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# ASYMPTOTIC DISTRIBUTION OF CERTAIN DEGENERATE V- AND U-STATISTICS WITH ESTIMATED PARAMETERS

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A PREPRINT

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## ABSTRACT

The asymptotic distribution of a wide class of V- and U-statistics with estimated parameters is derived in the case when the kernel is not necessarily differentiable along the parameter. The results have their application in goodness-of-fit problems.

**keywords:** degenerate V-statistics; characterization; goodness-of-fit

**MSC(2010):** 62E20; 62G20

## 1 Introduction

V- and U-statistics frequently appear in inferential procedures as estimators and test statistics. In many cases they depend on nuisance parameters, and, in order to keep the inference as broad as possible, estimators of such parameters are included in the statistic.

Although some earlier papers (see e.g. [25, 22, 9]) consider some particular cases, the first general study devoted to U-statistics with estimated parameters was done in [23], where the effect of estimating parameters in asymptotically normal (non-degenerate) U-statistics was examined. De Wet and Randles [7] studied a special case of degenerate V-statistics of order 2 obtained as an integrated square of a V-statistic of order 1. They proved that, under some regularity conditions, the limiting distribution is certain infinite linear combination of  $\chi_1^2$  random variables. Their method was subsequently used for examining particular goodness-of-fit tests in several papers (see e.g. [12, 26, 10]).

Here we extend the result from [7] by considering the case when the V-statistic of order 1 is replaced with a V-statistic of an arbitrary order. The practical value of this result is reflected in the fact that a class of characterization

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based test statistics are of this form. This class includes  $L^2$ -type tests employing V-empirical Laplace transforms of equidistributed random variables (see [5, 4]), but it is not limited to them. In fact, instead of V-empirical Laplace transforms one could employ V-empirical distribution functions (as e.g. in [6]), characteristic functions, densities, etc. Exploring the asymptotics of such tests is closely related to V- and U-empirical functions with estimated parameters. Additionally, it enables calculation of the approximate Bahadur efficiency, one of important tools for test comparison (see e.g. [3]). The exceptional performance of the tests from [5] set the motivation to generalize the result and hence facilitate further research on this topic.

In addition, we present the analogous result for U-statistics. This result, using different method of proof and slightly more general form, is also obtained in [1, Theorem 3].

Let  $X_1, \dots, X_n$  be a random sample from a distribution that depends on an unknown parameter  $\lambda$ . A V-statistic of order  $m$  with estimated parameter is given by

$$V_n(\hat{\lambda}_n) = \frac{1}{n^m} \sum_{i_1, \dots, i_m=1}^n \Phi(X_{i_1}, \dots, X_{i_m}; \hat{\lambda}_n). \quad (1)$$

If its kernel  $\Phi(x_1, \dots, x_m; \lambda)$  satisfies

$$\varphi_1(x) = E_\lambda(\Phi(x, X_2, \dots, X_m; \lambda)) = 0 \quad (2)$$

for all  $x$ , and

$$E_\lambda \varphi_2^2(X_1, X_2) > 0 \quad (3)$$

where

$$\varphi_2(x, y; \lambda) = E_\lambda(\Phi(x, y, X_3, \dots, X_m; \lambda)), \quad (4)$$

then (1) is a V-statistic with a weakly degenerate kernel.

If the parameter  $\lambda$  is known, then the asymptotic distribution is (see e.g. [24])

$$nV_n(\lambda) \xrightarrow{D} \binom{2m}{2} \sum_{k=1}^{+\infty} v_k Z_k^2, \quad (5)$$

where  $\{v_k\}, k = 1, 2, \dots$ , is the sequence of eigenvalues of the operator  $A$  defined on  $L^2(\mathbb{R}, F)$  as

$$Aq(x) = \int_R \varphi_2(x, y) q(y) dF(y), \quad (6)$$

and  $Z_k, k = 1, 2, \dots$ , are i.i.d random variables with standard normal distribution.

In [7] the following special case was considered.

$$V_n(\hat{\lambda}) = \int_{-\infty}^{\infty} \left( \frac{1}{n} \sum_{i=1}^n g(X_i, t; \hat{\lambda}) \right)^2 dM(t),$$

where  $g$  is a function satisfying some regularity conditions and  $M(t)$  is a finite measure. The kernel of  $V_n(\hat{\lambda})$  is then

$$\Phi(x_1, x_2, \hat{\lambda}) = \int_{-\infty}^{\infty} g(x_1, t; \hat{\lambda}) g(x_2, t; \hat{\lambda}) dM(t).$$

## 2 Main results

Generalizing the idea from [7], we consider the V-statistics of order  $2m$  with the following symmetrized kernel

$$\Phi(x_1, \dots, x_{2m}; \hat{\lambda}_n) = \frac{1}{(2m)!} \sum_{\pi \in \Pi(2m)} \int_{-\infty}^{+\infty} g(x_{\pi(1)}, \dots, x_{\pi(m)}; \hat{\lambda}_n) g(x_{\pi(m+1)}, \dots, x_{\pi(2m)}; \hat{\lambda}_n) dM(t), \quad (7)$$

where  $\Pi(2m)$  is the set of all permutations of  $\{1, 2, \dots, 2m\}$  and  $M(t)$  is a finite measure.

Next we state some regularity conditions imposed on the smoothness of the function  $g$  and the rate of convergence of the estimator  $\hat{\lambda}_n$ .

**Condition 1.** Suppose that  $\mu(t; \gamma) = E_{\lambda}(g(X_1, \dots, X_m, t; \gamma))$  exists for all  $\gamma$  in the neighbourhood of  $\lambda$  and it satisfies  $\mu(t; \lambda) \equiv 0$  for each  $t$ . Suppose additionally that for all  $\varepsilon > 0$  there exists a bounded sphere  $S$  in  $\mathbb{R}^p$  centered at  $\lambda$ , such that for  $\gamma \in S$  it holds

$$\frac{1}{\|\gamma - \lambda\|^2} \int_{-\infty}^{+\infty} (\mu(t; \gamma) - d_1 \mu(t; \lambda)^T (\gamma - \lambda))^2 dM(t) < \varepsilon,$$

where  $d_1 \mu(t; \lambda)$  is the vector of partial derivatives of  $\mu(t; \gamma) = E_{\lambda}(g(X_1, \dots, X_m, t; \gamma))$  at  $\gamma = \lambda$ , satisfying

$$\int_{-\infty}^{+\infty} (d_1 \mu(t; \lambda)_r)^2 dM(t) < \infty,$$

for  $r = 1, \dots, p$  where  $d_1 \mu(\cdot)_r$  is the  $r$ -th component of the vector  $d_1 \mu(\cdot)$ .

**Condition 2.** Suppose

$$\hat{\lambda}_n = \lambda + \frac{1}{n} \sum_{i=1}^n \alpha(X_i) + o_p\left(\frac{1}{\sqrt{n}}\right), \quad n \rightarrow \infty, \quad (8)$$

where  $E(\alpha(X_i)_r) = 0$  and  $E(\alpha(X_i)_r \alpha(X_i)_{r'}) < \infty$  for all  $1 \leq r \leq r' \leq p$ .

**Condition 3.** For any variation  $(i_1, \dots, i_m)$  of indices from  $\{1, 2, \dots, n\}$ , there exists a neighbourhood  $K(\lambda)$  of  $\lambda$  and a constant  $C > 0$ , such that for  $\gamma \in K(\lambda)$  and for a sphere  $D(\gamma, d)$  centered at  $\gamma$  with radius  $d$  such that  $D(\gamma, d) \subset K(\lambda)$ ,

it holds

$$\int_{-\infty}^{+\infty} E \left( \sup_{\gamma' \in D(\gamma, d)} |g(X_{i_1}, \dots, X_{i_m}, t; \gamma') - g(X_{i_1}, \dots, X_{i_m}, t; \gamma)| \right)^2 dM(t) \leq Cd^2.$$

In addition, for every  $\varepsilon > 0$  there exists  $d^* > 0$  such that for  $0 < d < d^*$  holds

$$\int_{-\infty}^{+\infty} E \left( \sup_{\gamma' \in D(\gamma, d)} |g(X_{i_1}, \dots, X_{i_m}, t; \gamma') - g(X_{i_1}, \dots, X_{i_m}, t; \gamma)| \right)^4 dM(t) < \varepsilon.$$

