# Gap structure of 1D cut and project Hamiltonians

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special thanks to Jean-Noël Fuchs

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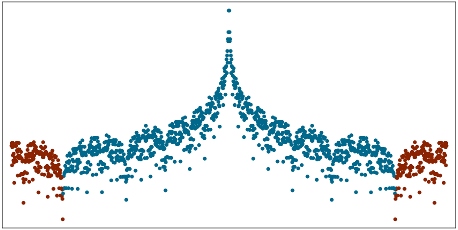
September 20, 2016



# Outline

# Electrons on quasiperiodic chains

Canonical cut and project method of slope $\alpha \to$ chain of two letters:

$$\ldots \text{ABAABABAABAABABAABABA} \ldots$$

Quantum model:

$$\cdots \bullet \overset{t_A}{—} \bullet \overset{t_B}{=} \bullet \overset{t_A}{—} \bullet \overset{t_A}{—} \bullet \overset{t_B}{=} \bullet \overset{t_A}{—} \bullet \overset{t_B}{=} \bullet \overset{t_A}{—} \bullet \cdots$$

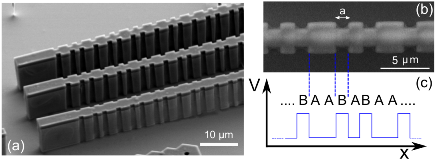
Hamiltonian: $H(\alpha) = \sum_{x \in \mathbb{Z}} t_{x,x+1} \left|x\right\rangle \left\langle x+1\right| + \text{h.c.}$

where $t_{x,x+1} = t_A$ or $t_B$ (see [Kohmoto 86])

$t_A/t_B$ is the only parameter of the model.

Experimental realization with cavity polaritons

[Tanese *et al* 2015]




$$\alpha = \frac{m}{n} \in \mathbb{Q} \implies \text{periodic chain of period } L = m + n$$

$$\alpha \in \mathbb{R} \setminus \mathbb{Q} \implies \text{quasiperiodic chain}$$

# The energy spectrum

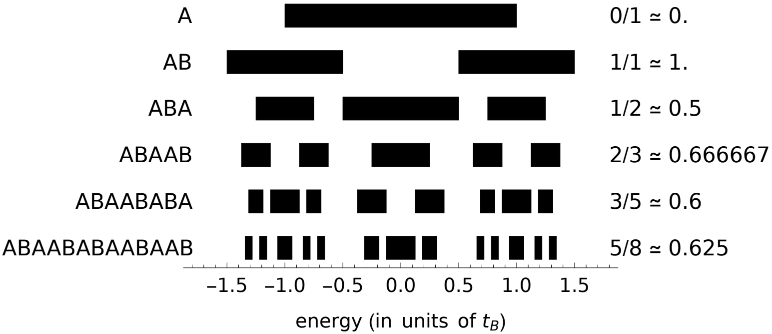
Well understood: electrons on *periodic* chains (Bloch's theory)

Idea: approach a QP chain $\alpha$ by a sequence of periodic *approximants*:

$$\alpha_l = \frac{m_l}{n_l} \xrightarrow[l \to \infty]{} \alpha$$

$\rightarrow$ energy spectrum consists of $m_l + n_l$ *energy bands*

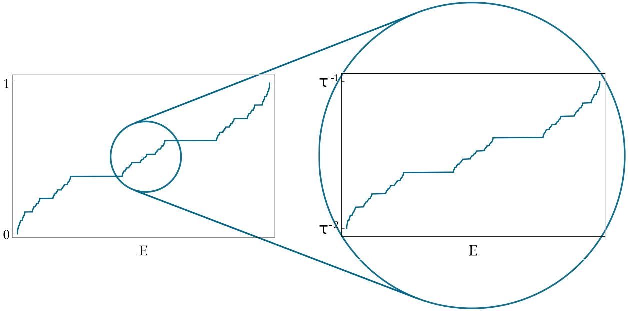
Fibonacci chain: $\alpha_l = F_l / F_{l+1} \xrightarrow[l \to \infty]{} \tau^{-1} = 0.618\ldots$



