Fractals in physics

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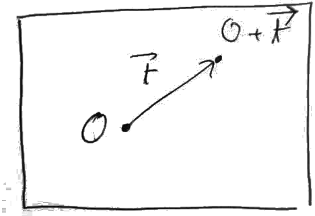
Continuous translation symmetry

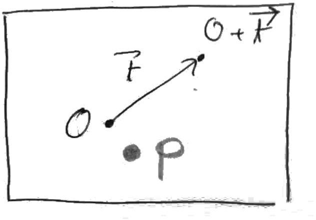
Setup: observer O in (empty) space



- Translated observer $O + \overrightarrow{t}$ makes the same observations as O
- $\implies$ Continuous translation symmetry
- Formally: $\forall \overrightarrow{t}, O + \overrightarrow{t} \sim O$

Symmetry breaking

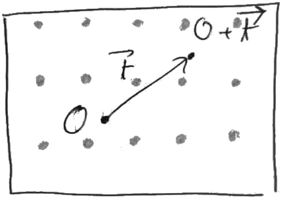
Add a reference point P



- Observer can measure distance to P: $O + \overrightarrow{t} \not\sim O$
- $\implies$ P breaks the symmetry

Discrete translation symmetry

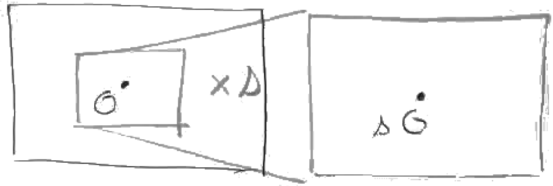
Can we partially restore translation symmetry?
$\rightarrow$ add a lattice of reference points



- $\exists \vec{t}, O + \vec{t} \sim O$
- $\implies$ Discrete translation symmetry

Continuous scaling symmetry

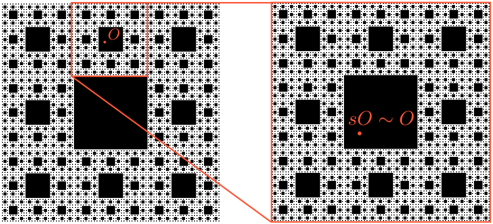
Back to empty space:



- Rescaled observer sO makes the same observations as O
- Continuous scaling symmetry
- $\forall s, sO \sim O$

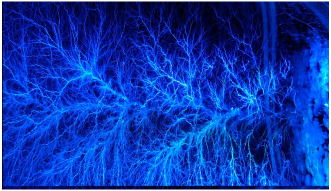
Can we break continuous scaling symmetry into discrete?

Fractals



[Menger sponge]

- Discrete scaling symmetry: $\exists s$, $sO \sim O$ (Menger sponge: $s = 3$)
- Fractals: discretely scale invariant objects
- $\implies$ fractals: **rough** objects

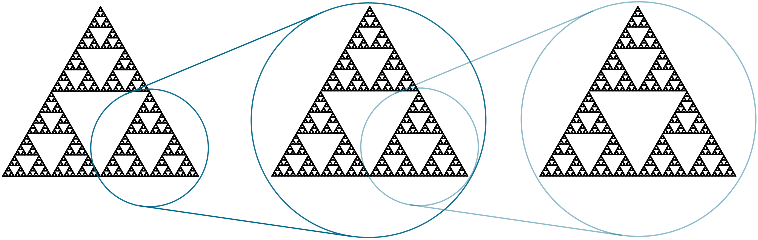


[Lichtenberg figure]

Symmetry: recap

- Empty space: continuous translation symmetry & continuous scaling symmetry
  - Lattice: discrete translational symmetry
  - Fractal: discrete scaling symmetry

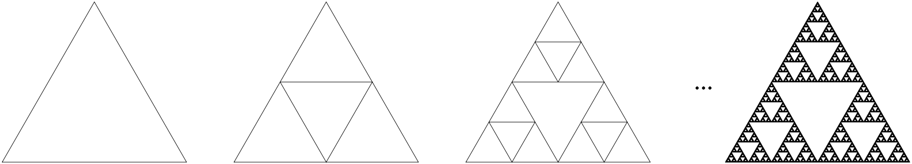
→ fractal: infinitely divisible object



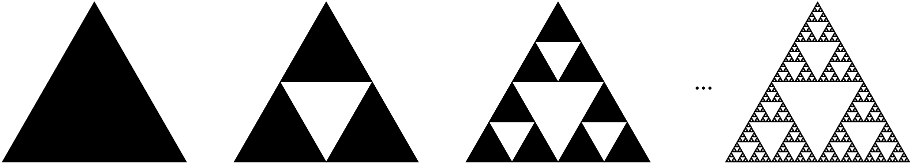
[Sierpiński triangle]

Geometric construction

A first example: the Sierpiński triangle



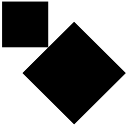
- independant of starting shape $\rightarrow$ only determined by the geometrical transformations used.