Following [7] we define a new statistic, that depends on the true value of the parameter

$$\begin{aligned} V_n^*(\lambda) &= \frac{1}{n^{2m}} \sum_{i_1, \dots, i_{2m}} \Phi_*(X_{i_1}, \dots, X_{i_{2m}}; \lambda) \\ &= \frac{1}{n^{2m}} \sum_{i_1, \dots, i_{2m}} \int_{-\infty}^{+\infty} \left( g(X_{i_1}, \dots, X_{i_m}, t; \lambda) + d_1 \mu(t; \lambda)^T \frac{1}{m} \sum_{i_j \in \{i_1, \dots, i_m\}} \alpha(X_{i_j}) \right) \\ &\quad \times \left( g(X_{i_{m+1}}, \dots, X_{i_{2m}}, t; \lambda) + d_1 \mu(t; \lambda)^T \frac{1}{m} \sum_{i_k \in \{i_{m+1}, \dots, i_{2m}\}} \alpha(X_{i_k}) \right) dM(t), \end{aligned} \quad (9)$$

where the function  $\alpha(\cdot)$  is defined in Condition 2. Let  $\varphi_1(x; \lambda)$  and  $\varphi_2(x, y; \lambda)$  be the first and second projection of the symmetrized version of the kernel  $\Phi_*$ . Notice that

$$\begin{aligned} \varphi_1^*(x; \lambda) &= E(\Phi_*(x, X_2, \dots, X_{2m}; \lambda)) \\ &= \int_{-\infty}^{+\infty} E \left( g(x, X_2, \dots, X_m, t; \lambda) + d_1 \mu(t; \lambda)^T \frac{1}{m} (\alpha(x) + \sum_{j \in \{2, \dots, m\}} \alpha(X_j)) \right) \\ &\quad \times E \left( g(X_{m+1}, \dots, X_{2m}, t; \lambda) + d_1 \mu(t; \lambda)^T \frac{1}{m} \sum_{k \in \{m+1, \dots, 2m\}} \alpha(X_k) \right) dM(t) \\ &= 0 \end{aligned}$$

which follows from Conditions 1 and 2. Moreover,

$$\begin{aligned} \varphi_2^*(x_1, x_{m+1}; \lambda) &= \frac{2m^2(2m-2)!}{(2m)!} E(\Phi_*(x_1, X_2, \dots, X_m, x_{m+1}, X_{m+2}, \dots, X_{2m}; \lambda)) \\ &= \frac{2m^2(2m-2)!}{(2m)!} \int_{-\infty}^{+\infty} \left( g_1(x_1, t; \lambda) + d_1 \mu(t; \lambda)^T \frac{1}{m} \alpha(x_1) \right) \left( g_1(x_{m+1}, t; \lambda) + d_1 \mu(t; \lambda)^T \frac{1}{m} \alpha(x_{m+1}) \right) dM(t), \end{aligned} \quad (10)$$

where  $g_1(x, t; \lambda) = E_\lambda(g(x, X_2, \dots, X_m, t; \lambda))$ .

Since  $V_n^*(\lambda)$  is a weakly degenerate V-statistic, it holds

$$nV_n^*(\lambda) \xrightarrow{d} \binom{2m}{2} \sum_{k=1}^{+\infty} v_k^* Z_k^2, \quad (11)$$

where  $\{v_k^*\}$  is the sequence of eigenvalues of integral operator  $A^*$  defined with

$$A^*q(x) = \int_R \varphi_2^*(x, y)q(y)dF(y), \quad (12)$$

and  $\{Z_k\}$  is the sequence of i.i.d. random variables with standard normal distribution. The following theorem gives the asymptotic distribution of a V-statistic with kernel (7).

**Theorem 2.1.** *Suppose Conditions 1-3 are satisfied. Additionally, suppose  $E(\Phi_*^2(X_1, \dots, X_{2m})) < +\infty$  and  $E(\Phi_*(X_{i_1}, \dots, X_{i_{2m}})) < +\infty$ , for all variations  $i_1, \dots, i_m, 1 \leq i_1, \dots, i_{2m} \leq n$ . Then*

$$n(V_n(\hat{\lambda}_n) - V_n^*(\lambda)) \xrightarrow{P} 0 \quad (13)$$

Moreover, if  $E(\varphi_2^*(X_1, X_2; \lambda))^2 > 0$ , then

$$nV_n(\hat{\lambda}_n) \xrightarrow{D} \binom{2m}{2} \sum_{k=1}^{+\infty} v_k^* Z_k^2,$$

where  $\{v_k^*\}$  is the sequence of eigenvalues of the operator  $A^*$  defined in (12) and  $\{Z_k\}$  is the sequence of i.i.d. random variables with standard normal distribution.

*Proof.* The idea is to show the equidistribution of  $nV_n(\hat{\lambda})$  and  $nV_n^*(\lambda)$ , from where, taking into account (11), will follow the statement of the theorem.

Without loss of generality we assume  $\int_{-\infty}^{+\infty} dM(t) = 1$ .

Define an auxillary statistic  $Y_n$  as

$$Y_n = \int_{-\infty}^{+\infty} \left( \frac{1}{n^m} \sum_{i_1, \dots, i_m} g(X_{i_1}, \dots, X_{i_m}, t; \lambda) + \mu(t; \hat{\lambda}_n) \right)^2 dM(t).$$

It is sort of midway from  $V_n^*(\lambda)$  to  $V_n(\hat{\lambda}_n)$ , depending both on the true value  $\lambda$  and its estimator  $\hat{\lambda}_n$ . We show that  $nY_n - nV_n^*(\lambda) \xrightarrow{P} 0$  and  $nV_n(\hat{\lambda}_n) - nY_n \xrightarrow{P} 0$ .

Consider first the difference between  $Y_n$  i  $V_n^*(\lambda)$ . Then, using the identity  $a^2 - b^2 = (a - b)^2 + 2b(a - b)$  we get

$$\begin{aligned} Y_n - V_n^*(\lambda) &= \int_{-\infty}^{+\infty} \left( \frac{1}{n^m} \sum_{i_1, \dots, i_m} g(X_{i_1}, \dots, X_{i_m}, t; \lambda) + \mu(t; \hat{\lambda}_n) \right)^2 dM(t) \\ &\quad - \int_{-\infty}^{+\infty} \left( \frac{1}{n^m} \sum_{i_1, \dots, i_m} \left( g(X_{i_1}, \dots, X_{i_m}, t; \lambda) + d_1 \mu(t; \lambda)^T \frac{1}{m} \sum_{i_k \in \{i_1, \dots, i_m\}} \alpha(X_{i_k}) \right) \right)^2 dM(t) \\ &= \int_{-\infty}^{+\infty} \left( \mu(t; \hat{\lambda}_n) - d_1 \mu(t; \lambda)^T \frac{1}{n} \sum_{k=1}^n \alpha(X_k) \right)^2 dM(t) + 2 \int_{-\infty}^{+\infty} \left( \frac{1}{n^m} \sum_{i_1, \dots, i_m} \left( g(X_{i_1}, \dots, X_{i_m}, t; \lambda) \right. \right. \\ &\quad \left. \left. + d_1 \mu(t; \lambda)^T \frac{1}{m} \sum_{i_k \in \{i_1, \dots, i_m\}} \alpha(X_{i_k}) \right) \right) \left( \mu(t; \hat{\lambda}_n) - d_1 \mu(t; \lambda)^T \frac{1}{n} \sum_{k=1}^n \alpha(X_k) \right) dM(t) \end{aligned}$$

Therefore, using the Cauchy-Schwarz inequality

$$\begin{aligned}
 n(Y_n - V_n(\lambda)) &\leq n \int_{-\infty}^{+\infty} \left( \mu(t; \hat{\lambda}_n) - d_1 \mu(t; \lambda)^T \frac{1}{n} \sum_{k=1}^n \alpha(X_k) \right)^2 dM(t) \\
 &\quad + 2 \left( \int_{-\infty}^{+\infty} n \left( \frac{1}{n^m} \sum_{i_1, \dots, i_m} g(X_{i_1}, \dots, X_{i_m}, t; \lambda) + d_1 \mu(t; \lambda)^T \frac{1}{n} \sum_{k=1}^n \alpha(X_k) \right)^2 dM(t) \right)^{\frac{1}{2}} \\
 &\quad \times \left( \int_{-\infty}^{+\infty} n \left( \mu(t; \hat{\lambda}_n) - d_1 \mu(t; \lambda)^T \frac{1}{n} \sum_{k=1}^n \alpha(X_k) \right)^2 dM(t) \right)^{\frac{1}{2}}. \\
 &= n \int_{-\infty}^{+\infty} \left( \mu(t; \hat{\lambda}_n) - d_1 \mu(t; \lambda)^T \frac{1}{n} \sum_{k=1}^n \alpha(X_k) \right)^2 dM(t) \\
 &\quad + 2\sqrt{nV_n^*(\lambda)} \left( \int_{-\infty}^{+\infty} n \left( \mu(t; \hat{\lambda}_n) - d_1 \mu(t; \lambda)^T \frac{1}{n} \sum_{k=1}^n \alpha(X_k) \right)^2 dM(t) \right)^{\frac{1}{2}}.
 \end{aligned} \tag{14}$$

From the Condition 1 it follows that

$$n \int_{-\infty}^{+\infty} \left( \mu(t; \hat{\lambda}_n) - d_1 \mu(t; \lambda)^T (\hat{\lambda}_n - \lambda) \right)^2 dM(t) < \varepsilon \|\sqrt{n}(\hat{\lambda}_n - \lambda)\|^2. \tag{15}$$

Since due to Condition 2  $\sqrt{n}(\hat{\lambda}_n - \lambda)$  is bounded in probability, the first summand of (14) tends to zero in probability. The second summand also tends to zero due to the Slutsky theorem and the fact that  $nV_n^*(\lambda)$  for V-statistics with kernels satisfying the conditions of the theorem are bounded in probability.

