AN EXPLICIT CONSTRUCTION OF THE DERIVED MODULI STACK OF HARDER-NARASIMHAN FILTRATIONS

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Abstract. In this article, we give an explicit construction of the derived moduli stack of Harder-Narasimhan filtrations on a connected projective scheme over an algebraically closed field k of characteristic 0 by using the methods in [BCFHR14]. Moreover, we describe the derived deforamtion theory of a filtered sheave on a connected projective scheme over k.

1. Introduction

Moduli spaces of sheaves are constructed by using the methods of geometric invariant theory. In the constructions, they are constructed as quotients of open subschemes of Quot schemes which parametrize semistable sheaves. The moduli schemes parametrize only semistable sheaves. Unstable sheaves have unique filtrations described by semistable sheaves, which are called Harder-Narasimhan(HN) filtration. Although moduli spaces parametrizing unstable sheaves are not constructed as schemes, but as stacks. Moreover, moduli stacks of HN filtrations with fixed HN types are constructed as quotient stacks of open subschemes of relative flag schemes (for example, see [Yos09]).

On the other hand, recent developments of derived algebraic geometry are remarkble. As you can see in [PTVV13], the theory of derived algebraic geometry provides a suitable framework for studying symplectic geometry of moduli stacks. Actually, the authers proved that the derived moduli stack of perfect complexes on a smooth proper Calabi-Yau variety admits a shifted symplectic strucuture.

We are interested in derived moduli stacks of HN-filtrations. In particular, the aim of this paper is to give explicit constructions of derived moduli stacks of HN-filtrations as quotient derives stacks. In the case of stable sheaves, explicit constructions of derived moduli spaces are studied in [BCFHR14] or [BSY20]. Moreover, explicit construction of derived Quot and Hilbert schemes are studied in [CFK02] and [CFK02], respectively. As a functorial approach, it is proved that derived moduli stacks of filtered perfect complexes on a proper smooth schemes are locally geometric in [DN17] by using a model strucuture of the category of filtered complexes of modules over rings.

We use the methods in [BCFHR14]. In the work, the authers explicitly describe the moduli space of semistable sheaves on a projective variety X as an open substack of the stack of truncated graded A-modules, where $A := \oplus_{i\geq 0}\Gamma(X, \mathscr{O}_X(i))$. In detail, they use a functor from the category of coherent sheaves on X to the category of $[p, q]$-graded A-modules defined by $\Gamma_{[p,q]}(\mathscr{F}) := \oplus_{i=p}^{q}\Gamma(X, \mathscr{F}(i))$, where $\mathscr{F}$ is a coherent sheaf on X and p, q are nonnegative integers. Then, a derived enhancement of the moduli of sheaves is constructed by using the structure of the derived moduli of graded A-modules from Hochschild cohomology. However, we need have some obstacles in order to use their methods. We need more detailed analysis of homological algebra of filtered (graded) modules which is developed by Năstăsescu and Van Oystaeyen ([NVO79]) et al. We also need to bridge between the deformation theory of filtered (graded) modules and that of filtered sheaves. We have to study the relation between HN-filtrations of sheaves and of graded A-modules to get the explicit description. For this, the work of Hoskins ([Hos18]) are useful. The auther study HN-filtrations of quiver representations. However, the definition of slopes of graded A-modules is a little different from that of quiver representations. In Section 2, we tackle these problems and describe the image of the moduli stack of HN-filtrations with a fixed HN-type on X inside the moduli stack of filtered $[p,q]$-graded A-modules explicitly. In Section 3, we construct a derived enhancement of moduli stacks of HN-filtrations and get the tangent complex at a point $[0=\mathscr{F}_0\subset\mathscr{F}_1\subset\cdots\subset\mathscr{F}_s]$ of the derived stack of HN-filtration on X.

Moreover, our construction is useful to study symplctic geometry of derived moduli stacks of HN-filtrations because a theory for studying symplectic geometry of derived quotient stacks have been developed by Yeung ([Yeu19], [Yeu21]), recently.

Notation and conventions. We work over an algebraically closed field k of characteristic zero. For $m,n\in\mathbb{N}$, " for $0\ll m\ll n$ " means $\exists m_0\forall m\geq m_0\exists n_0\forall n\geq n_0$.

