{t}{n}\right) = \prod_{j=1}^n \left[1 + \frac{1}{j} (\cos(\frac{jt}{n}) - 1) \right] \approx e^{\frac{1}{2} \frac{1}{j} (\cos(\frac{jt}{n}) - 1)}$$

$$\sum_{j=1}^n \frac{1}{j} (\cos(\frac{jt}{n}) - 1) \rightarrow \int_0^1 x^{-1} \cos(xt) - 1 dx \quad \text{so the result follows.}$$

↪ by Riemann sum.

#3.7.1

① $P(T > t) = e^{-\lambda t}$ for $t = 2^{-n}$.

$P(T > 1) = e^{-\lambda}$ by setting $\lambda = -\log(P(T > 1))$.

Use induction on n :

i) $n=1 \Rightarrow$ given above

ii) $P(T > 2^{-n}) = P(T > 2^{-n} | T > 2^{-(n+1)}) P(T > 2^{-(n+1)})$
 $+ P(T > 2^{-n} | T \leq 2^{-(n+1)}) P(T \leq 2^{-(n+1)})$
 $= P(T > 2^{-(n+1)})^2$

Thus, $P(T > 2^{-(n+1)})^2 = e^{-\lambda 2^{-n}} \Rightarrow P(T > 2^{-(n+1)}) = e^{-\lambda 2^{-n}/2} = e^{-\lambda 2^{-(n+1)}}$.

② $P(T > t) = e^{-\lambda t} \forall t = m 2^{-n}$

Use induction on m .

i) $m=1 \Rightarrow$ holds by ①

ii) $P(T > (m+1)2^{-n}) = P(T > (m+1)2^{-n} | T > m 2^{-n}) P(T > m 2^{-n})$
 $+ P(T > (m+1)2^{-n} | T \leq m 2^{-n}) P(T \leq m 2^{-n})$
 $= P(T > 2^{-n}) P(T > m 2^{-n})$
 $= e^{-\lambda 2^{-n}} e^{-\lambda m 2^{-n}} = e^{-\lambda(m+1)2^{-n}}.$

③ $P(T > t) = e^{-\lambda t}$ for $\forall t \geq 0$,

$P(T > t) =: h(t)$ is non-increasing, and $h(t) = e^{-\lambda t}$ for $\forall t = m 2^{-n}$.

$P(T \leq t) = 1 - h(t)$ is right continuous. $\Rightarrow h(t) = 1 - P(T \leq t)$ is also right cont.

For $t \geq 0$, $\exists t_k \in \{m 2^{-n} | m, n \in \mathbb{N}\}$ s.t. $t_k \geq t$ and $t_k \rightarrow t$.

Thus, $h(t) = \lim_{k \rightarrow \infty} h(t_k) = \lim_{k \rightarrow \infty} e^{-\lambda t_k} = e^{-\lambda t}$ (By continuity of $t \mapsto e^{-\lambda t}$)

#3.7.3

pdf of $\exp(\lambda)$ is $\lambda e^{-\lambda t}$ for $t \geq 0$.

cdf of $\exp(\lambda)$ is $1 - e^{-\lambda t}$ for $t \geq 0$

① Thus, $P(V > t) = P(T_i > t \ \forall i=1, \dots, n)$

$$\begin{aligned} &= \prod_{i=1}^n P(T_i > t) \\ &= \prod_{i=1}^n (1 - P(T_i \leq t)) \\ &= \prod_{i=1}^n e^{-\lambda_i t} = e^{-(\lambda_1 + \dots + \lambda_n)t} \quad \text{for } t \geq 0, \end{aligned} \quad (\text{By independence}).$$