Consider now the difference between  $V_n(\hat{\lambda}_n)$  and  $Y_n$ . Analogously to the previous case we get

$$\begin{aligned}
 n(V_n(\hat{\lambda}_n) - Y_n) &\leq \int_{-\infty}^{+\infty} n \left( \frac{1}{n^m} \sum_{i_1, \dots, i_m} \left( g(X_{i_1}, \dots, X_{i_m}, t; \hat{\lambda}_n) - g(X_{i_1}, \dots, X_{i_m}, t; \lambda) \right) - \mu(t; \hat{\lambda}_n) \right)^2 dM(t) \\
 &\quad + 2\sqrt{nY_n} \left( \int_{-\infty}^{+\infty} n \left( \frac{1}{n^m} \sum_{i_1, \dots, i_m} \left( g(X_{i_1}, \dots, X_{i_m}, t; \hat{\lambda}_n) - g(X_{i_1}, \dots, X_{i_m}, t; \lambda) \right) - \mu(t; \hat{\lambda}_n) \right)^2 dM(t) \right)^{\frac{1}{2}}.
 \end{aligned}$$

Since  $nY_n$  is bounded in probability, it suffices to prove that

$$\int_{-\infty}^{+\infty} n \left( \frac{1}{n^m} \sum_{i_1, \dots, i_m} \left( g(X_{i_1}, \dots, X_{i_m}, t; \hat{\lambda}_n) - g(X_{i_1}, \dots, X_{i_m}, t; \lambda) \right) - \mu(t; \hat{\lambda}_n) \right)^2 dM(t)$$

converges to zero in probability.

Define

$$Q_n(s) = \int_{-\infty}^{+\infty} \left( \frac{1}{n^m} \sum_{i_1, \dots, i_m} \left( g\left(X_{i_1}, \dots, X_{i_m}, t; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_{i_1}, \dots, X_{i_m}, t; \lambda) - \mu\left(t; \lambda + \frac{s}{\sqrt{n}}\right) \right) \right)^2 dM(t).$$

Then we need to show that

$$nQ_n(\sqrt{n}(\hat{\lambda}_n - \lambda)) \xrightarrow{P} 0.$$

Condition 2 ensures the existence of a sphere  $S$  in  $\mathbb{R}^p$  and  $\delta > 0$  such that for all  $n$

$$P\{\sqrt{n}(\hat{\lambda}_n - \lambda) \notin S\} \leq \frac{\delta}{2},$$

and for any  $\varepsilon' > 0$  we have

$$\begin{aligned} P\{nQ_n(\sqrt{n}(\hat{\lambda}_n - \lambda)) > \varepsilon'\} &= P\{nQ_n(\sqrt{n}(\hat{\lambda}_n - \lambda)) > \varepsilon', \sqrt{n}(\hat{\lambda}_n - \lambda) \in S\} \\ &\quad + P\{nQ_n(\sqrt{n}(\hat{\lambda}_n - \lambda)) > \varepsilon', \sqrt{n}(\hat{\lambda}_n - \lambda) \notin S\} \\ &\leq P\{\sup_{s \in S} nQ_n(s) > \varepsilon'\} + P\{\sqrt{n}(\hat{\lambda}_n - \lambda) \notin S\}. \end{aligned}$$

Hence it is enough to prove that  $\sup_{s \in S} nQ_n(s) \xrightarrow{P} 0$ . From the Chebyshev inequality it follows that

$$P\left\{\sup_{s \in S} nQ_n(s) > \varepsilon'\right\} \leq \frac{1}{\varepsilon'^2} E\left(\sup_{s \in S} nQ_n(s)\right)^2. \quad (16)$$

The sum  $nQ_n(s)$  can be decomposed as

$$\begin{aligned} nQ_n(s) &= n \int_{-\infty}^{+\infty} \left( \frac{1}{n^m} \sum_{i_1, \dots, i_m} \left( g\left(X_{i_1}, \dots, X_{i_m}, t; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_{i_1}, \dots, X_{i_m}, t; \lambda) - \mu\left(t; \lambda + \frac{s}{\sqrt{n}}\right) \right) \right)^2 dM(t) \\ &= \frac{n}{n^{2m}} \sum_{\{i_1, \dots, i_{2m}\} \in \mathcal{J}'} \int_{-\infty}^{+\infty} \left( g\left(X_{i_1}, \dots, X_{i_m}, t; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_{i_1}, \dots, X_{i_m}, t; \lambda) - \mu\left(t; \lambda + \frac{s}{\sqrt{n}}\right) \right) \\ &\quad \times \left( g\left(X_{i_{m+1}}, \dots, X_{i_{2m}}, t; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_{i_{m+1}}, \dots, X_{i_{2m}}, t; \lambda) - \mu\left(t; \lambda + \frac{s}{\sqrt{n}}\right) \right) dM(t) \\ &\quad + \frac{n}{n^{2m}} \sum_{\{i_1, \dots, i_{2m}\} \in \mathcal{J}''} \int_{-\infty}^{+\infty} \left( g\left(X_{i_1}, \dots, X_{i_m}, t; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_{i_1}, \dots, X_{i_m}, t; \lambda) - \mu\left(t; \lambda + \frac{s}{\sqrt{n}}\right) \right) \\ &\quad \times \left( g\left(X_{i_{m+1}}, \dots, X_{i_{2m}}, t; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_{i_{m+1}}, \dots, X_{i_{2m}}, t; \lambda) - \mu\left(t; \lambda + \frac{s}{\sqrt{n}}\right) \right) dM(t) \\ &= nQ'_n(s) + nQ''_n(s), \end{aligned}$$

where  $\mathcal{J}'$  is the set of  $2m$ -tuples of indices such that no entry repeats, i.e.  $i_1 \neq \dots \neq i_{2m}$ , and  $\mathcal{J}''$  is the complement of  $\mathcal{J}'$ , i.e. the set of  $2m$ -tuples of indices such that at least one entry repeats. Hence

$$\sup_{s \in S} nQ_n(s) \leq \sup_{s \in S} nQ'_n(s) + \sup_{s \in S} nQ''_n(s). \quad (17)$$

Consider now the first summand of the right hand side of (17). Defining

$$H(x_1, \dots, x_{2m}) = \sup_{s \in S} \int_{-\infty}^{+\infty} \left| \left( g\left(x_1, \dots, x_m, t; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t; \lambda + \frac{s}{\sqrt{n}}\right) - g(x_1, \dots, x_m, t; \lambda) \right) \right. \\ \left. \times \left( g\left(x_{m+1}, \dots, x_{2m}, t; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t; \lambda + \frac{s}{\sqrt{n}}\right) - g(x_{m+1}, \dots, x_{2m}, t; \lambda) \right) \right| dM(t) \quad (18)$$

we have that

$$\sup_{s \in S} n Q'_n(s) \leq \frac{n}{n^{2m}} \sum_{\{i_1, \dots, i_{2m}\} \in \mathcal{J}'} H(X_{i_1}, \dots, X_{i_{2m}}).$$

Consider the expectation

$$E\left(\frac{n}{n^{2m}} \sum_{\{i_1, \dots, i_{2m}\} \in \mathcal{J}'} H(X_{i_1}, \dots, X_{i_{2m}})\right)^2 = \frac{n^2}{n^{4m}} \sum_{\substack{\{i_1, \dots, i_{2m}\} \in \mathcal{J}' \\ \{i_{2m+1}, \dots, i_{4m}\} \in \mathcal{J}'}} E(H(X_{i_1}, \dots, X_{i_{2m}}) H(X_{i_{2m+1}}, \dots, X_{i_{4m}})) \\ = \frac{n^2}{n^{4m}} \sum_{c=0}^{2m} \binom{n}{c} \binom{n-c}{4m-2c} \Psi_n(c),$$

where exactly  $c$  indices from  $\{i_1, \dots, i_{2m}\}$  coincide with exactly  $c$  indices from  $\{i_{2m+1}, \dots, i_{4m}\}$ , all the others being different. Then each term of the sum above asymptotically behaves as  $\frac{1}{n^{c-2}} \Psi_n(c)$ .