## IDOS AND GAP LABELING

A convenient way to plot the spectrum: the integrated density of states (idos).

idos$(E)$ = fraction of states below energy $E$




idos of the Fibonacci Hamiltonian (devil's staircase)

- Electronic spectrum of quasiperiodic chains is hard to describe
- Rather: set of idos values in the gaps → **gap labeling theorem**

$$\text{idos}(E \in \text{gap}) = p + q\tau^{-2}$$

# The gap labeling theorem

The IDOS inside spectral gaps can be written

$$\mathrm{idos}(E \in \mathrm{gap}) = p + \frac{q}{1+\alpha}$$
$$\mathrm{idos}(E \in \mathrm{gap}) = \frac{q}{1+\alpha} \mod 1$$

where $p, q \in \mathbb{Z}$ are the *gap labels* see [Bellisard 89].

- The set of labels constrains the spectrum... but is not enough to reconstruct it
- The labels are model independent!
  - In particular, independent of $t_A$ and $t_B$
  - Gap labels are topological invariants

# The gap labeling theorem

$$\mathrm{idos}(E \in \mathrm{gap}) = \frac{q}{1+\alpha} \mod 1$$

- Has the gap label $q$ a physical interpretation?
- Can the theorem be applied to approximants?
- Does it help understanding the quasiperiodic limit?

# Gap labeling from Bloch's theory

Let $\alpha_l = \frac{m_l}{n_l} \to \alpha$ be a sequence of approximants.
Bloch's theorem: there are $L_l = m_l + n_l$ energy bands.

$$\mathrm{idos}(E \in \mathrm{gap}) = \frac{j(E)}{L_l}$$

We can find integers $p$, $q$ such that $j = pL_l + qn_l$.

$$\mathrm{idos}(E \in \mathrm{gap}) = \frac{qn_l}{m_l + n_l} \mod 1$$

$$\mathrm{idos}(E \in \mathrm{gap}) = \frac{q}{1 + \alpha_l} \mod 1$$

Letting $l \to \infty$,

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Letting $l \to \infty$,

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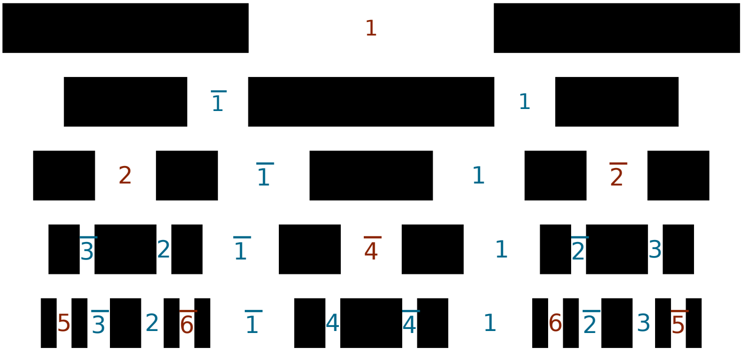
**Problem:**
At fixed $j$, $q$ depends on $l$.

# Transient and stable gaps

Gaps of successive approximants of the Fibonacci chain.

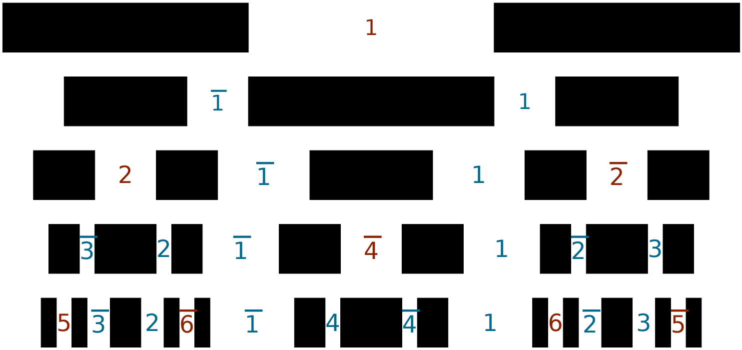



$\langle E \rangle_l$: mean energy of a gap, $\Delta_l(\langle E \rangle)$: its width.
Identify two gaps if they overlap:

$$0.5\Delta_l(\langle E \rangle) > |\langle E \rangle_l - \langle E' \rangle_{l+1}|$$

# Transient and stable gaps

Gaps of successive approximants of the Fibonacci chain.