2. Preliminaries

2.1. Derived moduli schemes associated to bundles of curved differential graded Lie algebras.

Definition 2.1. ([BCFHR14]) A curved differential graded Lie algebra is a quadruple $(L^\bullet,f,d,[\cdot,\cdot])$, where $(L^\bullet,[\cdot,\cdot])$ is a $\mathbb{Z}_{>0}-$graded Lie algebra, $f\in L^2$, and $d:L^\bullet\to L^\bullet$ is a degree one morphism of graded $k-$vector spaces such that (1) $d(f)=0$, (2) $d\circ d=[f,\cdot]$.

Remark 2.2. f is called the curving and d the twisted differential. If $f=0$, then a triple $(L^\bullet,d,[\cdot,\cdot])$ is called a differential graded Lie algebra (dgla).

Definition 2.3. ([BCFHR14]) Let $(L^\bullet,d,[\cdot,\cdot])$ be a dgla. Let $a\in L^1$. Then, a is a Maurer–Cartan element if a satisfies the equation

$$da+\frac{1}{2}[a,a]=0.$$

The set of Maurer-Cartan elements is denoted by $\mathrm{MC}(L)$.

Theorem 2.4. *([BCFHR14, BN18]) Let X be a scheme over k. Let $\mathscr{L}$ be a bundle of curved differential graded Lie algebra (curved dgla) over X. Then, we can associate a sheaf of differential graded algebras $\mathscr{R}_X$ by letting the underlying sheaf of graded $\mathscr{O}_X-$algebra be*

$$\mathscr{R}_X:=\mathrm{Sym}_{\mathscr{O}_X}\mathscr{L}[1]^\vee.$$

The derivation q on $\mathscr{R}_X$ is defined to be $q=q_1+q_2+q_3$, where $q_0:\mathscr{L}[1]^\vee\to\mathrm{Sym}^0_{\mathscr{O}_X}\mathscr{L}[1]^\vee$ is defined by the curving morphism , $q_1:\mathscr{L}[1]^\vee\to\mathrm{Sym}^1_{\mathscr{O}_X}\mathscr{L}[1]^\vee=\mathscr{L}[1]^\vee$ by the twisted differential and $q_2:\mathscr{L}[1]^\vee\to\mathrm{Sym}^2_{\mathscr{O}_X}\mathscr{L}[1]^\vee$ by the bracket.

This defines a differential graded (dg) scheme $(\mathscr{R}_X,q)$(about dg-schemes, see [CFK01]).

Example 2.5. *([BCFHR14, BN18]) Let $(L^\bullet,d,[\cdot,\cdot])$ be a dgla. $X:=L^1=\mathrm{Spec}(\mathrm{Sym}(L^{1\vee}))$. And, $\mathscr{L}^i\to L^1$ is the trivial vector bundle with fiber $L^i(i\geq 2)$. Then, $\{\mathscr{L}^i\}_{\geq 2}$ have a structure of bundles of curved dgla over L^1 by*

$$\begin{aligned}
&\textit{Curvatue} && f:L^1\to\mathscr{L}^2,\quad a\mapsto(a,da+\frac{1}{2}[a,a]),\\
&\textit{Differential} && d':\mathscr{L}^i\to\mathscr{L}^{i+1},\quad (\mu,b)\mapsto(\mu,db+[\mu,b]),\\
&\textit{Bracket} && [\cdot,\cdot]':\mathscr{L}^i\times\mathscr{L}^j\to\mathscr{L}^{i+j},\quad ((\mu,b),(\lambda,c))\mapsto(\mu+\lambda,[b,c]).
\end{aligned}$$

In addtion, we have an isomorphism $\mathrm{Spec}(H^0(\mathscr{R}_X))\simeq\mathrm{MC}(L)=Z(f)$.

