Fractals in physics

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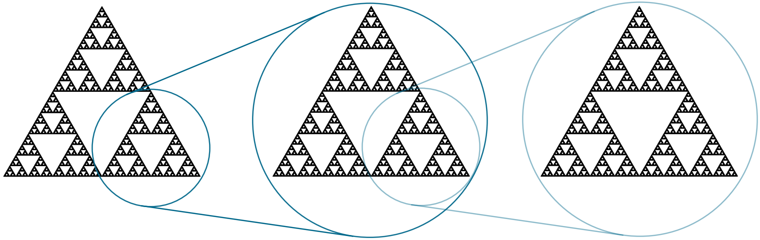
Euclidean space

Symmetry

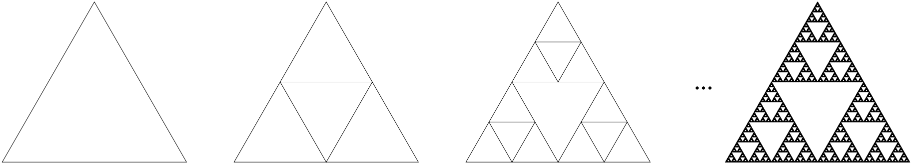
- Euclidean space (continuous translational symmetry & continuous scaling symmetry)
    - Crystallographic lattice (discrete translational symmetry)
    - Fractal set/manifold (discrete scaling symmetry)

$\rightarrow$ fractal: infinitely divisible object

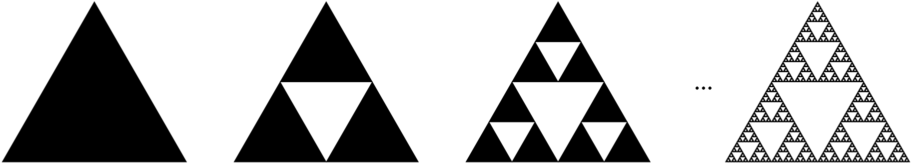


Geometric construction

A first example: the Sierpiński triangle



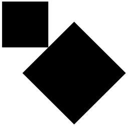
- independant of starting shape $\rightarrow$ only determined by the geometrical transformations used.

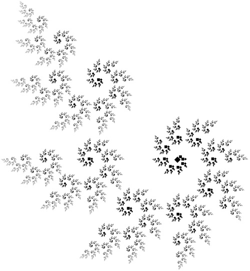


- The Sierpiński triangle is constructed by an Iterated Function System (IFS).

Geometric construction

Iterated function systems



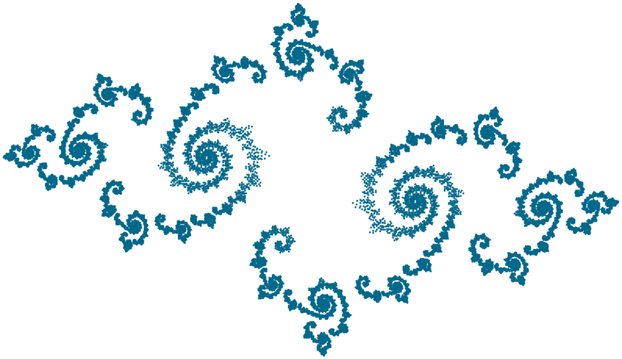


- Every fractal is approached by an IFS [Barnsley 1988].

Geometric construction

“non-trivial” fractals: Julia sets

- Define a recurrence $z_{n+1} = z_n^2 + c$
- Julia set: boundary of the convergence domain

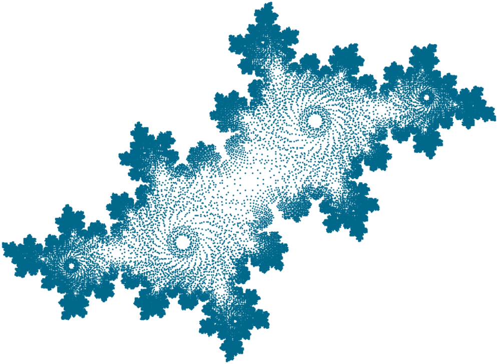


Julia set $c = -0.77 + 0.22i$

Geometric construction

"non-trivial" fractals: Julia sets

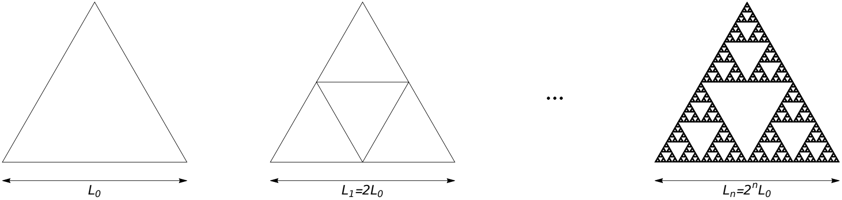
- Define a recurrence $z_{n+1} = z_n^2 + c$
- Julia set: boundary of the convergence domain



