Fractals and physics

Nicolas Macé (inspired by E. Akkermans)

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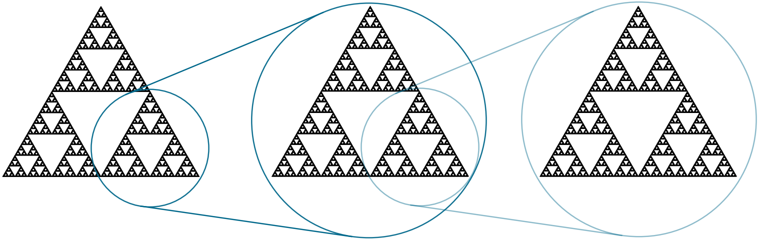
Symmetry

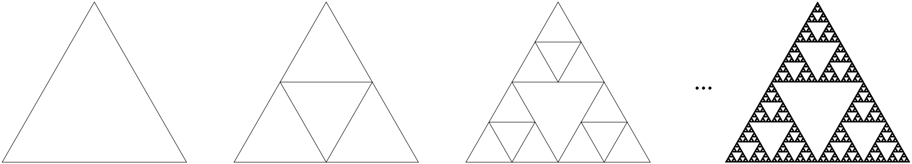
- Euclidean space (continuous translational symmetry & continuous scaling symmetry)
    - Crystallographic lattice (discrete translational symmetry)
    - Fractal set/manifold (discrete scaling symmetry)

$\rightarrow$ fractal: infinitely divisible object

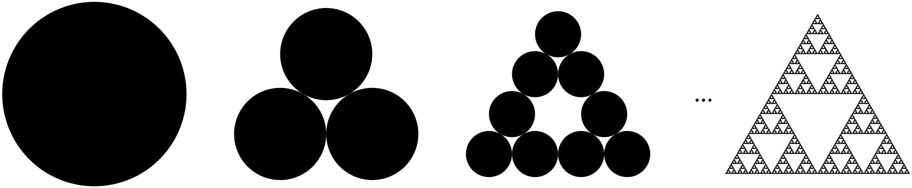


Geometric construction

A first example: the Sierpiński triangle



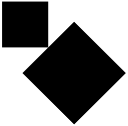
- independant of starting shape → only determined by the geometrical transformations used.

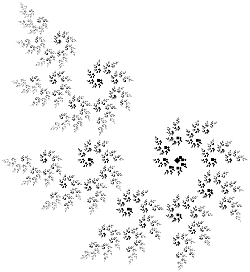


- The Sierpiński triangle is constructed by an Iterated Function System (IFS).

Geometric construction

Iterated function systems



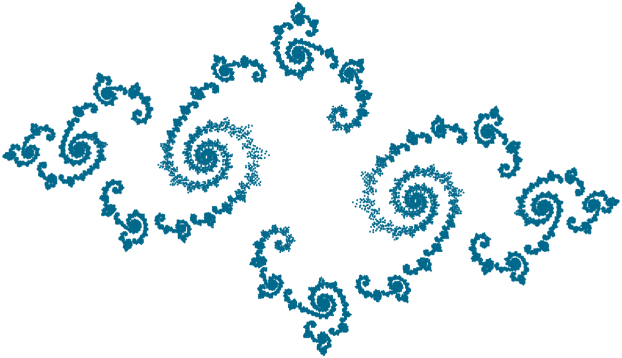


- Every fractal is approached by an IFS (Barnsley).

