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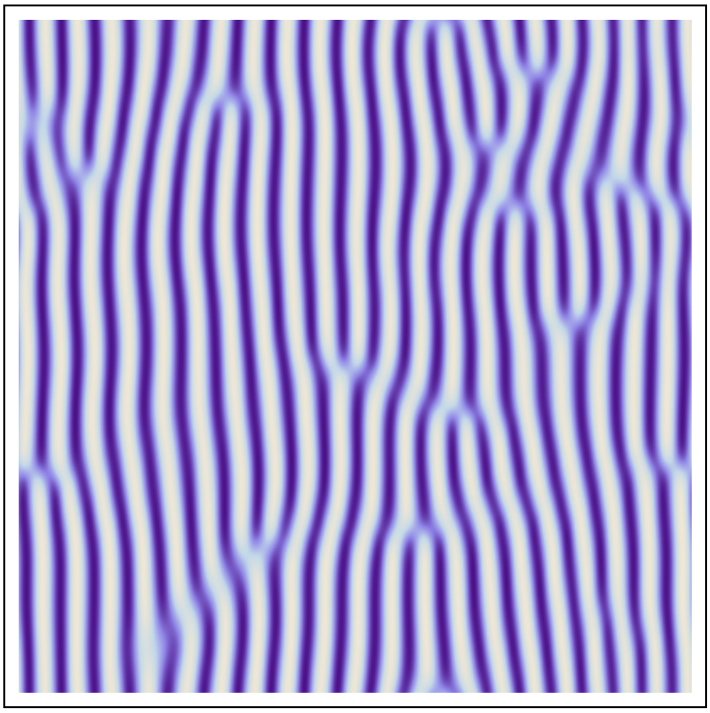
Nonpertubative renormalization group study of the Lifshitz critical point



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Abstract

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Résumé

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Introduction

The first chapter of the internship report is thus a very general introduction to granular materials and granular flows. At the end of this chapter, we give qualitative ideas about the segregation effect, ideas which are developed more rigorously in the appendix.

The second chapter will focus more specifically on the experiment we seek to model and understand, and on the approach we took to model it.

The third chapter give our results and discuss them.

Contents

Chapter 1

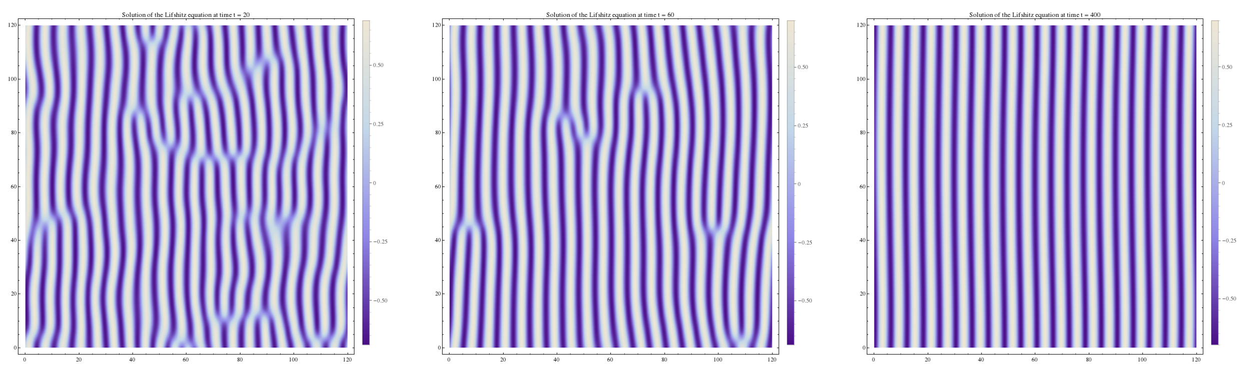
General presentation

The Lifshitz model

1.1 The Lifshitz model and its main features

1.1.1 The Lifshitz model

The modulated phase and the Lifshitz phase diagram.



(a) (b) (c)

Figure 1.1 – Time evolution of a field obeying the equation of movement derived from the (time dependant) Lifshitz action. We see that the field evolves toward a modulated steady state.

The Lifshitz model aims at describing a number of physical many-body systems. They share a common intriguing feature: having a so called modulated - or stripped - phase (fig. 1.1). In this phase, the order parameter is spatially periodic in one or several directions of space. The subspace spanned by these direction will from now be labelled $/\!/$. The hyperplan orthogonal to this modulation subspace will be labelled $\perp$.

Typically, the phase diagram of such a physical system will resemble the one presented in fig. 1.2.

A crucial feature of this phase diagram is the critical point (L in fig. 1.2), called the Lifshitz point. QUESTION : J'ai envie de dire que c'est l'un des rares exemples de point critique du second ordre qu'on trouve dans la nature, mais est-ce vrai ? The Lifshitz point is at the intersection of two *second order* phase transition lines. This is very seldomly encountered in nature. Therefore, the study of this critical point, and more precisely the determinantion of the critical exponents at this point is of particular interest.

Historically, the manganese phosphite (MnP) magnetic cristal was one of the first systems in which a Lifshitz point could be detected. Moreover, the entire phase diagram around the Lifshitz point of the MnP cristal could be inferred with high precision from experimental measurements [2]. In the case of the MnP magnetic cristal, the ρ_0 tunable parameter is an external magnetic field applied to the cristal, while the order parameter is the local magnetization of the atoms. In the modulated phase, it is the

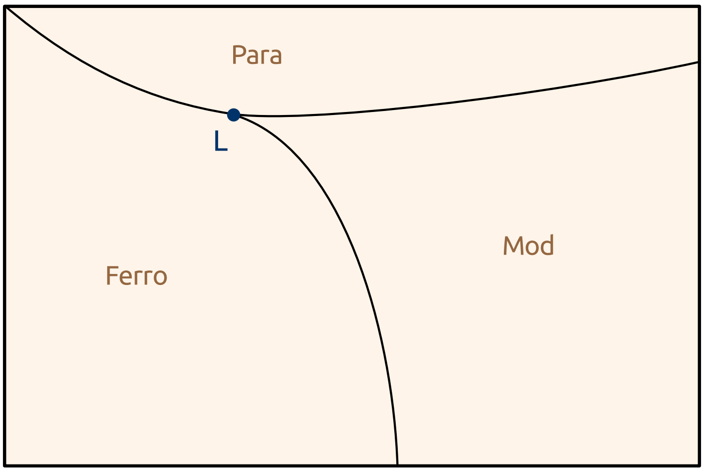


Figure 1.2 – Typical phase diagram of a system described by the Lifshitz model. The Lifshitz point is labelled L. "Para", "Ferro" and "Mod" are the abreviations for 'Paramagnetic", "Ferromagnetic" and "Modulated" respectively. Temperature varies along the vertical axis while the horizontal axis accounts for the variation of an extra parameter ρ_0 whose precise meaning depends on the physical nature of the studied system.

angle between the direction of the local magnetization and a direction of reference that is spatially modulated.

Experiments also provide evidence of Lifshitz point existence in ferroelectrics and liquid cristals.