When  $c = 0$  we get

$$\begin{aligned} \Psi_n(0) &= (4m)! E(H(X_1, \dots, X_{2m}) H(X_{2m+1}, \dots, X_{4m})) = (E(H(X_1, \dots, X_{2m})))^2 \\ &\leq (4m)! \left( \int_{-\infty}^{+\infty} \left( E \left[ \sup_{s \in S} \left| g\left(X_1, \dots, X_m, t; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_1, \dots, X_m, t; \lambda) \right| \right] \right)^2 dM(t) \right)^2 \\ &\leq (4m)! \left( 2 \int_{-\infty}^{+\infty} \left( E \left[ \sup_{s \in S} \left| g\left(X_1, \dots, X_m, t; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_1, \dots, X_m, t; \lambda) \right| \right] \right)^2 dM(t) \right. \\ &\quad \left. + 2 \int_{-\infty}^{+\infty} \left( \sup_{s \in S} \left| \mu\left(t; \lambda + \frac{s}{\sqrt{n}}\right) \right| \right)^2 dM(t) \right)^2 \\ &\leq 4(4m)! \left( C \frac{d^2}{n} + \int_{-\infty}^{+\infty} \left( \sup_{s \in S} \left| d_1 \mu\left(t; \lambda\right) \right|^T \frac{s}{\sqrt{n}} \right)^2 dM(t) \right)^2 \leq 4(4m)! \frac{d^4}{n^2} (C+K)^2, \end{aligned}$$

where

$$K = \int_{-\infty}^{+\infty} \sum_{r=1}^p (d_1 \mu(t; \lambda)_r)^2 dM(t), \quad (19)$$

which is finite due to Condition 1.



When  $c = 1$  we get

$$\begin{aligned}
 \Psi_n(1) &= 4m^2(4m-2)!E(H(X_1, X_2, \dots, X_{2m})H(X_1, X_{2m+1}, \dots, X_{4m-1})) \\
 &\leq 4m^2(4m-2)! \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E \left[ \sup_{s \in S} \left| g\left(X_1, \dots, X_m, t_1; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t_1; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_1, \dots, X_m, t_1; \lambda) \right| \right. \\
 &\quad \times \sup_{s \in S} \left| g\left(X_{m+1}, \dots, X_{2m}, t_1; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t_1; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_{m+1}, \dots, X_{2m}, t_1; \lambda) \right| \\
 &\quad \times \sup_{s \in S} \left| g\left(X_1, X_{2m+1}, \dots, X_{3m-1}, t_2; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t_2; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_1, X_{2m+1}, \dots, X_{3m-1}, t_2; \lambda) \right| \\
 &\quad \left. \times \sup_{s \in S} \left| g\left(X_{3m}, \dots, X_{4m-1}, t_2; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t_2; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_{3m}, \dots, X_{4m-1}, t_2; \lambda) \right| \right] dM(t_1) dM(t_2)
 \end{aligned}$$

Applying the Cauchy-Schwarz inequality and grouping back the integrals we get

$$\begin{aligned}
 \Psi_n(1) &\leq 4m^2(4m-2)! \int_{-\infty}^{+\infty} \left( E \left[ \sup_{s \in S} \left| g\left(X_1, \dots, X_m, t_1; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t_1; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_1, \dots, X_m, t_1; \lambda) \right| \right]^2 \right)^{\frac{1}{2}} \\
 &\quad \times E \left[ \sup_{s \in S} \left| g\left(X_{m+1}, \dots, X_{2m}, t_1; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t_1; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_{m+1}, \dots, X_{2m}, t_1; \lambda) \right| \right] dM(t_1) \\
 &\quad \times \int_{-\infty}^{+\infty} \left( E \left[ \sup_{s \in S} \left| g\left(X_1, X_{2m+1}, \dots, X_{3m-1}, t_2; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t_2; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_1, X_{2m+1}, \dots, X_{3m-1}, t_2; \lambda) \right| \right]^2 \right)^{\frac{1}{2}} \\
 &\quad \times E \left[ \sup_{s \in S} \left| g\left(X_{3m}, \dots, X_{4m-1}, t_2; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t_2; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_{3m}, \dots, X_{4m-1}, t_2; \lambda) \right| \right] dM(t_2) \\
 &= 4m^2(4m-2)! \left( \int_{-\infty}^{+\infty} \left( E \left[ \sup_{s \in S} \left| g\left(X_1, \dots, X_m, t_1; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t_1; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_1, \dots, X_m, t_1; \lambda) \right| \right]^2 \right)^{\frac{1}{2}} \right. \\
 &\quad \times E \left[ \sup_{s \in S} \left| g\left(X_{m+1}, \dots, X_{2m}, t_1; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t_1; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_{m+1}, \dots, X_{2m}, t_1; \lambda) \right| \right] dM(t_1) \Big)^2 \\
 &\leq 4m^2(4m-2)! \int_{-\infty}^{+\infty} E \left[ \sup_{s \in S} \left| g\left(X_1, \dots, X_m, t_1; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t_1; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_1, \dots, X_m, t_1; \lambda) \right| \right]^2 dM(t_1) \\
 &\quad \times \int_{-\infty}^{+\infty} \left( E \left[ \sup_{s \in S} \left| g\left(X_{m+1}, \dots, X_{2m}, t_1; \lambda + \frac{s}{\sqrt{n}}\right) - \mu\left(t_1; \lambda + \frac{s}{\sqrt{n}}\right) - g(X_{m+1}, \dots, X_{2m}, t_1; \lambda) \right| \right]^2 \right) dM(t_1).
 \end{aligned}$$

Applying now the inequality  $(a+b)^2 \leq 2a^2 + 2b^2$  to both factors of the product above we obtain

$$\begin{aligned}
 \Psi_n(1) &\leq 4m^2(4m-2)! \int_{-\infty}^{+\infty} E \left[ \sup_{s \in S} \left| g(X_1, \dots, X_m, t_1; \lambda + \frac{s}{\sqrt{n}}) - g(X_1, \dots, X_m, t_1; \lambda) \right| + \sup_{s \in S} \left| \mu(t_1; \lambda + \frac{s}{\sqrt{n}}) \right| \right]^2 dM(t_1) \\
 &\quad \times \int_{-\infty}^{+\infty} \left( E \left[ \sup_{s \in S} \left| g(X_{m+1}, \dots, X_{2m}, t_1; \lambda + \frac{s}{\sqrt{n}}) - g(X_{m+1}, \dots, X_{2m}, t_1; \lambda) \right| \right] + \sup_{s \in S} \left| \mu(t_1; \lambda + \frac{s}{\sqrt{n}}) \right| \right)^2 dM(t_1) \\
 &\leq 4m^2(4m-2)! \left( 2 \int_{-\infty}^{+\infty} E \left[ \sup_{s \in S} \left| g(X_1, \dots, X_m, t_1; \lambda + \frac{s}{\sqrt{n}}) - g(X_1, \dots, X_m, t_1; \lambda) \right| \right]^2 dM(t) \right. \\
 &\quad \left. + 2 \int_{-\infty}^{+\infty} \left( \sup_{s \in S} \left| \mu(t_1; \lambda + \frac{s}{\sqrt{n}}) \right| \right)^2 dM(t_1) \right) \\
 &\quad \times \left( 2 \int_{-\infty}^{+\infty} \left( E \left[ \sup_{s \in S} \left| g(X_{m+1}, \dots, X_{2m}, t_1; \lambda + \frac{s}{\sqrt{n}}) - g(X_{m+1}, \dots, X_{2m}, t_1; \lambda) \right| \right] \right)^2 dM(t_1) \right. \\
 &\quad \left. + 2 \int_{-\infty}^{+\infty} \left( \sup_{s \in S} \left| \mu(t_1; \lambda + \frac{s}{\sqrt{n}}) \right| \right)^2 dM(t_1) \right) \\
 &\leq 4m^2(4m-2)! \left( 2\varepsilon^{\frac{1}{2}} + 2\frac{d^2}{n}K \right) \left( 2\frac{d^2}{n}(C+K) \right) \\
 &\sim \varepsilon^{\frac{1}{2}} \frac{d^2}{n} + o_p\left(\frac{1}{n}\right),
 \end{aligned}$$

where  $K$  is defined in (19).