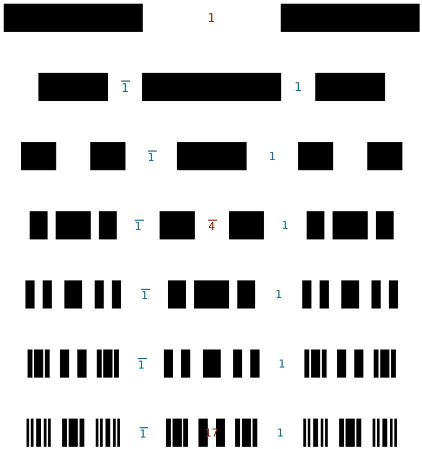

- Stable gaps have a fixed label, that does not depend on $l$.
- Transient gaps have a label that is $l$-dependent.

# Examples of transient and stable gaps

- The $E = 0$ gap
  - is transient
  - has the label

$$q_l = \left\lfloor \frac{(2+\sqrt{5})^{l/3}}{2\sqrt{5}} + \frac{1}{2} \right\rfloor$$

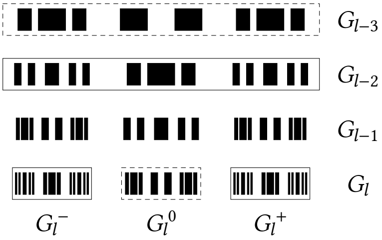
  - has vanishing width in the quasiperiodic limit.
    **True for all transient gaps**
- The two largest gaps
  - are stable
  - have label $q = \pm 1$
  - have a nonzero width in the quasiperiodic limit
    **True for all stable gaps**



# Recursive gap labeling

Let $G_l$ be the set of gap labels:
$G_l = \{(p, q) | \text{ids} = p + q/(1+\alpha)\}$
$G_l$ obeys the recursive relation:

$$
\begin{aligned}
G_l^- &= M^{-2} G_{l-2} \\
G_l^0 &= M^{-3} G_{l-3} + (1, -1) \\
G_l^+ &= M^{-2} G_{l-2} + (0, 1)
\end{aligned}
$$




Recursive construction of the spectrum [Niu, Nori 86]

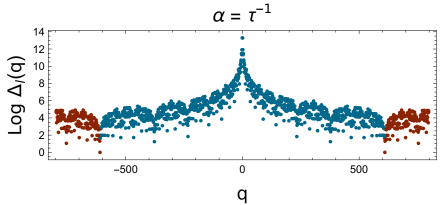
$M$ is the *inflation matrix*:

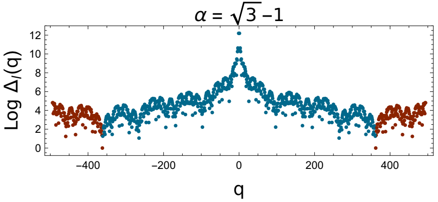
$$
M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \to M \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} AB \\ A \end{pmatrix}
$$

- Geometrical interpretation of the Fibonacci gap labeling
- Stable and transient gap are completely characterized:
  - Stable gaps are the iterates of the 2 largest gaps
  - Transient gaps are the iterates of the $E = 0$ gap.

# General case

- We can still distinguish (numerically) stable and transient gaps for various C&P chains. $\rightarrow$ the naive argument seem to work.
- We plot the gapwidth $\Delta_l$ as a function of the label:




- The label order gaps by width
  - The width behaves as a power-law of the label
- Above a critical label, all gaps are transient

$\rightarrow$ gap labels are physically meaningful for this model!

