Gap structure of 1D cut and project Hamiltonians

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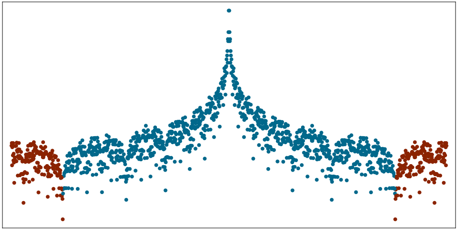
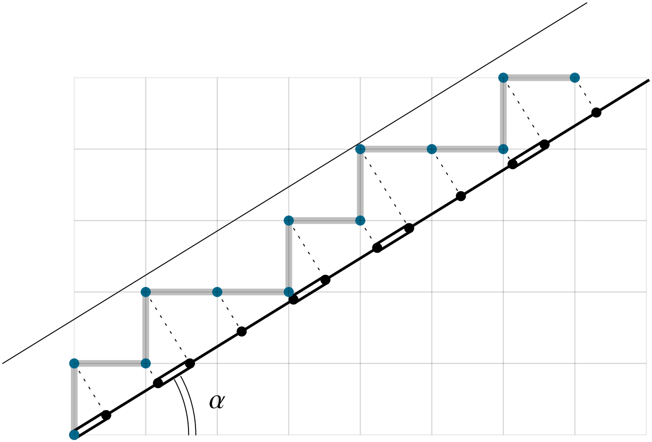


Outline

Electrons on quasiperiodic chains



Approximants:

$$\alpha_l = \frac{p_l}{q_l}$$

$$\alpha_l \to \alpha$$

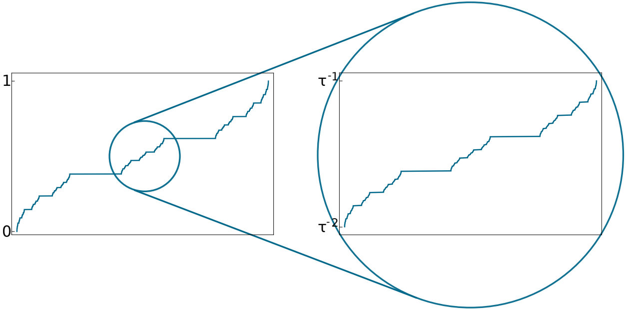
Hamiltonian: $H(\alpha) = \sum_{x\in\mathbf{Z}} t_{x,x+1} |x\rangle \langle x+1| + \text{h.c.}$

2 lengths $\to$ 2 jump amplitudes $t_{x,x+1} = t_1$ or t_2.

The electronic spectrum

A convenient way to plot the spectrum: the integrated density of states (idos).

idos(E) = fraction of states below energy E



idos of the Fibonacci Hamiltonian

- Electronic spectrum of quasiperiodic chains is hard to describe
- Rather: describe the idos in the gaps → **gap labeling theorem**

$$\text{idos}(E \in \text{gap}) = n\tau^{-2} \mod 1$$

The gap labeling theorem

The IDOS inside spectral gaps can be indexed using the irrational involved in the construction of the chain

$$\mathrm{idos}(E \in \mathrm{gap}) = \frac{n}{1+\alpha} \mod 1$$

- Constrains the spectrum... but is not enough to reconstruct it
- Model independent! (while the spectrum is model dependent) $\rightarrow$ a topological invariant

TODO: plot the idos for the diagonal and the on-site potential models

The gap labeling theorem

$$\mathrm{idos}(E \in \mathrm{gap}) = \frac{n}{1+\alpha} \mod 1$$

- Has the gap label n a physical interpretation?
- Can the theorem be applied to approximants?
- Does it help understanding the quasiperiodic limit?

A simple, but incorrect proof

Let $\alpha_l = \frac{p_l}{q_l} \to \alpha$ be a sequence of approximants.
$N_l = p_l + q_l$ is the maximum number of energy bands.

$$\mathrm{idos}(E \in \mathrm{gap}) = \frac{j(E)}{N_l}$$

We can find integers n, k such that $j = nq_l + kN_l$.

$$\mathrm{idos}(E \in \mathrm{gap}) = \frac{nq_l}{p_l + q_l} \mod 1$$

Letting $l \to \infty$,

$$\mathrm{idos}(E \in \mathrm{gap}) = \frac{n}{1 + \alpha} \mod 1$$

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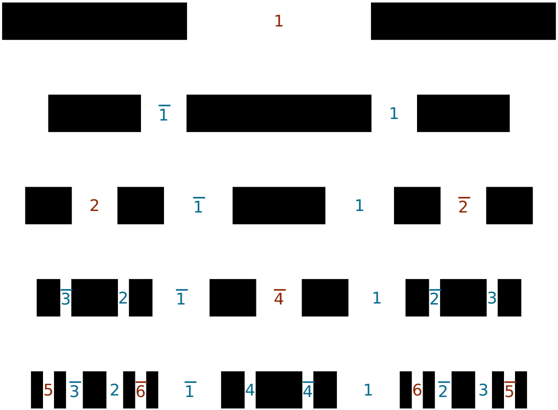
Problem

n may depend on l.