2.2. **Homological algebra of filtered modules.** Let R be a unital commutative ring

Definition 2.6. ([Luc71], [Kre14], [NVO79]) A filtered module M is a R-module M with a ascending chains $\{M_i \mid i \in \mathbb{Z}\}$ of A-submodule of M such that if $i \leq 0$, $M_i = 0$ and for $i \gg 0$, $M_i = M$. When we denote the minimum integer of the integers by s, we write

$$0 = M_0 \subset M_1 \subset \cdots \subset M_s = M$$

And, let M_1, M_2 be filtered $R-$modules. A homomorphism from M_1 to M_2 is a $R-$module homomorphism from M_1 to M_2 preserving their filtration. We denote the set of homomorphisms by $\mathrm{Hom}_{R-\mathrm{mod}_-}(M_1, M_2)$.

Definition 2.7. ([Luc71], [Kre14], [NVO79]) Let M be a filtered R-module. Then, associated graded module $\mathrm{gr}(M)$ is defined to be

$$\mathrm{gr}(M) := \oplus M_i/M_{i-1} \quad \mathrm{gr}_i(M) := M_i/M_{i-1}.$$

Definition 2.8. ([Luc71], [Kre14], [NVO79]) Let I be a filtered R-module. We say I is a filtered injective module if the components $\mathrm{gr}_i(I)$ of the associated graded module are injective R-modules.

Definition 2.9. ([Luc71], [Kre14], [NVO79]) Let M be a filtered R-module. Let $I^0, I^1, \cdots$ be filtered injective R-module. Then, a filtered injective resolution is a exact sequence of R-modules

$$0 \to M \to I^0 \to I^1 \to \cdots$$

such that induced sequences

$$0 \to \mathrm{gr}_i(M) \to \mathrm{gr}_i(I^0) \to \mathrm{gr}_i(I^1) \cdots$$

are the injective resolutions of $\mathrm{gr}_i(M)$.

Definition 2.10. ([DLP85, HL10],) Let M, N be a filtered R-module. Let $0 \to M \to I^\bullet$ be a filtered injective resolution of R. Then,

$$\mathrm{Ext}^i_{R,-}(N, M) := H^i(\mathrm{Hom}_{R,-}(N, I^\bullet)).$$

Remark 2.11. In the above definition, $\mathrm{Ext}_{R,-}(N, M)$ are independent of the choice of the filtered resolution of M. In [DLP85, HL10], $\mathrm{Ext}^i_-(N, M)$ are defined when M, N are filtered coherent sheaves on algebraic varieties.

Theorem 2.12. *([DLP85, HL10]) Let M, N be filtered R-modules.There are spectral sequences.*

$$\mathrm{Ext}^{p+q}_{R,-}(N, M) \Leftarrow E_1^{pq} = \begin{cases} 0 & p < 0 \\ \prod_i \mathrm{Ext}^{p+q}_R(\mathrm{gr}_i(N), \mathrm{gr}_{i-p}(M)) & p \geq 0 \end{cases}$$

2.3. **Harder-Narasimhan filtration of sheaves and modules.**

2.3.1. *For sheaves.* Let X be a projective k-scheme of finite type and $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X.

Definition 2.13. Let $P(t), Q(t) \in \mathbb{Q}[t]$. We say $P(t)(\succeq)Q(t)$ if

$$\frac{P(m)}{P(n)}(\geq)\frac{Q(m)}{Q(n)}$$

Definition 2.14. ([BCFHR14, Hos18]) Let $\mathcal{F}$ be a coherent sheaf on X. $\mathcal{F}$ is (semi)stable if for every proper nonzero subsheaf $\mathcal{E}$, $P(\mathcal{F}, t)(\succeq)P(\mathcal{E}, t)$, where $P(\mathcal{F}, t)$ and $P(\mathcal{E}, t)$ are the Hilbert polynomials of $\mathcal{F}$ and $\mathcal{E}$ with respect to $\mathcal{O}_X(1)$ respectively.

Definition 2.15. ([Hos18]) Let $\mathcal{F}$ be a coherent sheaf on X. The Harder-Narasimhan(HN) filtration of $\mathcal{F}$ is a filtration

$$0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^s = \mathcal{F}$$

such that the $\mathcal{F}_i := \mathcal{F}^i/\mathcal{F}^{i-1}$ are semistable and $P(\mathcal{F}_1, t) \succ P(\mathcal{F}_2, t) \succ \cdots \succ P(\mathcal{F}_s, t)$.