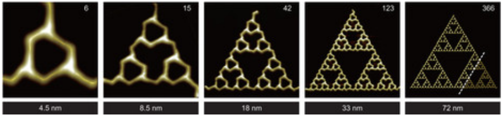
Julia set $c = -0.39 - 0.59i$

Scaling

- Give a physical meaning → give a length scale



- A natural way of doing it:

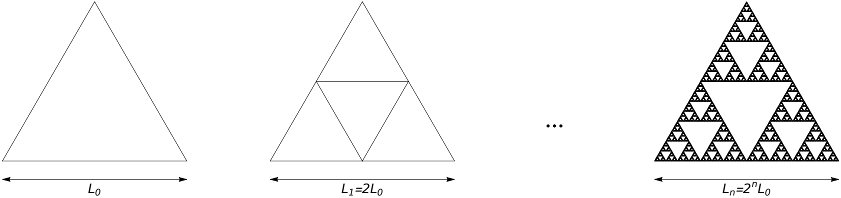


Assembling molecular Sierpiński triangle fractals, Nature Chemistry (2015)

- Scaling of physical quantities?
  - $M(L) \propto L^d$ on a d-dimensional Euclidean manifold... What happens on a fractal one?

The mass dimension

- $M(L) \propto L^d$ on a d-dimensional Euclidean manifold... What happens on a fractal one?

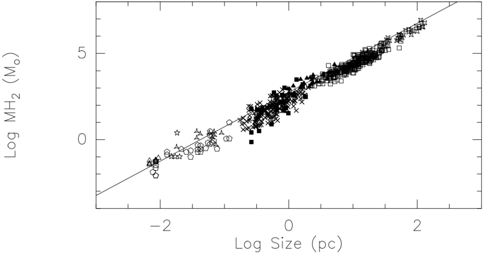


M_0 $\qquad$ $M_1 = 3M_0$ $\qquad$ $M_n = 3^n M_0$

$$M(L) \propto L^{d_M}, \text{ with } d_M = \log 3/\log 2$$

- d_M is the mass (or Hausdorff) dimension.
- $1 < d_M < 2$ different from $d = 1$, non-integer $\rightarrow$ signature of a fractal manifold.

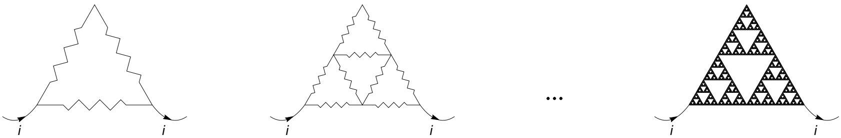
- Mass dimension: spot and characterize fractals, from large scales...



[Taken from *Astrophysical Fractals: Interstellar Medium and Galaxies*]

- ... to small ones

The electric dimension



R_0 $\quad$ $R_1 = ?R_0$ $\quad$ $R_n = ?^n R_0$

$$R(L) \propto L^{d_e}, \text{ with } d_e = \log ? / \log 2$$

- $d_M \neq d_e$, and they both reflect the structure of the fractal manifold.
- On an Euclidean manifold d_M and d_e would have been independant its structure, they would have only depended on d, its dimension.

The Fibonacci molecule

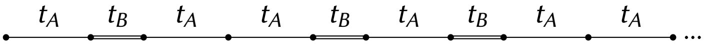
- The Fibonacci sequence:

  B

  A

  AB

- The Fibonacci molecule:



- The molecule is **not** a fractal!
- A quantity of interest: molecular spectrum

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  A

  AB

  ABA

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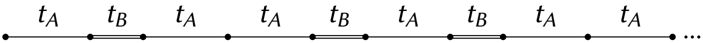

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  AB
  ABA
  ABAAB
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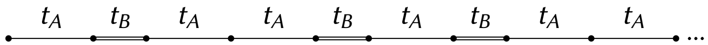

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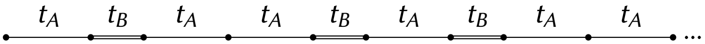

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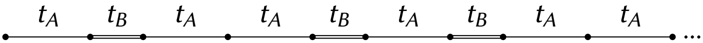

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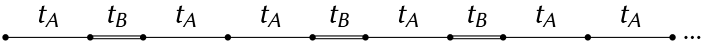

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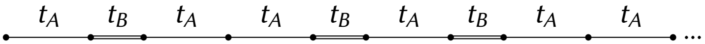

