Exact results on electronic wavefunctions of 2D quasicrystals

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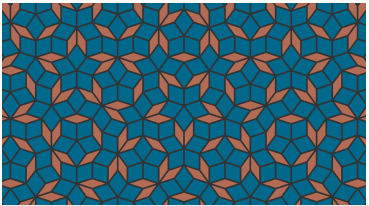
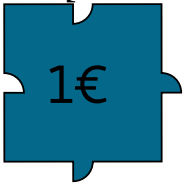
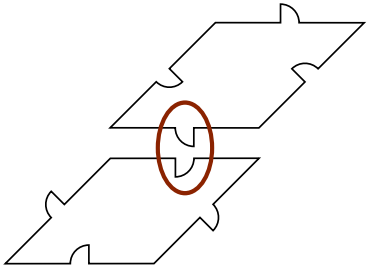


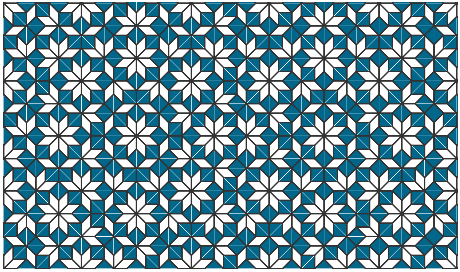
A quasiperiodic puzzle [Bédaride *et al.* 12]



Pay the squares, get the rhombuses for free!

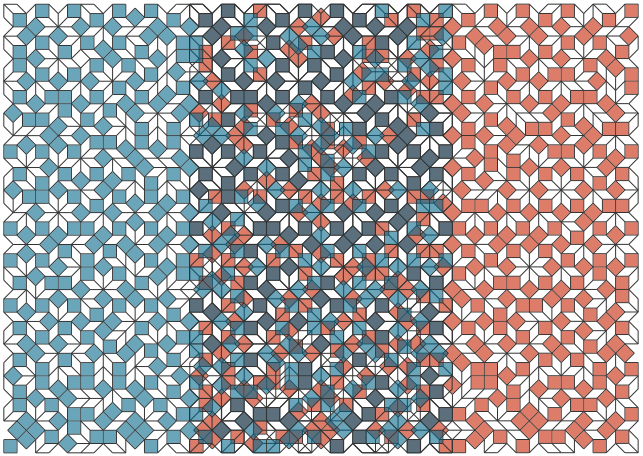


Forbidden configuration.



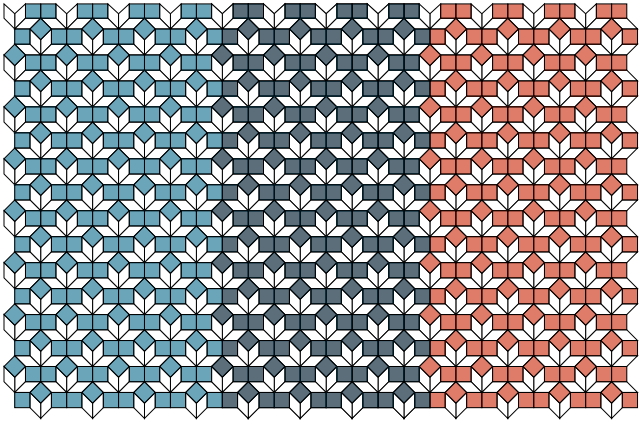
Patch of the Ammann-Beenker tiling.

Periodic, quasiperiodic and random



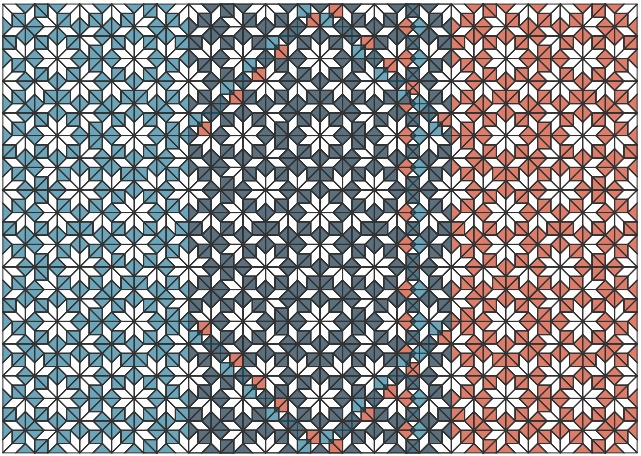
No long range order : random

Periodic, quasiperiodic and random



Perfect long range order : periodic

Periodic, quasiperiodic and random



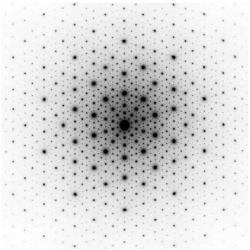
Long range order : quasiperiodic

(Math : **Meyer sets**, see Chap. 2 of [Grimm, Baake 13])