Theorem 2.16. *([Hos18]) Every coherent sheaf on X has a unique HN-filtration*

Remark 2.17. Our definitions of stability and HN-filtration are different from those in [HL10]. However, these definitions coincide when $\mathcal{F}$ is a pure sheaf. A benefit of our definitions is that we can consider all coherent sheaves within the non-pure sheaves.

2.3.2. *For modules.* Let A be a graded k-algebra which is all in nonnegative degrees and each graded piece is finite-dimensional and $A_0 = k$.

Definition 2.18. ([BCFHR14]) Let M be a $[p, q]$-graded k-module. Let $\lambda : A \otimes_k M \to M$ be a homomorphism of graded k-vector space. Then, we call M a λ-module and a k-submodule N of M is a λ-submodule of M if $\lambda(A \otimes_k N) \subset N$.

Remark 2.19.
- In the above definition, λ is determined by $\lambda|_{A_{[0,q-p]} \otimes_k M}$
- Any $[p, q]$-graded A-module M has a natural λ-module structure from the A-module structure of M.

Definition 2.20. Let M be a $[p, q]$-graded k-module with $\mathrm{dim} M_p + \mathrm{dim} M_q \neq 0$. Let $\theta_p, \theta_q \in \mathbb{Z}$, then

$$\mu_{(\theta_p, \theta q)}(M) := \frac{\theta_p \mathrm{dim} M_p + \theta_q \mathrm{dim} M_q}{\mathrm{dim} M_p + \mathrm{dim} M_q}.$$

We write μ for $\mu_{(\theta_p, \theta_q)}$ when θ_p, θ_q are obvious.

Definition 2.21. ([BCFHR14]) Let M be a $[p, q]$-graded λ-module . Let $\theta_p, \theta_q \in \mathbb{Z}$. M is (semi)stable with respect to (θ_p, θ_q) if for every nonzero proper λ-submodule N, $\mathrm{dim} N_p = \mathrm{dim} N_q = 0$ or " $\mathrm{dim} N_p + \mathrm{dim} N_q \neq 0$ and $\mu_{(\theta_p, \theta_q)}(N)(\leq)\mu_{(\theta_p, \theta_q)}(M)$ " holds.

Definition 2.22. Let M be a $[p, q]$-graded λ-module. Let $\theta_p := \mathrm{dim} M_q$ and $\theta_p := -\mathrm{dim} M_p$. The Harder-Narasimhan(HN) filtration of M is a filtration of λ-submodules of M

$$0 = M^0 \subset M^1 \subset \cdots \subset M^s = M$$

such that the M^i/M^{i-1} are semistable with respect to (θ_p, θ_q) and $(M^1/M^0) \succ (M^2/M^1) \succ \cdots \succ (M^s/M^{s-1})$.

For two $[p, q]$-graded λ-modules N^1, N^2, we say $N^1(\succeq)N^2$ if $\mathrm{dim} N^2_p = \mathrm{dim} N^2_q = 0$ or " $\mathrm{dim} N^1_p + \mathrm{dim} N^1_q \neq 0$ and $\mathrm{dim} N^2_p + \mathrm{dim} N^2_q \neq 0$ and $\mu_{(\theta_p, \theta_q)}(N^1)(\geq)\mu_{(\theta_p, \theta_q)}(N^2)$ ".

Remark 2.23. The above definition is similar to that of quiver representation, but different from it. The difference derives from the existence of the components $M_{p+1}, \cdots M_{q-1}$ (cf. [Rei03], [Zam14] or [Hos18])

Theorem 2.24. *Let M be a $[p, q]$-graded λ-module. There exists a HN-filtration of M*

Proof. We can prove this in the same way as in the proof of [Rei03, Proposition 2.5] or [Zam14, Theorem 2.6] □

Remark 2.25. Every $[p, q]$-graded λ-module dose not necessarily have a unique HN-filtration because the relation $\succ$ in Definition 2.22 is not a stability structure on [Rud97, Def 1.1] (i.e., the seesaw property does not hold).

