#1 (a)

```
% (a) setup
mu = 5;
n = 100;
sigma =  1/sqrt(n)
alpha = 0.05;
dist = makedist('Normal', "mu", mu, "sigma",sigma);
X = random(dist, n, 1);
```

(b),(c) $f(\mu)=1$

$$f(\mu|x^n) \propto \mathcal{L}(\mu) f(\mu)$$

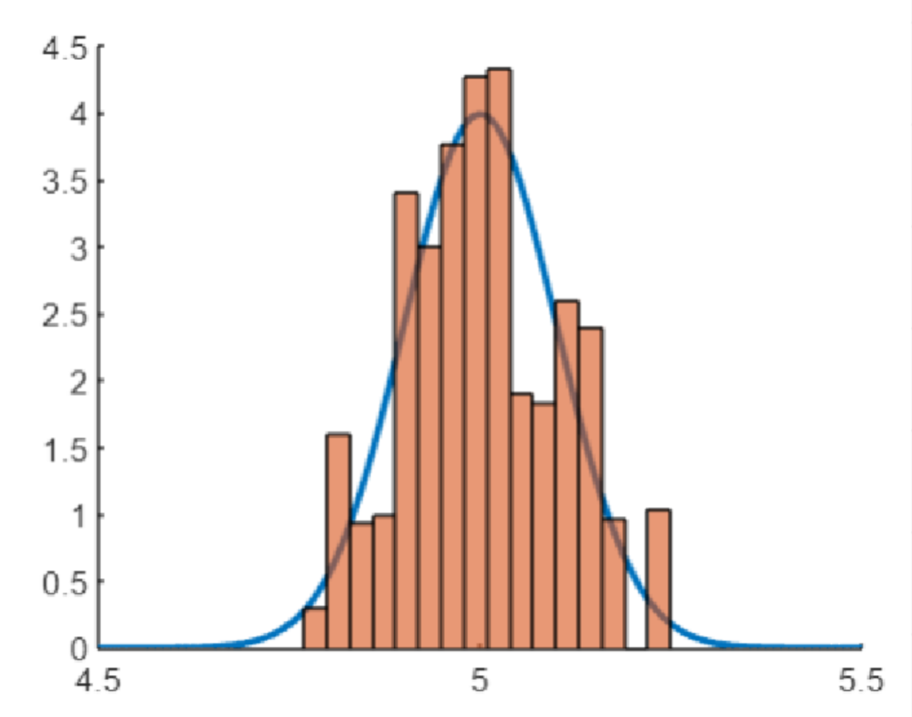
$$= \mathcal{L}(\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{1}{2}(x_i-\mu)^2} \right)$$

$$\propto e^{-\frac{n}{2}(\mu-\bar{x}_n)^2}.$$

Hence, $\mu | X^n \sim N(\bar{X}, \frac{1}{n})$

```
% (b) plot pdf
X1 = mu - 5*sigma:.01:mu + 5*sigma;
Y1 = pdf(dist,X1);
```

```
% (c) Simultate
B = 1000
T = zeros(B, 1);
for i = 1:B
    idx = unidrnd(n, 1);
    T(i) = X(idx);
end
figure
hold on
plot(X1,Y1,'LineWidth',2)
histogram(T, "Normalization",'pdf')
hold off
```



The code shows how to calculate the posterior density, and the righ figure shows the result. The curve is the density (for (b)) and the histogram is the result of simulation (for (c)).