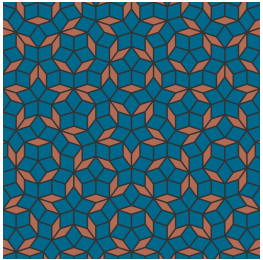
Quasicrystals

Quasicrystal → quasiperiodically arranged atoms :

- **aperiodicity**
- **long range order** (diffraction pattern exhibits sharp peaks).

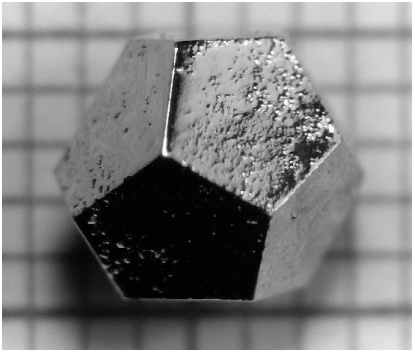


Diffraction pattern of a AlPdMn alloy
(Conradin Beeli group)

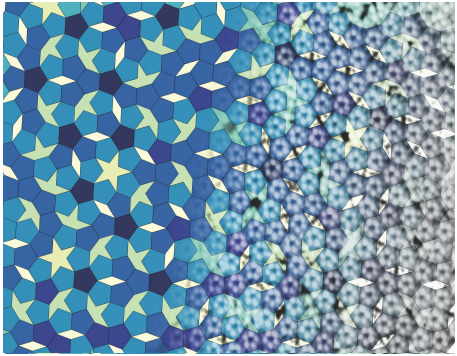


A patch of the quasiperiodic Penrose tiling,
used to model many quasicrystals.

Examples of quasicrystals



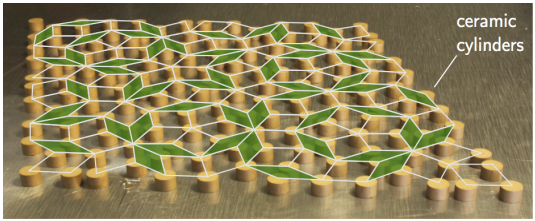
HoMgZn alloy in its icosahedral phase
(doi:10.1038/nmat1244)



A 2D molecular quasicrystal
(doi:10.1038/nature12993)

- many intermetallic alloys are quasiperiodic
- a single natural example : Khatyrka meteorite hosts quasicrystals (doi:10.1126/science.1170827).

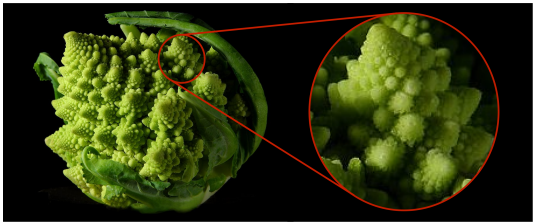
Engineered quasiperiodic structures



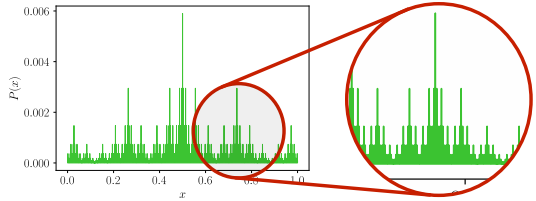
A network of dielectric resonators [Vignolo *et al.* 14]

- Plasmons in semiconductor stacks [Merlin *et al.* 85]
- Microwaves in perforated metallic films [Matsui *et al.* 07]
- Microwaves in dielectric resonator networks [Vignolo *et al.* 14]
- Light solitons [Freedman *et al.* 07]
- Cold atoms in laser potentials [Guidoni *et al.* 97]
- Polaritons in wire cavities [Tanese *et al.* 14]

Fractals



Romanesco broccoli (@ Wikimedia commons)



Electronic density along a quasiperiodic chain

- Broccoli & electronic density on a qp chain $\rightarrow$ discrete scale invariant objects.
- Discrete scale invariance $\rightarrow$ **fractal structure**.
- Fractals uncommon in general, but everywhere in the physics of quasicrystals!

