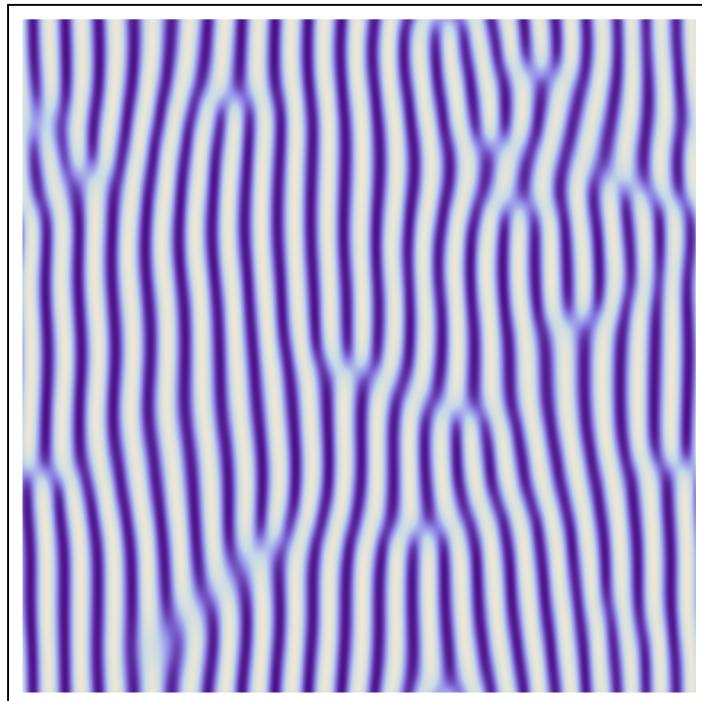


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Nonperturbative renormalization group study of the Lifshitz critical point



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Introduction

The Lifshitz model, is used to describe a wide range of materials including magnetic crystals, supraconductors, liquid crystals and ferroelectric materials. They share a common intriguing feature: having a so called modulated -or striped- phase. In this phase the order parameter is periodic in one direction of space, and constant in the other directions.

Though the Lifshitz model was proposed .. years ago, the value of the critical exponents of the critical point between the modulated, ferromagnetic and antiferromagnetic phases is still a matter of debate. During this internship we computed this critical exponents using the tools of the nonperturbative renormalization group. Though this has been done before [4] [1] our approach is new, and is thought to be more flexible. In particular it could be improved to compute the exponents with unpreceding accuracy.

The first chapter of this report is a very general introduction to the physics of the Lifshitz model. After an introduction to the ideas of renormalization, the second chapter focuses on the tool we used to compute the critical exponents of the model: the nonperturbative renormalization group. Finally, the third chapter gives our results and discuss them.

Useful notations

Through this report the following shorthand notations have been used:

$$\begin{array}{ccc} \int_x & \leftrightarrow & \int_{\mathbb{R}} dx \\ \int_q & \leftrightarrow & \int_{\mathbb{R}} \frac{dq}{(2\pi)^d} \\ \Gamma^{(N)}(x_1, \dots, x_N) & \leftrightarrow & \frac{\delta^N}{\delta\phi(x_1)\dots\delta\phi(x_N)}\Gamma[\phi] \\ f \cdot g & \leftrightarrow & \int_x f(x)g(x) \end{array}$$

The implicit summation on identical indices convention will be used.

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Chapter 1

The Lifshitz model: general presentation.

1.1 The Lifshitz model and its main features

1.1.1 The Lifshitz model

The modulated phase and the Lifshitz phase diagram.

The Lifshitz model aims at describing a number of physical many-body systems. They share a common intriguing feature: having a so called modulated - or striped - phase (fig. 1.1). In this phase, the order parameter is spatially periodic in one or several directions of space. We will label with a \parallel sign quantities living in these directions, and \perp the quantities living in the orthogonal subspace.

Typically, the phase diagram of such a physical system will resemble the one presented in fig. 1.2.

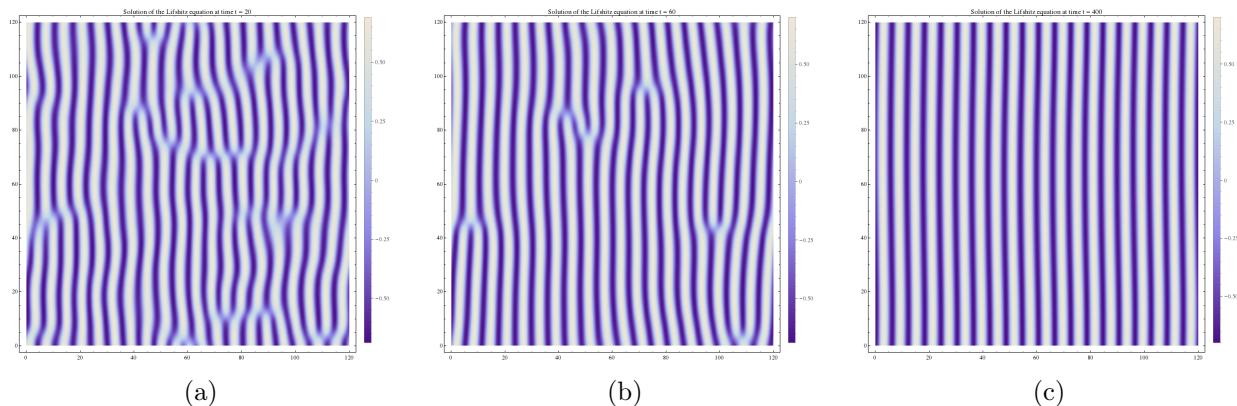


Figure 1.1 – Time evolution of a field obeying the equation of movement derived from the (time dependant) Lifshitz action. We see that the structure evolves toward a modulated steady state.

A crucial feature of this phase diagram is the intersection point of the three phases (L in fig. 1.2), called the Lifshitz point. Historically, the manganese phosphite (MnP) magnetic cristal was one of the first systems in which a Lifshitz point could be detected. Moreover, the entire phase diagram around the Lifshitz point of the MnP cristal could be inferred with high precision from experimental measurements [6]. In the case of the MnP magnetic cristal, the tunable parameter -which we will from now on call ρ_0 - is an external magnetic field applied to the crystal, while the order parameter is the local magnetization of the atoms. In the modulated phase, it is the angle between the direction of the local magnetization and a direction of reference that is spatially modulated.

Experiments also provide evidence of Lifshitz point existence in ferroelectrics and liquid cristals.

The Lifshitz model

The Lifshitz model is a field theory, describing a vector field ϕ whose components will be denoted ϕ_i . If we like to think in terms of magnetic systems, like the MnP cristal, we can say that $\phi(x)$ is the coarse-grained magnetization at position x . To write the action for the Lifshitz model, we chose

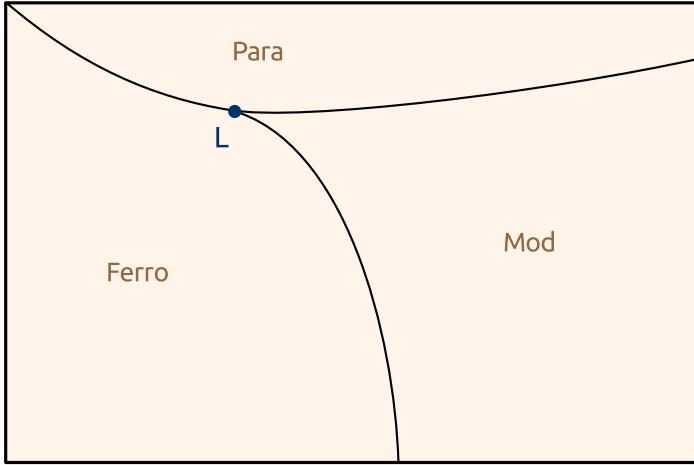


Figure 1.2 – Typical phase diagram of a system described by the Lifshitz model. The Lifshitz point is labelled L . “Para”, “Ferro” and “Mod” are the abbreviations for ‘Paramagnetic’, ‘Ferromagnetic’ and ‘Modulated’ respectively. Temperature varies along the vertical axis while the horizontal axis accounts for the variation of an extra parameter ρ_0 whose precise meaning depends on the physical nature of the studied system.

a basis $(\mathbf{e}_n)_{1 \leq n \leq d}$. We decide that this basis is such that its first m vectors span the m dimensional \parallel subspace, while of course the remaining $d - m$ base vector span the \perp subspace. In this basis, the Lifshitz action is

$$S = \int_x \sum_{i=1}^N \left(\frac{1}{2} \left(\sum_{n_\perp=m+1}^d \frac{\partial \phi_i}{\partial x_{n_\perp}} \mathbf{e}_{n_\perp} \right)^2 + \frac{\rho_0}{2} \left(\sum_{n_\parallel=1}^m \frac{\partial \phi_i}{\partial x_{n_\parallel}} \mathbf{e}_{n_\parallel} \right)^2 + \frac{\sigma_0}{2} \left(\sum_{n_\parallel=1}^m \frac{\partial^2 \phi_i}{\partial x_{n_\parallel}^2} \mathbf{e}_{n_\parallel} \right)^2 \right) + U(\phi) \quad (1.1)$$

As we want to model the broadest possible class of physical systems, we will say that U is an almost completely arbitrary potential. We only ask for it to have the $O(n)$ symmetry, ie to be a function of $\rho \stackrel{\text{def}}{=} \phi_i \phi_i / 2$.

From now on we will use the self-explanatory shorthand notation

$$H[\phi] = \int_x \left(\frac{1}{2} (\partial_\perp \phi)^2 + \frac{\rho_0}{2} (\partial_\parallel \phi)^2 + \frac{\sigma_0}{2} (\partial_\parallel^2 \phi)^2 + U(\rho) \right) \quad (1.2)$$

We see that this action closely resemble the well known action of the $O(n)$ model

$$S_{O(N)} = \int_x \left(\frac{1}{2} (\partial \phi)^2 + U(\rho) \right) \quad (1.3)$$

Namely, we recover it if we set $\rho_0 = 1$ and $\sigma_0 = 0$. We see that what differentiates the Lifshitz and $O(n)$ action is on one hand the presence a non trivial (ie different from 1) ρ_0 , breaking the $O(d)$ invariance, and on the other hand the presence of an extra term involving a Laplacian squared. Clearly, these two modifications must be responsible for the appearance of spatially modulated structures, but why exactly? We can gain a useful intuition of why a spatially modulated structure is closely linked to the existence of a laplacian squared term in the action by looking at a microscopic version of our model.

1.1.2 A discrete counterpart: the anisotropic Ising model

Stricto sensu the discrete counterpart of the Lifshitz model would be an anisotropic Heisenberg model, but to simplify things - without changing the essence of the argumentation - we consider an anisotropic Ising model instead.

First, let us consider a chain of Ising spins with the Hamiltonian

$$H_{\text{chain}} \stackrel{\text{def}}{=} -J \sum_i S_i S_{i+1} \quad (1.4)$$

We know that if J is positive, the interaction is ferromagnetic, whereas if J is negative, the interaction is antiferromagnetic. The antiferromagnetic order already shows some kind of spatial modulation, but it only exists at zero temperature! The idea to make a spatially modulated order survive at non zero temperatures is to consider a second neighbors *antiferromagnetic* interaction, together with a first neighbors *ferromagnetic* interaction:

$$H_{\text{chain}} = -J_1 \sum_i S_i S_{i+1} - J_2 \sum_i S_i S_{i+2} \quad (1.5)$$

However, for a long range order to exist at finite temperature, we need to work in two dimensions or more, *i.e.* to trade our spin chain for a spin lattice:

$$H_{\text{lattice}} \stackrel{\text{def}}{=} - \sum_i \left(J_0 \sum_{\delta_{\perp}} S_i S_{i+\delta_{\perp}} + J_1 \sum_{\delta_{\parallel}} S_i S_{i+\delta_{\parallel}} + J_2 \sum_{\delta_{\parallel}} S_i S_{i+2\delta_{\parallel}} \right) \quad (1.6)$$

Qualitatively, the competition between ferromagnetic and antiferromagnetic interactions will produce a spatial modulation of the spins at non zero temperatures, at least for some values of the interaction strengths ratio J_2/J_1 . The existence of a such stripped phases indeed is a well known feature of this model [5], called the ANNNI (axial next-nearest neighbour Ising) model.