The Lifshitz model

The Lifshitz model is a field theory, describing a vector field ϕ whose components will be denoted ϕ_i. If we like to think in terms of magnetic systems, like the MnP cristal, we can say that ϕ is the local magnetization. To write the action for the Lifshitz model, we chose a basis $(\mathbf{e_n})_{1\leq n\leq d}$. We decide that this basis is such that its first m vectors span the m dimensional $/\!\!/$ subspace, while of course the remaining $d-m$ base vector span the $\perp$ subspace. In this basis, the Lifshitz action is

$$S=\int_x \sum_{i=1}^{N}\left(\frac{1}{2}\left(\sum_{n_\perp=m+1}^{d}\frac{\partial\phi_i}{\partial x_{n_\perp}}\mathbf{e}_{\mathbf{n}_\perp}\right)^2+\frac{\rho_0}{2}\left(\sum_{n_{/\!/}=1}^{m}\frac{\partial\phi_i}{\partial x_{n_{/\!/}}}\mathbf{e}_{\mathbf{n}_{/\!/}}\right)^2+\frac{\sigma_0}{2}\left(\sum_{n_{/\!/}=1}^{m}\frac{\partial^2\phi_i}{\partial x_{n_{/\!/}}^2}\mathbf{e}_{\mathbf{n}_{/\!/}}\right)^2\right)+U(\phi) \tag{1.1}$$

As we want to model the broadest possible class of physical systems, we will say that U is an almost completely arbitrary potential. We only ask for it to have the $O(N)$ symmetry, ie to be a function of

$$\rho \stackrel{\text{def}}{=} \frac{\phi_i\phi_i}{2} \tag{1.2}$$

From now on we will use the self-explanatory shorthand notation

$$S=\int_x\left(\frac{1}{2}(\partial_\perp\phi)^2+\frac{\rho_0}{2}(\partial_{/\!/}\phi)^2+\frac{\sigma_0}{2}(\partial_{/\!/}^2\phi)^2+U(\rho)\right) \tag{1.3}$$

We see that this action closely ressemble the well known action of the $O(N)$ model

$$S_{O(N)}=\int_x\left(\frac{1}{2}(\partial\phi)^2+U(\rho)\right) \tag{1.4}$$

Namely, we recover it if we set $\rho_0=1$ and $\sigma_0=0$. We see that what differentiate the Lifshitz and $O(N)$ action is on one hand the presence a non trivial (*ie* different from 1) ρ_0, breaking the $O(N)$ invariance, and on the other hand the presence of an extra term involving a laplacian squared. Clearly, these two modifications must be responsible for the appearance of spatially modulated structures, but why exactly? We can gain a useful intuition of why a spatially modulated structure is closely linked to the existence of a laplacian squared term in the action by looking at a microscopic version of our model.

1.1.2 A discrete counterpart: the anisotropic Ising model

Stricto sensu the discrete counterpart of the Lifshitz model would be an anisotropic Heisenberg model, but to simplify things - without changing the essence of the argumentation - we consider an anisotropic Ising model instead.

First, let us consider a chain of Ising spins with the Hamiltonian

$$H_{\text{chain}} \stackrel{\text{def}}{=} -J \sum_i S_i S_{i+1} \tag{1.5}$$

We know that if J is positive, the interaction is ferromagnetic, whereas is J is negative, the interaction is antiferromegnetic. The antiferromagnetic order already shows some kind of spatial modulation, but it only exists at zero temperature! The idea to make a spatially modulated order survive at non zero temperatures is to consider a second neighbours *antiferromagnetic* interaction, together with a first neighbours *ferromagnetic* interaction:

$$H_{\text{chain}} = -J_1 \sum_i S_i S_{i+1} - J_2 \sum_i S_i S_{i+2} \tag{1.6}$$

The competition between ferromagnetic and antiferromagnetic interactions will produce a spatial modulation of the spins at non zero temperatures, at least for some values of the interaction strenghts ratio J_2/J_1. However, for a long range order to exist at finite temperature, we need to work in two dimensions or more, *ie* to trade our spin chain for a spin lattice:

$$H_{\text{lattice}} \stackrel{\text{def}}{=} -\sum_i \left(J_0 \sum_{\delta_\perp} S_i S_{i+\delta_\perp} + J_1 \sum_{\delta_/\!/} S_i S_{i+\delta_/\!/} + J_2 \sum_{\delta_/\!/} S_i S_{i+2\delta_/\!/} \right) \tag{1.7}$$

The existence of a stripped phase is a well known feature of this model [1], called the ANNNI (axial next-nearest neighbour Ising) model.

Now, what is the link between this discrete spin lattice hamiltonian, and our continuous action? First, note that a sum on nearest neighbours can be rewriten in terms of a discrete laplacian on the lattice, while a sum on next-nearest neighbours involves a discrete laplacian squared:

$$H_{\text{lattice}} = -\sum_i \left(\kappa S_i^2 + J_0 S_i \Delta_\perp S_i + (J_1 + 4J_2) S_i \Delta_/\!/ S_i - J_2 S_i \Delta_/\!/^2 S_i \right) \tag{1.8}$$

where we introduced the differencial operators on the lattice:

$$\Delta_/\!/ S_i = \sum_{\delta_/\!/} S_{i-\delta_/\!/} - 2S_i + S_{i+\delta_/\!/} \tag{1.9}$$

$$\Delta_/\!/^2 S_i = \sum_{\delta_/\!/} -S_{i-2\delta_/\!/} + 4S_{i-\delta_/\!/} - 4S_i + 4S_{i+\delta_/\!/} - S_{i+2\delta_/\!/} \tag{1.10}$$

This rewriting in terms of discrete differential operators makes it clear that this Hamiltonian is the discrete -microscopic- counterpart of the Lifshitz action. We now understand -at least intuitively- the origin of the spatially periodic structures (shown in fig. 1.1) the Lifshitz field exhibit. They exist because of the competition bewtween *nearest neightbours ferromagnetic interactions* (giving rise to the $\Delta_/\!/$ term in the Lifshitz action), and *next-nearest neightbours antiferromagnetic interactions* (giving rise to the $\Delta_/\!/^2$ term in the Lifshitz action).

At this point a question arises: why work with a Lifshitz coarse-grained field theory, since we have a better physical understanding of an underlying microscopic model? What is more, in passing to a continuous theory, we lose informations about the microscopic underlying lattice. This is actually not a problem since the statistical quantities we are interested in computing -namely the critical exponents of the phase transition- are universal; they do not depend on the specific microscopic model. Actually, passing to a field theory is even advantageous as it frees us of the irrelevant microscopic details. Even more crucial is the fact that field theories are the objects of choice for application of the powerful methods of the renormalization group, which we will now desribe.

Chapter 2

Introduction to the nonperturbative renormalization techniques

At the so-called Lifshitz critical point, three phases intersect. This is rather unusual, so we expect the physics of the vicinity of this point to be of special interest. To investigate it, we would like to compute the critical exponents associated to this transition point. To this end, we used the powerful machinery of the renormalization group, and more precisely of one particular implementation of the renormalization ideas: the nonpertubative renormalization group.

In this chapter we propose first a very general introduction to the ideas and concepts of renormalization. Then we focus on the nonperturbative renormalization group techniques.

2.1 Introduction to the renormalization group

2.1.1 The renormalization procedure

The idea of renormlization is to consider a given many-body system at different length scales. At a given length scale S the system is described by an effective Hamiltonian H_S. For a many-body system, two length scales play a special role:

— The microscopic length scale, a, which is for example for a cristal the typical distance between two neighbouring atoms. It is convenient to define the scale in the units of the microscopic lenghtscale a, and this is what we are going to do. So, for example, the system at $S = 3$ will mean the system at lengthscale $3a$.

— The macroscopic length scale L, which is the size of the system.

The Hamiltonian at the microscopic length scale is simply the microscopic Hamiltonian, *ie*

$$H_1 = H \tag{2.1}$$

while the Hamiltonian at the macroscopic length scale is called to effective action (and sometimes the Gibbs free energy) and is denoted Γ:

$$H_{L/a} = \Gamma \tag{2.2}$$

For the moment what we mean by "the Hamiltonian at a length scale S" is rather vague. To be more precise, let us imagine that we want to describe a magnetic cristal. The microscopic Hamiltonian will be in general a discrete sum of local observables O_α, depending on the value of the magnetization at site i, $\phi(i)$:

$$H[\phi] = \sum_i \sum_\alpha \kappa_\alpha O_\alpha[\phi(i), \nabla\phi(i), ...] \tag{2.3}$$

where κ_α is the coupling constant associated to the observable O_α. The partition function will be simply the sum over all possible configurations of the $\phi(i)$ of the Boltzmann weight associated to a given configuration:

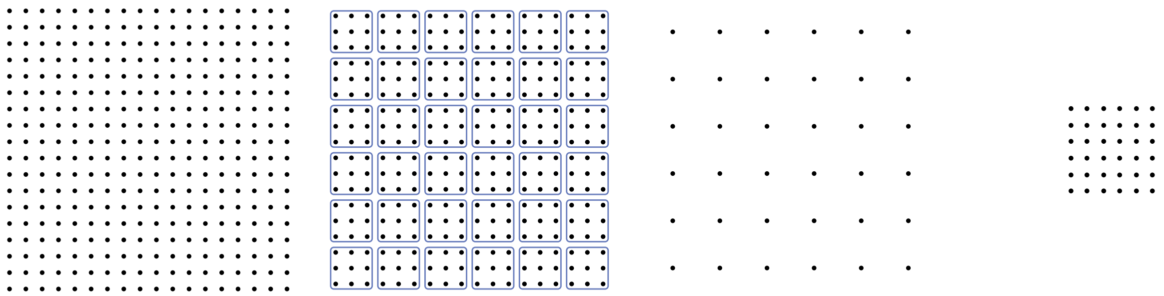
$$Z = \sum_{\text{conf } \phi} e^{-H[\phi]} \tag{2.4}$$

These equations describe the system at the microscopic scale $S = 1$. Now, if we want to describe it at a scale $S \geq 1$, surely we are no longer interested in knowing the fluctuations of the field over regions of size smaller than aS[1]. All we need is thus the average over regions of size aS of the field:

$$\tilde{\phi}(b) = \frac{1}{(aS)^d} \sum_{i \in B(b)} \phi(i) \tag{2.5}$$

where d is the dimension of space, and $B(b)$ is the set of sites i belonging to the block b.

Schematically what we do is group spins by blocks of size aS (fig. 2.1b), and average over these blocks.



(a) The initial lattice. (b) Block spin averaging. (c) The new lattice. (d) Rescaling.

Figure 2.1 – The renormalization procedure illustrated. Here we have chosen $S = 3$.

Now we replace the microscopic Hamiltonian by an effective Hamiltonian for the block spin field $\tilde{\phi}$:

$$H[\phi] \to \tilde{H}[\tilde{\phi}] \text{ such that } e^{-\tilde{H}[\tilde{\phi}]} = \sum_{\text{conf } \phi} \prod_b \delta\left(\tilde{\phi}(b) - \frac{1}{S^d} \sum_{i \in B(b)} \phi(i)\right) e^{-H[\phi]} \tag{2.6}$$

this Hamiltonian is designed such that

$$\sum_{\text{conf } \tilde{\phi}} e^{-\tilde{H}[\tilde{\phi}]} = Z \tag{2.7}$$

This Hamiltonian describe the new system depicted in fig. 2.1c. We are not done yet! To make the new Hamiltonian ressemble as much as possible the one we started from, we rescale all lengths (fig. 2.1d). We also rescale the field. Formally it means that we perform the change of variables

$$x' = x/S \tag{2.8}$$
$$\phi' = S^{\Delta} \tilde{\phi} \tag{2.9}$$

where x could be any length appearing in the Hamiltonian.

The Hamiltonian in the new variables $H'[\phi'] = \tilde{H}[\tilde{\phi}]$ is the effective Hamiltonian after the renormalization operation. It could seem strange that we rescaled the field as well as the lenghts. We do that in order for the new Hamiltonian to ressemble the old one as closely as possible. We are going to see on the example of the Lifshitz mean field theory how we can chose Δ for that purpose.

To conclude, the key ideas of the renormalization procedure are the averaging over block spins, and the rescaling of lengths and fields. We have described here the case of a discrete Hamiltonian, because it seemed more intuitive. But of course the ideas of renormalization are general and can as easily be applied to a continuous Hamiltonians.

1. Experimentally, we can imagine that looking at the system at scale S means probing it with devices having a spatial resolution of aS. Any measurement operation with such devices can be described mathematically by the convolution of an observable by an error function having a spatial support of diameter aS. This operation is roughly equivalement to averaging the observables (and therefore the field) on blocks of size aS.

2.1.2 The renormalization group

If the structure of the Hamiltonian is kept unchanged by the renormalization group procedure, *ie* if

$$H'[\phi'] = \sum_{i'} \sum_{\alpha} \kappa'_\alpha O_\alpha[\phi'(i'), \nabla'\phi'(i'), ...] \tag{2.10}$$

then the renormalization group action is a group action [2]. The group renormalization group is completely described by its action on the coupling contants:

$$\kappa_\alpha \to \kappa'_\alpha \stackrel{\text{def}}{=} g(\kappa_\alpha, S) \tag{2.11}$$

The renormalization group is a multiplicative, one parameter group:

$$g(g(\kappa_\alpha, S_1), S_2) = g(\kappa_\alpha, S_1 S_2) \tag{2.12}$$

These transformations are assumed to be continuous in the coupling constant. It is also very often possible to consider the scale S as a continuous parameter. Then the successive application of infintesimally close renormalization group transformations generates a continuous tranjectory in the space of coupling constants. This trajectory can be parametrized by $t \stackrel{\text{def}}{=} \log(S)$, an additive parameter playing the role of a time. It is often referred to as “the renormalization group time”.

Near a phase transition or critical point, fluctuations occur at all length scales, and thus one should expect the Hamiltonian to be scale invariant. In terms of the renormalization group action, scale invariance simply means that

$$g(\kappa_\alpha, S) = \kappa_\alpha \tag{2.13}$$

The fact that scale invariance has such a simple meaning in the renormalization group framework is an extremely good sign. It is a hint that renormalization group is a powerful tool to look for critical points.

To illustrate that, in appendix ??? (yet to be written!), we derive from very simple renormalization group arguments some useful formulas relating critical exponents.

2.1.3 Renormalization procedure applied to the mean field Lifshitz theory

As we have just seen, an operation from the renormalization group transforms our microscopic Hamiltonian H into an effective Hamiltonian at scale S, $H_g(S)$. We hope that this operation will not change the structure of our Hamiltonian, so that we can use the tools of the renormalization group. Since our Hamiltonian is not isotropic (it distinguishes between the direction of the modulation $/\!\!/$, and the orthogonal direction $\perp$), we expect an operation of the renormalization group to change lengthscales by two different amounts in the two unequivalent directions. Scales in the parallel direction will be changed by a factor $S_{/\!\!/} : x'_{/\!\!/} = (S_{/\!\!/})^{-1} x_{/\!\!/}$, while scales on the orthogonal direction will be changed by a factor $S_\perp : x'_\perp = (S_\perp)^{-1} x_\perp$.

To simplify things we can keep a single scale $S = S_\perp$, and define θ such that $S_{/\!\!/} = S^\theta$.Note that this is equivalent to changing the *units* in the parallel direction: if we say that lengths in the orthogonal direction are measured in meters, then lengths in the parallel direction are measured in (meters)$^\theta$. A volume, which is normally measured in (meters)^d will in our new units system be measured in (meters)$^{(d-m)+\theta m}$. It is as if d had been replaced by

$$d_m = d + m(\theta - 1) \tag{2.14}$$

We define two anomalous dimensions by

$$\langle \phi(p_{/\!\!/}) \phi(0) \rangle_{g^*} \propto |p_{/\!\!/}|^{\eta_{/\!\!/} - 4} \tag{2.15}$$

$$\langle \phi(p_\perp) \phi(0) \rangle_{g^*} \propto |p_\perp|^{\eta_\perp - 2} \tag{2.16}$$

2. Actually, invertibility cannot be guaranteed so it rather is a semigroup action, but the distinction is of no importance for us.

where g^* is a fixed point in the space of coupling constants, and the proportionality constant is independent of the scale.