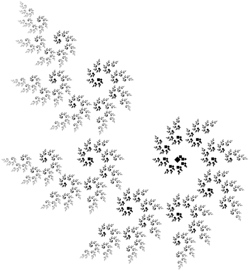


- The Sierpiński triangle is constructed by an Iterated Function System (IFS).

Geometric construction

Iterated function systems



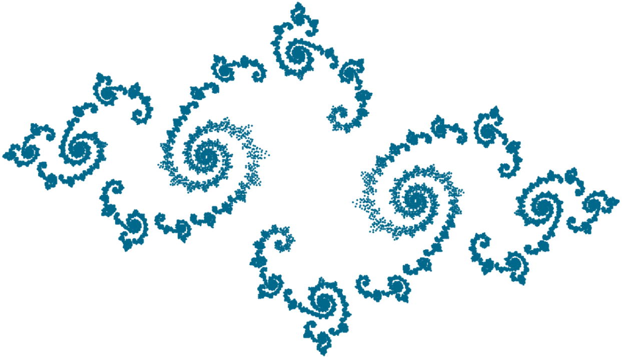


- Every fractal is approached by an IFS [Barnsley 1988].

Geometric construction

"non-trivial" fractals: Julia sets

- Define a recurrence $z_{n+1} = z_n^2 + c$
- (filled) Julia set: set of non-escaping points

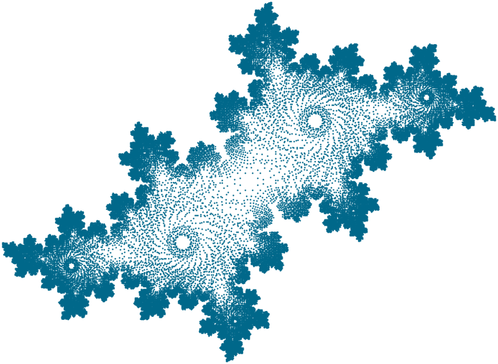


Julia set $c = -0.77 + 0.22i$

Geometric construction

"non-trivial" fractals: Julia sets

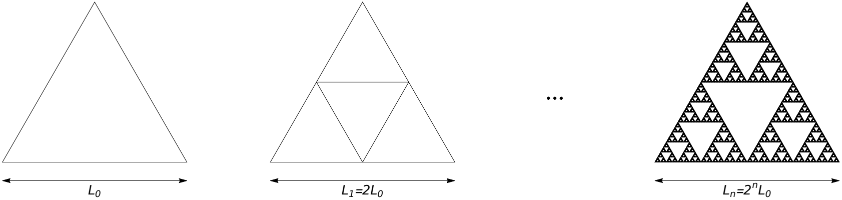
- Define a recurrence $z_{n+1} = z_n^2 + c$
- (filled) Julia set: set of non-escaping points



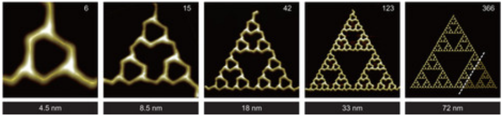
Julia set $c = -0.39 - 0.59i$

Scaling

- Give a physical meaning → give a length scale



- A natural way of doing it:

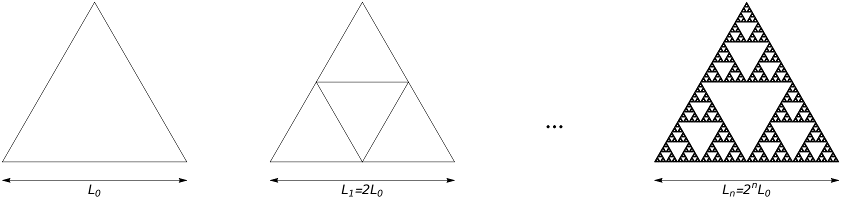


Assembling molecular Sierpiński triangle fractals, Nature Chemistry (2015)

- Scaling of physical quantities?
  - $M(L) \propto L^d$ on a d-dimensional Euclidean manifold... What happens on a fractal one?

The mass dimension

- $M(L) \propto L^d$ on a d-dimensional Euclidean manifold... What happens on a fractal one?