When  $c = 2$  we get

$$\Psi_n(2) = \left( 16 \binom{m}{2}^2 \varpi_1 + 4m^4 \varpi_2 + 16 \binom{m}{2} m^2 \varpi_3 \right) (4m-4)!,$$

where

$$\begin{aligned}
 \varpi_1 &= E(H(X_1, X_2, X_3, \dots, X_{2m})H(X_1, X_2, X_{2m+1}, \dots, X_{4m-2})), \\
 \varpi_2 &= E(H(X_1, X_2, \dots, X_{2m})H(X_1, X_{2m+1}, \dots, X_{3m-1}, X_{m+1}, X_{3m}, \dots, X_{4m-2})), \\
 \varpi_3 &= E(H(X_1, X_2, \dots, X_{2m})H(X_1, X_{2m+1}, \dots, X_{3m-1}, X_2, X_{3m}, \dots, X_{4m-2})).
 \end{aligned}$$

To assess the behaviour of these three functions we apply the same techniques as above. For  $\varpi_1$  we obtain

$$\begin{aligned}
 \varpi_1 &\leq \left( \int_{-\infty}^{+\infty} \left( E \left[ \sup_{s \in S} \left| g(X_1, \dots, X_m, t_1; \lambda + \frac{s}{\sqrt{n}}) - \mu(t_1; \lambda + \frac{s}{\sqrt{n}}) - g(X_1, \dots, X_m, t_1; \lambda) \right| \right] \right)^2 \right)^{\frac{1}{2}} \\
 &\quad \times E \left[ \sup_{s \in S} \left| g(X_{m+1}, \dots, X_{2m}, t_1; \lambda + \frac{s}{\sqrt{n}}) - \mu(t_1; \lambda + \frac{s}{\sqrt{n}}) - g(X_{m+1}, \dots, X_{2m}, t_1; \lambda) \right| \right]^2 dM(t_1) \\
 &\leq \varepsilon^{\frac{1}{2}} \frac{d^2}{n} + o_p\left(\frac{1}{n}\right),
 \end{aligned}$$

as the expression is identical to that of  $\Psi(1)$ .

For  $\varpi_2$  after some calculation we get

$$\begin{aligned}
 \varpi_2 &\leq \int_{-\infty}^{+\infty} \left( E \left[ \sup_{s \in S} \left| g(X_1, \dots, X_m, t_1; \lambda + \frac{s}{\sqrt{n}}) - g(X_1, \dots, X_m, t_1; \lambda) \right| + \sup_{s \in S} \left| \mu(t_1; \lambda + \frac{s}{\sqrt{n}}) \right| \right]^2 \right)^2 dM(t_1) \\
 &\leq 8 \left( \int_{-\infty}^{+\infty} E \left[ \sup_{s \in S} \left| g(X_1, \dots, X_m, t_1; \lambda + \frac{s}{\sqrt{n}}) - g(X_1, \dots, X_m, t_1; \lambda) \right| \right]^4 dM(t_1) \right. \\
 &\quad \left. + \int_{-\infty}^{+\infty} \left( \sup_{s \in S} \left| \mu(t_1; \lambda + \frac{s}{\sqrt{n}}) \right| \right)^4 dM(t_1) \right) \\
 &< 8 \left( \varepsilon + \frac{d^4}{n^2} \cdot K \right) = 8\varepsilon + o_p\left(\frac{1}{n}\right).
 \end{aligned}$$

For  $\varpi_3$  in a similar fashion we obtain the same result as for  $\varpi_2$ , i.e.

$$\begin{aligned}
 \varpi_3 &\leq \int_{-\infty}^{+\infty} \left( E \left[ \sup_{s \in S} \left| g(X_1, \dots, X_m, t_1; \lambda + \frac{s}{\sqrt{n}}) - g(X_1, \dots, X_m, t_1; \lambda) \right| + \sup_{s \in S} \left| \mu(t_1; \lambda + \frac{s}{\sqrt{n}}) \right| \right]^2 \right)^2 dM(t_1) \\
 &\leq 8\varepsilon + o_p\left(\frac{1}{n}\right).
 \end{aligned}$$

When  $c > 2$ ,  $\frac{1}{n^{c-2}} \Psi_n(c) = o_p\left(\frac{1}{n}\right)$  since the expectation of the corresponding functions is bounded, and hence, taking into account that for any  $\varepsilon > 0$ ,  $d$  can be made as small as desired, we conclude

$$E(\sup_{s \in S} nQ'_n(s))^2 < \text{const} \cdot \varepsilon + o_p(n^{-1}).$$

Consider now the second summand of (17). We have

$$\sup_{s \in S} nQ''_n(s) \leq \frac{n}{n^{2m}} \sum_{\{i_1, \dots, i_{2m}\} \in \mathcal{J}''} H(X_{i_1}, \dots, X_{i_{2m}})$$

where  $\mathcal{J}''$  is the set of  $2m$ -tuples of indices such that at least one entry repeats, and  $H$  is defined in (18). Let  $\mathcal{J}$  be the set of  $4m$ -tuples of indices such that  $\{i_1, \dots, i_{2m}\} \in \mathcal{J}''$  and  $\{i_{2m+1}, \dots, i_{4m}\} \in \mathcal{J}''$ . After some calculation we get

$$\begin{aligned}
 E \left( \frac{n}{n^{2m}} \sum_{\{i_1, \dots, i_{2m}\} \in \mathcal{J}''} H(X_{i_1}, \dots, X_{i_{2m}}) \right)^2 &= \frac{n^2}{n^{4m}} \sum_{\{i_1, \dots, i_{4m}\} \in \mathcal{J}} E \left( H(X_{i_1}, \dots, X_{i_{2m}}) H(X_{i_{2m+1}}, \dots, X_{i_{4m}}) \right) \\
 &\leq \frac{n^2}{n^{4m}} \sum_{\{i_1, \dots, i_{4m}\} \in \mathcal{J}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E \left( \sup_{s \in S} \left| g(X_{i_1}, \dots, X_{i_m}, t_1; \lambda + \frac{s}{\sqrt{n}}) - g(X_{i_1}, \dots, X_{i_m}, t_1; \lambda) - \mu(t_1; \lambda + \frac{s}{\sqrt{n}}) \right| \right. \\
 &\quad \times \sup_{s \in S} \left| g(X_{i_{m+1}}, \dots, X_{i_{2m}}, t_1; \lambda + \frac{s}{\sqrt{n}}) - g(X_{i_{m+1}}, \dots, X_{i_{2m}}, t_1; \lambda) - \mu(t_1; \lambda + \frac{s}{\sqrt{n}}) \right| \\
 &\quad \times \sup_{s \in S} \left| g(X_{i_{2m+1}}, \dots, X_{i_{3m}}, t_2; \lambda + \frac{s}{\sqrt{n}}) - g(X_{i_{2m+1}}, \dots, X_{i_{3m}}, t_2; \lambda) - \mu(t_2; \lambda + \frac{s}{\sqrt{n}}) \right| \\
 &\quad \left. \times \sup_{s \in S} \left| g(X_{i_{3m+1}}, \dots, X_{i_{4m}}, t_2; \lambda + \frac{s}{\sqrt{n}}) - g(X_{i_{3m+1}}, \dots, X_{i_{4m}}, t_2; \lambda) - \mu(t_2; \lambda + \frac{s}{\sqrt{n}}) \right| \right) dM(t_1) dM(t_2).
 \end{aligned}$$

Let us consider the case when only one index among the first  $2m$  indices repeats only once, and, analogously, only one index among the last  $2m$  repeats only once (all the other  $4m - 4$  indices being different). Out of many combinations of repeating patterns among indices, it is easy to show that this one has the highest order. The general term (labeled as  $(\diamond)$ ) is

$$(\diamond) = \frac{n^2}{n^{4m}} \binom{n}{2} \binom{n-2}{4m-4} \binom{2m}{2}^2 (4m-4)! \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E \left( \sup_{s \in S} \left| g \left( X_{i_1}, \dots, X_{i_m}, t_1; \lambda + \frac{s}{\sqrt{n}} \right) - g(X_{i_1}, \dots, X_{i_m}, t_1; \lambda) \right| \right).$$

When  $n \rightarrow \infty$ , applying the same techniques as before, we get

$$\begin{aligned} (\diamond) &\sim \left( \int_{-\infty}^{+\infty} E \left( \sup_{s \in S} \left| g \left( X_{i_1}, \dots, X_{i_m}, t_1; \lambda + \frac{s}{\sqrt{n}} \right) - g(X_{i_1}, \dots, X_{i_m}, t_1; \lambda) - \mu \left( t_1; \lambda + \frac{s}{\sqrt{n}} \right) \right| \right. \right. \\ &\quad \times \left. \left. \sup_{s \in S} \left| g \left( X_{i_{m+1}}, \dots, X_{i_{2m}}, t_1; \lambda + \frac{s}{\sqrt{n}} \right) - g(X_{i_{m+1}}, \dots, X_{i_{2m}}, t_1; \lambda) - \mu \left( t_1; \lambda + \frac{s}{\sqrt{n}} \right) \right| \right) dM(t_1) \right)^2 \\ &\leq 8 \int_{-\infty}^{+\infty} E \left( \sup_{s \in S} \left| g \left( X_{i_1}, \dots, X_{i_m}, t_1; \lambda + \frac{s}{\sqrt{n}} \right) - g(X_{i_1}, \dots, X_{i_m}, t_1; \lambda) \right| \right)^4 dM(t_1) + 8 \int_{-\infty}^{+\infty} \left( \sup_{s \in S} \left| \mu \left( t_1; \lambda + \frac{s}{\sqrt{n}} \right) \right| \right)^4 dM(t_1) \\ &< 8\varepsilon + o_p\left(\frac{1}{n}\right). \end{aligned}$$