But we also know that

$$\langle \phi(p_{/\!/})\phi(0)\rangle_{g^*} \propto |p_{/\!/}|^{\frac{2\Delta}{\theta}} \tag{2.17}$$

$$\langle \phi(p_\perp)\phi(0)\rangle_{g^*} \propto |p_\perp|^{2\Delta} \tag{2.18}$$

where Δ is the renormalization of the field : $\phi_{g(1)}(x) = S^{-\Delta}\phi_{g(S)}(Sx)$. Identifying these expressions for the renormalization of the correlation function with the previous ones, we can express Δ in two unequivalent ways, thus yielding a relation between $\eta_{/\!/}$ and $\eta_\perp$:

$$\theta = \frac{2-\eta_\perp}{4-\eta_{/\!/}} \tag{2.19}$$

Mean field analysis

We recall that the Lifshitz Hamiltonian is

$$H = \int_x \left(\frac{1}{2}(\partial_\perp\phi)^2 + \frac{\rho_0}{2}(\partial_{/\!/}\phi)^2 + \frac{\sigma_0}{2}(\partial_{/\!/}^2\phi)^2 + U(\rho)\right) \tag{2.20}$$

The mean field approximation consist in neglecting the fluctuations of the field. Therefore, the integration over the fluctuations performed during a renormalization group operation is trivial, and we can directly write the transformed Hamiltonian after a renormalization group operation $g(S)$:

$$H'[\phi'] = \int_{x'} S^{d_m}\left(S^{-2\Delta-2}(\partial'_\perp\phi')^2 + \sigma_0 S^{-2\Delta-4\theta}(\partial'^2_{/\!/}\phi)^2 + \rho_0 S^{-2\Delta-2\theta}(\partial'_{/\!/}\phi')^2 U'(\phi')\right) \tag{2.21}$$

This Hamiltonian must indentify with the previous one, so

$$\Delta = \frac{d_m-2}{2} \tag{2.22}$$

The Lifschitz point involves a non-trivial $\sigma_0(\partial_{/\!/}^2\phi)^2$, therefore at this critical point, σ_0 must not renormalize away, which is only possible if

$$\theta = \frac{1}{2} \tag{2.23}$$

In the mean field approximation, it is as if the physical dimension were

$$d_{\text{mean field}} = d - \frac{m}{2} \tag{2.24}$$

We immediately deduce that the upper critical dimension [3], which is "normally" 4, becomes

$$d_c^> = 4 + \frac{m}{2} \tag{2.25}$$

around the Lifschitz point.

In nature, the most common case is $m = 1$ (when in the modulated phase, the field is periodic in one direction of space). In that case the upper critical dimension is $d_c^> = 4.5$. When doing perturbation theory, one usually expand around the upper critical dimension, writing

$$d = d_c^> - \epsilon, \tag{2.26}$$

where ϵ is a supposedly small parameter. When $d = 3$, $\epsilon = 1.5$, which is not so small. So we expect the results from perturbation theory to be rather imprecise. To better tackle this problem, it would be great to have a method that is not perturbative in the dimension. This is what the nonperturbative renormalization group method provides, as we will now see.

3. The upper critical dimension is the dimension above which mean field theory is exact, in the sense that it gives the correct critical exponents.

2.2 The nonperturbative renormalization group

Starting from the microscopic Hamiltonian H, and applying successive transformations of the renormalization group, we construct a hierarchy of effective Hamiltonian describind the system at greater and greater distances. This process come to an end when we describe the system at the macroscopic distance. We then obtain the effective action Γ.

$$H \to H' \to H'' \to ... \to \Gamma \tag{2.27}$$

Now writing primes is cumbersome, so we devise a better notation system. We call H_k the effective Hamiltonian describing the system at lengthscale $1/k$. Then our previous hierarchy reads

$$H_\Lambda \to H_{\Lambda-\delta k} \to H_{\Lambda-2\delta k} \to ... \to H_0 \tag{2.28}$$

The idea of the nonperturbative renormalization group is to consider H_k *as a function of* k. We already know the boundary values H_Λ and H_0. If we could find a differential equation on H_k, it would be entirely determined!

Many exact differential equations can be derived from this starting point. They lead to different version of the nonperturbative renormalization group. We shall discuss here the so-called Wetterich, or Wegner-Wetterich equation.

2.2.1 The scale-dependant effective action

Let us try to build H_k explicitely. This Hamiltonian has to describe the system at (momentum) scale k, so we can build it from the microscopic Hamiltonian H, integrated over fluctuations of momentum $k \leq p \leq \Lambda$. To this end let us define

$$\Delta H_k[\varphi] = \frac{1}{2}\varphi \cdot R_k \cdot \varphi = \frac{1}{2}\int_{x,y} \varphi(x) R_k(x,y)\varphi(y) \tag{2.29}$$

or, in momentum space

$$\Delta H_k[\varphi] = \frac{1}{2}\int_{q,q'} \varphi(q) R_k(q,-q')\varphi(q') \tag{2.30}$$

and

$$H_k[\varphi] = H[\varphi] + \Delta H_k[\varphi] \tag{2.31}$$

We see that the ΔH_k term is a quadratic form on the field. Thus $R_k(q,q')$ acts as a scale-dependant and momentum-dependant squared mass term. Most of the time it is taken translation invariant

$$R_k(x,y) = R_k(x-y) \tag{2.32}$$

, and therefore diagonal in momentum space

$$R_k(q,-q') = R_k(q)\delta(q+q') \tag{2.33}$$

Fig. 2.2 give the typical shape of this regulator. This regulator is required to vanish at $k \to 0$ at to diverge for $k \to \infty$ (or $k \to \Lambda$), at fixed q^2. This can for example be achived by

$$R_k(q) = \frac{q^2}{e^{q^2/k^2} - 1} \tag{2.34}$$

Its role is to give a squared mass of order k^2 to the "slow" modes of momentum smaller than the scale k, while leaving "rapid" modes of momentum larger than k untouched. Qualitatively, in

$$\int \mathscr{D}\varphi e^{-H[\varphi]-\Delta H_k[\varphi]} \tag{2.35}$$

slow modes (ie spatial variations of the microscopic field on large scales) are protected from integration by the extra mass they gain from the regulator, while rapid modes (ie spatial variations of the field

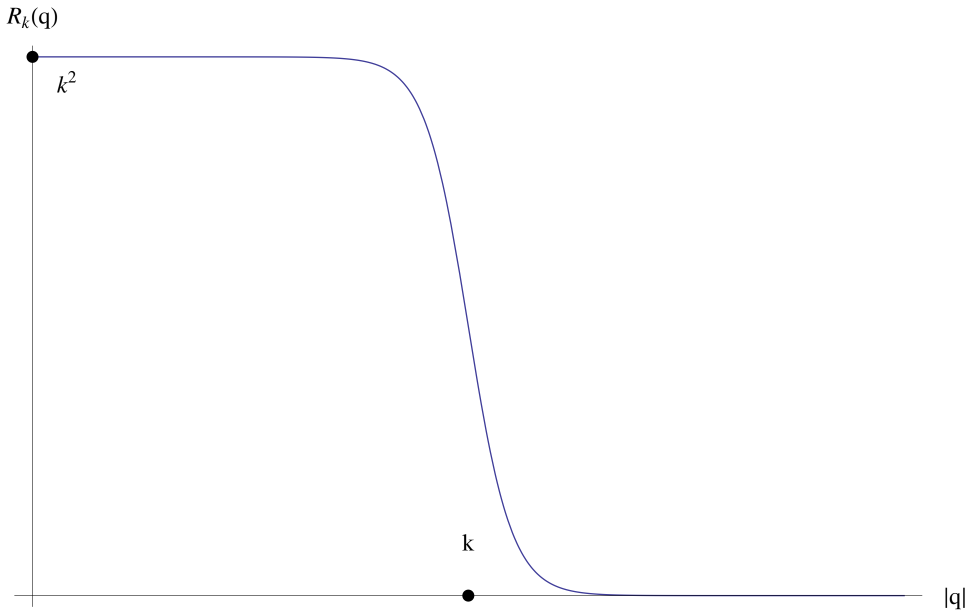


Figure 2.2 – Typical shape of the regulator function in momentum space $R_k(q)$.

on small scales) are integrated. We see that this regulator effectively implements the ideas of the renormalization group of integration. Following the idea of equation 2.6 we define teh scale dependant partition function

$$Z_k[h] = \int \mathscr{D}\varphi e^{-H[\varphi]-\Delta H_k[\varphi]+h\cdot\varphi} \tag{2.36}$$

This completely explicitely defines the Hamiltonian at scale k, and we could try to derive a differential equation for this object. However, as we shall see it is better to derive an equation for the scale dependant effective action instead.