Main matter

Link fractals and quasiperiodic geometry

The Fibonacci chain

Fibonacci numbers

A simple rule for generating numbers :

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 1 \\
F_2 &= F_1 + F_0 = 2 \\
F_3 &= F_2 + F_1 = 3 \\
F_4 &= F_3 + F_2 = 5 \\
F_5 &= F_4 + F_3 = 8 \\
&\vdots \\
F_{l+2} &= F_{l+1} + F_l
\end{aligned}
$$

Fibonacci words

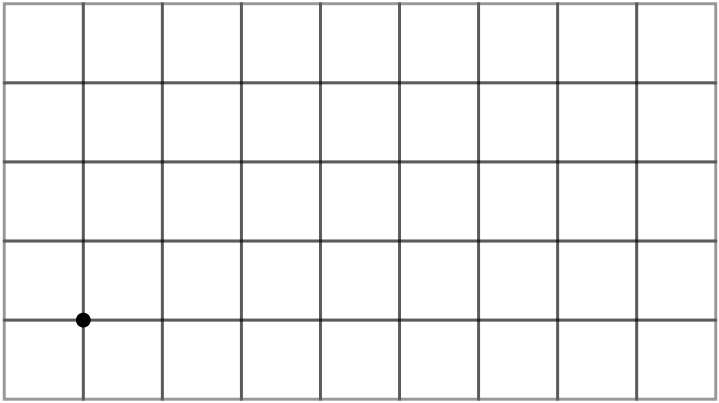
Letters instead of numbers, same rule :

$$
\begin{aligned}
C_0 &= \mathrm{B} \\
C_1 &= \mathrm{A} \\
C_2 &= C_1 C_0 = \mathrm{AB} \\
C_3 &= C_2 C_1 = \mathrm{ABA} \\
C_4 &= C_3 C_2 = \mathrm{ABAAB} \\
C_5 &= C_4 C_3 = \mathrm{ABAABABA} \\
&\vdots \\
C_{l+2} &= C_{l+1} C_l
\end{aligned}
$$

Fibonacci word from above

(Infinite) Fibonacci word : ABAABABAABAAB…

A ↔ horizontal step, B ↔ vertical step



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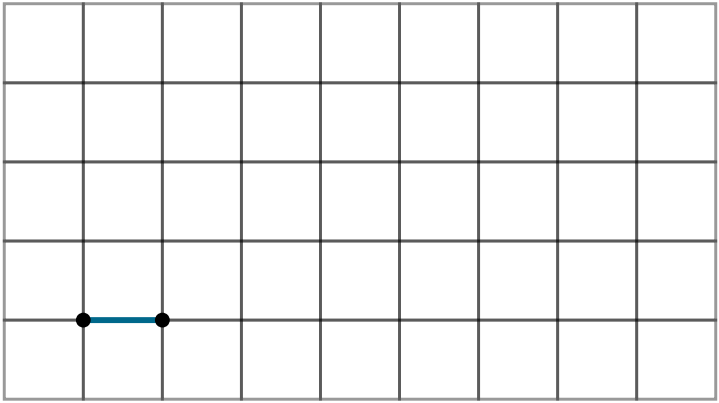

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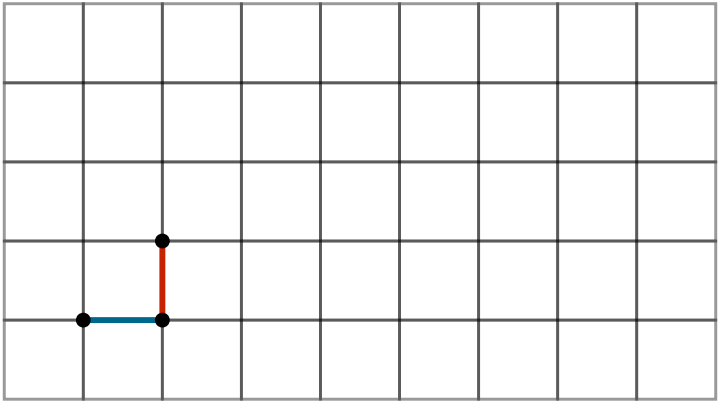

