M2 ICFP, parcours physique théorique

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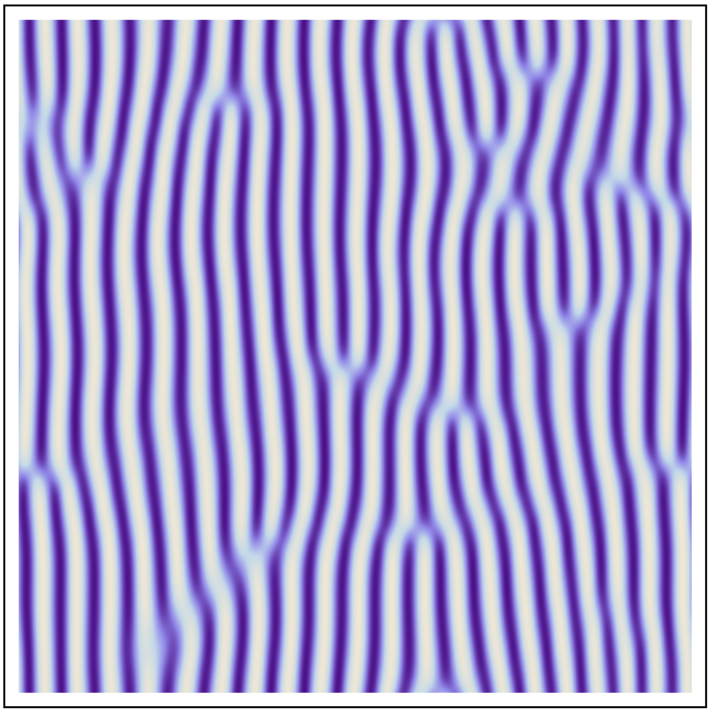
Nonpertubative renormalization group study of the Lifshitz critical point



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Abstract

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Résumé

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Introduction

The first chapter of the internship report is thus a very general introduction to granular materials and granular flows. At the end of this chapter, we give qualitative ideas about the segregation effect, ideas which are developed more rigorously in the appendix.

The second chapter will focus more specifically on the experiment we seek to model and understand, and on the approach we took to model it.

The third chapter give our results and discuss them.

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Chapter 1

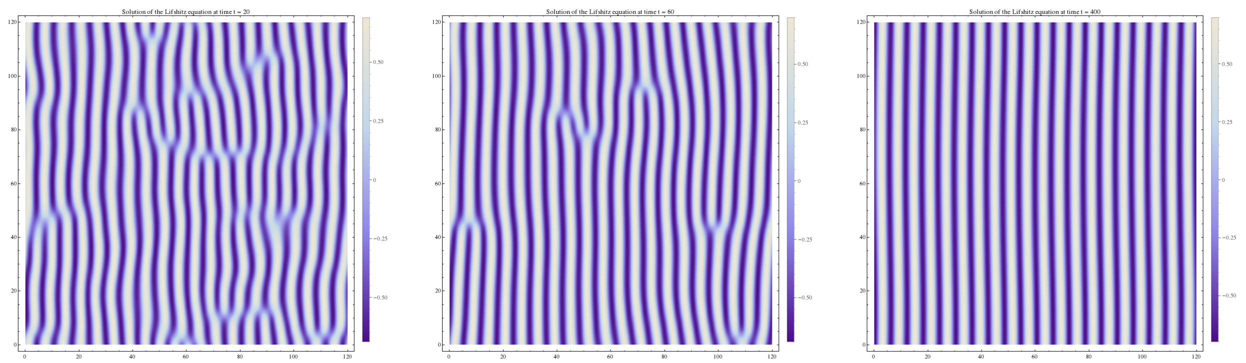
General presentation

The Lifshitz model

1.1 The Lifshitz model and its main features

1.1.1 The Lifshitz model

The modulated phase and the Lifshitz phase diagram.



(a) (b) (c)

Figure 1.1 – Time evolution of a field obeying the equation of movement derived from the (time dependant) Lifshitz action. We see that the field evolves toward a modulated steady state.

The Lifshitz model aims at describing a number of physical many-body systems. They share a common intriguing feature: having a so called modulated - or stripped - phase (fig. 1.1). In this phase, the order parameter is spatially periodic in one or several directions of space. The subspace spanned by these direction will from now be labelled $/\!/$. The hyperplan orthogonal to this modulation subspace will be labelled $\perp$.