Geometric construction

"non-trivial" fractals: Julia sets

- Define a recurrence $z_{n+1} = z_n^2 + c$
- Julia set: boundary of the convergence domain



Julia set $c = -0.77 + 0.22i$

Geometric construction

"non-trivial" fractals: Julia sets

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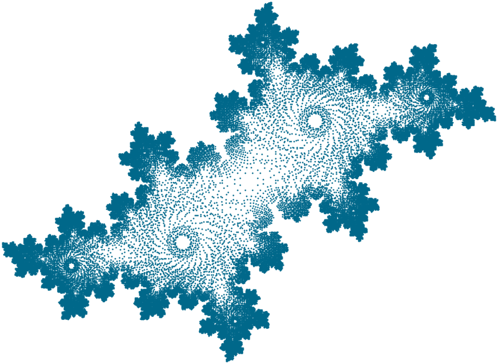
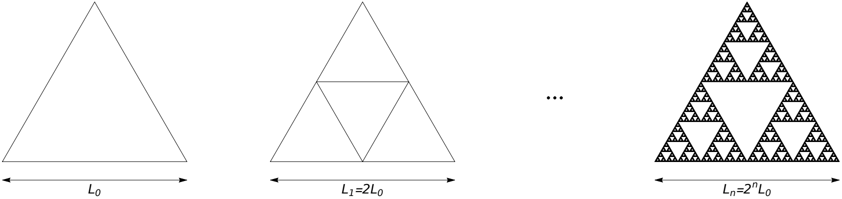


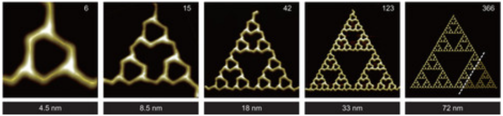
Julia set $c = -0.39 - 0.59i$

Scaling

- Give a physical meaning → give a length scale



- A natural way of doing it:



Assembling molecular Sierpiński triangle fractals, Nature Chemistry (2015)

- Scaling of physical quantities?
  - $M(L) \propto L^d$ on a d-dimensional Euclidean manifold... What happens on a fractal one?

The mass dimension

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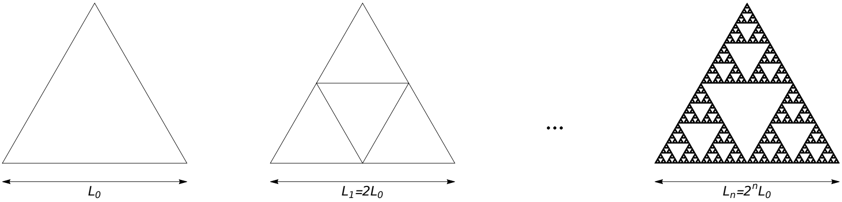
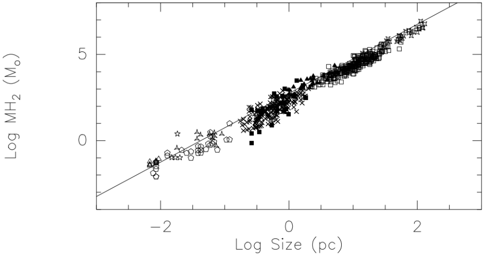


M_0 $\quad$ $M_1 = 3M_0$ $\quad$ $M_n = 3^n M_0$

$$M(L) \propto L^{d_M}, \text{ with } d_M = \log 3 / \log 2$$

- d_M is the mass (or Hausdorff) dimension.
- $1 < d_M < 2$ different from $d = 1$, non-integer $\rightarrow$ signature of a fractal manifold.

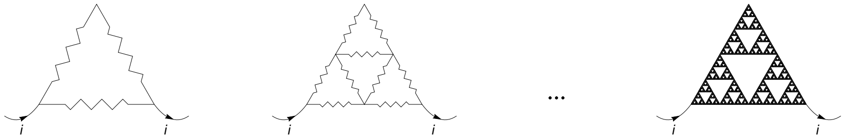
- Mass dimension: spot and characterize fractals, from large scales...



[Taken from *Astrophysical Fractals: Interstellar Medium and Galaxies*]

- ... to small ones

The electric dimension



R_0 $\qquad$ $R_1 = ?R_0$ $\qquad$ $R_n = ?^n R_0$

$$R(L) \propto L^{d_e}, \text{ with } d_e = \log ? / \log 2$$

- $d_M \neq d_e$, and they both reflect the structure of the fractal manifold.
- On an Euclidean manifold d_M and d_e would have been independant its structure, they would have only depended on d, its dimension.