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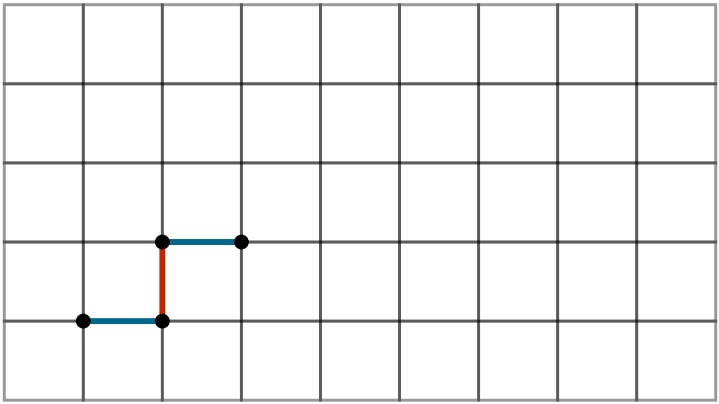

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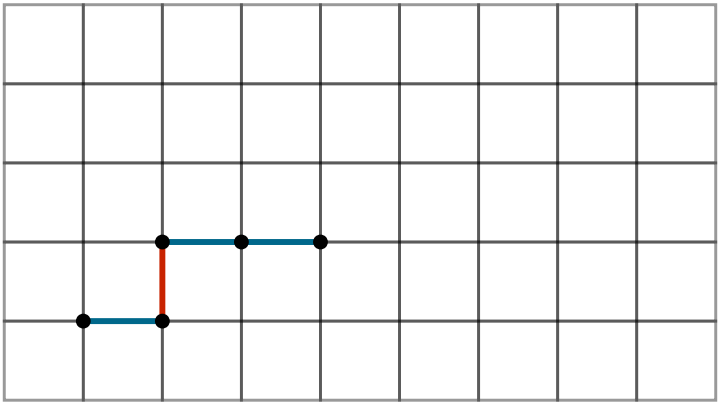

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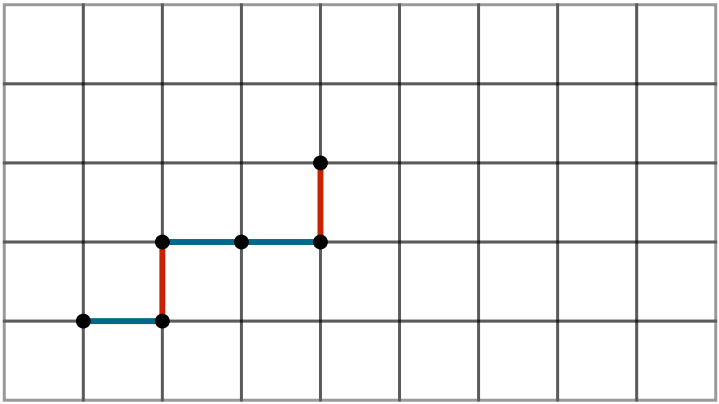

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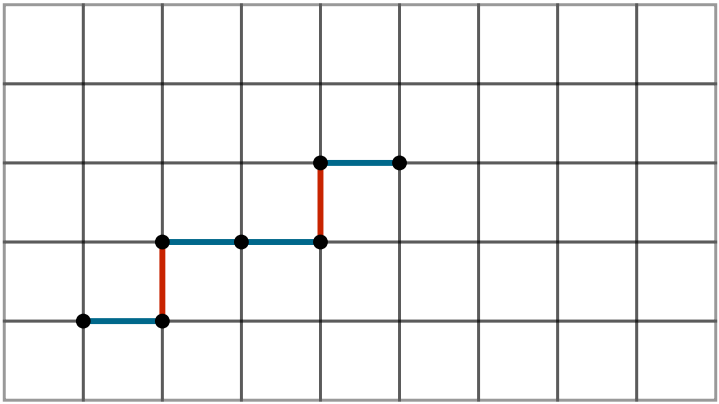