We can see that the two plots almost matches

(d) $\theta = e^{\mu}$

$\mu | X^n \sim N(\bar{X}, \frac{1}{n})$

$P(\theta \le t | X^n) = P(e^{\mu} \le t | X^n) = P(\mu \le \ln t \,| X^n)$

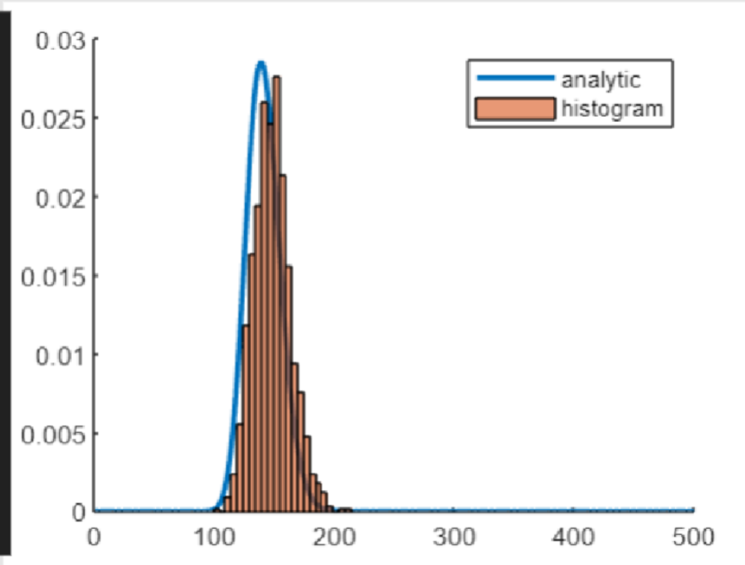
$= P(\sqrt{n}(\mu - \bar{X}) \le \sqrt{n}(\ln t - \bar{X}) \,| X^n)$

$= \Phi(\sqrt{n}(\ln t - \bar{X}))$

$$\boxed{\theta | X^n \sim \frac{1}{\sqrt{2\pi}} e^{-\frac{n}{2}(\ln\theta - \bar{x})^2} \cdot \frac{\sqrt{n}}{\theta}}$$ $\rightarrow$ Analytic result.

```
% (d) e^mu
X2 = 1:1:500;
Y2 = normpdf(log(X2),mean(X), 1/sqrt(n))./X2;
dist2 = makedist('Lognormal', "mu", mu, "sigma",1/sqrt(n));

S = zeros(B, 1);
for i = 1:B
    S(i) = random(dist2, 1, 1);
end
Y3 = pdf(dist2,X2);
figure
hold on
plot(X2,Y2,'LineWidth',2)
histogram(S, 'Normalization', 'pdf')
legend('analytic', 'histogram')
hold off
quantile(S, 0.025)
quantile(S, 0.975)
```



I plotted ftn given analytically and the simulated histogram

(e)

```
quantile(T, 0.025)
quantile(T, 0.975)
```

```
ans = 4.8169
ans = 5.2257
```

$\Rightarrow (3.05, 6.75)$ (Confidence interval obtained from the histogram)

(f)

```
quantile(S, 0.025)
quantile(S, 0.975)
```

```
ans = 121.6910
ans = 181.3348
```

$\Rightarrow (121.7, 181.3)$

#2 (a) $\lambda \sim \text{Gamma}(\alpha, \beta)$, $X_1, \dots, X_n \sim \text{Poisson}(\lambda)$

$$f(\lambda) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda/\beta}$$

$$\mathcal{L}(\lambda) = \prod_{i=1}^{n} f(x_i|\lambda) = \prod_{i=1}^{n} \left(\frac{\lambda^{x_i}}{x_i!} e^{-\lambda} \right)$$

$$= \frac{\lambda^{x_1+\dots+x_n}}{x_1! \cdots x_n!} e^{-n\lambda}.$$

$$f(\lambda|x^n) \propto f(\lambda)\mathcal{L}(\lambda)$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda/\beta} \cdot \frac{\lambda^{\sum_{i=1}^n x_i}}{x_1! \cdots x_n!} e^{-n\lambda}$$

$$\propto \lambda^{\alpha+\sum_{i=1}^n x_i - 1} e^{-(\frac{1}{\beta}+n)\lambda}.$$

$$= \lambda^{\alpha'-1} e^{-\lambda/\beta'},$$

where $\alpha' = \alpha + \sum_{i=1}^n x_i$

$1/\beta' = (\frac{1}{\beta}+n) \Rightarrow \beta' = \frac{1}{\frac{1}{\beta}+n} = \frac{\beta}{1+\beta n}.$

Thus, $f(\lambda|x^n) = \text{Gamma}(\alpha', \beta')$

i.e. $\lambda|x^n \sim \text{Gamma}(\alpha', \beta')$.