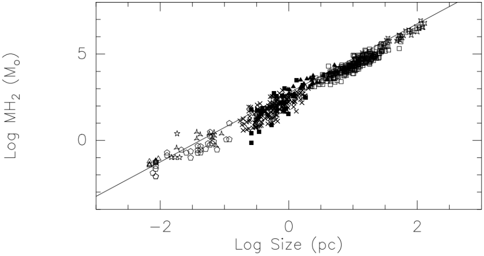


M_0 $\quad$ $M_1 = 3M_0$ $\quad$ $M_n = 3^n M_0$

$$M(L) \propto L^{d_M}, \text{ with } d_M = \log 3 / \log 2$$

- d_M is the mass (or Hausdorff) dimension.
- $1 < d_M < 2$ different from $d = 1$, non-integer $\rightarrow$ signature of a fractal manifold.

- Mass dimension: spot fractals, from large scales...

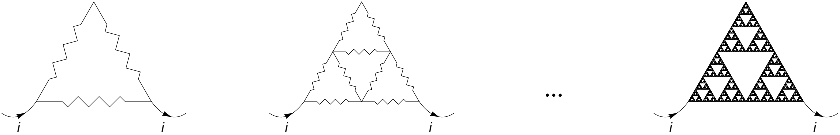


Insterstallar medium fractal over 4 to 6 orders of magnitude [Fig. from Combes 1999]

- ... to small ones

The electric dimension

[Akkermans, 2015 lectures]



R_0 $\qquad$ $R_1 = ?R_0$ $\qquad$ $R_n = ?^n R_0$

$$R(L) \propto L^{d_e}, \text{ with } d_e = \log ? / \log 2$$

- $d_M \neq d_e$, both reflect the structure of the fractal manifold.
- On an Euclidean manifold $d_M = d_e = d$.

The Fibonacci molecule

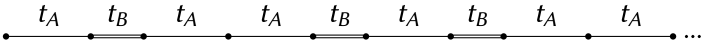
- The Fibonacci sequence:

  B

  A

  AB

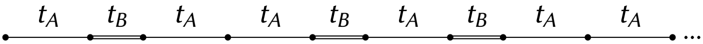
- The Fibonacci molecule:



- The molecule is **not** a fractal!
- A quantity of interest: molecular spectrum

The Fibonacci molecule

- The Fibonacci sequence:
  A
  AB
  ABA
- The Fibonacci molecule:



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  AB
  ABA
  ABAAB
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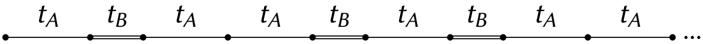

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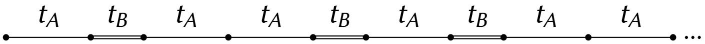

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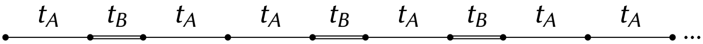

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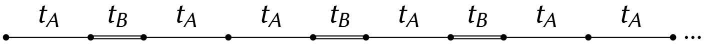

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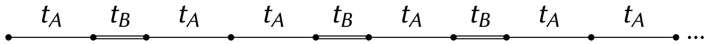
- The Fibonacci molecule:



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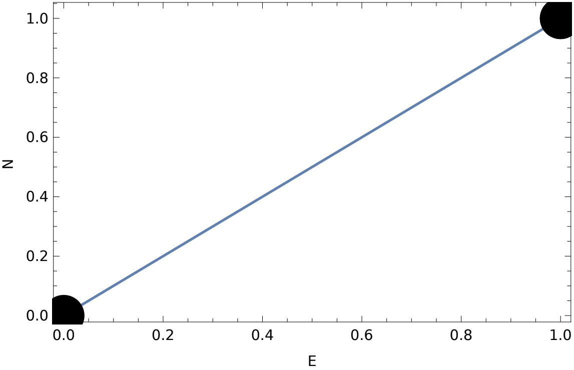
Spectrum of Fibonacci molecules

ABAABABAABAABABAABABAABAABABAABAABABAABABAABAABABAABABAABAABABAABAABABAABABAABAABABAABAAB



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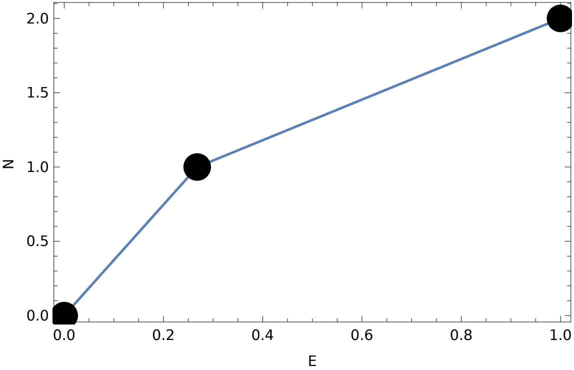