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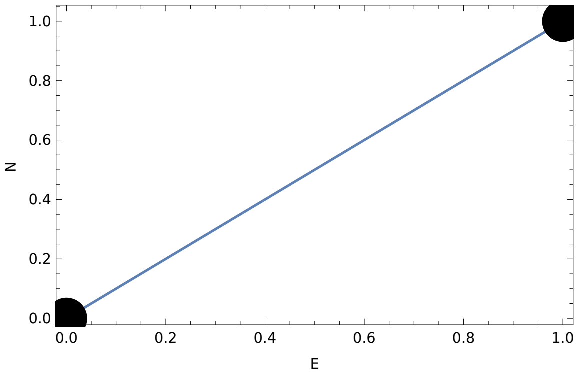
Spectrum of Fibonacci molecules

ABAABABAABAABABAABABAABAABABAABAABABAABABAABAABABAABABAABAABABAABAABABAABABAABAABABAABAAB



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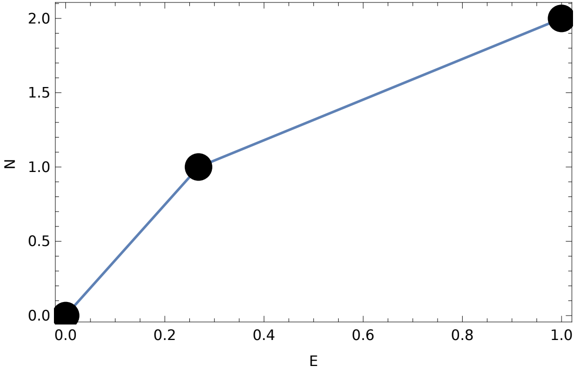

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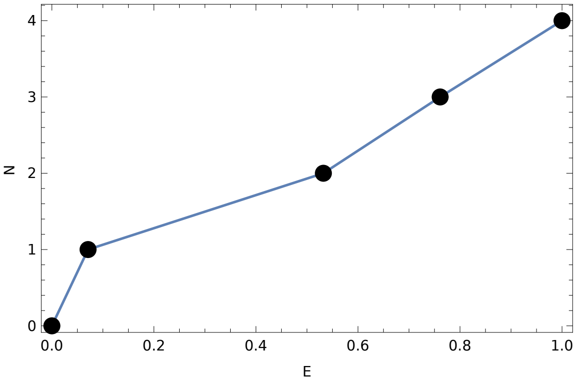

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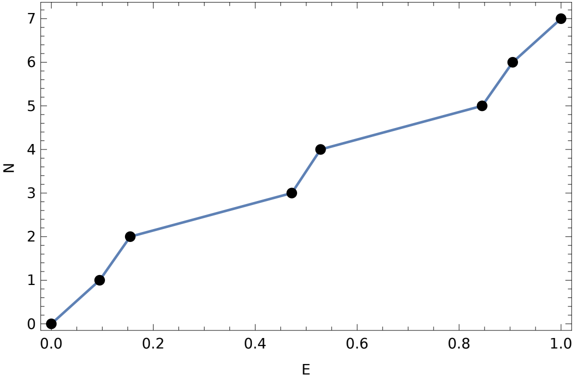

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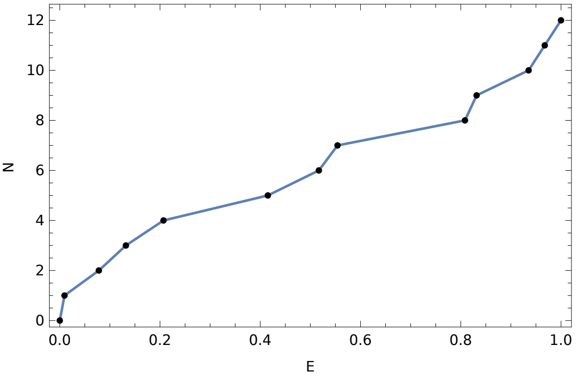

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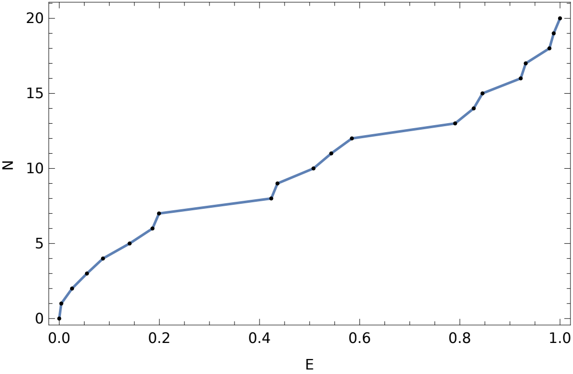