Now, what is the link between this discrete spin lattice Hamiltonian, and our continuous action? Note that a sum on nearest neighbours can be rewritten in terms of a discrete Laplacian on the lattice, while a sum on next-nearest neighbors involves a discrete Laplacian squared:

$$H_{\text{lattice}} = - \sum_i \left(\kappa S_i^2 + J_0 S_i \Delta_{\perp} S_i + (J_1 + 4J_2) S_i \Delta_{\parallel} S_i - J_2 S_i \Delta_{\parallel}^2 S_i \right) \quad (1.7)$$

where we introduced the differential operators on the lattice:

$$\Delta_{\parallel} S_i = \sum_{\delta_{\parallel}} S_{i-\delta_{\parallel}} - 2S_i + S_{i+\delta_{\parallel}} \quad (1.8)$$

$$\Delta_{\perp} S_i = \sum_{\delta_{\perp}} S_{i-\delta_{\perp}} - 2S_i + S_{i+\delta_{\perp}} \quad (1.9)$$

$$\Delta_{\parallel}^2 S_i = \sum_{\delta_{\parallel}} -S_{i-2\delta_{\parallel}} + 4S_{i-\delta_{\parallel}} - 4S_i + 4S_{i+\delta_{\parallel}} - S_{i+2\delta_{\parallel}} \quad (1.10)$$

This rewriting in terms of discrete differential operators makes it clear that this Hamiltonian is the discrete -microscopic- counterpart of the Lifshitz action. We now understand -at least intuitively- the origin of the spatially periodic structures (shown in fig. 1.1) the Lifshitz field exhibit. They exist because of the competition between *nearest neighbours ferromagnetic interactions* (giving rise to the Δ_{\parallel} term in the Lifshitz action), and *next-nearest neighbours antiferromagnetic interactions* (giving rise to the Δ_{\parallel}^2 term in the Lifshitz action).

At this point a question arises: why work with a Lifshitz coarse-grained field theory, since we have a better physical understanding of an underlying microscopic model? What is more, in passing to a continuous theory, we lose information about the microscopic underlying lattice. This is actually not a problem since the statistical quantities we are interested in computing -namely the critical exponents of the phase transition- are universal; they do not depend on the specific microscopic model, if the symmetries of this model were carefully implemented in the continuum limit. Actually, passing to a field theory is even advantageous as it frees us of the irrelevant microscopic details. Even more crucial is the fact that field theories are the objects of choice for application of the powerful methods of the renormalization group, which we will now describe.

Chapter 2

Introduction to the nonperturbative renormalization techniques

At the so-called Lifshitz critical point, three phases intersect. This is rather unusual, so we expect the physics of the vicinity of this point to be of special interest. To investigate it, we would like to compute the critical exponents associated to this transition point. To this end, we used the powerful machinery of the renormalization group, and more precisely of one particular implementation of the renormalization ideas: the nonperturbative renormalization group.

In this chapter we propose first a very general introduction to the ideas and concepts of renormalization. Then we focus on the nonperturbative renormalization group techniques.

2.1 Introduction to the renormalization group

2.1.1 The renormalization procedure

The idea of renormalization is to consider a given many-body system at different length scales. At a given length scale S the system is described by an effective Hamiltonian H_S .

For the moment what we mean by “the Hamiltonian at a length scale S ” is rather vague. To be more precise, let us imagine that we want to describe a magnetic crystal. The microscopic Hamiltonian will be in general a discrete sum of local observables O_α , depending on the value of the magnetization at site i , $\phi(i)$:

$$H[\phi] = \sum_i \sum_\alpha \kappa_\alpha O_\alpha[\phi(i), \nabla\phi(i), \dots] \quad (2.1)$$

where κ_α is the coupling constant associated to the observable O_α . This Hamiltonian describes the system at the microscopic scale $S = 1$. Now, if we want to describe it at a scale $S \geq 1$, surely we are no longer interested in knowing the fluctuations of the field over regions of size smaller than aS^1 . All we need is thus the average over regions of size aS of the field:

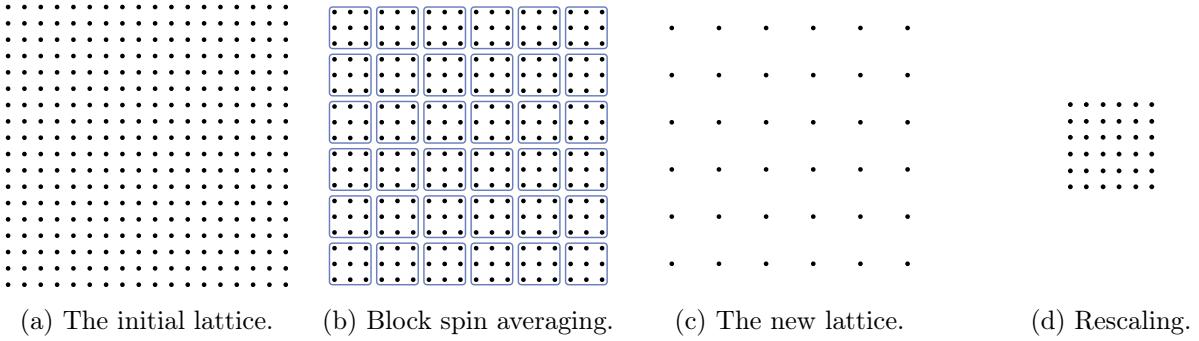
$$\tilde{\phi}(b) = \frac{1}{(aS)^d} \sum_{i \in B(b)} \phi(i) \quad (2.2)$$

where d is the dimension of space, and $B(b)$ is the set of sites i belonging to the block b .

Schematically what we do is group spins by blocks of size aS (fig. 2.1b), and average over these blocks.

Now we replace the microscopic Hamiltonian by an effective Hamiltonian for the block spin field $\tilde{\phi}$. This Hamiltonian describes the new system depicted in fig. 2.1c. We are not done yet! To make the

1. Experimentally, we can imagine that looking at the system at scale S means probing it with devices having a spatial resolution of aS . Any measurement operation with such devices can be described mathematically by the convolution of an observable by an error function having a spatial support of diameter aS . This operation is roughly equivalent to averaging the observables (and therefore the field) on blocks of size aS .



(a) The initial lattice. (b) Block spin averaging. (c) The new lattice. (d) Rescaling.

Figure 2.1 – The renormalization procedure illustrated. Here we have chosen $S = 3$.

new system look as much as possible like the one we started from, we rescale all lengths (fig. 2.1d). We also rescale the field. Formally it means that we perform the change of variables

$$x' = x/S \quad (2.3)$$

$$\phi' = S^\Delta \tilde{\phi} \quad (2.4)$$

where x could be any length appearing in the Hamiltonian.

The Hamiltonian in the new variables $H'[\phi'] = \tilde{H}[\tilde{\phi}]$ is the effective Hamiltonian after the renormalization operation. It could seem strange that we had to rescale the field as well as the lengths. We do that in order for the new Hamiltonian to resemble the old one as closely as possible. We are going to see on the example of the Lifshitz mean field theory how we can chose Δ for that purpose.

To conclude, the key ideas of the renormalization procedure are the averaging over block spins, and the rescaling of lengths and fields. We have described here the case of a discrete Hamiltonian, because it seemed more intuitive. But of course the ideas of renormalization are general and can as easily be applied to a continuous Hamiltonians.

2.1.2 The renormalization group

If the structure of the Hamiltonian is kept unchanged by the renormalization group procedure, *ie* if

$$H'[\phi'] = \sum_{i'} \sum_{\alpha} \kappa'_{\alpha} O_{\alpha}[\phi'(i'), \nabla' \phi'(i'), \dots] \quad (2.5)$$

then the renormalization group action is a group action². We can define a group action g , transforming the coupling constants at scale 1 in the coupling constant at scale S : The group renormalization group is completely described by its action on the coupling constants:

$$\kappa_{\alpha} \rightarrow \kappa'_{\alpha} \stackrel{\text{def}}{=} g(\kappa_{\alpha}, S) \quad (2.6)$$

The renormalization group is a multiplicative, one parameter group:

$$g(g(\kappa_{\alpha}, S_1), S_2) = g(\kappa_{\alpha}, S_1 S_2) \quad (2.7)$$

These transformations are assumed to be continuous in the coupling constant. It is also very often possible to consider the scale S as a continuous parameter. Then the successive application of infinitesimally close renormalization group transformations generates a continuous trajectory in the space of coupling constants. This trajectory can be parametrized by $t \stackrel{\text{def}}{=} \log(S)$, an additive parameter playing the role of a time. It is often referred to as “the renormalization group time”.

Near a phase transition or critical point, fluctuations occur at all length scales, and thus one should expect the Hamiltonian to be scale invariant. In terms of the renormalization group action, scale invariance simply means that

$$g(\kappa_{\alpha}, S) = \kappa_{\alpha} \quad (2.8)$$

2. Actually, invertibility cannot be guaranteed so it rather is a semigroup action, but the distinction is of no importance for us.

The fact that scale invariance has such a simple meaning in the renormalization group framework is an extremely good sign. It is a hint that renormalization group is a powerful tool to look for critical points.

As an application of these ideas, imagine that we are close to a critical point. Then eq. 2.6 can be linearized:

$$\frac{d \log g(\kappa_\alpha, S)}{d \log S} = \lambda_\alpha \quad (2.9)$$

If λ_α is positive, then the fixed point is repulsive in direction α , and the coupling is said to be *relevant*. On the other hand if λ_α is negative, the fixed point is attractive in this direction and the coupling is *irrelevant*.

Let us imagine that we are at a point g_0 in the space of coupling constants, very close to a fixed point. If we do a small change in scale, taking us from g_0 to $g(S)$, then, because we are close to a fixed point, we have moved at first order in the direction of the most relevant coupling, let us say it is direction $\alpha = 0$. Then we have $g(S) = g_0 S^{\lambda_0}$. Under the same infinitesimal change in scale, the correlation length ξ is transformed according to $\xi_{g_0} = S \xi_{g(S)} \propto S^{-1/\lambda_0}$. Usually, and also in the case of the Lifshitz model, the most relevant direction is the direction of temperature. Then the ν critical exponent is given by $\xi_{g_0} \propto |T - T_c|^{-\nu}$. By simple comparison with our result from the renormalization group, we have thus obtained the very important result

$$\nu = \frac{1}{\lambda_0} \quad (2.10)$$

relating ν , the critical exponent of the correlation length, to λ_0 , the most relevant direction around a given fixed point.

2.1.3 Renormalization procedure applied to the mean field Lifshitz theory

As we have just seen, an operation from the renormalization group transforms our microscopic Hamiltonian H into an effective Hamiltonian at scale S , $H_g(S)$. We hope that this operation will not change the structure of our Hamiltonian, so that we can use the tools of the renormalization group. Since our Hamiltonian is not isotropic (it distinguishes between the direction of the modulation \parallel , and the orthogonal direction \perp), we expect an operation of the renormalization group to change lengthscales by two different amounts in the two inequivalent directions. Scales in the parallel direction will be changed by a factor $S_\parallel : x'_\parallel = (S_\parallel)^{-1} x_\parallel$, while scales on the orthogonal direction will be changed by a factor $S_\perp : x'_\perp = (S_\perp)^{-1} x_\perp$.

To simplify things we can keep a single scale $S = S_\perp$, and define θ such that $S_\parallel = S^\theta$. Note that this is equivalent to changing the *units* in the parallel direction: if we say that lengths in the orthogonal direction are measured in meters, then lengths in the parallel direction are measured in (meters) $^\theta$. A volume, which is normally measured in (meters) d will in our new units system be measured in (meters) $^{(d-m)+\theta m}$. It is as if d had been replaced by

$$d_m = d + m(\theta - 1) \quad (2.11)$$

We define two anomalous dimensions by

$$\langle \phi(p_\parallel) \phi(0) \rangle_{g^*} \propto |p_\parallel|^{\eta_\parallel - 4} \quad (2.12)$$

$$\langle \phi(p_\perp) \phi(0) \rangle_{g^*} \propto |p_\perp|^{\eta_\perp - 2} \quad (2.13)$$

where g^* is a fixed point in the space of coupling constants, and the proportionality constant is independent of the scale.