Typically, the phase diagram of such a physical system will resemble the one presented in fig. 1.2.

A crucial feature of this phase diagram is the critical point (L in fig. 1.2), called the Lifshitz point. QUESTION : J'ai envie de dire que c'est l'un des rares exemples de point critique du second ordre qu'on trouve dans la nature, mais est-ce vrai ? The Lifshitz point is at the intersection of two *second order* phase transition lines. This is very seldomly encountered in nature. Therefore, the study of this critical point, and more precisely the determinantion of the critical exponents at this point is of particular interest.

Historically, the manganese phosphite (MnP) magnetic cristal was one of the first systems in which a Lifshitz point could be detected. Moreover, the entire phase diagram around the Lifshitz point of the MnP cristal could be inferred with high precision from experimental measurements [2]. In the case of the MnP magnetic cristal, the ρ_0 tunable parameter is an external magnetic field applied to the cristal, while the order parameter is the local magnetization of the atoms. In the modulated phase, it is the

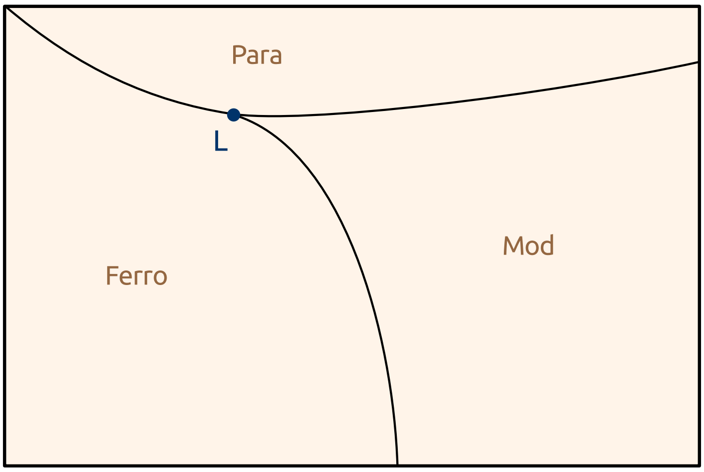


Figure 1.2 – Typical phase diagram of a system described by the Lifshitz model. The Lifshitz point is labelled L. "Para", "Ferro" and "Mod" are the abreviations for 'Paramagnetic", "Ferromagnetic" and "Modulated" respectively. Temperature varies along the vertical axis while the horizontal axis accounts for the variation of an extra parameter ρ_0 whose precise meaning depends on the physical nature of the studied system.

angle between the direction of the local magnetization and a direction of reference that is spatially modulated.

Experiments also provide evidence of Lifshitz point existence in ferroelectrics and liquid cristals.

The Lifshitz model

The Lifshitz model is a field theory, describing a vector field ϕ whose components will be denoted ϕ_i. If we like to think in terms of magnetic systems, like the MnP cristal, we can say that ϕ is the local magnetization. To write the action for the Lifshitz model, we chose a basis $(\mathbf{e_n})_{1\leq n\leq d}$. We decide that this basis is such that its first m vectors span the m dimensional $/\!\!/$ subspace, while of course the remaining $d-m$ base vector span the $\perp$ subspace. In this basis, the Lifshitz action is

$$S=\int_x\sum_{i=1}^{N}\left(\frac{1}{2}\left(\sum_{n_\perp=m+1}^{d}\frac{\partial\phi_i}{\partial x_{n_\perp}}\mathbf{e}_{\mathbf{n}_\perp}\right)^2+\frac{\rho_0}{2}\left(\sum_{n_{/\!\!/}=1}^{m}\frac{\partial\phi_i}{\partial x_{n_{/\!\!/}}}\mathbf{e}_{\mathbf{n}_{/\!\!/}}\right)^2+\frac{\sigma_0}{2}\left(\sum_{n_{/\!\!/}=1}^{m}\frac{\partial^2\phi_i}{\partial x_{n_{/\!\!/}}^2}\mathbf{e}_{\mathbf{n}_{/\!\!/}}\right)^2\right)+U(\phi) \tag{1.1}$$