We start by defining the scale dependant Legendre transform of the free energy,

$$\Gamma_k^{\text{Leg}}[\phi] \overset{\text{def}}{=} -W_k[h] + h\cdot\phi \tag{2.37}$$

where the free energy is defined as $W_k[h] = \log(Z_k[h])$. We note in passing that eq. 2.37 implies

$$\phi(x) = \frac{\delta W_k[h]}{\delta h(x)} = \frac{1}{Z_k[\varphi]}\int \mathscr{D}\varphi e^{-H_k[\varphi]+h\cdot\varphi}\varphi \overset{\text{def}}{=} \langle\varphi(x)\rangle_k \tag{2.38}$$

This tells us that ϕ is the background field (at scale k), *ie* the mean value of the field φ (at scale k).

We define the scale-dependant effective action as

$$\Gamma_k[\phi] \overset{\text{def}}{=} \Gamma_k^{\text{Leg}}[\phi] - \Delta H_k[\phi] \tag{2.39}$$

This object has the right properties to be the effective action at scale k. To begin with, it verifies

$$\Gamma_0 = \Gamma \tag{2.40}$$

as it should. This property follows trivially from the definition. Less trivially, we also have

$$\Gamma_\Lambda = H \tag{2.41}$$

again, as we should. This property is show is appendix ??? (to be written!).

Finally, the scale-dependant effective action verifies the differential equation

$$\partial_k\Gamma_k[\phi] = \frac{1}{2}\int_{x,y} \partial_k\left(R_k(x,y)\right)\left(\Gamma_k^{(2)}(x,y) + R_k(x,y)\right)^{-1} \tag{2.42}$$

The demonstration can be found in appendix A. We can use the fact that R_k is invariant by translation, to write the right hand side of this integro-differential equation as a single integral over momentum q:

$$\partial_k \Gamma_k[\phi] = \frac{1}{2}\int_q \partial_k \left(R_k(q)\right)\left(\Gamma_k^{(2)}(q,-q) + R_k(q)\right)^{-1} \tag{2.43}$$

Moreover, we shall make the change of variable $k \to t = \log(k/\Lambda)$, to take advantage of the additive properties of t [4]. In order to simplify further the Wetterich equation, we define an operator $\hat{\partial}_t$ as the differenciation with respect to t acting only on R_t. That is,

$$\hat{\partial}_t \stackrel{\text{def}}{=} \frac{\partial R_t(\,\cdot\,)}{\partial t}\frac{\partial}{\partial R_t(\,\cdot\,)} \tag{2.44}$$

where $R_t(\,\cdot\,)$ is a shorthand notation for the function $q \mapsto R_t(q)$. We have then

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\hat{\partial}_t \mathrm{tr}\left(\log\left(\Gamma_t^{(2)}(q,-q) + R_t(q)\right)\right)\ {}^{5} \tag{2.45}$$

This is the form of the Wetterich equation we will use from now on. Even if the field is vectorial, eq. 2.45 is still valid, provided that the trace acts on the vector space of the field as well [6].

Note that no approximations were made in the course of deriving the Wetterich equation. It is thus an exact equation describing the (renormlization group) time dependance of the scale-dependant effective action Γ_t, interpolating between $\Gamma_\Lambda = H$ and $\Gamma_0 = \Gamma$.

We we can use the Wetterich equation as a starting point for finding the critical properties of the Lifshitz model.

4. As we have seen before, t can be regarded as a time parametrizing the renormalization group trajectory in the coucpling constants space.

5. Here we have used the formula for the trace on continuous indices in Fourier space: $\mathrm{tr}\,(A) = \int_x A(x,x) = \int_q A(q,-q)$.

6. we refer to appendix A for a sketch of proof

Chapter 3

The Lifshitz model

As we have seen, the Wetterich equation governs the flow of the scale-dependant effective action Γ_t. Solving directly this differential equation to find Γ_t, though in theory possible, is in practice very difficult. Inded the Γ_t depends on the (background) field ϕ, a function of the momentum. Determining completely Γ_t requieres knowing its value for all functions ϕ.

So, instead of solving directly the Wetterich equation, we start from a reasonable functional form for Γ_t with unknown coefficients, we plug it in the Wetterich equation to get flow equations for this coefficients, and we solve these equations.

If we assume that it is analytic in the potential, the effective action can be written as the infinite sum

$$\Gamma_t[\phi] = \int_x \left(m_0(\partial\phi(x))^2 + m_1(\partial\phi(x)^2)^2 + ... + n_0(\partial^2\phi(x))^2 + ... + l_0\phi(x)^2 + l_1\phi(x)^3 + ...\right) \tag{3.1}$$

This expression can be further simplified if the system under study has some invariance properties. The simplest model having such an invariance is the Ising one, whose microscopic Hamiltonian is invariant under the action of the $\mathbb{Z}_2$ symmetry group. If we chose a regulator that preserves this symmetry, it is clear that the effective action Γ_t must also be invariant under $\mathbb{Z}_2$. The most general expression for Γ_t that preserves this symmetry is

$$\Gamma_t^{\text{Ising}}[\phi] = \int_x \sum_{i,j\in\mathbb{N}} Z_{t,ij}(\partial\phi(x))^{2i}\phi(x)^{2j} \tag{3.2}$$

The strategy is now simple:
- Compute $\Gamma_t^{(2)}(x,y)$ from 3.2
- Plug the result in the right hand side of the Wetterich equation
- Deduce the flow equation for $Z_{t,ij}$: $\partial_t Z_{t,ij} = F_{i,j}\left(Z_{t,kl}\right)$

Following this procedure we went from a functional differential equation to a set of simple differential equations. Solving this set of flow equations give us a complete knowledge of the model. In particular solving the fixed point equations $0 = F_{i,j}\left(Z^*_{t,kl}\right)$ will give us access to the critical properties of the Ising model: critical exponents, critical potential,... Note that for the moment no approximations have been made!

In practice however we know that the lowest degree terms in $\partial\phi$ and ϕ will have the most significant contribution to the flow. Therefore we can make the approximation of cutting the developement at some order. This has the great practical advantage of reducing the number of flow equations to solve to a finite number. Note that other approximation schemes are possible (development in powers of the field, BMW scheme...). We will not talk about these.

Now, let us apply this approximation scheme to the Lifshitz model.

3.1 An Anzatz for the Lifshitz scale-dependant effective action

We recall that the Lifshitz microscopic Hamiltonian is

$$H[\phi] = \int_x \left(\frac{1}{2}(\partial_\perp \phi)^2 + \frac{\rho_0}{2}(\partial_{/\!/} \phi)^2 + \frac{\sigma_0}{2}(\partial_{/\!/}^2 \phi)^2 + V(\rho) \right) \tag{3.3}$$

where $\rho = \phi_i \phi_i / 2$. The Lifshitz Hamiltonian has the $O(n)$ symmetry [1], therefore the scale-deendant effective action should also have this symmetry. Moreover the kinetik term in the Lifshitz Hamiltonian decomposes into two parts:

— $\int_x \frac{\rho_0}{2}(\partial_{/\!/}\phi)^2 + \frac{\sigma_0}{2}(\partial_{/\!/}^2\phi)^2$, invariant under $\left(x_{/\!/}, x_\perp\right) \rightarrow \left(\mathcal{R}(m)x_{/\!/}, x_\perp\right)$

— $\int_x \frac{1}{2}(\partial_\perp\phi)^2$, invariant under $\left(x_{/\!/}, x_\perp\right) \rightarrow \left(x_{/\!/}, \mathcal{R}(d-m)x_\perp\right)$

Since the scale-dependant effective action must satisfy these symmetry properties, its most general form is

$$\Gamma_k[\phi] = \int_x U(\rho) + \left(\frac{1}{2}Z_\perp(\rho)(\partial_\perp\phi)^2 + \frac{1}{4}Y_\perp(\rho)(\partial_\perp\rho)^2 + ... \right) + \left(\frac{1}{2}\rho_0(\rho)(\partial_{/\!/}\phi)^2 + ... \right) + \left(\frac{1}{2}Z_{/\!/}(\rho)(\partial_{/\!/}^2\phi)^2 + ... \right) \tag{3.4}$$