But we also know that

$$\langle \phi(p_\parallel) \phi(0) \rangle_{g^*} \propto |p_\parallel|^{\frac{2\Delta}{\theta}} \quad (2.14)$$

$$\langle \phi(p_\perp) \phi(0) \rangle_{g^*} \propto |p_\perp|^{2\Delta} \quad (2.15)$$

where Δ is the renormalization of the field : $\phi_{g(1)}(x) = S^{-\Delta}\phi_{g(S)}(Sx)$. Identifying these expressions for the renormalization of the correlation function with the previous ones, we can express Δ in two inequivalent ways, thus yielding a relation between η_{\parallel} and η_{\perp} :

$$\theta = \frac{2 - \eta_{\perp}}{4 - \eta_{\parallel}} \quad (2.16)$$

Mean field analysis

We recall that the Lifshitz Hamiltonian is

$$H = \int_x \left(\frac{1}{2}(\partial_{\perp}\phi)^2 + \frac{\rho_0}{2}(\partial_{\parallel}\phi)^2 + \frac{\sigma_0}{2}(\partial'^2_{\parallel}\phi)^2 + U(\rho) \right) \quad (2.17)$$

The mean field approximation consists in neglecting the fluctuations of the field. Therefore, the integration over the fluctuations performed during a renormalization group operation is trivial, and we can directly write the transformed Hamiltonian after a renormalization group operation $g(S)$:

$$H'[\phi'] = \int_{x'} S^{d_m} \left(S^{-2\Delta-2}(\partial'_{\perp}\phi')^2 + \sigma_0 S^{-2\Delta-4\theta}(\partial'^2_{\parallel}\phi')^2 + \rho_0 S^{-2\Delta-2\theta}(\partial'_{\parallel}\phi')^2 U'(\phi') \right) \quad (2.18)$$

This Hamiltonian must indentify with the previous one, and because we have chosen the normalization of the field, so we must have $S^{d_m+2\Delta-2} = 1$. This sets the renormalization of the field at the mean field level.

As we have seen -at least qualitatively, stripped pattern appear because of a nontrivial Laplacian squared term. Therefore at the Lifschitz point σ_0 must be nontrivial, and in particular must not renormalize away, which is only possible if

$$\theta = \frac{1}{2} \quad (2.19)$$

In the mean field approximation, it is as if the physical dimension were

$$d_{\text{mean field}} = d - \frac{m}{2} \quad (2.20)$$

We immediately deduce that the upper critical dimension³, which is “normally” 4, becomes

$$d_c^> = 4 + \frac{m}{2} \quad (2.21)$$

around the Lifschitz point.

In nature, the most common case is $m = 1$ (when in the modulated phase, the field is periodic in one direction of space). In that case the upper critical dimension is $d_c^> = 4.5$. When doing perturbation theory, one usually expand around the upper critical dimension, writing

$$d = d_c^> - \epsilon, \quad (2.22)$$

where ϵ is a supposedly small parameter. When $d = 3$, $\epsilon = 1.5$, which is not so small. So we expect the results from perturbation theory to be rather imprecise. To better tackle this problem, it would be great to have a method that is not perturbative in the dimension. This is what the nonperturbative renormalization group method provides, as we will now see.

2.2 The nonperturbative renormalization group

The idea of the nonperturbative renormalization group is to find an exact evolution equation of a scale-dependent object, which could possibly be the microscopic Hamiltonian, or the free energy, or the Legendre transform of the Hamiltonian: the effective action.

We shall discuss here the so-called Wetterich, or Wegner-Wetterich equation, whose object of study is the scale dependent *effective action* at scale k , Γ_k . The effective action Γ takes into account fluctuations on all length scales (ie on all momentum scales), while Γ_k only takes into account fluctuations of momentum (ie inverse length scale) between k and Λ .

3. The upper critical dimension is the dimension above which mean field theory is exact, in the sense that it gives the correct critical exponents.

2.2.1 The scale-dependent effective action

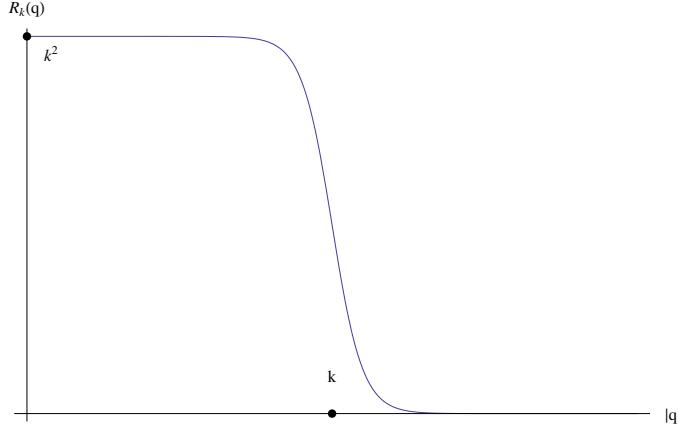


Figure 2.2 – Typical shape of the regulator function in momentum space $R_k(q)$.

Let us try to build Γ_k explicitly. To this end let us define the *regulator* $R_k(x, y) = R(x)\delta(x - y)$, whose role is to give a mass to the modes of momentum smaller than k , while leaving the other modes untouched. The typical shape of the regulator in momentum space is given by fig. 2.2. Moreover, the regulator is required to vanish at $k \rightarrow 0$ and to diverge for $k \rightarrow \infty$ (or $k \rightarrow \Lambda$), at fixed q^2 . This can for example be achieved by

$$R_k(q) = \frac{q^2}{e^{q^2/k^2} - 1} \quad (2.23)$$

We now introduce a modified, scale dependent Hamiltonian:

$$H_k[\varphi] \stackrel{\text{def}}{=} H[\varphi] + \Delta H_k[\varphi], \text{ with } \Delta H_k[\varphi] \stackrel{\text{def}}{=} \frac{1}{2}\varphi \cdot R_k \cdot \varphi = \frac{1}{2} \int_{x,y} \varphi(x) R_k(x, y) \varphi(y) \quad (2.24)$$

We see that the ΔH_k term is a quadratic on the field. Thus R_k indeed acts as a scale-dependent and momentum-dependent squared mass term, giving a mass of order k^2 to the “slow” modes of momentum smaller than k .

We can now define the Legendre transform of the scale dependent free energy,

$$\Gamma_k^{\text{Leg}}[\phi] \stackrel{\text{def}}{=} -W_k[h] + h \cdot \phi \quad (2.25)$$

where the free energy is defined as $W_k[h] = \log Z_k$, and $Z_k = \int \mathcal{D}\varphi e^{-H_k[\varphi]+h \cdot \varphi}$. Qualitatively, since the slow modes are given a mass in W_k , the Legendre transform only acts on the rapid mode, and we understand that $\Gamma_k^{\text{Leg}}[\phi]$ indeed has the property of taking into accounts only rapid modes.⁴

Finally, we introduce the scale-dependent effective action:

$$\Gamma_k[\phi] \stackrel{\text{def}}{=} \Gamma_k^{\text{Leg}}[\phi] - \Delta H_k[\phi] \quad (2.27)$$

This object has the right properties to be the effective action at scale k . Indeed it can be shown to verify $\Gamma_0 = \Gamma$, $\Gamma_\Lambda = H$.⁵

Finally, the scale-dependent effective action verifies the differential equation

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_k (R_k(x, y)) \left(\Gamma_k^{(2)}(x, y) + R_k(x, y) \right)^{-1} \quad (2.28)$$

4. We note in passing that eq. 2.25 implies

$$\phi(x) = \frac{\delta W_k[h]}{\delta h(x)} = \frac{1}{Z_k[\varphi]} \int \mathcal{D}\varphi e^{-H_k[\varphi]+h \cdot \varphi} \varphi \stackrel{\text{def}}{=} \langle \varphi(x) \rangle_k \quad (2.26)$$

This tells us that ϕ is the background field (at scale k), ie the mean value of the field φ (at scale k).

5. The first identity is trivial, given the properties of the regulator function. Proving the second one requires slightly more work. It can be shown using that $\exp -\Gamma_k[\phi] = \int \mathcal{D}\varphi \exp -S[\varphi] + \frac{\delta \Gamma}{\delta \varphi} \cdot (\varphi - \phi) - \frac{1}{2}(\varphi - \phi) \cdot R_k \cdot (\varphi - \phi)$, and remembering that $R_k \rightarrow \infty$ when $k \rightarrow \Lambda$.

This equation is often called the Wetterich equation. The demonstration can be found in appendix A. We can use the fact that R_k is invariant by translation, to write the right hand side of this integro-differential equation as a single integral over momentum q :

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_q \partial_k (R_k(q)) \left(\Gamma_k^{(2)}(q, -q) + R_k(q) \right)^{-1} \quad (2.29)$$

Moreover, we shall make the change of variable $k \rightarrow t = \log(k/\Lambda)$, to take advantage of the additive properties of t ⁶. In order to simplify further the Wetterich equation, we define an operator $\hat{\partial}_t$ as the differentiation with respect to t acting only on R_t . That is,

$$\hat{\partial}_t \stackrel{\text{def}}{=} \frac{\partial R_t(\cdot)}{\partial t} \frac{\partial}{\partial R_t(\cdot)} \quad (2.30)$$

where $R_t(\cdot)$ is a shorthand notation for the function $q \mapsto R_t(q)$. We have then

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \hat{\partial}_t \text{tr} \left(\log \left(\Gamma_t^{(2)}(q, -q) + R_t(q) \right) \right)^7 \quad (2.31)$$

This is the form of the Wetterich equation we will use from now on. Even if the field is vectorial, eq. 2.31 is still valid, provided that the trace acts on the vector space of the field as well⁸.

Note that no approximations were made in the course of deriving the Wetterich equation. It is thus an exact equation describing the (renormalization group) time dependence of the scale-dependent effective action Γ_t , interpolating between $\Gamma_\Lambda = H$ and $\Gamma_0 = \Gamma$.

We can use the Wetterich equation as a starting point for finding the critical properties of the Lifshitz model.

6. As we have seen before, t can be regarded as a time parameterizing the renormalization group trajectory in the coupling constants space.

7. Here we have used the formula for the trace on continuous indices in Fourier space: $\text{tr}(A) = \int_x A(x, x) = \int_q A(q, -q)$.

8. we refer to appendix A for a sketch of proof

Chapter 3

The Lifshitz critical point: methods and results

As we have seen, the Wetterich equation governs the flow of the scale-dependent effective action Γ_t . Solving directly this differential equation to find Γ_t , though in theory possible, is in practice very difficult. Indeed the Γ_t depends on the (background) field ϕ , a function of the momentum. Determining completely Γ_t requires knowing its value for all functions ϕ .

So, instead of solving directly the Wetterich equation, we start from a reasonable functional form for Γ_t with unknown coefficients, we plug it in the Wetterich equation to get flow equations for this coefficients, and we solve these equations.

If we assume that it is analytic in the potential, the effective action can be written as the infinite sum

$$\Gamma_t[\phi] = \int_x (m_0(\partial\phi(x))^2 + m_1(\partial\phi(x)^2)^2 + \dots + n_0(\partial^2\phi(x))^2 + \dots + l_0\phi(x)^2 + l_1\phi(x)^3 + \dots) \quad (3.1)$$

This expression can be further simplified if the system under study has some invariance properties. The simplest model having such an invariance is the Ising one, whose microscopic Hamiltonian is invariant under the action of the \mathbb{Z}_2 symmetry group. If we chose a regulator that preserves this symmetry, it is clear that the effective action Γ_t must also be invariant under \mathbb{Z}_2 . The most general expression for Γ_t that preserves this symmetry is

$$\Gamma_t^{\text{Ising}}[\phi] = \int_x \sum_{i,j \in \mathbb{N}} Z_{t,ij} (\partial\phi(x))^{2i} \phi(x)^{2j} \quad (3.2)$$

The strategy is now simple:

- Compute $\Gamma_t^{(2)}(x, y)$ from 3.2
- Plug the result in the right hand side of the Wetterich equation
- Deduce the flow equation for $Z_{t,ij}$: $\partial_t Z_{t,ij} = F_{i,j}(Z_{t,kl})$

Following this procedure we went from a functional differential equation to a set of simple differential equations. Solving this set of flow equations give us a complete knowledge of the model. In particular solving the fixed point equations $0 = F_{i,j}(Z_{t,kl}^*)$ will give us access to the critical properties of the Ising model: critical exponents, critical potential,... Note that for the moment no approximations have been made!

In practice however we know that the lowest degree terms in $\partial\phi$ and ϕ will have the most significant contribution to the flow. Therefore we can make the approximation of cutting the development at some order. This has the great practical advantage of reducing the number of flow equations to solve to a finite number. Note that other approximation schemes are possible (development in powers of the field, BMW scheme...). We will not talk about these.

Now, let us apply this approximation scheme to the Lifshitz model.