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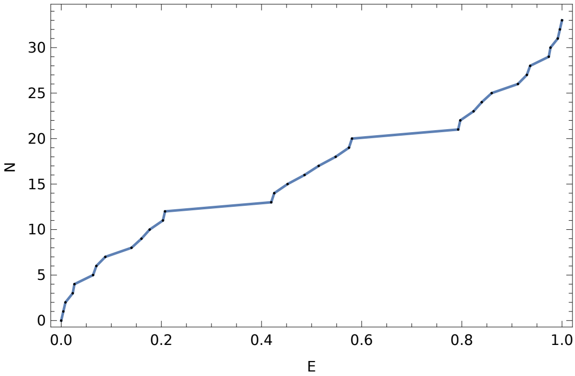

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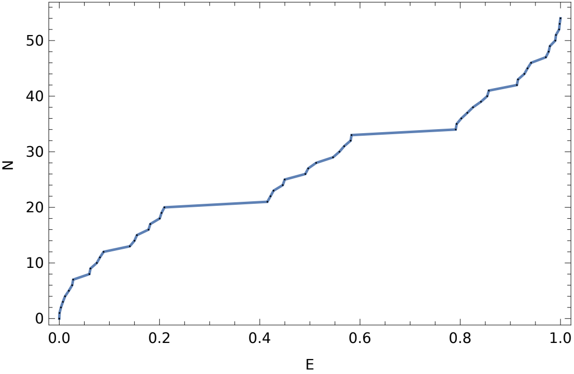

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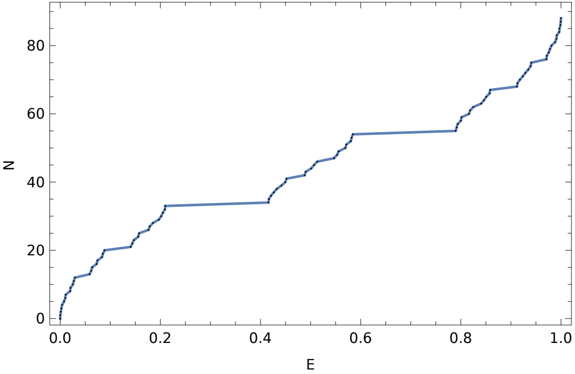
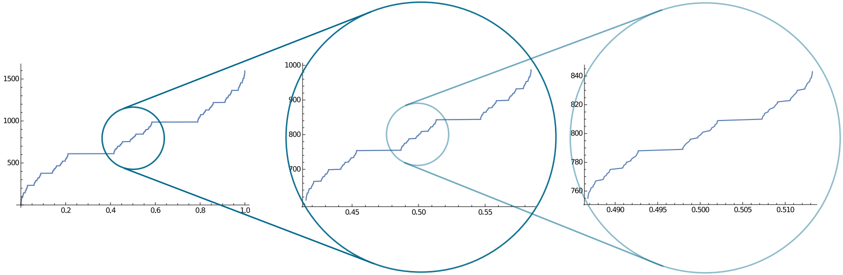


Hidden fractals: the Fibonacci chain

- No obvious fractal nature, but...



... the graph of the density of states is a fractal!

- What can we say about a scale invariant function?

Interlude: discrete and continuous scale invariance

Continuous scale invariance

$$\forall a,\ f(ax) = b(a)f(x)$$

Then $f(x) = Cx^{\alpha}$

Discrete scale invariance

$$\exists a,\ f(ax) = b(a)f(x)$$

Then $f(x) = ?$

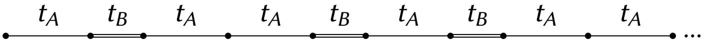
$$f(x) = G\left(\frac{\ln x}{\ln a}\right) x^{\alpha}$$ [Saleur, Sornette 1996]

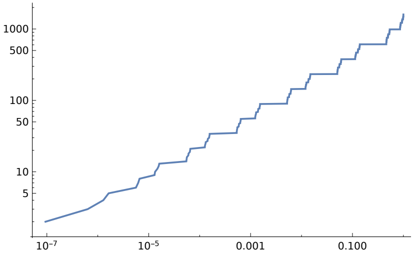
First order expansion:

$$f(x) \simeq \left[G_0 + G_1 \cos\left(2\pi \frac{\ln x}{\ln a}\right)\right] x^{\alpha}$$

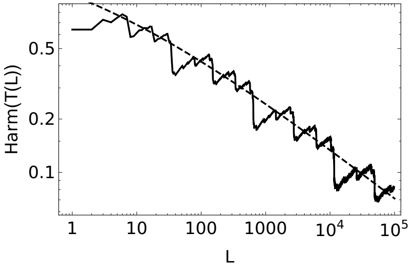
$\rightarrow$ **log-periodic oscillations**

Fibonacci and log–periodic oscillations





Spectrum of the molecule



Transmission probability [Macé *et al* 2017]

Summary

- Fractal manifolds and the study of discrete scaling symmetry is a fasinating mathematical topic in itself
- Fractals arise in surprisingly diverse areas of physics
- ... but, always, they are beautiful!