Therefore,

$$E(\sup_{s \in S} n Q_n''(s))^2 < 8\varepsilon + o_p(n^{-1}),$$

and from (17) we get

$$E\left(\sup_{s \in S} n Q_n(s)\right)^2 \leq 2\left(\sup_{s \in S} n Q_n'(s)\right)^2 + 2\left(\sup_{s \in S} n Q_n''(s)\right)^2 \sim \text{const} \cdot \varepsilon + o_p(1),$$

which in turn means that

$$E\left(\sup_{s \in S} n Q_n(s)\right)^2 \rightarrow 0,$$

and inserting it in (16) ends the proof.  $\square$

## 2.1 U-statistics

We now present the analogous theorem for U-statistics with estimated parameters. Define

$$U_n(\hat{\lambda}_n) = \frac{1}{\binom{n}{2m}} \sum_{i_1 < \dots < i_{2m}} \Phi(X_{i_1}, \dots, X_{i_{2m}}; \hat{\lambda}_n),$$

where  $\Phi(x_1, \dots, x_{2m}; \hat{\lambda}_n)$  is defined in (7) and the corresponding auxiliary statistic

$$U_n^*(\lambda) = \frac{1}{\binom{n}{2m}} \sum_{i_1 < \dots < i_{2m}} \Phi_*(X_{i_1}, \dots, X_{i_{2m}}; \lambda),$$

where  $\Phi_*(x_1, \dots, x_{2m}; \lambda)$  is defined in (9).

**Theorem 2.2.** Let  $X_1, \dots, X_n$  be a random sample with distribution function  $F$ . Let all condition of Theorem 2.1 hold, and, additionally,

$$E\left(\int_{-\infty}^{+\infty} |g_1(X_1, t; \lambda)|^2 dM(t)\right) < \infty. \quad (20)$$

Then

$$nU_n(\hat{\lambda}_n) \xrightarrow{D} \frac{2m(2m-1)}{2} \sum_{k=1}^{\infty} (v_k^* Z_k^2 - v_k),$$

where  $\{v_k^*\}$  and  $\{v_k\}$ ,  $k = 1, 2, \dots$ , are sequences of eigenvalues of operators  $A^*$  and  $A$ , defined in (12) and (6), respectively, while  $Z_k, k = 1, 2, \dots$ , are i.i.d. standard normal random variables.

*Proof.* Using the result on limiting distribution of degenerate U-statistics (see e.g. [13]), we get that

$$nU_n^*(\lambda) \xrightarrow{d} \frac{2m(2m-1)}{2} \sum_{k=1}^{\infty} v_k^* (Z_k^2 - 1).$$

Thus it is enough to prove that

$$n(U_n(\hat{\lambda}_n) - U_n^*(\lambda)) \xrightarrow{P} \left(\frac{2m}{2}\right) \sum_{k=1}^{\infty} (v_k - v_k^*). \quad (21)$$

Consider the difference

$$\begin{aligned} n(U_n(\hat{\lambda}_n) - U_n^*(\lambda)) &= \frac{n}{n(n-1) \cdots (n-2m+1)} \sum_{\{i_1, \dots, i_{2m}\} \in \mathcal{J}'} \left( \Phi(X_{i_1}, \dots, X_{i_{2m}}; \hat{\lambda}_n) - \Phi_*(X_{i_1}, \dots, X_{i_{2m}}; \lambda) \right) \\ &= \frac{n}{n(n-1) \cdots (n-2m+1)} \left( n^{2m} (V_n(\hat{\lambda}_n) - V_n^*(\lambda)) \right. \\ &\quad \left. - \sum_{\{i_1, \dots, i_{2m}\} \in \mathcal{J}''} \left( \Phi(X_{i_1}, \dots, X_{i_{2m}}; \hat{\lambda}_n) - \Phi_*(X_{i_1}, \dots, X_{i_{2m}}; \lambda) \right) \right), \end{aligned}$$

where the sets  $\mathcal{J}'$  and  $\mathcal{J}''$  were defined in the proof of Theorem 2.1. From the proof Theorem 2.1 we know that  $n(V_n(\hat{\lambda}_n) - V_n^*(\lambda)) \xrightarrow{P} 0$ , so it remains to find the limit in probability of

$$\begin{aligned} \Xi_n &= \frac{1}{(n-1) \cdots (n-2m+1)} \sum_{\{i_1, \dots, i_{2m}\} \in \mathcal{J}''} \left( \Phi(X_{i_1}, \dots, X_{i_{2m}}; \hat{\lambda}_n) - \Phi^*(X_{i_1}, \dots, X_{i_{2m}}; \lambda) \right) \\ &= \frac{1}{(n-1) \cdots (n-2m+1)} \sum_{\{i_1, \dots, i_{2m}\} \in \mathcal{J}_1''} \left( \Phi(X_{i_1}, \dots, X_{i_{2m}}; \hat{\lambda}_n) - \Phi^*(X_{i_1}, \dots, X_{i_{2m}}; \lambda) \right) \\ &\quad + \frac{1}{(n-1) \cdots (n-2m+1)} \sum_{\{i_1, \dots, i_{2m}\} \in \mathcal{J}'' \setminus \mathcal{J}_1''} \left( \Phi(X_{i_1}, \dots, X_{i_{2m}}; \hat{\lambda}_n) - \Phi^*(X_{i_1}, \dots, X_{i_{2m}}; \lambda) \right) \\ &= \Xi_n^{(1)} + \Xi_n^{(2)}, \end{aligned}$$

where  $\mathcal{J}_1''$  is the set of all  $2m$ -tuples of indices such that only one entry repeats only once and all the others are different.

For the first summand we have

$$\begin{aligned}\Xi_n^{(1)} &= \frac{n}{n-2m+1} \binom{2m}{2} \frac{1}{\binom{n}{2m-1}} \sum_{i_1 \neq i_2 \neq \dots \neq i_{2m-1}} \left( \Phi(X_{i_1}, X_{i_1}, X_{i_2}, \dots, X_{i_{2m-1}}; \hat{\lambda}_n) - \Phi_*(X_{i_1}, X_{i_1}, X_{i_2}, \dots, X_{i_{2m-1}}; \lambda) \right) \\ &= \frac{n}{n-2m+1} \binom{2m}{2} \frac{1}{2m-1} \frac{1}{\binom{n}{2m-1}} \sum_{i_1 \neq i_2 \neq \dots \neq i_{2m-1}} \left( \phi(X_{i_1}, X_{i_2}, \dots, X_{i_{2m-1}}; \hat{\lambda}_n) - \phi_*(X_{i_1}, X_{i_2}, \dots, X_{i_{2m-1}}; \lambda) \right),\end{aligned}$$

where  $\phi(X_{i_1}, \dots, X_{i_{2m-1}}; \hat{\lambda}_n)$  and  $\phi_*(X_{i_1}, \dots, X_{i_{2m-1}}; \lambda)$  are symmetrized versions of  $\Phi(X_{i_1}, X_{i_1}, X_{i_2}, \dots, X_{i_{2m-1}}; \hat{\lambda}_n)$  and  $\Phi_*(X_{i_1}, X_{i_1}, X_{i_2}, \dots, X_{i_{2m-1}}; \lambda)$ , respectively.