3.1 An Anzatz for the Lifshitz scale-dependant effective action

We recall that the Lifshitz microscopic Hamiltonian is

$$H[\phi] = \int_x \left(\frac{1}{2}(\partial_{\perp}\phi)^2 + \frac{\rho_0}{2}(\partial_{\parallel}\phi)^2 + \frac{\sigma_0}{2}(\partial_{\parallel}^2\phi)^2 + V(\rho) \right) \quad (3.3)$$

where $\rho = \phi_i\phi_i/2$. The Lifshitz Hamiltonian has the $O(n)$ symmetry¹, therefore the scale-dependant effective action should also have this symmetry. Moreover the kinetic term in the Lifshitz Hamiltonian decomposes into two parts:

- $\int_x \frac{\rho_0}{2}(\partial_{\parallel}\phi)^2 + \frac{\sigma_0}{2}(\partial_{\parallel}^2\phi)^2$, invariant under $(x_{\parallel}, x_{\perp}) \rightarrow (\mathcal{R}(m)x_{\parallel}, x_{\perp})$
- $\int_x \frac{1}{2}(\partial_{\perp}\phi)^2$, invariant under $(x_{\parallel}, x_{\perp}) \rightarrow (x_{\parallel}, \mathcal{R}(d-m)x_{\perp})$

Since the scale-dependant effective action must satisfy these symmetry properties, its most general form is

$$\Gamma_k[\phi] = \int_x U(\rho) + \left(\frac{1}{2}Z_{\perp}(\rho)(\partial_{\perp}\phi)^2 + \frac{1}{4}Y_{\perp}(\rho)(\partial_{\perp}\rho)^2 + \dots \right) + \left(\frac{1}{2}\rho_0(\rho)(\partial_{\parallel}\phi)^2 + \dots \right) + \left(\frac{1}{2}Z_{\parallel}(\rho)(\partial_{\parallel}^2\phi)^2 + \dots \right) \quad (3.4)$$

It turns out to be a very good approximation to cut the derivative expansion to first order². That is to say, we make the approximation

$$\frac{1}{2}Z_{\perp}(\rho)(\partial_{\perp}\phi)^2 + \frac{1}{4}Y_{\perp}(\rho)(\partial_{\perp}\rho)^2 + \dots \simeq \frac{1}{2}Z_{\perp}(\rho)(\partial_{\perp}\phi)^2 \quad (3.5)$$

$$\frac{1}{2}\rho_0(\rho)(\partial_{\parallel}\phi)^2 + \dots \simeq \frac{1}{2}\rho_0(\rho)(\partial_{\parallel}\phi)^2 \quad (3.6)$$

$$\frac{1}{2}Z_{\parallel}(\rho)\partial_{\parallel}^2\phi)^2 + \dots \simeq \frac{1}{2}Z_{\parallel}(\rho)\partial_{\parallel}^2\phi)^2 \quad (3.7)$$

Moreover, we make the approximation that the field renormalizations $Z_{\perp}(\rho)$, $\rho_0(\rho)$ and $Z_{\parallel}(\rho)$ do not depend on the field. So, the definitive form of the Anzatz we chose for Γ_t is

$$\Gamma_t[\phi_i] = \int_x \left(\frac{Z_{\perp}}{2}(\partial_{\perp}\phi)^2 + \frac{\rho_0}{2}(\partial_{\parallel}\phi)^2 + \frac{Z_{\parallel}}{2}(\partial_{\parallel}^2\phi)^2 + U(\rho) \right) \quad (3.8)$$

This is the form of the Lifshitz scale-dependent effective action we have worked on during this internship. Though this does not appear explicitly, the field renormalizations and the effective potential U depend on the renormalization time t . As we are going to see, the t dependence of Z_{\perp} and Z_{\parallel} gives rise to non trivial anomalous dimensions. The anomalous dimensions are expected to act as a small correction to the value of the critical exponents. At first order, we can thus assume trivial anomalous dimensions. This amounts to forgetting the t dependence of Z_{\parallel} and Z_{\perp} in the effective action. This approximation is called the *local potential approximation*.

We will now make use of the Wetterich equation to know how these quantities change through the renormalization process. Then we will use this knowledge to derive the critical exponents.

3.2 The Lifshitz renormalization flows

3.2.1 Dimension-driven versus fluctuations-driven flows

We recall that looking at the mean field version of a theory consists in neglecting all fluctuations of the field. Therefore, in the case of a mean field theory, the renormalization procedure - average on fluctuations up to a certain scale, definition of an effective Hamiltonian and rescaling - only requires rescaling. The conclusion is that the renormalization flows of mean field theories' coupling constants are only due to their dimension.

1. Meaning that a rotation of the n -dimensional field ϕ leaves the Hamiltonian invariant: $H[\mathcal{R}(n)\phi] = H[\phi]$ if \mathcal{R} is an orthogonal $n \times n$ matrix.

2. To compute critical exponents, one does not need to take into account the full momentum dependence of the theory. However for computing correlation function the momentum dependence is essential. Were we to compute these quantities, we will probably have to go further in the derivative expansion

If we wish to go beyond the mean field approximation, we have to take into account the fluctuations of the field. They are often small compared to the mean value of the field, meaning that their contribution to the flows will be small compared to the contribution coming from the mean field theory. In other words, the fluctuation-driven part of a renormalization flow is generally small compared to its dimension-driven part. Wishing to concentrate on the small non-trivial part of the flow: the fluctuation-driven part, we define *dimensionless coupling constants*, that are by construction subject to a fluctuation-driven flow only.³

With these ideas in mind, we define the following dimensionless quantities

$$\boxed{\begin{array}{|c|c|c|} \hline q_{\parallel}^2 = k^{2\theta} y_{\parallel} & q_{\perp}^2 = Z_{\parallel} Z_{\perp}^{-1} k^{4\theta} y_{\perp} & \rho_0 = Z_{\parallel} k^{2\theta} \bar{\rho}_0 \\ \hline R(q_{\perp}^2, q_{\parallel}^2) = Z_{\parallel} k^{4\theta} y_{\parallel}^2 r(y_{\perp}, y_{\parallel}) & U(\rho) = Z_{\parallel} k^{d_m} u(\bar{\rho}) & \rho = Z_{\parallel}^{-1} k^{-4\theta + d_m} \bar{\rho} \\ \hline \end{array}}$$

Note that given the relation

$$\theta = \frac{2 - \eta_{\perp}}{4 - \eta_{\parallel}} \quad (3.9)$$

we have a certain liberty relative to the adimensionning of the physical quantities. For example we can define $\rho_0 = Z_{\parallel} k^{2\theta} \bar{\rho}_0$ or $\rho_0 = Z_{\perp}^{1/2} Z_{\parallel}^{1/2} k \bar{\rho}_0$. These two definitions lead to two different $\bar{\rho}_0$ functions, but the two functions have *the same dimensional flow*. Here we chose the adimensionning such that Z_{\parallel} simplifies everywhere in the propagators.

3.2.2 Flow of the potential

From the shape of the potential, we can tell in which phase we are. Therefore knowing how the potential changes with t is of paramount importance. We shall thus start with the derivation of the flow equation for the potential.

We see that

$$U(\rho_0) = \delta(0)^{-1} \Gamma_t[\phi]|_{\phi(x)=\phi_0} \quad (3.10)$$

where ϕ_0 is some uniform (in direct space) configuration of the field, and where $\rho_0 = \phi_{0i} \phi_{0i}/2$. Note that if we were to work on a finite-size system, the $\delta(0)^{-1}$ term would be replaced by the volume of the system. Therefore it plays the role of the system volume for the infinite size system we consider here.

We take a derivative with respect to t to get an expression for the flow of the potential, and we plug in the Wetterich equation on the right hand side:

$$\partial_t U(\rho_0) = \delta(0)^{-1} \partial_t \Gamma = \frac{1}{2\delta(0)} \hat{\partial}_t \text{tr} \left(\log \left(\Gamma_t^{(2)} + R_t \right) \right) \quad (3.11)$$

This is the flow equation for the potential!⁴

From eq. 3.8 we compute the second derivative of the effective action in Fourier space:

$$\Gamma_{ij}^{(2)}(p, q) = \left(\delta_{ij} \left(Z_{\perp} p_{\perp}^2 + \rho_0 p_{\parallel}^2 + Z_{\parallel} (p_{\parallel}^2)^2 + U'(\rho(x)) \right) + \phi_i(p) \phi_j(q) U''(\rho(x)) \right) \delta(p + q) \quad (3.12)$$

We can decompose the $\Gamma_{ij}^{(2)}$ function on a orthogonal the projector on the direction of the field $\Pi_{i,j}^a \stackrel{\text{def}}{=} \delta_{ij} - \frac{\phi_i \phi_j}{2\rho}$ and on the projector orthogonal to it $\Pi_{i,j}^r = \frac{\phi_i \phi_j}{2\rho}$. This allows us to easily rewrite the regularized full propagator appearing in the Wetterich equation:

$$\left(\Gamma_{ij}^{(2)}(p, q) + R_t(p, q) \right)^{-1} = (G_a(q) \Pi_{ij}^a + G_r(q) \Pi_{ij}^r) \delta(p + q) \quad (3.13)$$

3. Numerically, if we want to track the contribution to the flow of fluctuations 10^{12} weaker than the mean field contribution, we must achieve at least 12 digits precision, which is often rendered impossible by rounding errors. In this context using dimensionless coupling constant indeed seems an excellent idea!

4. It should be noted that we only derived the flow equation of the potential for a constant field configuration. This is of no importance as we do not need the *functional* dependence of the potential, $(x \mapsto \rho(x)) \mapsto U(\rho(x))$ but only its “digital” dependence $\rho_0 \mapsto U(\rho_0)$ to characterize it completely.

where we used the radial and angular propagators:

$$G_r(q) \stackrel{\text{def}}{=} \frac{1}{Z_{\parallel}q_{\parallel}^4 + Z_{\perp}q_{\perp}^2 + \rho_0 q_{\parallel}^2 + R_t(q) + U'(x) + 2\rho U^{(2)}(\rho)} \quad (3.14)$$

$$G_a(q) \stackrel{\text{def}}{=} \frac{1}{Z_{\parallel}q_{\parallel}^4 + Z_{\perp}q_{\perp}^2 + \rho_0 q_{\parallel}^2 + R_t(q) + U'(\rho)} \quad (3.15)$$

We can now rewrite in a more explicit way the Wetterich equation:

$$\partial_t \Gamma_t = \frac{1}{2} \int_q (G_a(q)\Pi_{ij}^a + G_r(p)\Pi_{ij}^r) \partial_t R_{t,ij}(q) = \frac{1}{2} \int_q (G_a(q) + (n-1)G_r(p)) \partial_t R_t(q) \quad (3.16)$$

Note that the radial (massive) propagator appears once whereas the angular (massless) propagator appears $n-1$ times in the flow equation. This is of course because as soon as we choose a direction for the field ϕ , the $O(n)$ invariance is no longer explicit (though the equation is of course still $O(n)$ invariant). Therefore, a massive Goldstone mode and $n-1$ massless modes appear, in accordance with Goldstone's theorem.

Imposing now that the field is constant, we have for the flow of the potential

$$\partial_t U(\phi_0) = \frac{1}{2} \int_q \left(G_r(q) \Big|_{\text{unif}} + (n-1)G_a(q) \Big|_{\text{unif}} \right) \partial_t R_t(q) \quad (3.17)$$

This is as far as we can get without giving explicitly a form to the regulator. At this point, it is a customary procedure to introduce standard functions called *threshold functions* in order to lighten the notations. Their definition can be found in appendix A.2.3.

Passing to dimensionless quantities the flow of the potential can be expressed in terms of the l threshold function:

$$d_t u_t(\rho) = -d_m u_t(\rho) + (\theta\eta_{\parallel} + d_m - 4\theta)\rho u'_t(\rho) + 8v_m v_{d-m} \left(l_0^{dm} (u'_t(\rho) + 2\rho u''_t(\rho)) + (n-1)l_0^{dm} (u'_t(\rho)) \right) \quad (3.18)$$

This is a second order, nonlinear differential equation, that can be solved at least numerically, once we have chosen an explicit form for the regulator (in order to compute the l_0 threshold function), and once we know the values of ρ_0 , η_{\parallel} and $\theta = (4 - \eta_{\parallel})/(2 - \eta_{\perp})$. As we are going to see now, the values of η_{\parallel} and η_{\perp} are related to the flows of Z_{\parallel} and Z_{\perp} .

3.2.3 Flow of the field renormalizations.

What we call field renormalizations are the prefactors Z_{\parallel} and Z_{\perp} . We will briefly explain why $\partial_t \log Z_{\perp} \sim -\eta_{\perp}$. The argument is of course exactly the same for η_{\parallel} .