A concrete example

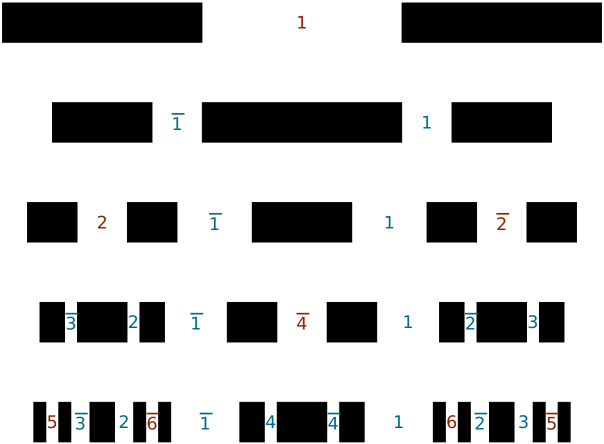
Gaps of successive approximants of the Fibonacci chain.



Call $\langle E \rangle_l$ the mean energy of a gap, and $\Delta_l(\langle E \rangle)$ its width. I identify two gaps if they overlap:

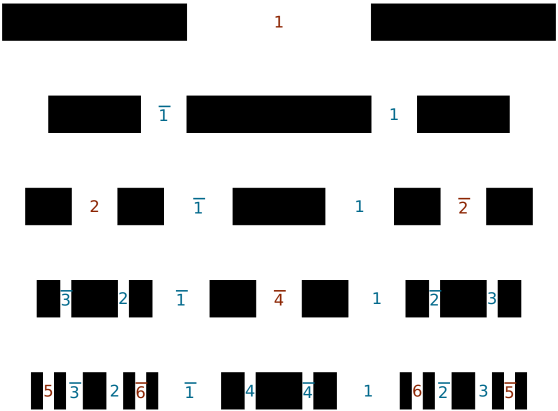
$$0.5\Delta_l(\langle E \rangle) > |\langle E \rangle_l - \langle E' \rangle_{l+1}|$$

Stable and transient gaps



- Red labeled gaps closes as $l \to \infty$ $\to$ transient gaps.
- Blue labeled gaps stay open $\to$ stable gaps.

Stable and transient gaps



- Blue gaps have a well-defined label. The naive proof works!
- Red gaps have an ill-defined label but disappear.

$\to$ the naive proof works and correctly labels the gaps.

Recursive gap labeling

Recursive construction of the spectrum...

...translates into recursive construction of the gaps!

$$G_l^{\text{left}} = M^{-2} G_{l-2}$$
$$G_l^{0} = M^{-3} G_{l-3} + \mathbf{g}_1$$
$$G_l^{\text{right}} = M^{-2} G_{l-2} + \mathbf{g}_2$$

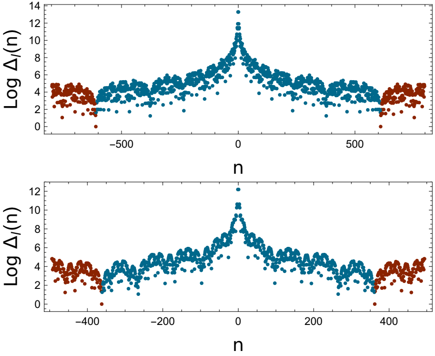
Where G_l is the set of labels:
$G_l = \{(m, n) | \text{ids} = n/(1+\alpha) + m\}$

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

- Stable gaps are the iterates of the 2 main gaps
- Transient gaps are the iterates of the central gap.

General case

We plot the gapwidth Δ_l as a function of the label for various quasiperiodic chains:



- The width decreases as a power-law of the label
- Above a critical label, all gaps are transient
- Recursive gap labeling using the Hofstadter rules? [Rüdinger, Piéchon 98]

Conclusion and perspectives

- The gap labeling theorem can be extended to approximants
- The price to pay is the introduction of transient gaps, absent in the quasiperiodic case
- Gap labels have a physical meaning:
  - It orders gap by decreasing width
  - It separates stable from transient gaps
  - It can be interpreted as a Chern number [Levy *et al* 2015]

Perspectives:

- Understand the gap width behavior with the gap label
- Investigate to 2D quasicrystals, which also have gaps [Prunelé *et al* 2002]