Using the law of large numbers for U-statistics (see e.g. [13]), and the law of large numbers for U-statistics with estimated parameters (see [11]), we get that

$$\Xi_n^{(1)} \xrightarrow{P} \binom{2m}{2} \frac{1}{2m-1} (E\phi(X_{i_1}, \dots, X_{i_{2m-1}}; \hat{\lambda}_n) - E\phi_*(X_{i_1}, X_{i_2}, \dots, X_{i_{2m-1}}; \lambda)).$$

We then have

$$\begin{aligned}E\phi_*(X_{i_1}, X_{i_2}, \dots, X_{i_{2m-1}}; \lambda) &= \frac{m^2(2m-1)!}{(2m)!} E \left( \int_{-\infty}^{+\infty} \left( g(X_1, X_2, \dots, X_{i_m}, t; \lambda) + d_1 \mu(t; \lambda)^T \frac{1}{m} \sum_{j \in \{1, \dots, m\}} \alpha(X_j) \right) \right. \\ &\quad \times \left. \left( g(X_1, X_{m+1}, \dots, X_{2m-1}, t; \lambda) + d_1 \mu(t; \lambda)^T \frac{1}{m} \sum_{k \in \{1, m+1, \dots, 2m-1\}} \alpha(X_k) \right) dM(t) \right) \\ &= \frac{m^2}{2m} \int_{-\infty}^{+\infty} E \left( g_1(X_1, t; \lambda) + d_1 \mu(t; \lambda)^T \frac{1}{m} \alpha(X_1) \right)^2 dM(t) \\ &= (2m-1) E\varphi_2(X_1, X_1) = (2m-1) \sum_{k=1}^{\infty} v_k^*,\end{aligned}$$

provided that the operator  $A^*$  is nuclear, i.e.  $\sum_{k=1}^{\infty} |v_k^*| < \infty$ . This will be the case if  $E|\varphi_2^*(X_1, X_1)| < \infty$ . Indeed,

$$\begin{aligned}E|\varphi_2^*(X_1, X_1)| &= E \left| \int_{-\infty}^{+\infty} \left( g_1(X_1, t; \lambda) + d_1 \mu(t; \lambda)^T \frac{1}{m} \alpha(X_1) \right)^2 dM(t) \right| \\ &\leq 2E \int_{-\infty}^{+\infty} \left| g_1(X_1, t; \lambda) \right|^2 dM(t) + \frac{2}{m^2} E \left| \alpha(X_1) \right|^2 \cdot \int_{-\infty}^{+\infty} \left| d_1 \mu(t; \lambda)^T \right|^2 dM(t),\end{aligned}$$

which is finite due to (20) and Conditions 1 and 2. Analogously  $E\phi(X_{i_1}, \dots, X_{i_{2m-1}}; \hat{\lambda}) = (2m-1) \sum_{k=1}^{\infty} v_k$ , and hence

$$\Xi_n^{(1)} \xrightarrow{P} \binom{2m}{2} \sum_{k=1}^{\infty} (v_k - v_k^*).$$

Next we show that  $\Xi_n^{(2)} \xrightarrow{P} 0$ . Let  $1 < r < 2m$  and let  $2m - r$  of  $2m$  indices in the expression for  $\Xi_n^{(2)}$  be different. Then, the corresponding terms of the sum are proportional to

$$\frac{n}{(n-2m+1) \cdots (n-2m+r)} \cdot \frac{1}{\binom{n}{2m-r}} \sum_{i_1 \neq i_2 \neq \cdots \neq i_{2m-r}} \left( \phi^{(r)}(X_{i_1}, X_{i_2}, \dots, X_{i_{2m-r}}; \hat{\lambda}_n) - \phi_*^{(r)}(X_{i_1}, X_{i_2}, \dots, X_{i_{2m-r}}; \lambda) \right),$$

where  $\phi^{(r)}(X_{i_1}, X_{i_2}, \dots, X_{i_{2m-r}}; \hat{\lambda}_n)$  and  $\phi_*^{(r)}(X_{i_1}, X_{i_2}, \dots, X_{i_{2m-r}}; \lambda)$  are obtained using the appropriate symmetrization of initial kernels. Using the law of large numbers for U-statistics and the Slutsky theorem, we get that the term above tends to zero in probability. This holds for all  $r$ , hence  $\Xi_n^{(2)} \xrightarrow{P} 0$  and (21) follows.  $\square$

### 3 Application

The main application of Theorems 2.1 i 2.2 lies in goodness-of-fit tests based on equidistribution-type characterizations. Such characterizations have the following form. Let  $X_1, \dots, X_m$  be independent copies of random variable  $X$  and let  $\omega_1(\cdot)$  and  $\omega_2(\cdot)$  be two functions such that

$$\omega_1(X_1, \dots, X_m) \stackrel{d}{=} \omega_2(X_1, \dots, X_m) \quad (22)$$

if and only if  $X \sim F$ . The tests based on  $L^2$  distance of estimators of V- or U-empirical functions of  $\omega_1(\cdot)$  and  $\omega_2(\cdot)$  often have the form of degenerate V- or U-statistics of order higher than 2.

One of the examples are test statistics based on V-empirical Laplace transforms from [5],[4]. There asymptotic distributions can be obtained using Theorem 2.1. However, the kernel in this case is differentiable, and it is possible to obtain the asymptotics directly using the mean value theorem. Here we present two examples when this is not possible and the application of our result is called for.

#### Example 1

Let  $X_1, \dots, X_n$  be a random sample from distribution  $F$ . Consider testing the composite null hypothesis  $F(x) = F_0(x, \lambda)$  where  $\lambda$  is scale parameter. A bunch of scale families of distributions can be characterized with equidistribution-type characterizations of the form (22) where  $\omega_1(\cdot)$  and  $\omega_2(\cdot)$  are two homogenous functions, i.e.

$$\omega_i(\lambda X_1, \dots, X_m) = \lambda \omega_i(X_1, \dots, X_m), \quad \lambda > 0.$$

Some examples of the characterization of this type are given in [8],[18],[15].

The most natural approach to construct test is to estimate distribution functions of  $\omega_1(\cdot)$  and  $\omega_2(\cdot)$  respectively and to based a test on their difference. A natural approach which yields a degenerate V-statistics is to integrate the squared difference over an empirical measure, i.e.

$$W_n = \int (G_n^{(1)}(t) - G_n^{(2)}(t))^2 dF_n(t),$$

where  $F_n(t)$  is empirical distribution function. Examples of such tests can be found in [6].

Another possibility, currently not explored yet, is to consider

$$\tilde{W}_n(\hat{\lambda}) = \int (\tilde{G}_n^{(1)}(t) - \tilde{G}_n^{(2)}(t))^2 dM(t),$$

where  $\tilde{G}^{(i)}$  is U-(V-) empirical distribution function of scaled sample and  $\{M(t)\}$  is finite measure. Without loss of generality we may assume that  $dM(t)$  is a density function of some random variable.

Applying the Theorem 2.1 we get that the distribution of  $n\tilde{W}_n(\hat{\lambda})$  coincides with  $n\tilde{W}_n(\lambda)$  which doesn't depend on  $\lambda$ .

This follows from the fact that

$$\begin{aligned} \tilde{W}_n(\hat{\lambda}) &= \frac{1}{n^{2m}} \sum_{i_1, \dots, i_{2m}} \int (I\{\hat{\lambda}\omega_1(X_{i_1}, \dots, X_{i_m}) < t\} - I\{\hat{\lambda}\omega_2(X_{i_1}, \dots, X_{i_m}) < t\})(I\{\hat{\lambda}\omega_1(X_{i_{m+1}}, \dots, X_{i_{2m}}) < t\} \\ &\quad - I\{\hat{\lambda}\omega_2(X_{i_{m+1}}, \dots, X_{i_{2m}}) < t\}) dM(t) \end{aligned}$$

and, under  $H_0$ ,

$$\begin{aligned} \mu(t; \gamma) &= E_\lambda(g(X_1, \dots, X_m, t; \gamma)) \\ &= E_\lambda(I\{\gamma\omega_1(X_1, \dots, X_m) < t\} - I\{\gamma\omega_2(X_1, \dots, X_m) < t\}) \\ &= P_\lambda\{\gamma\omega_1(X_1, \dots, X_m) < t\} - P_\lambda\{\gamma\omega_2(X_1, \dots, X_m) < t\} = 0 \end{aligned}$$

Last equality holds for each  $\gamma$  due to the characterization. Therefore the first derivative  $d_1\mu(t; \lambda)$  will be also equal to zero.

Consider now the particular case of the Puri-Rubin characterization [21], i.e. when  $\omega_1(X_1, X_2) = |X_1 - X_2|$  and  $\omega_2(X_1) = X_1$ , which implies that  $F$  is exponential distribution with some scale parameter  $\lambda$ , and let  $dM(t) = e^{-t} dt$ .

Then the corresponding test statistic is

$$\tilde{W}_n^{\text{PR}}(\hat{\lambda}) = \frac{1}{n^4} \sum_{i_1, \dots, i_4} \int (I\{\hat{\lambda}|X_{i_1} - X_{i_2}| < t\} - I\{\hat{\lambda}X_{i_1} < t\})(I\{\hat{\lambda}|X_{i_3} - X_{i_4}| < t\} - I\{\hat{\lambda}X_{i_3} < t\}) e^{-t} dt,$$

where  $\hat{\lambda} = \bar{X}^{-1}$ .