Posterior mean of λ:

$$\bar{\lambda}_n = \int \lambda f(\lambda | x^n) \, d\theta$$

$$= \text{expectation of Gamma}(\alpha', \beta')$$

$$= \alpha'\beta' \text{ (from chapter 3)}$$

$$= \left(\alpha + \sum_{i=1}^{n} x_i\right) \frac{\beta}{1+\beta n}.$$

(b) $f(x|\lambda) = \frac{\lambda^x}{x!} e^{-\lambda}$. $\log f(x|\lambda) = x \log\lambda - \lambda - \log x!$

$$\frac{\partial^2 \log f(x|\lambda)}{\partial \lambda^2} = -\frac{x}{\lambda^2}$$

$$I(\theta) = -\mathbb{E}_\lambda\left(\frac{\partial^2 \log f(x|\lambda)}{\partial \lambda^2}\right)$$

$$= \mathbb{E}_\lambda\left(\frac{x}{\lambda^2}\right) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}.$$

Jeffreys' prior $f(\lambda) \propto \sqrt{I(\lambda)} = \frac{1}{\sqrt{\lambda}}$.

$$f(\lambda | x^n) \propto f(\lambda) \mathcal{L}(\lambda)$$

$$= \lambda^{-1/2} \frac{\lambda^{\sum x_i}}{x_1! \cdots x_n!} e^{-n\lambda}$$

$$\propto \lambda^{-1/2 + \sum x_i} e^{-n\lambda}$$

$$\Rightarrow \lambda | x^n \sim \text{Gamma}\left(\frac{1}{2} + \sum x_i, \frac{1}{n}\right) \text{ (Posterior)}$$

#3 (a) $X \sim \text{Binomial}(n, p)$, $p \sim \text{Beta}(\alpha, \beta)$.

$X_1, \cdots, X_N \sim \text{Binomial}(n, p)$

$$f(p|x) \propto f(p)\mathcal{L}(p)$$

$$\propto p^{\alpha-1}(1-p)^{\beta-1} \cdot p^{\sum x_i}(1-p)^{\sum(n-x_i)}$$

$$= p^{(\alpha+\sum x_i)-1}(1-p)^{(\beta+nN-\sum x_i)-1}$$

$$p|X \sim \text{Beta}(\alpha+\sum X_i, \beta+nN-\sum X_i)$$

$$\hat{p}(x) = \mathbb{E}(p|X=x) = \boxed{\frac{\alpha+\sum x_i}{\alpha+\beta+nN} \text{ (Bayes' estimator)}}$$

Bayes' risk:

$$R(p, \hat{p}) = \mathbb{V}_p(\hat{p}) + \text{bias}_p^2(\hat{p})$$

$$\mathbb{V}_p(\hat{p}) = \frac{1}{(\alpha+\beta+nN)^2}\mathbb{V}_p(\sum X_i)$$

$$= \frac{1}{(\alpha+\beta+nN)^2} N np(1-p).$$

$$\mathbb{E}(\hat{p}(X)) = \frac{\alpha+nNp}{\alpha+\beta+nN}.$$

$$\Rightarrow \text{bias}_p^2(\hat{p}(X)) = \left(\frac{\alpha + nNp}{\alpha+\beta+nN} - p\right)^2$$

$$= \left(\frac{(1-p)\alpha + p\beta}{\alpha+\beta+nN}\right)^2$$

Hence, $R(p,\hat{p}) = \dfrac{((1-p)\alpha+p\beta)^2 + Nnp(1-p)}{(\alpha+\beta+nN)^2}$

$$r(f,\hat{p}) = \int R(p,\hat{p}) f(p)\,dp$$

$$= \int \frac{((\alpha-\beta)^2 - nN)p^2 + (nN - 2\alpha(\alpha-\beta))p + \alpha^2}{(\alpha+\beta+nN)^2} f(p)\,dp$$

$$= \boxed{\frac{A\,\mathbb{E}[p^2] + B\,\mathbb{E}[p] + \alpha^2}{(\alpha+\beta+nN)^2}, \text{ expectation taken over priori.}}$$

where $A = (\alpha-\beta)^2 - nN$, $B = nN - 2\alpha(\alpha-\beta)$

$$\mathbb{E}[p^2] = \mathbb{V}(p) + \mathbb{E}(p)^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \left(\frac{\alpha}{\alpha+\beta}\right)^2$$

$$\mathbb{E}[p] = \frac{\alpha}{\alpha+\beta}.$$

For the case with one observation ($X \sim \text{Binomial}(n,p)$), take $N=1$.