3.2.4 The anomalous dimension in the orthogonal direction.

Flow of the field renormalization and anomalous dimension

From the inverse full propagator (given by eq. 3.13) we can retrieve the field renormalization in the \perp direction:

$$\frac{d}{dp_{\perp}^2} \Gamma_{ij}^{(2)}(p, q) = \delta_{ij} \delta(p+q) Z_{\perp} \quad (3.19)$$

and because $\Gamma^{(2)}$ flows accordingly to the Wetterich equation, from the above equation we can compute the flow of Z_{\perp} . Note that this flow will depend on the external impulsion p . From dimensional analysis it is easy to see that

$$Z_{\perp} \xrightarrow[p \rightarrow 0, k > 0]{} C k^{-\eta_{\perp}} \quad (3.20)$$

where C is a dimensionless constant factor. So, at vanishing p , the flow of Z_{\perp} is linked to the anomalous dimension:

$$-\partial_t \log Z_{\perp} \xrightarrow[p \rightarrow 0, k > 0]{} \eta_{\perp} \quad (3.21)$$

We shall use this relation to compute the anomalous dimension from the flow of Z_\perp . To express η_{perp} we can simply take the limit of vanishing external moment of equation 3.19. This gives

$$\eta_\perp = -\frac{1}{\delta(0)Z_\perp} \partial_t \lim_{p \rightarrow 0} \frac{d}{dp_\perp^2} \left(\Gamma_{ii}^{(2)}(p, -p) \Big|_{\text{unif}} \right) \quad (3.22)$$

The index i is still not fixed. To set things, we decide that 1 is the direction of the uniform field ϕ_0 . Then we can either chose $i = 1$, or $i \neq 1$ (all directions orthogonal to the field are equivalent). We made here the choice $i \neq 1$, because we know that in that case the expression we obtain for η_\perp is exact in the $n \rightarrow \infty$ limit. Explicitly, we chose $i = 2$. Now all we need to do to obtain the explicit expression of η_\perp in terms of the propagators is compute $\partial_t \Gamma_{22}^{(2)}(p, -p) \Big|_{\text{unif}}$, derive it with respect to p_\perp and take the limit of vanishing p . As the computation is rather lengthy, we will only give the important intermediate steps.

Computation of the flow

Notice that we can intertwine ∂_t and $\Big|_{\text{unif}}$ since the uniform field does not flow. Therefore all we have to compute is $(\partial_t \Gamma)_{22}^{(2)} \Big|_{\text{unif}}$, the flow of the effective action derived twice with respect to the field. This double functional derivative gives rise to two terms, one involving a 4-points function $\Gamma^{(4)}$ and one involving two 3-points functions $\Gamma^{(3)}$.

All the dependence on the external momentum p of the first term is held by the 4-points function. Here, because the kinetic terms in the effective action only involve two fields, every n -points function for $n \geq 3$ is momentum independent. Therefore this first term does not contribute to the anomalous dimensions. So we keep only the second term, whose contribution to $\partial_t \left(\Gamma_{2,2}^{(2)}(q, -q) \Big|_{\text{unif}} \right)$ we call $D(p)$. Noting symbolically the indices that are summed upon with dots, we have

$$D(p) = \frac{-1}{2} \hat{\partial}_t \text{tr} \left(G_{..}(\cdot) \Gamma_{.2.}^{(3)}(\cdot, \cdot + p) G_{..}(\cdot + p) \Gamma_{.2.}^{(3)}(\cdot + p, \cdot) \right) \quad (3.23)$$

where $G_{ij}(p, q) = \left(\Gamma_{ij}^{(2)}(p, q) + R(p) \delta(p + q) \delta_{ij} \right)^{-1}$ is the full propagator.

Computing explicitly the Γ_{i2j}^3 function, which, like the 4-points function, does not depend on the extrernal momentum, we can rewrite the anomalous dimension in a compact way:

$$\eta_\perp = \frac{1}{Z_\perp} \text{tr} \left(\Pi_{..}^r \Gamma_{.2.}^{(3)} \Pi_{..}^a \Gamma_{.2.}^{(3)} \right) \hat{\partial}_t \lim_{p \rightarrow 0} \frac{d}{dp_\perp^2} \int_q G_r(q) G_a(p + q) \quad (3.24)$$

This is a very general expression, holding as soon as the effective action has no kinetic terms of order ≥ 3 in the field. In particular, it also holds for the $O(n)$ model. More specifically here we have $\text{tr} \left(\Pi_{..}^r \Gamma_{.2.}^{(3)} \Pi_{..}^a \Gamma_{.2.}^{(3)} \right) = 2\rho \left(U^{(2)}(\rho) \right)^2$

Action of the derivative with respect to the momentum

Computing the derivative is a lengthy, error-prone computation. We used Mathematica to do it. The procedure is explained in appendix ??? to be written. We finally obtained for η_\perp the following expression, using an m threshold function defined in appendix A.2.3.

$$\eta_\perp = 64 \frac{v_m v_{d-m}}{d-m} \rho \left(u^{(2)}(\rho) \right)^2 m_{0,2220} \quad (3.25)$$

3.2.5 The anomalous dimension in the parallel direction, and the flow of ρ_0 .

The two other computations are very similar. In particular the flow of ρ_0 should be given by the flow of Z_\perp with the role of the \perp and \parallel directions exchanged, because of the symmetry between ρ_0

and Z_\perp . The formula for η_{\parallel} is more complicated as it involves fourth powers of the momentum.

$$d_t \rho_0 = -\theta (2 - \eta_{\parallel}) \rho_0 - 64 \frac{v_{d-m} v_m}{m} \rho \left(u^{(2)}(\rho) \right)^2 m_{1,2202} \quad (3.26)$$

$$\eta_{\parallel} = 32 v_{d-m} v_m \rho \left(u^{(2)}(\rho) \right)^2 \left[m_{1,3100} - \frac{1}{2} k_{1,2100} + \frac{1}{m} (-12 s_{1,4102} + 12 v_{1,31002} - 2 w_{1,2102}) \right] \quad (3.27)$$

$$+ \frac{1}{m(m+2)} (48 t_{1,5104} - 72 z_{1,4104} + 12 u_{1,3104} + 16 y_{1,3104} - 2 x_{1,2104}) \quad (3.28)$$

Equations 3.18, 3.25, 3.26, 3.27 tell us how the potential, ρ_0 and the anomalous dimensions are modified under the action of the renormalization group. With these equation, we can now compute the critical exponents of the Lifshitz critical point.

3.2.6 Solving the fixed point equations

To compute the critical exponents at the Lifshitz point now that we have derived the flow equations of the couplings, several approaches are possible. They all rely on the flow equation for the potential,

$$d_t u_t(\rho) = -d_m u_t(\rho) + (\theta \eta_{\parallel} + d_m - 4\theta) \rho u'_t(\rho) + 8 v_m v_{d-m} \left(l_0^{dm} (u'_t(\rho) + 2\rho u''_t(\rho)) + (n-1) l_0^{dm} (u'_t(\rho)) \right) \quad (3.29)$$

This equation is perhaps the most important of the three flow equations we have. Indeed the shape of the potential tells us were we are in the phase diagram (fig. 1.2).

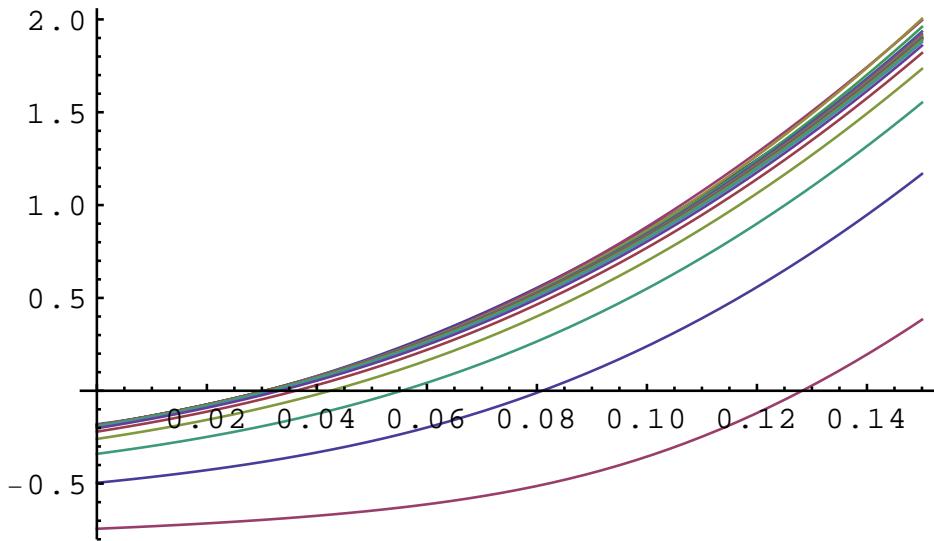


Figure 3.1 – The potential for the Ising model at different times (local potential approximation was used). The accumulation of shapes tells us that we started close from a fixed point.

An idea is to start from an initial shape for the potential, plug it in the flow equation to see how the potential evolve in time. If at first the potential is almost stationary, that means we have started close from a critical point. By trial and error, we can start closer and closer from the critical point. Once we have a sufficient precision, we can computing the critical exponents associated to this critical point. I tried this method with some success on the Ising model (fig. 3.1). Several drawbacks entail this method. First, we have to solve a partial differential equation, which take some time. Secondly and more importantly, once we have found a critical point, it is hard to know which one it is if there exist several ones (which is the case for the Lifshitz model in $d = 3$).

For all this reasons, I used another approach: solving directly the fixed point equation, which is only a differential equation. This approach has, as we are going to see, the extra advantage of letting one locate with ease a fixed point of interest.

The equation we are interested in solving is

$$0 = -d_m u(\rho) + (\theta \eta_{\parallel} + d_m - 4\theta) \rho u'(\rho) + 8v_m v_{d-m} \left(l_0^{dm} (u'(\rho) + 2\rho u''(\rho)) + (n-1) l_0^{dm} (u'(\rho)) \right) \quad (3.30)$$

Recalling that the l_0 functions are defined by an integral, we see that this is a nonlinear differential algebraic equation, whose general solution is not known. One may wonder if there exists a clever choice of regulator that let us compute the integral, thus turning the nonlinear differential algebraic equation into a simple nonlinear differential equation. For the Ising model (in the local potential approximation), such a form of regulator is indeed known⁵. For the Lifshitz model, it is not known if there exists a regulator allowing one to compute the integral explicitly. However I found a form of regulator allowing one to compute the integral approximately. Under this approximation - which we will explicit and justify later - the fixed point equation becomes

$$0 = u(\rho) - a(\theta, \eta) \rho u'(\rho) + b_1(\theta, \eta) b_2(\theta, \eta, \rho_0) \left(\frac{1}{1 + \rho_0 + u'(\rho) + 2\rho u''(\rho)} + \frac{n-1}{1 + \rho_0 + u'(\rho)} \right) \quad (3.31)$$

This is a second order non-linear differential equation that depends on three parameters: η , θ and ρ_0 .

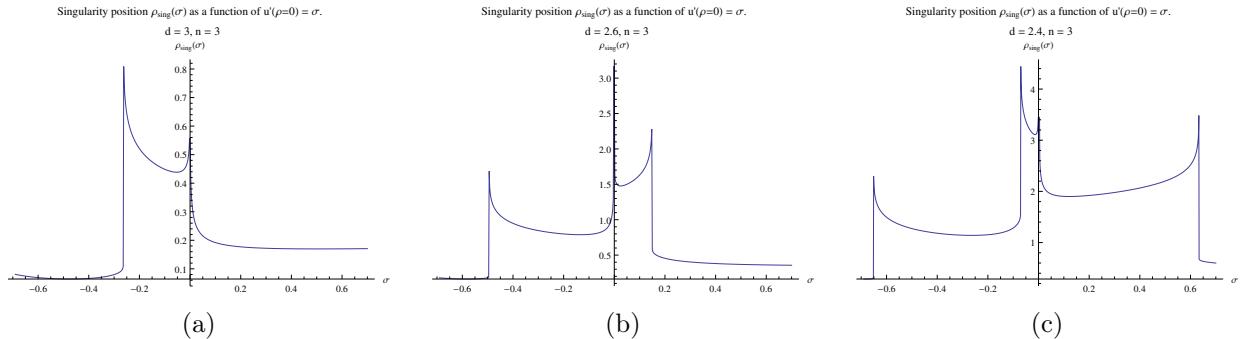


Figure 3.2 – Plot of the (numerically computed) explosion time for the $O(n)$ model in the local potential approximation. We see that in $d = 3$ only two physical solutions exist. They are the Gaussian fixed point, at $\sigma = 0$, and the Wilson-Fisher fixed point at its left. A second fixed point appears at $d = 5/7 \simeq 2.6$, and a third one at $d =$.

