Gap labels

In the gaps, the integrated density of states can only take values in the set $\omega\mathbb{Z}+\mathbb{Z}\cap[0,1]$. This is the well known gap labelling theorem, applied to the Fibonacci chain. Said differently, we can write the idos in a gap in the form

$$g(q) = \omega q \mod 1 = \omega q + p \tag{1}$$

were q is an integer, labelling the gap in which the idos takes the value g.

For the n$^{\text{th}}$ approximant, the gap labelling becomes

$$g_n(q) = \omega_n q + p \tag{2}$$

with $\omega_n = F_{n-1}/F_n$, and $q \in [1, F_n)$. As we have seen, in the strong modulation limit the spectrum has a hierarchical, trinary tree structure, and therefore the gaps should have this structure as well. Indeed, if we call G_n the set of gap values for the n$^{\text{th}}$ approximant, then we have respectively for the set of gap values in the bonding/atomic/antibonding clusters:

$$G_n^- = d_-(G_{n-2}) = \omega_n^2 G_{n-2} \tag{3}$$
$$G_n^0 = d_0(G_{n-3}) = \omega_n^3 G_{n-3} + \omega_n^2 \tag{4}$$
$$G_n^+ = d_+(G_{n-2}) = \omega_n^2 G_{n-2} + \omega_n \tag{5}$$

These relations have a geometrical interpretation that can be best seen if we replace the gap $g_n(q) = \omega_n q + p$ by the vector $\mathbf{g} = (p, q)$. Then the above relations are replaced by the affine transformations

$$d_-(\mathbf{g}) = \Omega^{-2}\mathbf{g} \tag{6}$$
$$d_0(\mathbf{g}) = \Omega^{-3}\mathbf{g} + (1, -1) \tag{7}$$
$$d_+(\mathbf{g}) = \Omega^{-2}\mathbf{g} + (0, 1) \tag{8}$$

where Ω is nothing but the substitution matrix

$$\Omega = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{9}$$

that generates the Fibonacci sequence by acting repetitively of the letters A and B.

Some of the $F_n - 1$ gaps of the n$^{\text{th}}$ approximant are *transient*: they will close in the quasiperiodic limit $n \to \infty$. The gap at $E = 0$ zero that appears when F_n is even is an example of such a transient gap. The non-transient gaps we call *stable*. Once they appear, they stay open all the way to the quasiperiodic limit. The two main gaps that separate the molecular clusters from the atomic cluster are examples of such stable gaps. From the recursive construction of the gap labels (3), it is apparent that the stable gaps are the iterates of the two main gaps, while the transient gaps are the iterates of the $E = 0$ gap.

TODO:

— Peut-être peut-on calculer quelques quantités explicitement ? Par exemple le nombre de gaps transitoires et stables à l'étape n, ou alors leur distribution en fonction de q (je m'attends à ce que les gaps stables soient pour q petit et les gaps transitoires pour q grand, peut être même y a-t-il un q critique (augmentant avec n) au-delà duquel tous les gaps sont transitoires ?).

— On peut sans doute faire de jolies figures, comme celle ci-dessous, que je ne sais pas comment interpréter...

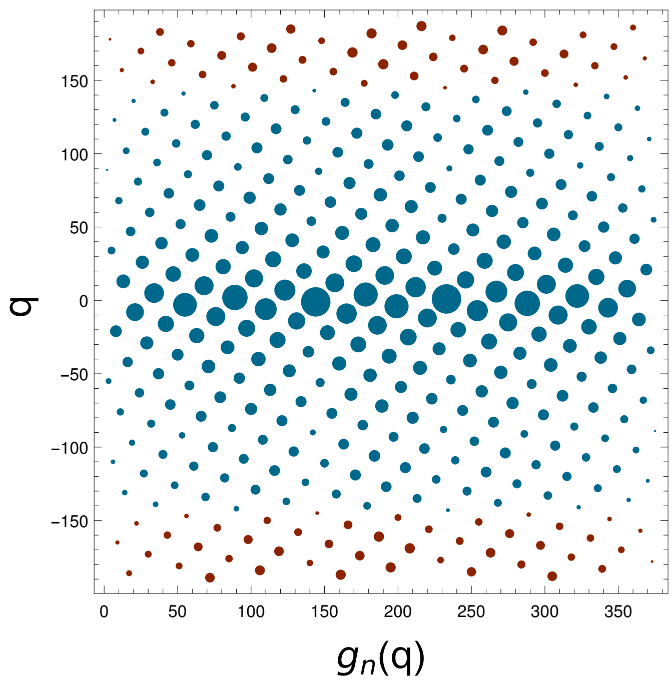

Figure 1 – The $(q, g_n(q))$ plane for an approximant ($\rho = 0.5$). The dots actually form a square lattice tilted by ω_n. The size of a dot is proportionnal to the log of the width of the corresponding gap. Blue: stable gaps, red: transient gaps.