(b) $X \sim \text{Poisson}(\lambda)$, $\lambda \sim \text{Gamma}(\alpha, \beta)$

$$f(\lambda) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda/\beta}.$$

By #2 of this homework, $\lambda | x \sim \text{Gamma}(\alpha', \beta')$

with $\alpha' = \alpha + x$, $\beta' = \frac{\beta}{1+\beta}$

$$\Rightarrow \hat{\lambda}(x) = \mathbb{E}(\lambda | X = x) = \alpha' \beta' = \boxed{\frac{(\alpha+x)\beta}{1+\beta}. \text{ (Bayes' estimator)}}$$

Bayes' risk:

$$R(\lambda, \hat{\lambda}) = \mathbb{V}(\hat{\lambda}(X)) + \text{bias}^2_\lambda(\hat{\lambda}(X))$$

$$= \left(\frac{\beta}{1+\beta}\right)^2 \mathbb{V}(X) + \left(\mathbb{E}(\hat{\lambda}(X)) - \lambda\right)^2$$

$$= \left(\frac{\beta}{1+\beta}\right)^2 \lambda + \left(\frac{(\alpha+\lambda)\beta}{1+\beta} - \lambda\right)^2$$

$$= \frac{\beta^2 \lambda + (\alpha\beta - \lambda)^2}{(1+\beta)^2} = \frac{1}{(1+\beta)^2}\left(\lambda^2 + (\beta^2 - 2\alpha\beta)\lambda + \alpha^2\beta^2\right)$$

$$r(f, \hat{\lambda}) = \int R(\lambda, \hat{\lambda}) f(\lambda) d\lambda$$

$$= \frac{1}{(1+\beta)^2}\left[\mathbb{E}[\lambda^2] + (\beta^2 - 2\alpha\beta)\mathbb{E}[\lambda] + \alpha^2\beta^2\right]$$

$$\mathbb{E}[\lambda^2] = \mathbb{V}(\lambda) + \mathbb{E}[\lambda]^2 = \alpha\beta^2 - \alpha^2\beta^2$$

$$\mathbb{E}[\lambda] = \alpha\beta$$

$$\Rightarrow r(f, \hat{\lambda}) = \frac{1}{(1+\beta)^2}\left(\alpha\beta^2 - \alpha^2\beta^2 + (\beta^2 - 2\alpha\beta)\alpha\beta + \alpha^2\beta^2\right)$$

$$= \frac{1}{(1+\beta)^2}(\alpha\beta^2 + \alpha\beta^3) = \boxed{\frac{\alpha\beta^2}{1+\beta}}$$

(c) $X \sim N(\theta, \sigma^2)$, σ^2 known, $\theta \sim N(a, b^2)$

$$f(\theta|x) \propto L(\theta) f(\theta)$$

$$\propto \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2} \cdot \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2b^2}(\theta-a)^2}$$

$$\propto e^{-\frac{1}{2\sigma^2}(x-\theta)^2 - \frac{1}{2b^2}(\theta-a)^2}$$

$$\frac{1}{2\sigma^2}(x-\theta)^2 + \frac{1}{2b^2}(\theta-a)^2 = \left(\frac{1}{2\sigma^2} + \frac{1}{2b^2}\right)\left(\theta - \frac{\frac{1}{2\sigma^2}x + \frac{1}{2b^2}a}{\frac{1}{2\sigma^2} + \frac{1}{2b^2}}\right)^2 + k$$

Let $\frac{1}{\tau^2} = \frac{1}{\sigma^2} + \frac{1}{b^2}$. Then, $\frac{1}{2\sigma^2}(x-\theta)^2 + \frac{1}{2b^2}(\theta-a)^2$

$$= \frac{1}{2\tau^2}(\theta - \bar{\theta}) + k,$$

where $\bar{\theta} = wx + (1-w)a$, $w = \frac{\frac{1}{\sigma^2}}{\frac{1}{\sigma^2} + \frac{1}{b^2}}$.