We let define $\sigma \stackrel{\text{def}}{=} u'(0)$. Then, assuming that $u''(\rho)$ is regular around $\rho = 0$ (which is always the case for a physical solution), we have $u(0) = nb_1b_2/(1 + \sigma)$. The knowledge of $u(0)$ and $u'(0)$ uniquely define a solution $u_\sigma(\rho)$. Now, because of the nonlinearities of the differential equation, we expect most solutions to exhibit a “finite time blow-up” behavior, *ie* we expect the u_σ or one of its derivative to diverge at some finite “time” $\rho = \rho_\sigma^d$. This is indeed what happens for all values of σ except a finite number of them. We must not forget that u is the physical potential. There are no reasons it should not be defined for all values of the field. Therefore the set of physical solutions must included in the (finite) set of solutions defined for all values of the field⁶. Hunting for critical points is thus extremely easy with this method. Before applying it to the Lifshitz model, we tried the method with the $O(n)$, whose critical points structure is well known. The results are in perfect agreement with our expectations, see fig. 3.2.

For the Lifshitz model, things are a bit more complicated as the equation has three parameters: θ , η_{\parallel} and ρ_0 . In the local potential approximation, which we have considered here, the first two parameters are taken to their mean field value: $\theta \simeq \theta_{\text{mean field}} = 1/2$, $\eta_{\parallel} \simeq \eta_{\parallel \text{ mean field}} = 0$. Only one parameter, ρ_0 , remain. We recall that this parameter is driving the Lifshitz transition (it is the horizontal axis of the phase diagram 1.2). Therefore the physics of the Lifshitz model is essentially encoded in the value of this parameter, and we surely cannot approximate it by its mean field value $\rho_0 \text{ mean field} = 0$.

We have adopted the following strategy:

5. It is the so-called θ regulator, $R_t(q) = (k^2 - q^2)\theta(1 - q^2/k^2)$

6. Actually, it turns out that every solution defined for all values of the field is a physical solution. See for example [2] (Ising model), and [3] ($O(n)$ model).

- Start at $d = d_c^>$, where mean field is exact in the sense that $\rho_0 = 0$.
- Slightly decrease the dimension: $d \rightarrow d - \delta d$.
- Find the how the Lifshitz point has been displaced and the value of ρ_0 modified by this shift in dimension.
- Go back to step two.

We follow this algorithm until we reach $d = 3$, at which point our results concerning critical exponents can be compared with the one obtained from other works.

3.2.7 Fixed point potential at the Lifshitz critical point

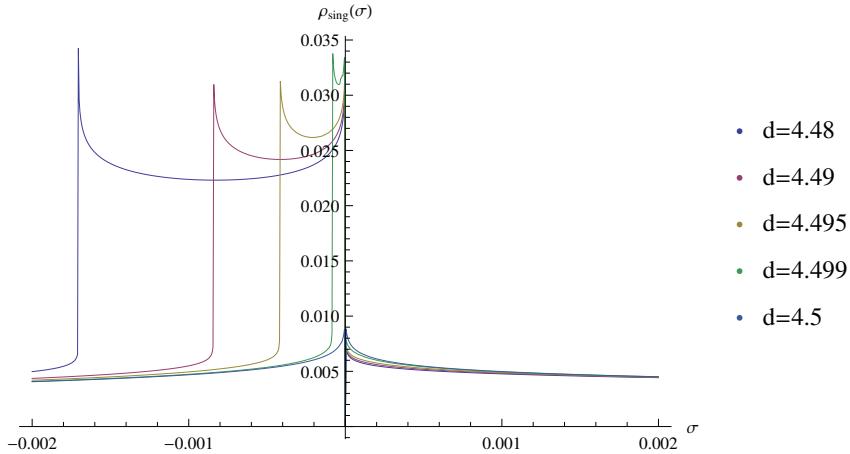


Figure 3.3 – Emergence of the Lifshitz fixed point (at the left of the $\sigma = 0$ Gaussian fixed point), when d is slightly lowered, starting from $d_c^>$. Here we have chosen $m = 1$ (and thus $d_c^> = 4.5$) and $n = 3$.

As we have seen, the fixed point equation of the Lifshitz model depend on an extra parameter ρ_0 , contrary to the fixed point equation of the $O(n)$ model. Because this parameter varies locally, a global picture of the explosion time such as the one shown in fig. 3.2 does not exist for the Lifshitz model. However close to the upper critical dimension, it is possible to make a local picture of the emergence of the Lifshitz fixed point (fig. 3.3).

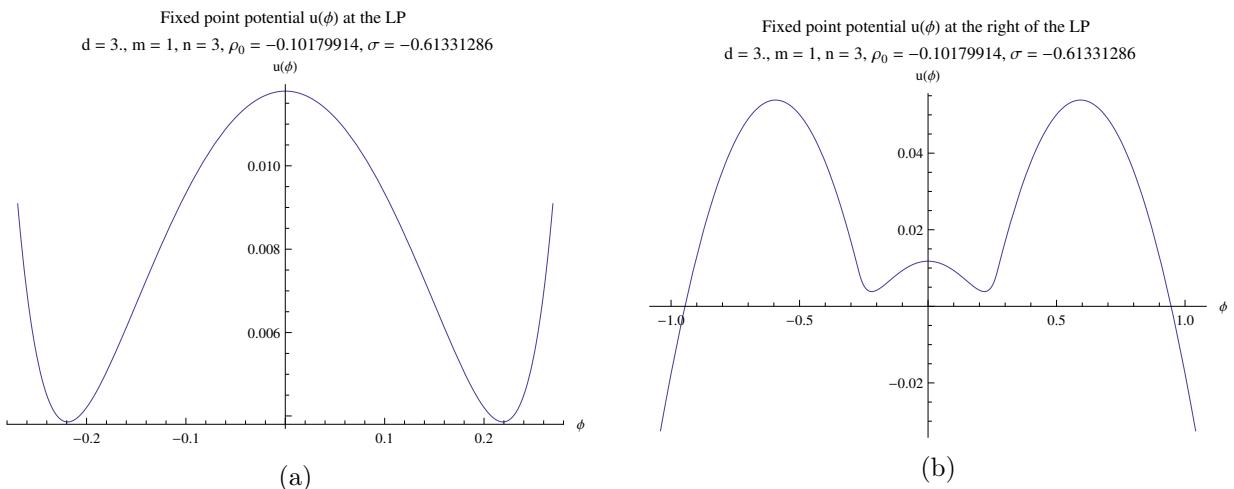


Figure 3.4 – Shape of the potential at the Lifshitz point (left image) and and close to it (right image). ϕ is the module of the field (here tridimensional): $\phi \stackrel{\text{def}}{=} \sqrt{\phi_i \phi_i}$.

Figure 3.4a shows the shape of the dimensionless potential at the Lifshitz point. It has the standard “Mexican-hat” structure. Note that for an initial condition σ very close to the Lifshitz point initial condition, the shape of the potential is completely different (fig. 3.4b). In particular it has the

unphysical properties of going to $-\infty$ in the large field limit, meaning that it is not a confining potential. This high sensitivity to the variations of σ in the neighborhood of the Lifshitz (in fact any) fixed point allowed us to fine-tune the value of σ_{Lifshitz} up to 12 decimal places very easily. This fine-tuning is essential in order to obtain accurate critical exponents.

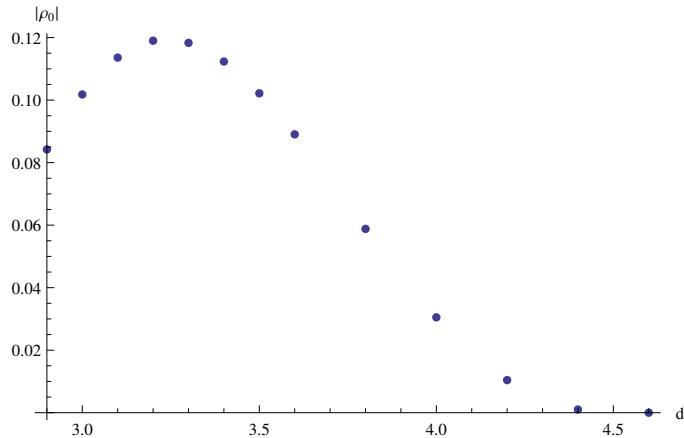


Figure 3.5 – The parameter ρ_0 as a function of the space dimension d .

To compute ρ_0 , we make use of the its flow equation (eq. 3.26). At a fixed point, it simply is an algebraic equation on ρ_0 , depending parametrically on the fixed point potential, which depends parametrically on ρ_0 . This means that formally we can write the problem as the self-consistent equation $\rho = f(\rho, u(\rho))$. To solve the problem we simply compute iteratively the sequence $\rho^{(n+1)} = f(\rho^{(n)}, u(\rho^{(n)}))$ until we reach a fixed point. By starting close to $d_c^>$ and progressively going down in dimension we thus obtain ρ_0 at the Lifshitz critical point as a function on d . The result is shown in fig. 3.5.

It is interesting to note that ρ_0 is not monotonically increasing in module as we go down in dimension: it reaches a maximum at $d \simeq 3.2$ and goes down again. This non-monotony could be a precursor at the local potential level (*ie* with the approximation that $\eta_{\parallel} = \eta_{\perp} = 0$) of the non-monotony of the anomalous dimensions. Indeed, as was pointed out in [4], not only are the anomalous dimensions of the Lifshitz model non-monotonous, but even more interestingly, the anomalous dimension η_{\parallel} vanishes below the upper critical dimension. This very peculiar behavior seemingly inherent to the Lifshitz model is yet to be understood on physical grounds.

Before we go on with the computation of the critical exponents, we will say a word on the numerical methods we used. For solving the nonlinear differential equation on the potential (eq. 3.31), a simple Runge-Kutta method did not prove sufficiently accurate, because of the stiffness of the equation. Two methods were found well adapted to this stiff equation: Burlisch-Stoer's extrapolation scheme and the backward differentiation formula method. The latter proved the most accurate in our case. We therefore ended up using an order 5 BDF method to solve our fixed point equation.⁷ To solve the algebraic equation on ρ_0 we simply used Newton's method.

3.2.8 Critical exponents

We recall that close to criticality the correlation length behaves as $\xi \propto |T - T_c|^{-\nu}$. In the case of the Lifshitz model, the existence of two nonequivalent directions implies the existence of two correlation lengths, and of two critical exponents traditionally denoted ν_4 and ν_2 , corresponding to correlation lengths in the \parallel and \perp directions respectively.

If we had numerically computed the fixed point potential $u(\rho)$ with infinite accuracy, then, when plugging it as an initial condition of the flow equation for the potential (eq. 3.18), it should not evolve at all. That is to say, we should have $u_t(\rho) = u(\rho)$ for all t . Of course, we always have a

7. The BDF implicitly express $u(\rho + \Delta\rho)$ in terms of $u(\rho)$, $u(\rho - \Delta\rho)$, $u(\rho - 2\Delta\rho)$... At each “time step” $\Delta\rho$, this implicit equation must be solved to get $\rho + \Delta\rho$. We used Newton's method for that purpose.