It turns out to be a very good approximation to cut the derivative expansion to first order [2]. That is to say, we make the approximation

$$\frac{1}{2}Z_\perp(\rho)(\partial_\perp\phi)^2 + \frac{1}{4}Y_\perp(\rho)(\partial_\perp\rho)^2 + ... \simeq \frac{1}{2}Z_\perp(\rho)(\partial_\perp\phi)^2 \tag{3.5}$$

$$\frac{1}{2}\rho_0(\rho)(\partial_{/\!/}\phi)^2 + ... \simeq \frac{1}{2}\rho_0(\rho)(\partial_{/\!/}\phi)^2 \tag{3.6}$$

$$\frac{1}{2}Z_{/\!/}(\rho)\partial_{/\!/}^2\phi)^2 + ... \simeq \frac{1}{2}Z_{/\!/}(\rho)\partial_{/\!/}^2\phi)^2 \tag{3.7}$$

Moreover, we make the approximation that the field renormalizations $Z_\perp(\rho)$, $\rho_0(\rho)$ and $Z_{/\!/}(\rho)$ do not depend on the field. So, the definitive form of the Anzatz we chose for Γ_t is

$$\Gamma_t[\phi_i] = \int_x \left(\frac{Z_\perp}{2}(\partial_\perp\phi)^2 + \frac{\rho_0}{2}(\partial_{/\!/}\phi)^2 + \frac{Z_{/\!/}}{2}(\partial_{/\!/}^2\phi)^2 + U(\rho) \right) \tag{3.8}$$

This is the form of the Lifshitz scale-dependent effective action we have worked on during this internship. Though this does not appear explicitly, the field renormalizations and the effective potential U depend on the renormalization time t. We will now make use of the Wetterich equation to know how this quantities change through the renormalization process. Then we will use this knowledge to derive the critical exponents.

3.2 The Lifshitz renormalization flows

3.2.1 Dimension-driven versus fluctuations-driven flows

We recall that looking at the mean field version of a theory consists in neglecting all fluctuations of the field. Therefore, in the case of a mean field theory, the renormalization procedure - average on fluctuations up to a certain scale, definition of an effective Hamiltonian and rescaling - only requires rescaling. The conclusion is that the renormalization flows of mean field theories' coupling constants are only due to their dimension.

If we wish to go beyond the mean field approximation, we have to take into account the fluctuations of the field. They are often small compared to the mean value of the field, meaning that their contribution to the flows will be small compared to the contribution coming from the mean field theory. In other words, the fluctuation-driven part of a renormalization flow is generally small compared to its dimension-driven part. Wishing to concentrate on the small non-trivial part of the flow: the

1. Meaning that a rotation of the n-dimensional field ϕ leaves the Hamiltonian invariant: $H[\mathcal{R}(n)\phi] = H[\phi]$ if $\mathcal{R}$ is an orthogonal $n \times n$ matrix.

2. To compute critical exponents, one does not need to take into account the full momentum dependence of the theory. However for computing correlation function the momentum dependence is essential. Were we to compute these quantities, we will probably have to go further in the derivative expansion

fluctuation-driven part, we define *dimensionless coupling constants*, that are by construction subject to a fluctuation-driven flow only. [3]

With these ideas in mind, we define the following dimensionless quantities

$$y_{/\!/} = \frac{q_{/\!/}^2}{k^{2\theta}} \tag{3.9}$$

$$q_\perp^2 = \frac{Z_{/\!/}}{Z_\perp} k^{4\theta} y_\perp \tag{3.10}$$

$$R(q_\perp^2, q_{/\!/}^2) = Z_{/\!/} k^{4\theta} y_{/\!/}^2 r(y_\perp, y_{/\!/}) \tag{3.11}$$

$$\rho_0 = Z_{/\!/} k^{2\theta} \bar{\rho}_0 \tag{3.12}$$

$$m^2 = Z_{/\!/} k^{4\theta} \bar{m}^2 \tag{3.13}$$

$$U(\rho) = Z_{/\!/} k^{d_m} u(\bar{\rho}) \tag{3.14}$$

$$\rho = Z_{/\!/}^{-1} k^{-4\theta + d_m} \bar{\rho} \tag{3.15}$$

Note that given the relation

$$\theta = \frac{2 - \eta_\perp}{4 - \eta_{/\!/}} \tag{3.16}$$

we have a certain liberty relative to the adimensionning of the physical quantities. For example we can define $\rho_0 = Z_{/\!/} k^{2\theta} \bar{\rho}_0$ or $\rho_0 = Z_\perp^{1/2} Z_{/\!/}^{1/2} k \bar{\rho}_0$. These two definitions lead to two different $\bar{\rho}_0$ functions, but the two functions have *the same dimensional flow.*. Here we chose the adimensionning such that $Z_{/\!/}$ simplifies everywhere in the propagators.

3.2.2 Flow of the potential

From the shape of the potential, we can tell in which phase we are. Therefore knowing how the potential changes with t is of paramount importance. We shall thus start with the derivation of the flow equation for the potential.

We see that

$$U(\rho_0) = \delta(0)^{-1} \Gamma_t[\phi]|_{\phi(x)=\phi_0} \tag{3.17}$$

where ϕ_0 is some uniform (in direct space) configuration of the field, and where $\rho_0 = \phi_{0i}\phi_{0i}/2$. Note that if we were to work on a finite-size system, the $\delta(0)^{-1}$ term would be replaced by the volume of the system. Therefore it plays the role of the system volume for the infinite size system we consider here.

We take a derivative with respect to t to get an expression for the flow of the potential, and we plug in the Wetterich equation on the right hand side:

$$\partial_t U(\rho_0) = \delta(0)^{-1} \partial_t \Gamma = \frac{1}{2\delta(0)} \hat{\partial}_t \mathrm{tr} \left(\log \left(\Gamma_t^{(2)} + R_t \right) \right) \tag{3.18}$$

This is the flow equation for the potential! [4]

From eq. 3.8 we compute the first functional derivative of the Lifshitz effective action:

$$\frac{\delta \Gamma_t}{\delta \phi_i(x)}[\phi] = -Z_\perp \Delta_\perp \phi_i(x) - \rho_0 \Delta_\perp \phi_i(x) + Z_{/\!/} \Delta_{/\!/}^2 \phi_i(x) + U'(\rho)\phi_i(x) \tag{3.19}$$

taking a second functional derivative with respect to the field:

$$\frac{\delta^2 \Gamma_t}{\delta \phi_i(x) \delta \phi_j(y)}[\phi] = \left(\delta_{ij} \left(-Z_\perp \Delta_\perp - \rho_0 \Delta_\perp + Z_{/\!/} \Delta_{/\!/}^2 + U'(\rho) \right) + U''(\rho)\phi_i(x)\phi_j(y) \right) \delta(x-y) \tag{3.20}$$

3. Numerically, if we want to track the contribution to the flow of fluctuations 10^{12} weaker than the mean field contribution, we must achieve at least 12 digits precision, which is often rendered impossible by rounding errors. In this context using dimensionless coupling constant indeed seems an excellent idea!