② $P(I=i) = P(V=T_i) = P(T_j \geq T_i \ \forall j=1, \dots, n)$

$$= P(T_j \geq T_i \ \forall j \neq i)$$

$$= \int_0^\infty \left(\int_{[t_i, \infty)^{n-1}} \prod_{j \neq i} \lambda_j e^{-\lambda_j t_j} dt_1 \dots dt_{i-1} dt_{i+1} \dots dt_n \right) \lambda_i e^{-\lambda_i t_i} dt_i$$

$$= \int_0^\infty \left\{ \prod_{j \neq i} \left(\int_{t_i}^\infty \lambda_j e^{-\lambda_j t_j} dt_j \right) \right\} \lambda_i e^{-\lambda_i t_i} dt_i$$

$$= \int_0^\infty \left\{ \prod_{j \neq i} (e^{-\lambda_j t_i}) \right\} \lambda_i e^{-\lambda_i t_i} dt_i$$

$$= \lambda_i \int_0^\infty e^{-(\lambda_1 + \dots + \lambda_n)t} dt$$

$$= \frac{\lambda_i}{\lambda_1 + \dots + \lambda_n}.$$

③ V and I are independent.

$$P(V > t, I=i) = P(T_i > t, T_j \geq T_i \ \forall j \neq i)$$

$$= \int_t^\infty \left(\prod_{j \neq i} \int_{t_i}^\infty \lambda_j e^{-\lambda_j t_j} dt_j \right) \lambda_i e^{-\lambda_i t_i} dt_i$$

$$= \int_t^\infty \left(\prod_{j \neq i} e^{-\lambda_j t_i} \right) \lambda_i e^{-\lambda_i t_i} dt_i$$

$$= \frac{\lambda_i}{\lambda_1 + \dots + \lambda_n} e^{-(\lambda_1 + \dots + \lambda_n)t}$$

$$= P(I=i) P(V > t).$$

By this, V, I are independent.

#3.8.4

(i) X symmetric stable with index α , $p < \alpha$.

$$P(|X| > \frac{2}{\alpha}) \leq u^{-1} \int_{-u}^u (1 - \varphi(t)) dt.$$

$$\varphi(t) = e^{-bt|t|^\alpha} \approx |t - b|t|^\alpha \text{ so that}$$

$$P(|X| > x) \leq \frac{x}{2} \int_{-\frac{x}{2}}^{\frac{x}{2}} b|t|^\alpha dt = C|x|^{-\alpha} \text{ for some } C$$

$$u = \frac{2}{\alpha}$$

$$E|X|^p = \int_0^\infty p x^{p-1} P(|X| > x) dx$$

$$\leq \int_0^1 p x^{p-1} dx + \int_1^\infty p x^{p-\alpha-1} dx < \infty, \text{ as } p - \alpha - 1 < -1.$$

(ii) $P(|X| \geq x) \geq Cx^{-\alpha}$.

$$E|X|^\alpha = \int_0^\infty \alpha x^{\alpha-1} P(|X| > x) dx$$

$$\geq \int_0^\infty \alpha x^{\alpha-1} C x^{-\alpha} dx = \alpha C \int_0^\infty \frac{1}{x} dx = \infty.$$

Thus, $E|X|^\alpha = \infty$.

$$Ee^{itX} \sim 1 - b|t|^\alpha.$$

#3.8.5

(i) By thm 3.8.8, \exists iid X_1, X_2, \dots s.t. $\frac{X_1 + \dots + X_k - b_k}{a_k} \Rightarrow Y$

X_{i1}, X_{i2}, \dots s.t. $\frac{X_{i1} + \dots + X_{ik} - b_{ik}}{a_{ik}} \Rightarrow Y_i$

$$a_n = \inf \{x \mid P(|X_1| > x) \leq \frac{1}{n}\}$$

$$P(|X_1| > a_n) = a_n^{-\alpha} L(a_n) = \frac{1}{n} \Rightarrow a_n \sim n^{1/\alpha}$$

$$a_{nk} \sim (nk)^{1/\alpha}$$

Thus, $\frac{a_{nk}}{a_n} \rightarrow k^{1/\alpha}$ as $n \rightarrow \infty$.

By pf of thm 3.8.8, $\alpha_k = \frac{a_{nk}}{a_n} \rightarrow k^{1/\alpha}$.

(ii) If $\alpha < 1$, by #3.8.1, we can let $b_n = 0$.

$$-\frac{k b_n - b_{nk}}{a_n} = 0 \text{ so that } \beta_k = 0.$$