As we want to model the broadest possible class of physical systems, we will say that U is an almost completely arbitrary potential. We only ask for it to have the $O(N)$ symmetry, ie to be a function of

$$\rho\stackrel{\text{def}}{=}\frac{\phi_i\phi_i}{2} \tag{1.2}$$

From now on we will use the self-explanatory shorthand notation

$$S=\int_x\left(\frac{1}{2}(\partial_\perp\phi)^2+\frac{\rho_0}{2}(\partial_{/\!\!/}\phi)^2+\frac{\sigma_0}{2}(\partial_{/\!\!/}^2\phi)^2+U(\rho)\right) \tag{1.3}$$

We see that this action closely ressemble the well known action of the $O(N)$ model

$$S_{O(N)}=\int_x\left(\frac{1}{2}(\partial\phi)^2+U(\rho)\right) \tag{1.4}$$

Namely, we recover it if we set $\rho_0=1$ and $\sigma_0=0$. We see that what differentiate the Lifshitz and $O(N)$ action is on one hand the presence a non trivial (*ie* different from 1) ρ_0, breaking the $O(N)$ invariance, and on the other hand the presence of an extra term involving a laplacian squared. Clearly, these two modifications must be responsible for the appearance of spatially modulated structures, but why exactly? We can gain a useful intuition of why a spatially modulated structure is closely linked to the existence of a laplacian squared term in the action by looking at a microscopic version of our model.

1.1.2 A discrete counterpart: the anisotropic Ising model

Stricto sensu the discrete counterpart of the Lifshitz model would be an anisotropic Heisenberg model, but to simplify things - without changing the essence of the argumentation - we consider an anisotropic Ising model instead.

First, let us consider a chain of Ising spins with the Hamiltonian

$$H_{\text{chain}} \stackrel{\text{def}}{=} -J \sum_i S_i S_{i+1} \tag{1.5}$$

We know that if J is positive, the interaction is ferromagnetic, whereas is J is negative, the interaction is antiferromegnetic. The antiferromagnetic order already shows some kind of spatial modulation, but it only exists at zero temperature! The idea to make a spatially modulated order survive at non zero temperatures is to consider a second neighbours *antiferromagnetic* interaction, together with a first neighbours *ferromagnetic* interaction:

$$H_{\text{chain}} = -J_1 \sum_i S_i S_{i+1} - J_2 \sum_i S_i S_{i+2} \tag{1.6}$$

The competition between ferromagnetic and antiferromagnetic interactions will produce a spatial modulation of the spins at non zero temperatures, at least for some values of the interaction strenghts ratio J_2/J_1. However, for a long range order to exist at finite temperature, we need to work in two dimensions or more, *ie* to trade our spin chain for a spin lattice:

$$H_{\text{lattice}} \stackrel{\text{def}}{=} -\sum_i \left(J_0 \sum_{\delta_\perp} S_i S_{i+\delta_\perp} + J_1 \sum_{\delta_{/\!/}} S_i S_{i+\delta_{/\!/}} + J_2 \sum_{\delta_{/\!/}} S_i S_{i+2\delta_{/\!/}} \right) \tag{1.7}$$

The existence of a stripped phase is a well known feature of this model [1], called the ANNNI (axial next-nearest neighbour Ising) model.

Now, what is the link between this discrete spin lattice hamiltonian, and our continuous action? First, note that a sum on nearest neighbours can be rewriten in terms of a discrete laplacian on the lattice, while a sum on next-nearest neighbours involves a discrete laplacian squared:

$$H_{\text{lattice}} = -\sum_i \left(\kappa S_i^2 + J_0 S_i \Delta_\perp S_i + (J_1 + 4J_2) S_i \Delta_{/\!/} S_i - J_2 S_i \Delta_{/\!/}^2 S_i \right) \tag{1.8}$$

where we introduced the differencial operators on the lattice:

$$\Delta_{/\!/} S_i = \sum_{\delta_{/\!/}} S_{i-\delta_{/\!/}} - 2S_i + S_{i+\delta_{/\!/}} \tag{1.9}$$

$$\Delta_{/\!/}^2 S_i = \sum_{\delta_{/\!/}} -S_{i-2\delta_{/\!/}} + 4S_{i-\delta_{/\!/}} - 4S_i + 4S_{i+\delta_{/\!/}} - S_{i+2\delta_{/\!/}} \tag{1.10}$$