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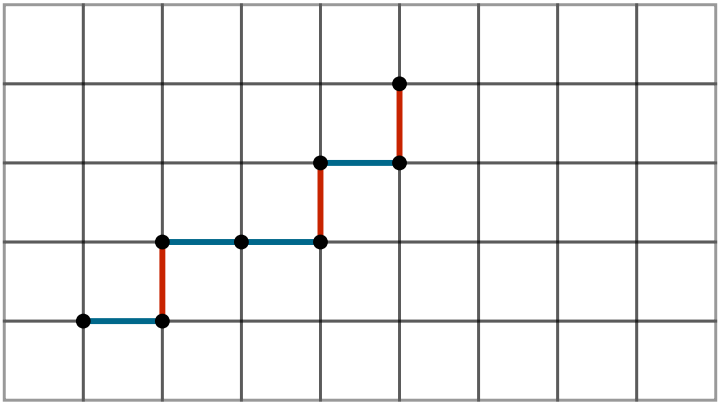

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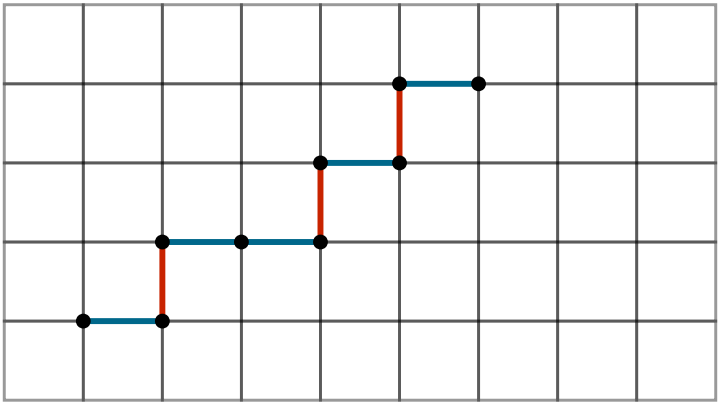

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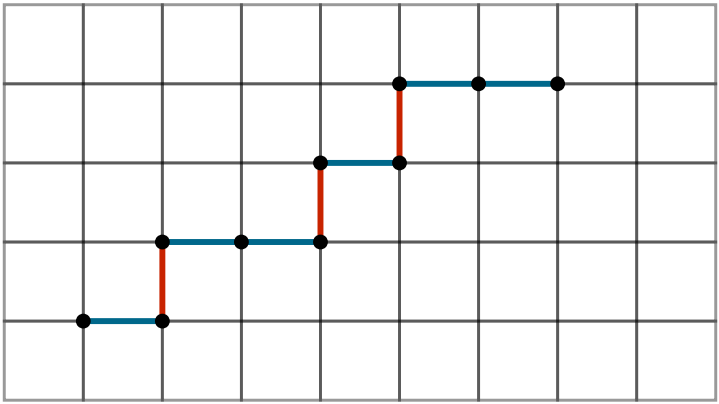

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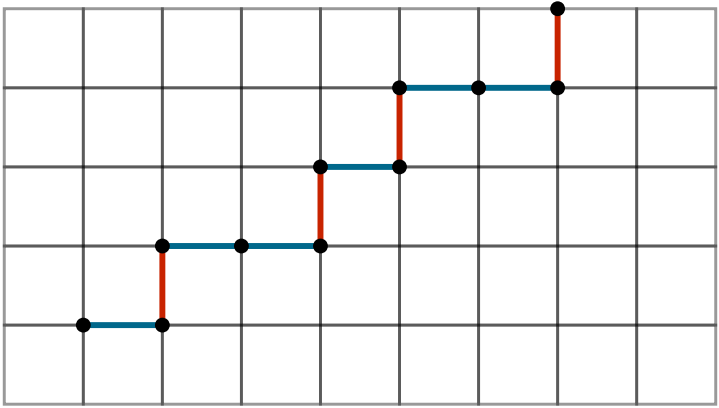