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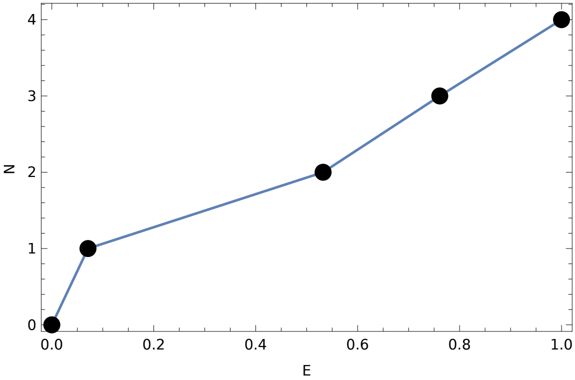

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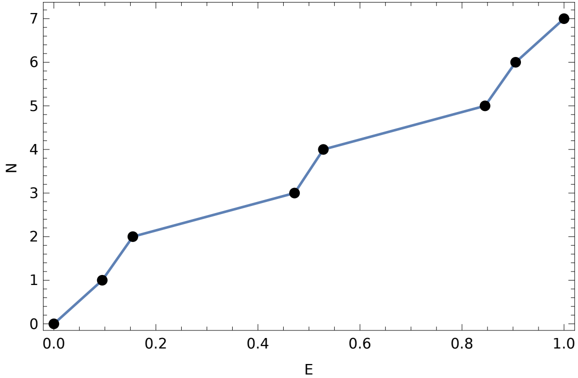

Spectrum of Fibonacci molecules

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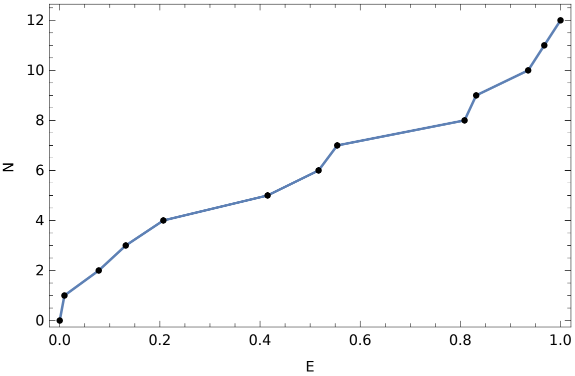

Spectrum of Fibonacci molecules

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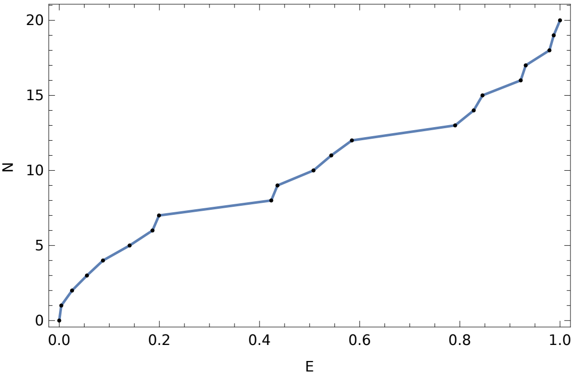

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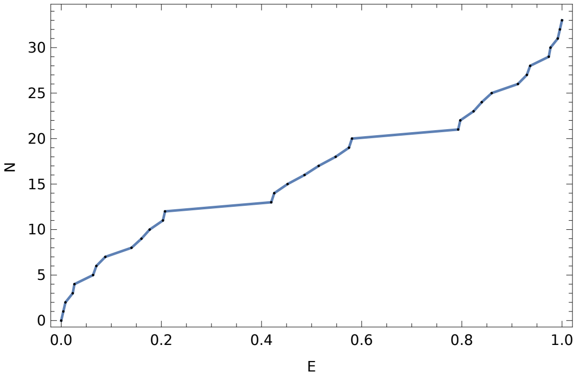

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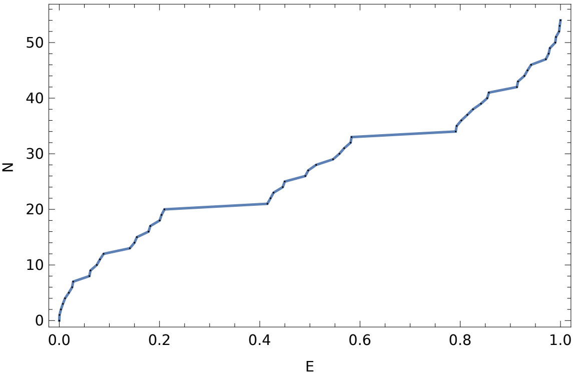