This rewriting in terms of discrete differential operators makes it clear that this Hamiltonian is the discrete -microscopic- counterpart of the Lifshitz action. We now understand -at least intuitively- the origin of the spatially periodic structures (shown in fig. 1.1) the Lifshitz field exhibit. They exist because of the competition bewtween *nearest neightbours ferromagnetic interactions* (giving rise to the $\Delta_{/\!/}$ term in the Lifshitz action), and *next-nearest neightbours antiferromagnetic interactions* (giving rise to the $\Delta_{/\!/}^2$ term in the Lifshitz action).

At this point a question arises: why work with a Lifshitz coarse-grained field theory, since we have a better physical understanding of an underlying microscopic model? What is more, in passing to a continuous theory, we lose informations about the microscopic underlying lattice. This is actually not a problem since the statistical quantities we are interested in computing -namely the critical exponents of the phase transition- are universal; they do not depend on the specific microscopic model. Actually, passing to a field theory is even advantageous as it frees us of the irrelevant microscopic details. Even more crucial is the fact that field theories are the objects of choice for application of the powerful methods of the renormalization group, which we will now desribe.

Chapter 2

The renormalization procedure

Introduction to the nonperturbative renormalization techniques

At the so-called Lifshitz critical point, three phases intersect. This is rather unsual, so we expect the physics of the vicinity of this point to be of special interest. To investigate it, we would like to compute the critical exponents associated to this transition point. To this end, we used the powerful machinery of the renormalization group, and more precisely of one particular implementation of the renormalization ideas: the nonpertubative renormalization group.

In this chapter we propose first a very general introduction to the ideas and concepts of the renormalization. Then we focus on the nonperturbative renormalization group techniques.

2.1 Introduction to the renormalization group

2.2 The nonperturbative renormalization group

Appendix A

Derivation of the Wetterich equation

We want to arrive at the formula,

$$\partial_k \Gamma_k = \frac{1}{2} \int_{x,y} R_k(x,y) \left(\Gamma_k^{(2)}(x,y) + R_k(x,y) \right)^{-1} \tag{A.1}$$

A.1 Some useful relations

We recall that if W_k is a function of h (the external field), then we can define its Legendre transform Γ_k^{Leg}, a function of ϕ (the background field), by

$$\Gamma_k^{\mathrm{Leg}}[\phi] = h \cdot \phi - W_k[h] \tag{A.2}$$

or equivalently,

$$\phi = \frac{\delta W_k[h]}{\delta h} \tag{A.3}$$

$$h = \frac{\delta \Gamma_k^{\mathrm{Leg}}[\phi]}{\delta \phi} \tag{A.4}$$

We see that we can *either* consider ϕ as being a function of h, *or* the reverse. That is,

$$\frac{\delta \phi}{\delta h} = \left(\frac{\delta h}{\delta \phi} \right)^{-1} \tag{A.5}$$

Using (A.3) and (A.4), we deduce that

$$\Gamma_k^{\mathrm{Leg}\ (2)}(x,y) = W_k^{(2)}(x,y)^{-1} \tag{A.6}$$

But we also know that the free energy W_k is the generating function of the connected n points correlation functions. In particular, we have thus

$$\Gamma_k^{\mathrm{Leg}\ (2)}(x,y) = G_c^2(x,y)^{-1} \tag{A.7}$$

This result will be useful later.

We also recall that we defined the effective action at scale k as

$$\Gamma_k[\phi] = \Gamma_k^{\mathrm{Leg}}[\phi] - \Delta H_k[\phi] \tag{A.8}$$

with

$$\Delta H_k[\phi] = \frac{1}{2} \phi \cdot R_k \cdot \phi \tag{A.9}$$

A.2 The derivation

On one hand we have

$$\partial_k e^{W_k[h]} = e^{W_k[h]}\partial_k W_k[h] = Z_k[\varphi]W_k[h] \tag{A.10}$$

while, on the other hand we alse have

$$\partial_k e^{W_k[h]} = \partial_k Z_k[\varphi] = -\frac{1}{2}\int_{x,y}\partial_k R_k(x,y)\underbrace{\int \mathscr{D}\varphi\varphi(x)\varphi(y)e^{-H_k[varphi]-\Delta H_k[\varphi]+h\cdot\varphi}}_{=Z_k[\varphi]\langle\varphi(x)\varphi(y)\rangle_k} \tag{A.11}$$

putting these two expressions together we deduce that

$$\partial_k W_k[h] = -\frac{1}{2}\int_{x,y}\partial_k R_k(x,y)\langle\varphi(x)\varphi(y)\rangle_k \tag{A.12}$$