4. It should be noted that we only derived the flow equation of the potential for a constant field configuration. This is of no importance as we do not need the *functional* dependence of the potential, $(x \mapsto \rho(x)) \mapsto U(\rho(x))$ but only its "digital" dependence $\rho_0 \mapsto U(\rho_0)$ to characterize it completely.

and passing to Fourier space:

$$\frac{\delta^2\Gamma_t}{\delta\phi_i(p)\delta\phi_j(q)}[\phi] \stackrel{\text{def}}{=} \Gamma^{(2)}_{ij}(p,q) = \Big(\delta_{ij}\left(Z_\perp p_\perp^2 + \rho_0 p_{/\!/}^2 + Z_{/\!/}(p_{/\!/}^2)^2 + U'(\rho(x))\right) + \phi_i(p)\phi_j(q)U''(\rho(x))\Big)\,\delta(p+q) \tag{3.21}$$

We can decompose the two-points 1-particle irreducible function $\Gamma^{(2)}_{ij}$ on a orthogonal projectors along the direction of the field and orthogonal to it:

$$\Pi^a_{i,j} = \delta_{ij} - \frac{\phi_i\phi_j}{2\rho} \tag{3.22}$$

$$\Pi^r_{i,j} = \frac{\phi_i\phi_j}{2\rho} \tag{3.23}$$

thus allowing us to easily write the regularized propagator appearing in the Wetterich equation:

$$\left(\Gamma^{(2)}_{ij}(p,q) + R_t(p,q)\right)^{-1} = \left(G_a(q)\Pi^a_{ij} + G_r(q)\Pi^r_{ij}\right)\delta(p+q) \tag{3.24}$$

where we used the radial and angular propagators:

$$G_r(q)_{i,j} \stackrel{\text{def}}{=} \frac{1}{Z_{/\!/}q_{/\!/}^4 + Z_\perp q_\perp^2 + \rho_0 q_{/\!/}^2 + R_t(q) + U'(x) + 2\rho U^{(2)}(\rho)} \tag{3.25}$$

$$G_a(q)_{i,j} \stackrel{\text{def}}{=} \frac{1}{Z_{/\!/}q_{/\!/}^4 + Z_\perp q_\perp^2 + \rho_0 q_{/\!/}^2 + R_t(q) + U'(\rho)} \tag{3.26}$$

and chosen the regulator to be diagonal: $R_{t,ij}(q) = R_t(q)\delta_{ij}$. We can now rewrite in a more explicit way the Wetterich equation:

$$\partial_t\Gamma_t = \frac{1}{2}\int_q \left(G_a(q)\Pi^a_{ij} + G_r(p)\Pi^r_{ij}\right)\partial_t R_{t,ij}(q) = \frac{1}{2}\int_q \left(G_a(q) + (n-1)G_r(p)\right)\partial_t R_t(q) \tag{3.27}$$

Note that the radial (massive) propagator appears once whereas the angular (massless) propagator appears $n-1$ times in the flow equation. This is of course because as soon as we choose a direction for the field ϕ, the $O(n)$ invariance is no longer explicit (though the equation is of course still $O(n)$ invariant). Therefore, a massive Goldstone mode and $n-1$ massless modes appear, in accordance with Goldstone's theorem.

Imposing now that the field is constant, we have for the flow of the potential

$$\partial_t U(\phi_0) = \frac{1}{2}\int_q \left(G_r(q)\Big|_{\text{unif}} + (n-1)G_a(q)\Big|_{\text{unif}}\right)\partial_t R_t(q) \tag{3.28}$$

This is as far as we can get without giving explicitly a form to the regulator. At this point, it is a customary procedure to introduce standard functions called *threshold functions* in order to lighten the notations.

The flow of the potential can be expressed in terms of the l threshold function: [5]

$$\partial_t U = 8v_m v_{d-m} k^{d_m}\left(l_0^{dm}\left(u'(\bar\rho) + 2\bar\rho u''(\bar\rho)\right) + (N-1)l_0^{dm}\left(u'(\bar\rho)\right)\right) \tag{3.29}$$

Recall that $u(\bar\rho) = k^{-d_m}U(\rho)$ and $\bar\rho = Z_{/\!/}k^{4\theta-d_m}\rho$, so that

$$d_t u(\bar\rho) = -d_m u(\bar\rho) + (\theta\eta_{/\!/} + d_m - 4\theta)\bar\rho u'(\bar\rho) + \partial_t u(\bar\rho) \tag{3.30}$$

Dropping the bars everywhere, we have for the dimensionless potential,

$$d_t u(\rho) = -d_m u(\rho) + (\theta\eta_{/\!/} + d_m - 4\theta)\rho u'(\rho) + 8v_m v_{d-m}\left(l_0^{dm}\left(u'(\rho) + 2\rho u''(\rho)\right) + (n-1)l_0^{dm}\left(u'(\rho)\right)\right) \tag{3.31}$$

5. Definitions of the threshold functions used here can be found in appendix A.2.3.

The fixed point potential therefore verifies

$$0 = u(\rho) - a(d_m, \theta, \eta_{/\!/})\rho u'(\rho) - b(d,m)\left(l_0^{dm}\left(u'(\rho) + 2\rho u''(\rho)\right) + (n-1)l_0^{dm}\left(u'(\rho)\right)\right) \tag{3.32}$$

where

$$a = \frac{\theta\eta_{/\!/} + d_m - 4\theta}{d_m} \tag{3.33}$$

$$b = \frac{8v_m v_{d-m}}{d_m} \tag{3.34}$$

Numerically, the the fixed point potential equation is not nice because it is an implicit differential equation $0 = F(\rho; u, u', u'')$. However we can put it the form $u(\rho) = f(\rho; u', u'')$. This indicates that, by differentiation with respect to ρ, we can produce an *explicit* equation on $v(\rho) \stackrel{\text{def}}{=} u'(\rho)$:

$$0 = (1-a)v - a\rho v' + b\left(l_1\left(v + 2\rho v'\right)\left(3v' + 2\rho v''\right) + (N-1)l_1\left(v\right)v'\right) \tag{3.35}$$

a second order, explicit differential equation on the derivative of the potential.

Appendix A

Derivation of the Wetterich equation

We want to arrive at the formula,

$$\partial_k \Gamma_k = \frac{1}{2} \int_{x,y} R_k(x,y) \left(\Gamma_k^{(2)}(x,y) + R_k(x,y) \right)^{-1} \tag{A.1}$$

A.1 Some useful relations

We recall that if W_k is a function of h (the external field), then we can define its Legendre transform Γ_k^{Leg}, a function of ϕ (the background field), by

$$\Gamma_k^{\mathrm{Leg}}[\phi] = h \cdot \phi - W_k[h] \tag{A.2}$$

or equivalently,

$$\phi = \frac{\delta W_k[h]}{\delta h} \tag{A.3}$$

$$h = \frac{\delta \Gamma_k^{\mathrm{Leg}}[\phi]}{\delta \phi} \tag{A.4}$$

We see that we can *either* consider ϕ as being a function of h, *or* the reverse. That is,

$$\frac{\delta \phi}{\delta h} = \left(\frac{\delta h}{\delta \phi} \right)^{-1} \tag{A.5}$$

Using (A.3) and (A.4), we deduce that

$$\Gamma_k^{\mathrm{Leg}\ (2)}(x,y) = W_k^{(2)}(x,y)^{-1} \tag{A.6}$$

But we also know that the free energy W_k is the generating function of the connected n points correlation functions. In particular, we have thus

$$\Gamma_k^{\mathrm{Leg}\ (2)}(x,y) = G_c^2(x,y)^{-1} \tag{A.7}$$

This result will be useful later.

We also recall that we defined the effective action at scale k as

$$\Gamma_k[\phi] = \Gamma_k^{\mathrm{Leg}}[\phi] - \Delta H_k[\phi] \tag{A.8}$$

with

$$\Delta H_k[\phi] = \frac{1}{2} \phi \cdot R_k \cdot \phi \tag{A.9}$$

A.2 The derivation

On one hand we have

$$\partial_k e^{W_k[h]} = e^{W_k[h]}\partial_k W_k[h] = Z_k[\varphi]W_k[h] \tag{A.10}$$

while, on the other hand we alse have

$$\partial_k e^{W_k[h]} = \partial_k Z_k[\varphi] = -\frac{1}{2}\int_{x,y}\partial_k R_k(x,y)\underbrace{\int \mathscr{D}\varphi\varphi(x)\varphi(y)e^{-H_k[\varphi]-\Delta H_k[\varphi]+h\cdot\varphi}}_{=Z_k[\varphi]\langle\varphi(x)\varphi(y)\rangle_k} \tag{A.11}$$

putting these two expressions together we deduce that

$$\partial_k W_k[h] = -\frac{1}{2}\int_{x,y}\partial_k R_k(x,y)\langle\varphi(x)\varphi(y)\rangle_k \tag{A.12}$$

We already almost have the Wetterich equation! Actually all that remains to be done is reexpressing the left and right hand sides of the previous expression in terms of the scale-dependant effective action Γ_k.