(Refer example 11.2 with $n=1$)

Hence, it follows that $\theta | X \sim N(\bar{\theta}, \tau^2)$,

where $\bar{\theta}, \tau$ defined as above.

Bayes' estimator $= \mathbb{E}(\theta | X = x) = \bar{\theta} = wx + (1-w)a$

$$= \boxed{\frac{\frac{1}{\sigma^2}}{\frac{1}{\sigma^2}+\frac{1}{b^2}}x + \frac{\frac{1}{b^2}}{\frac{1}{\sigma^2}+\frac{1}{b^2}}a} = \hat{\theta}$$

Bayes' Risk:

$$R(\hat{\theta},\theta) = \mathbb{V}_\theta(\hat{\theta}) + \text{bias}_\theta^2(\hat{\theta})$$

$$= \left(\frac{\frac{1}{\sigma^2}}{\frac{1}{\sigma^2}+\frac{1}{b^2}}\right)^2 \mathbb{V}(X) + \left(\frac{\frac{1}{\sigma^2}}{\frac{1}{\sigma^2}+\frac{1}{b^2}}\theta + \frac{\frac{1}{b^2}}{\frac{1}{\sigma^2}+\frac{1}{b^2}}a - \theta\right)^2$$

$$= \frac{1}{\left(\frac{1}{\sigma^2}+\frac{1}{b^2}\right)^2}\left[\frac{1}{\sigma^2} + \left(\frac{a}{b^2} - \frac{\theta}{b^2}\right)^2\right]$$

$$= \frac{1}{\left(\frac{1}{\sigma^2}+\frac{1}{b^2}\right)^2}\left(\frac{1}{\sigma^2} + \frac{(\theta-a)^2}{b^4}\right)$$

$$r(f,\hat{\theta}) = \int R(\hat{\theta},\theta) f(\theta)d\theta$$

$$= \frac{1}{\left(\frac{1}{\sigma^2}+\frac{1}{b^2}\right)^2}\left(\frac{1}{\sigma^2} + \frac{1}{b^4}\left(\mathbb{E}(\theta^2) - 2a\mathbb{E}(\theta) + a^2\right)\right)$$

$$= \frac{1}{\left(\frac{1}{\sigma^2}+\frac{1}{b^2}\right)^2}\left(\frac{1}{\sigma^2} + \frac{1}{b^4}\left(b^2 + a^2 - 2a\cdot a + a^2\right)\right)$$

$$= \frac{1}{\left(\frac{1}{\sigma^2}+\frac{1}{b^2}\right)^2}\left(\frac{1}{\sigma^2} + \frac{1}{b^2}\right) = \boxed{\frac{1}{\frac{1}{\sigma^2}+\frac{1}{b^2}}}$$

#4 Suppose $\hat{\theta} = \bar{X}$ is inadmissible.
Then, $\exists \hat{\theta}'$ such that $R(\theta, \hat{\theta}') \le R(\theta, \hat{\theta}) \ \forall \theta$
and $R(\theta, \hat{\theta}') < R(\theta, \hat{\theta})$ for at least one θ.

$$\Rightarrow R(\theta, \hat{\theta}) = E_\theta(L(\theta, \hat{\theta}))$$
$$= E_\theta\left(\frac{(\theta - \bar{X})^2}{\sigma^2}\right) = \frac{1}{\sigma^2} \cdot \frac{\sigma^2}{n} = \frac{1}{n} \text{ (constant)}$$
$$= \frac{1}{\sigma^2} E_\theta[(\theta - \bar{X})^2] = \frac{1}{\sigma^2} R'(\theta, \hat{\theta}),$$

where $R'(\theta, \hat{\theta})$ is square error loss.

Take θ s.t. $R(\theta, \hat{\theta}') < R(\theta, \hat{\theta})$. Then, $R'(\theta, \hat{\theta}') < R'(\theta, \hat{\theta})$.
Also, $R'(\theta, \hat{\theta}') \le R'(\theta, \hat{\theta}) \ \forall \theta$.
Thus, $\hat{\theta}$ is inadmissible w.r.t. square error loss R'.
This contradicts theorem 12.20. Thus, $\bar{X}$ is admissible w.r.t. R.

By theorem 12.21, $\bar{X}$ is minimax.