We already almost have the Wetterich equation! Actually all that remains to be done is reexpressing the left and right hand sides of the previous expression in terms of the scale-dependant effective action Γ_k.

A.2.1 The left hand side

We start from $W_k = h\cdot\phi - \Gamma_k^{\text{Leg}}$, and we deduce that

$$\partial_k W_k = h\cdot\partial_k\phi - \partial_k\Gamma_k^{\text{Leg}} \tag{A.13}$$

Moreover

$$\partial_k\Gamma_k^{\text{Leg}}[\phi] = \partial_k\left(\Gamma_k[\phi] + \frac{1}{2}\phi\cdot R_k\cdot\phi\right) \tag{A.14}$$

Now, we must be careful not to forget that $\Gamma_k[\phi]$ depends explicitely on k, but also implicitely through ϕ! so we can write, somehow abusively,

$$\partial_k\Gamma_k[\phi] = \partial_k\Gamma_k[\phi] + \partial_k\phi\cdot\frac{\delta\Gamma_k[\phi]}{\delta\phi} \tag{A.15}$$

We conclude that

$$\partial_k\Gamma_k^{\text{Leg}}[\phi] = \partial_k\Gamma_k[\phi] + pk\phi\cdot h + \frac{1}{2}\phi\cdot\partial_k R_k\cdot\phi \tag{A.16}$$

and therefore that

$$\partial_k W_k[\phi] = -\partial_k\Gamma_k[\phi] - \frac{1}{2}\phi\cdot\partial_k R_k\cdot\phi \tag{A.17}$$

A.2.2 The right hand side

To transform the right hand side, we simply note that the connected two points correlation function is

$$G^2_{c,k}(x,y) = \langle\varphi(x)\varphi(y)\rangle_k - \langle\varphi(x)\rangle_k\langle\varphi(y)\rangle_k \tag{A.18}$$

so that

$$\partial_k W_k[\phi] = -\frac{1}{2}\int_{x,y}\partial_k R_k(x,y)\langle\varphi(x)\varphi(y)\rangle_k = -\frac{1}{2}\int_{x,y}\partial_k R_k(x,y)G^2_{c,k}(x,y) - \frac{1}{2}R_k\cdot\phi\cdot\phi \tag{A.19}$$

A.2.3 Conclusion

Putting equations A.17 and A.19 together, we finally have

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_k R_k(x,y) G_{c,k}^2(x,y) \tag{A.20}$$

Remembering that the two point connected correlation function is related to the effective action by A.7, we conclude that

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_k R_k(x,y) \left(\Gamma_k^{(2)}(x,y) + R_k(x,y) \right)^{-1} \tag{A.21}$$

This is the Wetterich equation for a scalar field ϕ.

Passing to the Wetterich equation for a vector field ϕ_i is very simple, as soon as we realize that all we have to do is to perform the following replacements:

$$\phi \to \phi_i \tag{A.22}$$

$$\Gamma_k^{(2)}(x,y) \to \Gamma_k^{(2)}(x,y)_{i,j} \tag{A.23}$$

$$R_k(x,y) \to R_k(x,y)_{i,j} \tag{A.24}$$

The Wetterich equation is thus transformed into

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_k R_k(x,y)_{i,j} \left(\Gamma_k^{(2)}(x,y) + R_k(x,y) \right)_{i,j}^{-1} \tag{A.25}$$

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[1] Walter Selke. The {ANNNI} model — theoretical analysis and experimental application. *Physics Reports*, 170(4):213 – 264, 1988.

[2] Y. Shapira, C. C. Becerra, N. F. Oliveira, and T. S. Chang. Phase diagram, susceptibility, and magnetostriction of mnp: Evidence for a lifshitz point. *Phys. Rev. B*, 24:2780–2806, Sep 1981.