A.2.1 The left hand side

We start from $W_k = h\cdot\phi - \Gamma_k^{\text{Leg}}$, and we deduce that

$$\partial_k W_k = h\cdot\partial_k\phi - \partial_k\Gamma_k^{\text{Leg}} \tag{A.13}$$

Moreover

$$\partial_k\Gamma_k^{\text{Leg}}[\phi] = \partial_k\left(\Gamma_k[\phi] + \frac{1}{2}\phi\cdot R_k\cdot\phi\right) \tag{A.14}$$

Now, we must be careful not to forget that $\Gamma_k[\phi]$ depends explicitely on k, but also implicitely through ϕ! We have

$$\partial_k = \partial_{k,\ \phi\text{ fixed}} + \partial_k\phi\cdot\frac{\delta}{\delta\phi} \tag{A.15}$$

so that

$$\partial_k\Gamma_k[\phi] = \partial_{k,\ \phi\text{ fixed}}\Gamma_k[\phi] + \partial_k\phi\cdot\frac{\delta\Gamma_k[\phi]}{\delta\phi} \tag{A.16}$$

Dropping the "ϕ fixed" labels, we conclude that

$$\partial_k\Gamma_k^{\text{Leg}}[\phi] = \partial_k\Gamma_k[\phi] + \partial_k\phi\cdot h + \frac{1}{2}\phi\cdot\partial_k R_k\cdot\phi \tag{A.17}$$

and therefore that

$$\partial_k W_k[\phi] = -\partial_k\Gamma_k[\phi] - \frac{1}{2}\phi\cdot\partial_k R_k\cdot\phi \tag{A.18}$$

A.2.2 The right hand side

To transform the right hand side, we simply note that the connected two points correlation function is

$$G^2_{c,k}(x,y) = \langle\varphi(x)\varphi(y)\rangle_k - \langle\varphi(x)\rangle_k\langle\varphi(y)\rangle_k \tag{A.19}$$

so that

$$\partial_k W_k[\phi] = -\frac{1}{2}\int_{x,y}\partial_k R_k(x,y)\langle\varphi(x)\varphi(y)\rangle_k = -\frac{1}{2}\int_{x,y}\partial_k R_k(x,y)G^2_{c,k}(x,y) - \frac{1}{2}R_k\cdot\phi\cdot\phi \tag{A.20}$$

A.2.3 Conclusion

Putting equations A.18 and A.20 together, we finally have

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_k R_k(x,y) G^2_{c,k}(x,y) \tag{A.21}$$

Remembering eq. A.7, we conclude that

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_k R_k(x,y) \left(\Gamma^{(2)}_k(x,y) + R_k(x,y) \right)^{-1} \tag{A.22}$$

This is the Wetterich equation for a scalar field ϕ.

Passing to the Wetterich equation for a vector field ϕ_i is very simple, as soon as we realize that all we have to do is to perform the following replacements:

$$\phi \rightarrow \phi_i \tag{A.23}$$

$$\Gamma^{(2)}_k(x,y) \rightarrow \Gamma^{(2)}_k(x,y)_{i,j} \tag{A.24}$$

$$R_k(x,y) \rightarrow R_k(x,y)_{i,j} \tag{A.25}$$

The Wetterich equation is thus transformed into

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_k R_k(x,y)_{i,j} \left(\Gamma^{(2)}_k(x,y) + R_k(x,y) \right)^{-1}_{i,j} \tag{A.26}$$

Appendix B

Threshold functions

B.0.4 The l function

We define

$$l_n^{(d,m)}(\omega,\bar{\rho}_0) = -\frac{n+\delta_{n,0}}{4}\int_0^\infty dq_{/\!/}\int_0^\infty dq_\perp y_{/\!/}^{m/2+1} y_\perp^{(d-m)/2-1}\frac{\theta\eta_{/\!/} r + 2\theta y_{/\!/} r^{(1,0)} + 2y_\perp r^{(0,1)}}{\left[y_{/\!/}^2(1+r)+y_\perp+\bar{\rho}_0 y_{/\!/}+\omega\right]^{n+1}} \tag{B.1}$$

For $n>0$, we see that we have

$$-\hat{\partial}_t\int\frac{dq_{/\!/}^m dq_\perp^{d-m}}{(2\pi)^d}\frac{1}{\left[Z_{/\!/}q_{/\!/}^4+Z_\perp q_\perp^2+\rho_0 q_{/\!/}^2+R+m^2\right]^n} = 16 v_m v_{d-m} Z_{/\!/}^{-n} k^{d_m-4\theta n} l_n^d(\bar{m}^2) \tag{B.2}$$

and for $n=0$

$$\hat{\partial}_t\int\frac{dq_{/\!/}^m dq_\perp^{d-m}}{(2\pi)^d}\log\left(Z_{/\!/}q_{/\!/}^4+Z_\perp q_\perp^2+\rho_0 q_{/\!/}^2+R+m^2\right) = 16 v_m v_{d-m} k^{d_m} l_0^d(\bar{m}^2) \tag{B.3}$$

B.0.5 The n function

Dimensionful:

$$N_{1,ab\alpha\beta} = -\frac{1}{16 v_m v_{d-m}}\hat{\partial}_t\int_q q_\perp^\alpha q_{/\!/}^\beta \frac{d}{dq_{/\!/}^2} P G_a(q)^a G_r(q)^b \tag{B.4}$$

adimensionning and spherical integration:

$$N_{1,ab00} = -\frac{1}{4} Z_{/\!/}^{1-a-b} k^{d_m-2\theta+4\theta(1-a-b)}\hat{\partial}_t\int dy_{/\!/} dy_\perp y_{/\!/}^{m/2-1} y_\perp^{(d-m)/2-1}\frac{d}{dy_{/\!/}}\bar{P}\bar{G}_a(y)^a\bar{G}_r(y)^b \tag{B.5}$$

B.0.6 The m function

Dimensionful:

$$M_{1,ab\alpha\beta} = -\frac{1}{16 v_m v_{d-m}}\hat{\partial}_t\int_q q_\perp^\alpha q_{/\!/}^\beta\left(\frac{d}{dq_{/\!/}^2}P\right)^2 G_a(q)^a G_r(q)^b \tag{B.6}$$

adimensionning and spherical integration:

$$M_{1,ab00} = -\frac{1}{4} Z_{/\!/}^{1-a-b} k^{d_m-4\theta+4\theta(1-a-b)}\hat{\partial}_t\int dy_{/\!/} dy_\perp y_{/\!/}^{m/2-1} y_\perp^{(d-m)/2-1}\left(\frac{d}{dy_{/\!/}}\bar{P}\right)^2 \bar{G}_a(y)^a\bar{G}_r(y)^b \tag{B.7}$$

B.0.7 The k function

Dimensionful:

$$K_{1,ab\alpha\beta} = -\frac{1}{16 v_m v_{d-m}} \hat{\partial}_t \int_q q_\perp^\alpha q_\sslash^\beta \left(\frac{d^2}{d(q_\sslash^2)^2} P \right) G_a(q)^a G_r(q)^b \tag{B.8}$$

adimensionning and spherical integration:

$$K_{1,ab00} = -\frac{1}{4} Z_\sslash^{1-a-b} k^{d_m - 4\theta + 4\theta(1-a-b)} \hat{\partial}_t \int dy_\sslash dy_\perp y_\sslash^{m/2-1} y_\perp^{(d-m)/2-1} \left(\frac{d^2}{dy_\sslash^2} \bar{P} \right) \bar{G}_a(y)^a \bar{G}_r(y)^b \tag{B.9}$$

Bibliography

[1] Walter Selke. The {ANNNI} model — theoretical analysis and experimental application. *Physics Reports*, 170(4):213 – 264, 1988.

[2] Y. Shapira, C. C. Becerra, N. F. Oliveira, and T. S. Chang. Phase diagram, susceptibility, and magnetostriction of mnp: Evidence for a lifshitz point. *Phys. Rev. B*, 24:2780–2806, Sep 1981.