Hidden fractals: the Fibonacci chain

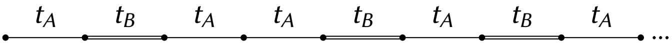
- The Fibonacci sequence:

  B

  A

  AB

- The Fibonacci (tight binding) chain:



- Whenever $t_A \neq t_B$, the chain is quasiperiodic.
  - Spectral properties?

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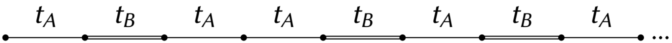

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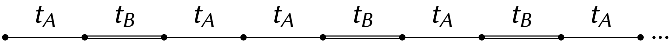

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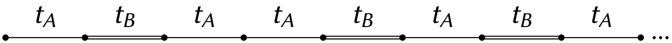

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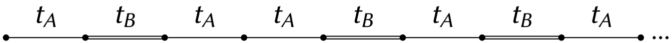

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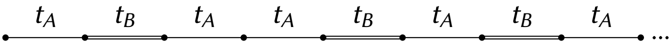

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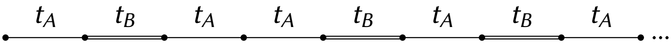

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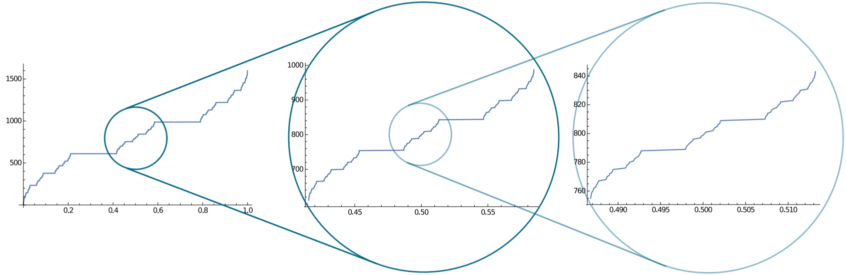
Hidden fractals: the Fibonacci chain

- No obvious fractal nature, but...



... the graph of the density of states is a fractal!

- What can we say about a scale invariant function?

Interlude: discrete and continuous scale invariance

Continuous scale invariance

$$\forall a,\ f(ax) = b(a)f(x)$$

Then $f(x) = Cx^{\alpha}$

Discrete scale invariance

$$\exists a,\ f(ax) = b(a)f(x)$$

Then $f(x) = ?$

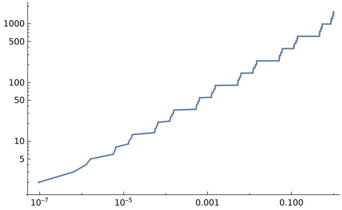
A tool to analyze scale invariant functions: the Mellin tranform.

$$\{f\}(z) = \int_{\mathbb{R}^+} f(x)x^{-z}\frac{\mathrm{d}x}{x}$$

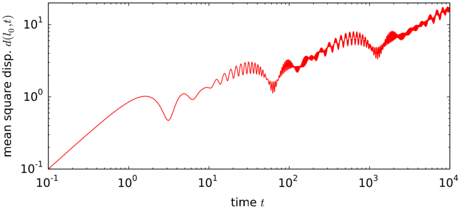
$$\{x^{\alpha}\}(z) = \frac{1}{z+\alpha}$$

$\rightarrow$ the scaling factor(s) of a function are given by the poles of its Mellin transform.

Fibonacci and log–periodic oscillations



- A *discretely* scale invariant function exhibits log–periodic oscillations...



[*Origin of the* log*–periodic oscillations in the quantum dynamics [...]* Thiem (2015)]

Summary

- Fractal manifolds and the study of discrete scaling symmetry is a fasinating mathematical topic in itself
- Fractals arise in surprisingly diverse areas of physics
- ... but, always, they are beautiful!