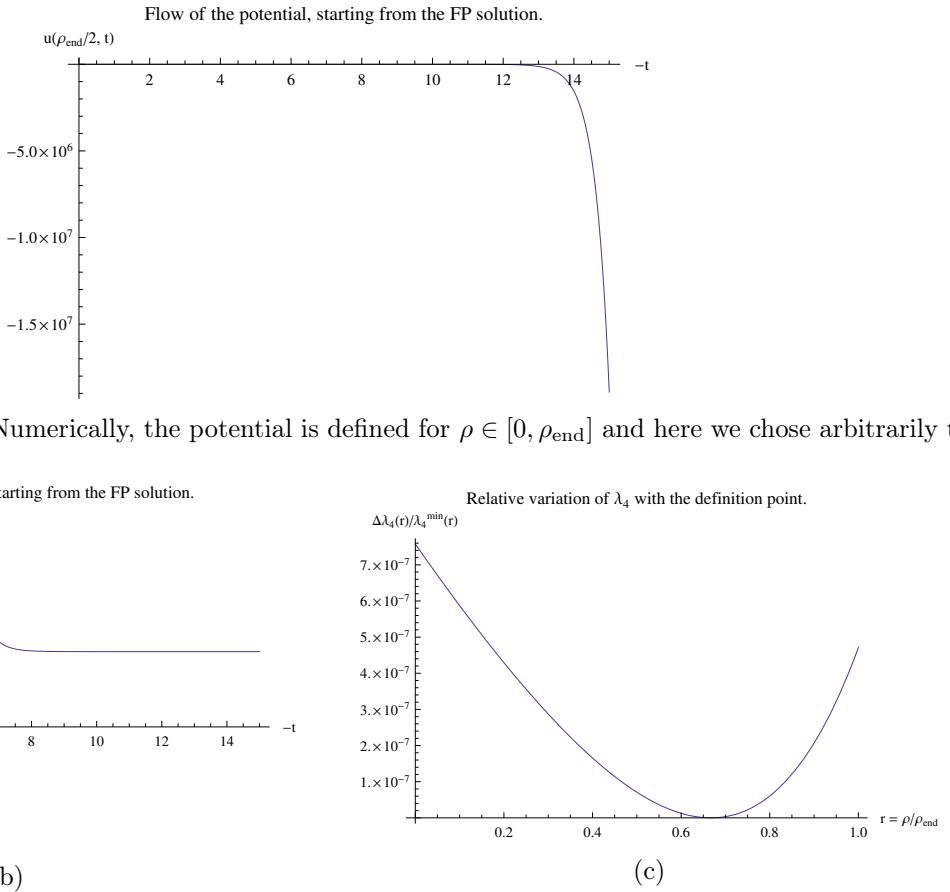


Figure 3.6

finite accuracy on the fixed point potential, and when plugging it as an initial condition of the flow equation, the potential evolve in time, slowly going away from the critical surface. In the Lifshitz case two “directions” in the space of coupling constants are relevant ; one corresponding to ν_4 and one corresponding to ν_2 . We thus have,

$$u_t(\rho) = u(\rho) + \delta u_4(\rho) e^{\lambda_4 t} + \delta u_2(\rho) e^{\lambda_2 t} \quad (3.32)$$

at the dominant order. As we have seen, $\nu_4 = 1/\lambda_4$ and $\nu_2 = 1/\lambda_2$. Since $\nu_4 = \theta\nu_2 \leq \nu_2$, we have

$$u_t(\rho) \simeq u(\rho) + \delta u_4(\rho) e^{\lambda_4 t} \quad (3.33)$$

in the long time limit. That is to say, we should have $\partial_t \log u_t(\rho) \simeq \lambda_4$ when t is sufficiently large. This is exactly what we obtain (fig. 3.6b). This is how we measured λ_4 and thus ν_4 . We found

$$\nu_4 = 0.3999 \quad (3.34)$$

$$\nu_2 = 0.7999 \quad (3.35)$$

We may wonder if these results depend on the point ρ at which we compute $\partial_t \log u_t(\rho)$. We computed the relative variation of λ_4 with respect to the definition point (fig. 3.6c), to see that these variations only affect the sixth decimal place in the exponent. They are therefore completely negligible compared to other sources of errors.

The values of the critical exponents we found are in very good agreement with the perturbative expansion at order ϵ^2 (see for example [4]). They also agree well with results obtained with another non-perturbative approach using the so-called Wilson-Polchinski equation (see [1]).

So, apparently, the study conducted during this internship was only successful in finding the same results with a new method! However, contrary to the Wilson-Polchinski approach, ours can be very easily extended to take into account the anomalous dimensions. In fact, computing the anomalous

dimensions is as simple as computing ρ_0 . All we have to do is use the fixed point (algebraic) equations on η_{\parallel} and η_{\perp} we have already derived, to find the values of the anomalous dimensions, in a fashion completely similar to what have been done concerning ρ_0 . This has not been done during the internship simply because of a lack of time.

Taking the anomalous dimensions into account completely modifies the values of the critical exponents (cf. [4]). So, they seem to play a crucial role in the physics of the Lifshitz model. In that respect, it is interesting to note that continuation of the work began during this internship should provide us with values of the anomalous dimensions, computed with unpreceding precision.⁸

Conclusion

Before starting this internship, I thought of the renormalization group as a conceptually important object, yet nearly impossible to implement in practice except in a handful of cases. I was thus as surprised as pleased to discover in Wetterich equation a powerful and very systematic way of implementing the ideas of the renormalisation group. From a more practical point of view, I enjoyed doing both analytical computations -mainly to derive the flow equations of the model, as well as numerical simulations.

Though the work I did during these eight weeks cannot be considered as an exceedingly original contribution, it was satisfying for me to work -with some modest success, on a new approach to the Lifshitz model. Indeed, during the internship we were able to retrieve the already known critical exponents of the model, with satisfying precision. The approach we opted for: namely, solving directly the fixed point equation, has not been much used yet. The easiness with which we were able to solve the fixed point equation is a good omen: this approach could very well be fruitful when applied to other models. In particular, it should be powerful to probe complicated phase diagrams in search of fixed point.

Concerning the Lifshitz model, there is of course still many things to do.

First of all, we could get rid of the approximation we made, simply by using at the fixed point equation on the derivative of the potential, instead of solving the equation on the potential. Indeed, the equation on the derivative is an *explicit* differential equation, while the equation on the potential is in general *implicit*. Unfortunately, this idea occurred to us only at the end of the internship, and while we had enough time to test it successfully on the $O(n)$ model, we lacked time to implement it in the case of the Lifshitz model.

Secondly, including the anomalous dimensions in the game would probably be very interesting, since as we have said, they likely play an important yet poorly understood role in the Lifshitz model.

⁸. The anomalous dimensions have already been computed within the nonperturbative framework, at order 12 in the field. Our method should give us more precise results as it takes into account all orders in the field.

Appendix A

Derivation of the Wetterich equation

We want to arrive at the formula,

$$\partial_k \Gamma_k = \frac{1}{2} \int_{x,y} R_k(x,y) \left(\Gamma_k^{(2)}(x,y) + R_k(x,y) \right)^{-1} \quad (\text{A.1})$$

A.1 Some useful relations

We recall that if W_k is a function of h (the external field), then we can define its Legendre transform Γ_k^{Leg} , a function of ϕ (the background field), by

$$\Gamma_k^{\text{Leg}}[\phi] = h \cdot \phi - W_k[h] \quad (\text{A.2})$$

or equivalently,

$$\phi = \frac{\delta W_k[h]}{\delta h} \quad (\text{A.3})$$

$$h = \frac{\delta \Gamma_k^{\text{Leg}}[\phi]}{\delta \phi} \quad (\text{A.4})$$

We see that we can *either* consider ϕ as being a function of h , *or* the reverse. That is,

$$\frac{\delta \phi}{\delta h} = \left(\frac{\delta h}{\delta \phi} \right)^{-1} \quad (\text{A.5})$$

Using (A.3) and (A.4), we deduce that

$$\Gamma_k^{\text{Leg}}(x,y) = W_k^{(2)}(x,y)^{-1} \quad (\text{A.6})$$

But we also know that the free energy W_k is the generating function of the connected n points correlation functions. In particular, we have thus

$$\Gamma_k^{\text{Leg}}(x,y) = G_c^2(x,y)^{-1} \quad (\text{A.7})$$

This result will be useful later.

We also recall that we defined the effective action at scale k as

$$\Gamma_k[\phi] = \Gamma_k^{\text{Leg}}[\phi] - \Delta H_k[\phi] \quad (\text{A.8})$$

with

$$\Delta H_k[\phi] = \frac{1}{2} \phi \cdot R_k \cdot \phi \quad (\text{A.9})$$

A.2 The derivation

On one hand we have

$$\partial_k e^{W_k[h]} = e^{W_k[h]} \partial_k W_k[h] = Z_k[\varphi] W_k[h] \quad (\text{A.10})$$

while, on the other hand we also have

$$\partial_k e^{W_k[h]} = \partial_k Z_k[\varphi] = -\frac{1}{2} \int_{x,y} \partial_k R_k(x,y) \underbrace{\int \mathcal{D}\varphi \varphi(x)\varphi(y) e^{-H_k[\varphi]-\Delta H_k[\varphi]+h\cdot\varphi}}_{=Z_k[\varphi]\langle\varphi(x)\varphi(y)\rangle_k} \quad (\text{A.11})$$

putting these two expressions together we deduce that

$$\partial_k W_k[h] = -\frac{1}{2} \int_{x,y} \partial_k R_k(x,y) \langle\varphi(x)\varphi(y)\rangle_k \quad (\text{A.12})$$

We already almost have the Wetterich equation! Actually all that remains to be done is reexpressing the left and right hand sides of the previous expression in terms of the scale-dependant effective action Γ_k .

A.2.1 The left hand side

We start from $W_k = h \cdot \phi - \Gamma_k^{\text{Leg}}$, and we deduce that

$$\partial_k W_k = h \cdot \partial_k \phi - \partial_k \Gamma_k^{\text{Leg}} \quad (\text{A.13})$$

Moreover

$$\partial_k \Gamma_k^{\text{Leg}}[\phi] = \partial_k \left(\Gamma_k[\phi] + \frac{1}{2} \phi \cdot R_k \cdot \phi \right) \quad (\text{A.14})$$

Now, we must be careful not to forget that $\Gamma_k[\phi]$ depends explicitly on k , but also implicitly through ϕ ! We have

$$\partial_k = \partial_{k, \phi \text{ fixed}} + \partial_k \phi \cdot \frac{\delta}{\delta \phi} \quad (\text{A.15})$$

so that

$$\partial_k \Gamma_k[\phi] = \partial_{k, \phi \text{ fixed}} \Gamma_k[\phi] + \partial_k \phi \cdot \frac{\delta \Gamma_k[\phi]}{\delta \phi} \quad (\text{A.16})$$

Dropping the “ ϕ fixed” labels, we conclude that

$$\partial_k \Gamma_k^{\text{Leg}}[\phi] = \partial_k \Gamma_k[\phi] + \partial_k \phi \cdot h + \frac{1}{2} \phi \cdot \partial_k R_k \cdot \phi \quad (\text{A.17})$$

and therefore that

$$\partial_k W_k[\phi] = -\partial_k \Gamma_k[\phi] - \frac{1}{2} \phi \cdot \partial_k R_k \cdot \phi \quad (\text{A.18})$$

A.2.2 The right hand side

To transform the right hand side, we simply note that the connected two points correlation function is

$$G_{c,k}^2(x,y) = \langle \varphi(x)\varphi(y) \rangle_k - \langle \varphi(x) \rangle_k \langle \varphi(y) \rangle_k \quad (\text{A.19})$$

so that

$$\partial_k W_k[\phi] = -\frac{1}{2} \int_{x,y} \partial_k R_k(x,y) \langle \varphi(x)\varphi(y) \rangle_k = -\frac{1}{2} \int_{x,y} \partial_k R_k(x,y) G_{c,k}^2(x,y) - \frac{1}{2} R_k \cdot \phi \cdot \phi \quad (\text{A.20})$$

A.2.3 Conclusion

Putting equations A.18 and A.20 together, we finally have

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_k R_k(x, y) G_{c,k}^2(x, y) \quad (\text{A.21})$$

Remembering eq. A.7, we conclude that

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_k R_k(x, y) \left(\Gamma_k^{(2)}(x, y) + R_k(x, y) \right)^{-1} \quad (\text{A.22})$$

This is the Wetterich equation for a scalar field ϕ .

Passing to the Wetterich equation for a vector field ϕ_i is very simple, as soon as we realize that all we have to do is to perform the following replacements:

$$\phi \rightarrow \phi_i \quad (\text{A.23})$$

$$\Gamma_k^{(2)}(x, y) \rightarrow \Gamma_k^{(2)}(x, y)_{i,j} \quad (\text{A.24})$$

$$R_k(x, y) \rightarrow R_k(x, y)_{i,j} \quad (\text{A.25})$$

The Wetterich equation is thus transformed into

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_k R_k(x, y)_{i,j} \left(\Gamma_k^{(2)}(x, y) + R_k(x, y) \right)_{i,j}^{-1} \quad (\text{A.26})$$

Appendix B

Threshold functions

B.0.4 The l function

We define

$$l_n^{(d,m)}(\omega, \bar{\rho}_0) = -\frac{n + \delta_{n,0}}{4} \int_0^\infty dq_{\parallel} \int_0^\infty dq_{\perp} y_{\parallel}^{m/2+1} y_{\perp}^{(d-m)/2-1} \frac{\theta \eta_{\parallel} r + 2\theta y_{\parallel} r^{(1,0)} + 2y_{\perp} r^{(0,1)}}{\left[y_{\parallel}^2 (1+r) + y_{\perp} + \bar{\rho}_0 y_{\parallel} + \omega \right]^{n+1}} \quad (\text{B.1})$$

For $n > 0$, we see that we have

$$-\hat{\partial}_t \int \frac{dq_{\parallel}^m dq_{\perp}^{d-m}}{(2\pi)^d} \frac{1}{\left[Z_{\parallel} q_{\parallel}^4 + Z_{\perp} q_{\perp}^2 + \rho_0 q_{\parallel}^2 + R + m^2 \right]^n} = 16v_m v_{d-m} Z_{\parallel}^{-n} k^{d_m - 4\theta n} l_n^d(\bar{m}^2) \quad (\text{B.2})$$

and for $n = 0$

$$\hat{\partial}_t \int \frac{dq_{\parallel}^m dq_{\perp}^{d-m}}{(2\pi)^d} \log \left(Z_{\parallel} q_{\parallel}^4 + Z_{\perp} q_{\perp}^2 + \rho_0 q_{\parallel}^2 + R + m^2 \right) = 16v_m v_{d-m} k^{d_m} l_0^d(\bar{m}^2) \quad (\text{B.3})$$

This allows us to easily identify our expression for the flow of the potential with the l_0 threshold function.