# Conclusion and perspectives

- The gap labeling theorem can be extended to approximants
- The price to pay is the introduction of transient gaps, absent in the quasiperiodic case
- Gap label has a physical meaning:
  - It orders gap by decreasing width
  - It separates stable from transient gaps
  - It can be interpreted as a winding number of edge states inside the gaps [Levy *et al* 2015]

Perspectives:

- Investigate rigorously the general case
  - Recursive gap labeling using the Hofstadter rules [Rüdinger, Piéchon 98]
- Investigate to 2D quasicrystals, which also have gaps [Prunelé *et al* 2002]

# The gap labeling theorem: precise statement

Let $w$ be a cut-and-project word. Consider the Hamiltonian:

$$H(w) = \sum_{x,y} t(T^{-y} w, x - y) \left|x\right\rangle \left\langle y\right|$$

Interactions must be local:

$$\sup_{u \in \mathrm{Hull}(w)} \sum_{x} |t(u, x)| < \infty$$

Gap labeling theorem:
The idos in a gap is a linear combination of frequencies of local environments of $w$.

taken from *The non-commutative geometry of aperiodic solids*, Bellissard 2003.

# Cases where the naive proof fails

Consider approximants to the Fibonacci chain. Gaps are labeled by

$$\mathrm{idos}(q) = \frac{q}{1+\alpha_l} \mod 1$$

Consider the sequence of gap labels

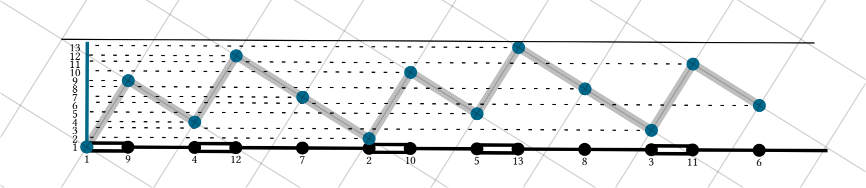
$$q_{l=3k} = \left[ \frac{(2+\sqrt{5})^k}{2\sqrt{5}} \right]$$

We have $\mathrm{idos}(q_l) = 1/2$ (it labels the $E = 0$ gap). Taking $l \to \infty$, we could – incorrectly – conclude that $1/2$ is a gap.
However, there is no finite $q$ such that

$$\frac{1}{2} = q\tau^{-2} \mod 1$$

# Conumbering and gap labeling

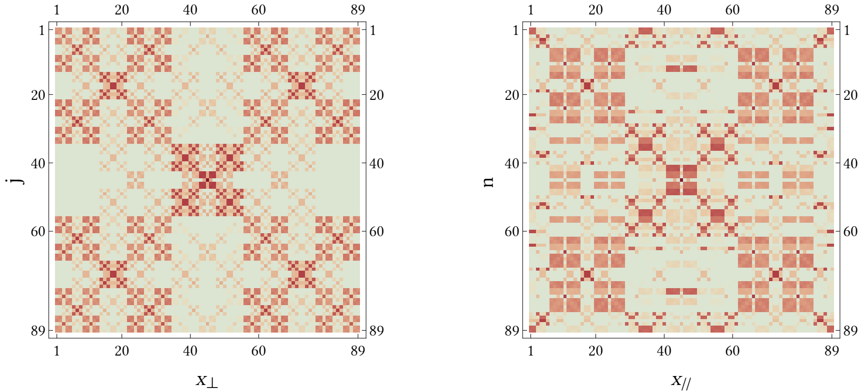
Conumbering: labeling of the atoms according to their internal space position.




see Mosseri & Sire 1990.

$$\text{idos} = \frac{j}{L_l} \quad \xleftarrow{\text{conumbering}} \quad \xrightarrow[\text{normal numbering}]{} \quad \text{idos} = \frac{q}{1+\alpha_l} \mod 1$$

# Conumbering and gap labeling




Plotting the local density of states makes the symmetry between gap labels and atomic labels evident.