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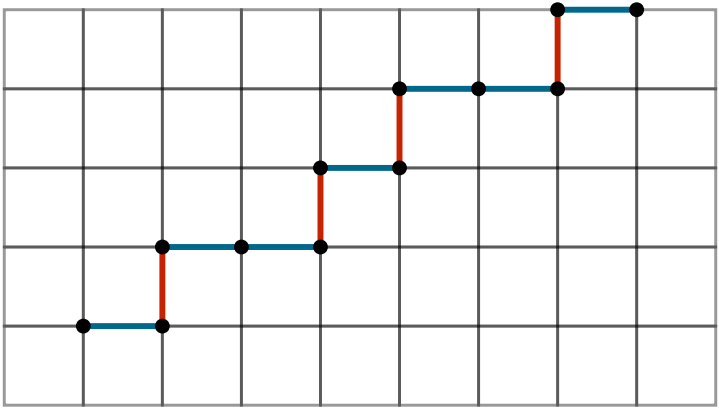

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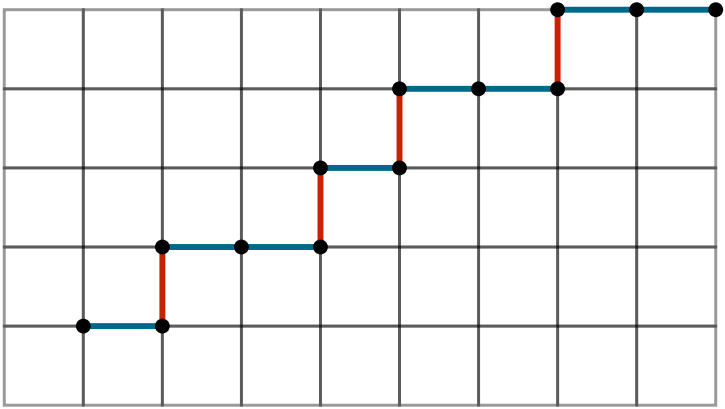
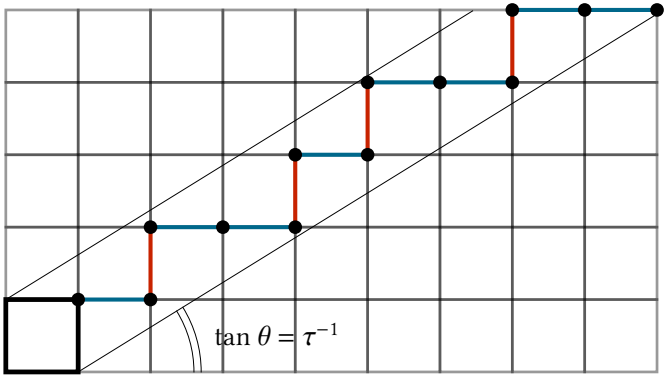


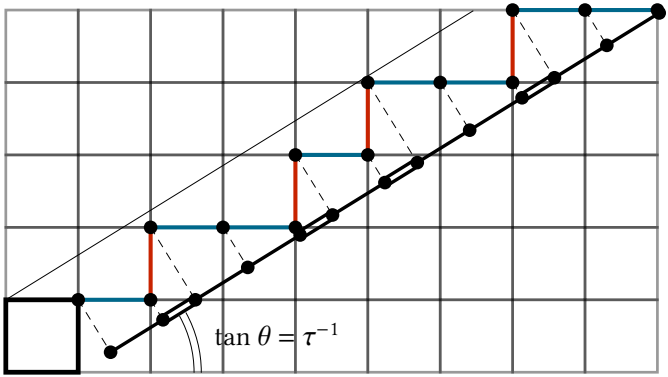
Quasiperiodicity of the Fibonacci word



- average slope = inverse of the golden ratio ($\tau \simeq 1.6$)
- bounded fluctuations

$\rightarrow$ similar environments everywhere
$\rightarrow$ quasiperiodicity [Duneau, Katz 85]

Cut-and-project



C&P $\rightarrow$ quasiperiodic (or periodic) tiling!

The cut-and-project algorithm :

1. choose a hypercubic lattice (here $\mathbb{Z}^2$)
2. choose a "physical plane" $E_{\parallel}$ (here a slope)
3. select points by translating the unit hypercube along $E_{\parallel}$
4. project them onto $E_{\parallel}$.

From letters to atoms

- The Fibonacci word : ABAABABA...
- The Fibonacci (tight-binding) chain of atoms :

Hamiltonian :

$$\hat{H} = -\sum_m t_m \left|m-1\right\rangle \left\langle m\right| + \text{H.c}$$

Schrödinger equation for the eigenstate of energy E :

$$E\psi(m) = -t_m\psi(m-1) - t_{m+1}\psi(m+1)$$

A fractal state