B.0.5 The m function

The m threshold function was defined as

$$Z_{\parallel}^{1-a-b+\alpha/2} Z_{\perp}^{-\alpha/2} k^{d_m + 4\theta(1-a-b+\alpha/2+\beta/4)} m_{1,ab\alpha\beta} = -\frac{1}{16v_m v_{d-m}} \hat{\partial}_t \int_q q_{\perp}^{\alpha} q_{\parallel}^{\beta} \left(\frac{d}{dq_{\parallel}^2} P \right)^2 G_a(q)^a G_r(q)^b \quad (\text{B.4})$$

where P is the inverse of the full propagator.

The other functions appearing in the expression of η_{\parallel} are defined in a similar fashion. We do not explicitly give their definition here, as we do not need them.

Appendix C

The approximated regulator

C.1 Why do we need an approximation?

The problem is to compute the double integrals appearing in the threshold functions. In the case of the $O(n)$ model threshold functions are simple integrals, and the choice of the so-called θ regulator:

$$R_k(q) = (k^2 - q^2)\theta\left(1 - \frac{q^2}{k^2}\right) \quad (\text{C.1})$$

makes it possible to compute the integral explicitly. In the case of the Lifshitz model, we did not find a form of the regulator that made it possible to compute the double integral explicitly. However, we used the following multiplicative regulator based on the θ regulator:

$$R_t(q_{\parallel}^2, q_{\perp}^2) = R_{\parallel}(q_{\parallel}^2)r(q_{\perp}^2) \quad (\text{C.2})$$

with

$$r(q_{\perp}^2) = \left(1 - \frac{q_{\perp}^2}{k^2}\right)\theta\left(1 - \frac{q_{\perp}^2}{k^2}\right) \quad (\text{C.3})$$

All the integrals we will encounter are of the type

$$I_n(w) = v_m v_{d-m} \int_0^\infty dq_{\parallel} q_{\parallel}^{m-1} \int_0^\infty dq_{\perp} q_{\perp}^{(d-m)-1} \frac{\partial_t R_t(q_{\parallel}, q_{\perp})}{\left[a(q_{\parallel}^2) + Z_{\perp} q_{\perp}^2 + R_t + w\right]^n} \quad (\text{C.4})$$

This gives rise to integrals in the \perp direction which are all of the type

$$J_{nD} = \int_0^1 dq_{\perp} \frac{q_{\perp}^D}{\left[a(q_{\parallel}^2) + Z_{\perp} q_{\perp}^2 + R_{\parallel}(q_{\parallel}^2) \left(1 - \frac{q_{\perp}^2}{k^2}\right)\right]^n} \quad (\text{C.5})$$

The good thing is that these integrals can be expressed “explicitly” in terms of an hypergeometric function

$$J_{nD} = \frac{k^{D+1} {}_2F_1\left((1+D)/2, n; (3+D)/2; -\frac{b}{a}\right)}{(D+1)a^n} \quad (\text{C.6})$$

with

$$a = a(q_{\parallel}^2) + R_{\parallel} + w \quad (\text{C.7})$$

$$b = Z_{\perp} k^2 - R_{\parallel}(q_{\parallel}^2) \quad (\text{C.8})$$

The ratio a/b actually only depends on the dimensionless quantities $\bar{\rho}_0$ and y_{\parallel} . Dropping the bar on the dimensionless ρ_0 , we define the function of dimensionless variables

$$h(D, n, y_{\parallel}, \rho_0) \stackrel{\text{def}}{=} {}_2F_1\left((1+D)/2, n; (3+D)/2; -\frac{b}{a}\right) \quad (\text{C.9})$$

We could compute the integral in the \perp direction, but we still need to compute the integral in the \parallel direction. Sadly, the function h depends on y_{\parallel} , which makes it impossible to compute analytically the integral. At this point we had the choice either to compute the integral numerically at the moment we solve the flow equations, or to try to find a good approximation that let us “almost” compute the integral analytically. We chose the second option because it facilitates the numerical integration of the flow equations, and of course also because the approximation we found is acceptable!

C.2 The approximation

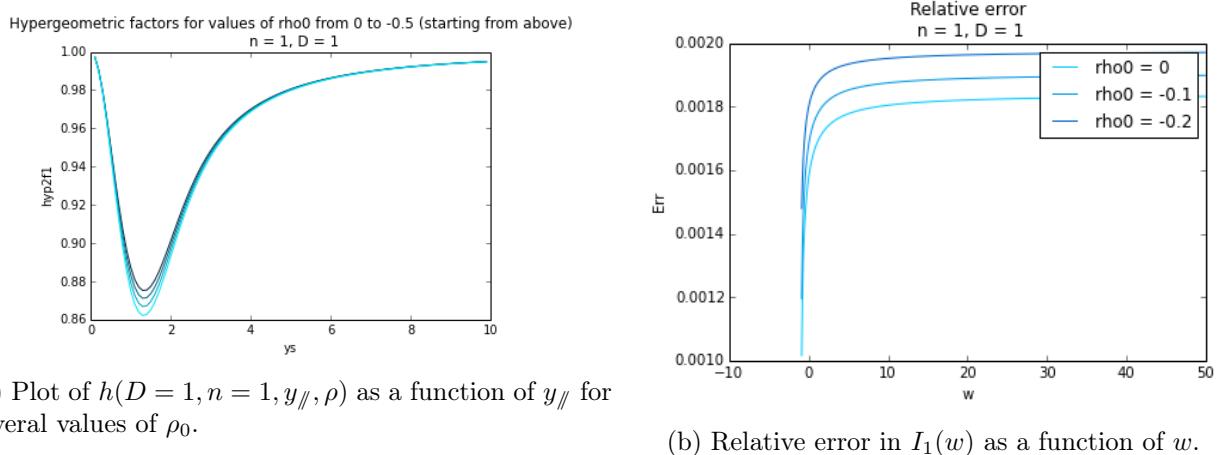


Figure C.1

We made the approximation $h \simeq 1$, which at first sight can seem rather brutal. As we are going to see, it can be quite well justified!

The approximation consist in replacing the curves appearing in fig. C.1a by the straight line $h = 1$. The greater D and n are, the deeper the “well” we see in the figure is deep. Therefore the worst case for doing the approximation is $n = 1$ and $D = 1$. In this case, we have plotted the function for several values of ρ_0 , to make it clear that the greater ρ_0 is in module, the worse the approximation will be. From numerical computations (see the last section of the report for the details) we have seen that $\rho_0 \in [-0.2, 0]$. Therefore $\rho_0 = -0.2$ is the worst case for us.

To see how good our approximation is, we have computed numerically the integral $I_{n=1}$, with and without the approximation. The graph of the relative error as a function of the argument w of the threshold function is shown in fig. C.1b. In all our computations, w is either $u'(\rho)$ (massless modes) or $u'(\rho) + 2\rho u''(\rho)$ (massive mode). It can be shown, invoking the convexity of the Γ_t^{Leg} function, that these arguments can never be less than -1 . This is why we focused on $w \in [-1, \infty]$.

We see that in the worst case the error is of approximately 0.2 %, which is good, given the precision we seek on the critical exponents. With this approximation, all the computations can be done analytically, leading to flow equations we very easily solved numerically.

Appendix D

Some Mathematica scripts

In this appendix, we give as examples of the work that was done during the internship, some “representative” excerpts from our Mathematica scripts.

D.1 Computation of the anomalous dimensions

We showed in section 3.2.4 that

$$\eta_\perp = \frac{1}{Z_\perp} \text{tr} \left(\Pi^r \Gamma^{(3)}_{2\cdot} \Pi^a \Gamma^{(3)}_{2\cdot} \right) \hat{\partial}_t \lim_{p \rightarrow 0} \frac{d}{dp_\perp^2} \int_q G_r(q) G_a(p+q) \quad (\text{D.1})$$

We still need to let the derivative with respect to p^2 act on the massless propagator $G_a(p+q)$, and then to take the limit $p \rightarrow 0$. This is extremely cumbersome to do by hand, so we used Mathematica to do the expansion.

We start by defining our own scalar product, *scal*. It is defined to be symmetric in its two arguments, and linear. These are the only properties of a scalar product we are going to need.

```
SetAttributes [scal , Orderless];
scal[a_ b_, c_] /; NumberQ[a] := a scal[b, c];
scal[a_ + b_, c_] := scal[a, c] + scal[b, c];
scal[0, a_] := 0;
```

The scalar products appears inside an integral over an internal momentum q . Since the integrand is invariant by rotation, the scalar products can be simplified. We designed replacement rules to take into account this simplifications:

```
integr1 = {scal[ps, qs]^2 -> 1/m scal[ps, ps] scal[qs, qs],
           scal[ps, qs]^4 -> 3/(m (m + 2)) (scal[ps, ps] scal[qs, qs])^2};
integr2 = {scal[ps, qs] -> 0};
```

Then we give to Mathematica the expression of the massless propagator

```
G = (Zs scal[ps + qs, ps + qs]^2 + Zp scal[qp, qp] + rho0 scal[ps + qs, ps + qs]
     + R[scal[ps + qs, ps + qs], scal[qp, qp]] + U')^-1;
```

We are now ready to expand the propagators in powers of the scalar product, to take the derivative with respect to p^2 and take the limit of vanishing external momentum:

```
Series [Series [G, {scal[ps, ps], 0, 2}] // Normal,
       {scal[ps, qs], 0, 4}] // Normal;
% /. integr1;
% /. integr2;
(1/2)*D[% , {scal[ps, ps], 2}];
% /. ps -> 0;
```

After that, we obtain an expression that can still be simplified a little bit by integration by parts, to give the final form of eq. 3.25.

D.2 Integration of the fixed point equations

As a second example, we give the loop used to produce graphs of the “explosion time” of the potential as a function of the initial condition on the fixed point equation. The integration is stopped when the ”*Event*” is triggered. Here it is when denominator of the massive propagator explodes; as this was found to be the condition met at the explosion time.

```
Do[
  xsing = 0;
  NDSolve[{0 == equ[x, rho0, 1/2, 0],
            u[0] == (n b1[1/2, 0] b2[1/2, 0, rho0])/(1 + sig),
            u'[0] == sig}, u, {x, 0, xfin},
            Method -> {"EventLocator",
              "Event" -> (-1 + n)/(1 + rho0 + u'[x]) +
                (-u[x] + x a[1/2, 0] u'[x])/(b1[1/2, 0] b2[1/2, 0, rho0]) },
              "EventAction" :> {Throw[xsing = x, "StopIntegration"]},
              Method -> "BDF"}, AccuracyGoal -> 30];
  AppendTo[xs, {sig, xsing}];
  , {sig, sigList}]
```

This produces the graph below, showing the neighborhood of the Lifshitz point in $d = 3$.

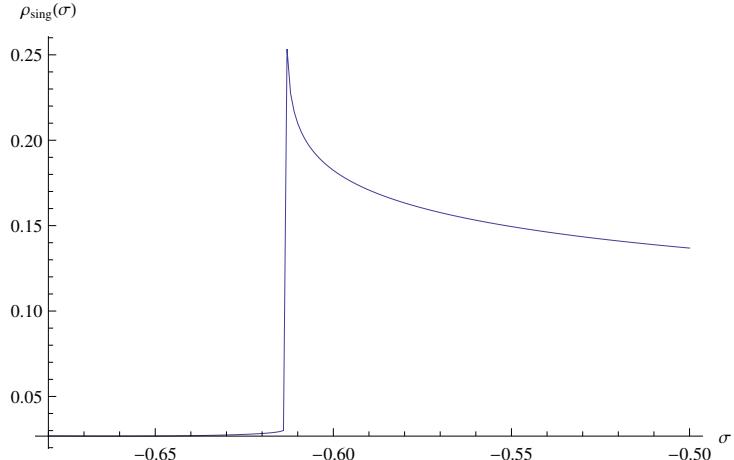


Figure D.1 – The explosion time ρ_{sing} as a function of the initial condition $u'(\rho = 0) = \sigma$, in the neighborhood of the Lifshitz point, with $d = 3$, $m = 1$.

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