Spectrum of Fibonacci molecules

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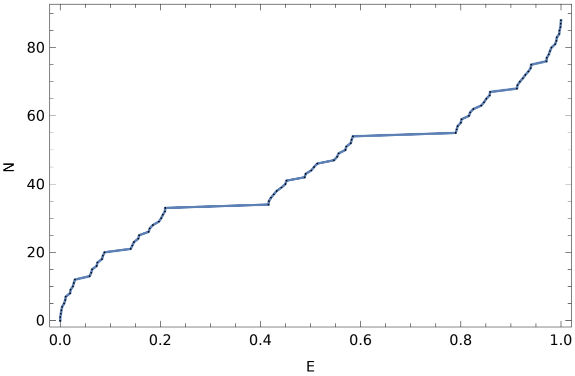
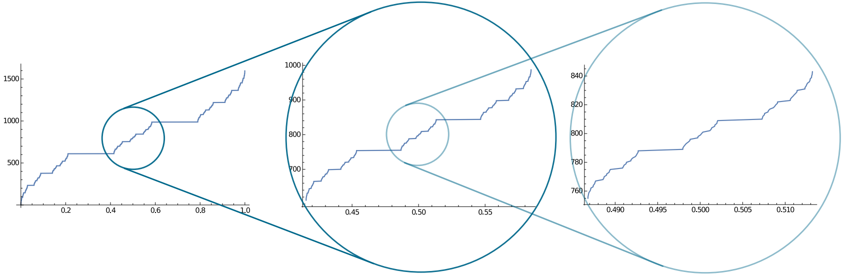


Hidden fractals: the Fibonacci chain

- No obvious fractal nature, but...



... the graph of the density of states is a fractal!

- What can we say about a scale invariant function?

Interlude: discrete and continuous scale invariance

Continuous scale invariance

$$\forall a,\ f(ax) = b(a)f(x)$$

Then $f(x) = Cx^{\alpha}$

Discrete scale invariance

$$\exists a,\ f(ax) = b(a)f(x)$$

Then $f(x) = ?$

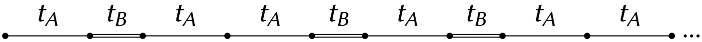
$$f(x) = G\left(\frac{\ln x}{\ln a}\right) x^{\alpha}$$ [Saleur, Sornette 1996]

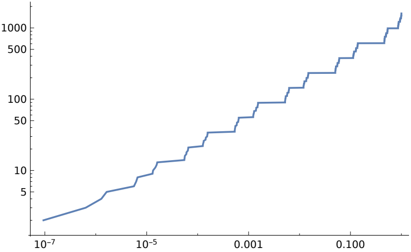
First order expansion:

$$f(x) \simeq \left[G_0 + G_1 \cos\left(2\pi \frac{\ln x}{\ln a}\right)\right] x^{\alpha}$$

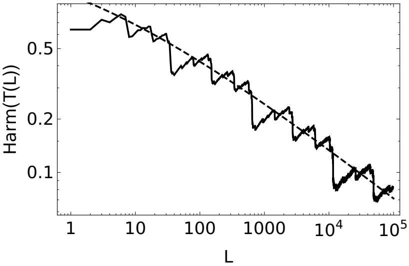
$\rightarrow$ **log-periodic oscillations**

Fibonacci and log–periodic oscillations





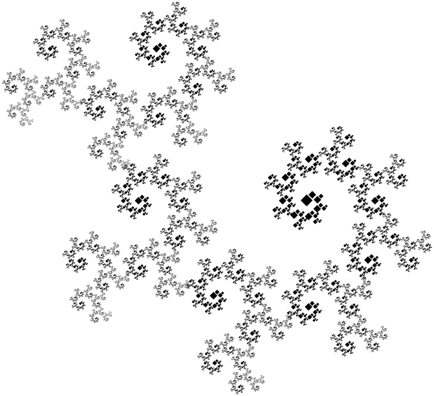
Spectrum of the molecule



Transmission probability [Macé *et al* 2017]

Fractals:

- break continuous scaling symmetry into discrete
- are rough objects, mathematically interesting
- arise in surprisingly diverse areas of physics
- ... and, always, are beautiful!



[Slides and scripts: github.com/Yukee/talks]