According to the argument above, it is enough to obtain the asymptotic distribution of  $n\tilde{W}_n^{\text{PR}}(\lambda)$ . The symmetrized kernel of  $\tilde{W}_n^{\text{PR}}(\lambda)$  is

$$\begin{aligned} \Phi_n^{\text{PR}}(x_1, x_2, x_3, x_4; \lambda) &= \frac{1}{4!} \sum_{\pi \in \Pi(4)} \left( e^{-\lambda \max(|x_{\pi(1)} - x_{\pi(2)}|, |x_{\pi(3)} - x_{\pi(4)}|)} - e^{-\lambda \max(x_{\pi(3)}, |x_{\pi(1)} - x_{\pi(2)}|)} \right. \\ &\quad \left. - e^{-\lambda \max(x_{\pi(1)}, |x_{\pi(3)} - x_{\pi(4)}|)} + e^{-\lambda \max(x_{\pi(1)}, x_{\pi(3)})} \right) \end{aligned}$$



and its second projection is equal to

$$\begin{aligned}\varphi_2^{\text{PR}}(s, t) &= \frac{1}{18} + \frac{1}{2}(e^{-2s-t} + e^{-s-2t}) - \frac{1}{4}(e^{-2t} + e^{-2s}) - \frac{16}{9}e^{-s-t} \\ &\quad + \frac{1}{9}e^{-\min(s,t)}(2 - 3\min(s,t)) + \frac{1}{18}e^{-\max(s,t)}(19 - 6\min(s,t)),\end{aligned}$$

Conditions 1 and 2 obviously hold and Condition 3 follows from the fact that the kernel is a linear combination of indicators.

Hence, the asymptotic distribution follows from (5) with the corresponding integral operator being

$$A^{\text{PR}}q(x) = \int_R \varphi_2^{\text{PR}}(x, y)q(y)e^{-y}dy.$$

The eigenvalues of  $A^{\text{PR}}$  cannot be obtained analytically, however they can be approximated numerically using the method from [2].

## Example 2

The second example comes from testing hypothesis within the location family. In goodness-of-fit tests based on equidistribution characterizations, it is often the case that estimating a location parameter, unlike the scale one, changes the asymptotic distribution (see e.g. [16]).

Let  $X_1, \dots, X_n$  be a random sample from distribution  $F$ . Consider testing the composite null normality hypothesis  $F(x) = \Phi(\frac{x-\theta}{\sigma})$ , where both  $\theta$  and  $\sigma$  are unknown, based on the famous Polya's characterization [20], arguably the first ever published equidistribution-type characterization. It states that if  $X_1$  and  $X_2$  are i.i.d. random variables with distribution function  $F$ , then the equidistribution

$$\frac{X_1 + X_2}{\sqrt{2}} \stackrel{d}{=} X_1 \tag{23}$$

implies that  $F$  is normal with zero mean and arbitrary variance. Some normality tests based on this characterization can be found in [17] and [14].

Let  $F_n(t)$  be the usual empirical distribution function and let

$$G_n(t) = \frac{1}{n^2} \sum_{i,j=1}^n \mathbf{I}\left\{\frac{X_i + X_j}{\sqrt{2}} \leq t\right\}$$

be the V-empirical distribution function associated with the random variable in (23).

Here we consider an  $\omega^2$ -type test statistic

$$T_n(\hat{\theta}) = \int_{-\infty}^{\infty} (\tilde{G}_n(t) - \tilde{F}_n(t))^2 d\tilde{F}_n(t), \tag{24}$$

where  $\tilde{F}_n(t)$  and  $\tilde{G}_n(t)$  are the aforementioned empirical d.f.'s applied to the shifted sample  $X_i^* = X_i - \hat{\theta}$  and  $\hat{\theta} = \bar{X}$ , the sample mean. It is easy to see that the statistic is location and scale free. After transformation we obtain

$$T_n(\hat{\theta}) = \int_{-\infty}^{\infty} (G_n(s - \hat{\theta} + \hat{\theta}\sqrt{2}) - F_n(s))^2 dF_n(s).$$

However, in order to apply Theorem 2.1 we need to get rid of the empirical measure. Define

$$\bar{T}_n(\hat{\theta}) = \int_{-\infty}^{\infty} (G_n(s - \hat{\theta} + \hat{\theta}\sqrt{2}) - F_n(s))^2 dF(s),$$

where  $F(s) = \Phi(\frac{s-\theta}{\sigma})$ , where  $\theta$  and  $\sigma$  are true parameter values. Since the test statistic is location-scale invariant, we assume  $\theta = 0$  and  $\sigma = 1$ .

Put  $W_n(s) = (G_n(s - \hat{\theta} + \hat{\theta}\sqrt{2}) - F_n(s))$ . Then, using the law of large numbers for V-statistics, the fact that  $\sqrt{n}W_n(s)$  converges to a centered Gaussian process, continuous mapping theorem, and Donsker theorem, we get

$$nT_n(\hat{\theta}) - n\bar{T}_n(\hat{\theta}) = \int \frac{1}{\sqrt{n}} nW_n^2(s) d(\sqrt{n}(F_n(s) - F(s))) \xrightarrow{d} \int 0 d\mathbb{G}_F = 0,$$

where  $\mathbb{G}_F = B \circ F$  and  $\{B(s)\}$  the standard Brownian bridge. Hence the statistics are asymptotically equivalent.

The statistic  $\bar{T}_n(\hat{\theta})$  is a V-statistic with kernel of the form (7)

$$\Psi(x_1, \dots, x_4; \hat{\lambda}_n) = \frac{1}{4!} \sum_{\pi \in \Pi(4)} \int_{-\infty}^{+\infty} g(x_{\pi(1)}, x_{\pi(2)}, t; \hat{\theta}_n) g(x_{\pi(3)}, x_{\pi(4)}, t; \hat{\theta}_n) dM(t),$$

where

$$g(x_1, x_2, s; \gamma) = I\left\{\frac{x_1 + x_2}{\sqrt{2}} \leq s + \gamma(\sqrt{2} - 1)\right\} - \frac{1}{2}I\{x_1 \leq s\} - \frac{1}{2}I\{x_2 \leq s\}.$$

We now show that the conditions for applying Theorem 2.1 are fulfilled. Taking into account that  $\theta = 0$ ,

$$\mu(s; \theta) = E_{\theta}(g(x_1, x_2, s; \gamma))|_{\gamma=\theta} = (\Phi(s + \gamma(\sqrt{2} - 1)) - \Phi(s))|_{\gamma=\theta} = 0.$$

In addition

$$d_1\mu(s; \gamma) = \phi(s + \gamma(\sqrt{2} - 1)) \cdot (\sqrt{2} - 1),$$

where  $\phi(x)$  is the standard normal density. Therefore,

$$\int_{-\infty}^{+\infty} (\phi(s) \cdot (\sqrt{2} - 1))^2 \phi(s) ds < \infty,$$

and

$$\begin{aligned}
 & \frac{1}{\gamma^2} \int_{-\infty}^{+\infty} (\Phi(s + \gamma(\sqrt{2} - 1)) - \Phi(s) - \phi(s) \cdot (\sqrt{2} - 1) \cdot \gamma)^2 \phi(s) ds \\
 &= \frac{1}{\gamma^2} \int_{-\infty}^{+\infty} \left( \frac{\gamma^2(\sqrt{2} - 1)^2}{2} \phi'(s + \xi(s)) \right)^2 \phi(s) ds \\
 &= \gamma^2 \frac{(\sqrt{2} - 1)^4}{4} \int_{-\infty}^{+\infty} (\phi'(s + \xi(s)))^2 \phi(s) ds < \varepsilon,
 \end{aligned}$$

where the last inequality follows from the boundness of the function  $\phi'(s)$ . Hence, Condition 1 is satisfied. Condition 2 holds obviously for the sample mean. Condition 3 is straightforward to verify using the properties of the normal density and the finiteness of the first and second moment of the kernel  $\Psi_*$ .

The second projection is equal to

$$\begin{aligned}
 \psi_2^*(x_1, x_3; \theta) &= \frac{16}{24} E(\Psi_*(x_1, X_2, x_3, X_4; \theta)) \\
 &= \frac{2}{3} \int_{-\infty}^{+\infty} \left( g_1(x_1, s; \theta) + d_1 \mu(s; \theta) \frac{x_1}{m} \right) \left( g_1(x_3, s; \theta) + d_1 \mu(s; \theta) \frac{x_3}{m} \right) \phi(s) ds \\
 &= \frac{2}{3} \int_{-\infty}^{+\infty} \left( \Phi(\sqrt{2}s - x_1) - \frac{1}{2} \Phi(s) - \frac{1}{2} I\{x_1 < s\} + \phi(s) \cdot (\sqrt{2} - 1) \frac{x_1}{m} \right) \\
 &\quad \times \left( \Phi(\sqrt{2}s - x_3) - \frac{1}{2} \Phi(s) - \frac{1}{2} I\{x_3 < s\} + \phi(s) \cdot (\sqrt{2} - 1) \frac{x_3}{m} \right) \phi(s) ds
 \end{aligned}$$

The expression above can be calculated and expressed as a function of bivariate normal distributions using the formulae from [19].

The asymptotic distribution of  $nT_n(\hat{\theta})$  now follows from Theorem 2.1 and the corresponding eigenvalues can be obtained numerically using the method from [2].

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