3. Moduli stacks of Harder-Narasimhan filtrations

Notation 3.1. • X : a connected projective scheme over k.

- $\mathscr{O}_X(1)$: a very ample imbertible sheaf on X.
- $A := \Gamma_*(\mathscr{O}_X) = \oplus_{i\geq 0}\Gamma(X, \mathscr{O}(i))$.
- $\mathcal{C}\mathrm{oh}_\alpha(X)$: the stack of coherent sheaves with Hilbert polynomial α on X.
- $\mathcal{F}\mathcal{C}\mathrm{oh}_{(\alpha_1,\cdots,\alpha_s)}(X)$: the stack of filtered coherent sheaves on X of type $(\alpha_1,\cdots,\alpha_s)$
- $\mathcal{F}\mathcal{C}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X)$: the stack of Harder-Narasimhan filtration (of sheaves) of type $(\alpha_1,\cdots,\alpha_s)$ on X.
- $\mathcal{M}\mathrm{od}^{[p,q]}_\alpha(A)$: the stack of graded A-modules of type $\alpha|_{[p,q]}$ in degree $[p,q]$.
- $\mathcal{F}\mathcal{M}\mathrm{od}^{[p,q]}_{(\alpha_1,\cdots,\alpha_s)}(A)$: the stack of filtered graded A-modules of type $(\alpha_1|_{[p,q]},\cdots,\alpha_s|_{[p,q]})$ in degree $[p,q]$, where $\alpha_i|_{[p,q]}$ means a tuple $(\alpha_i(p),\cdots,\alpha_i(q))$.
- $\mathcal{F}\mathcal{M}\mathrm{od}^{[p,q],\mathrm{sfg}}_{(\alpha_1,\cdots,\alpha_s)}(A)$: the stack of strict finitely generated graded A-modules of type $(\alpha_1|_{[p,q]},\cdots,\alpha_s|_{[p,q]})$ in degree $[p,q]$.
- $\mathrm{Coh}(Y)$: the category of coherent sheaves on a scheme Y.
- $\mathrm{FCoh}(Y)$: the category of filtered coherent sheaves on a scheme Y.
- $\mathrm{Mod}^{[p,q]}(A\otimes\mathscr{O}_Y)$: the category of graded coherent $A\otimes\mathscr{O}_Y$-modules on Y in the degree $[p,q]$.
- $\mathrm{FMod}^{[p,q]}(A\otimes\mathscr{O}_Y)$: the category of filtered graded coherent $A\otimes\mathscr{O}_Y$-modules on Y in the degree $[p,q]$.
- $\Gamma_{[p,q]}(\mathscr{F}) := \oplus_{i=p}^q \pi_*(\mathscr{F}(i))$ for a coherent sheaf $\mathscr{F}$ on $X\times_k Y$ and the natural projection $\pi : X\times_k Y \to Y$.

Remark 3.2. When considering any filtered $[p,q]$-graded A-modules $0 = M_0 \subset M_1 \subset \cdots \subset M_s = M$, it is strictly finitely generated if each M_{i+1}/M_i is graded A-module which is generated in degree p (thus each M_i is generated in degree p).

3.1. **Open embeddings.** In this subsection, we construct an open immersion from the moduli stack of filtered sheaves to that of filtered modules over A

First, we can define the following morphism

$$\Gamma^{\mathrm{fil}}_{[p,q]} : \mathrm{FCoh}(X\times_k Y) \to \mathrm{FMod}^{[p,q]}(A\otimes\mathscr{O}_Y) \tag{1}$$

by $\Gamma^{\mathrm{fil}}_{[p,q]}(\mathscr{F}) := \Gamma_{[p,q]}(\mathscr{F}) = \Gamma_{[p,q]}(\mathscr{F}_s) \supset \cdots \supset \Gamma_{[p,q]}(\mathscr{F}_1) \supset \Gamma_{[p,q]}(\mathscr{F}_0) = 0$ for any object $\mathscr{F} = \mathscr{F}_s \supset \cdots \supset \mathscr{F}_1 \supset \mathscr{F}_0 = 0$ because pushforwards are left exact.

The morphism of category

$$\Gamma_{[p,q]} : \mathrm{Coh}(X\times_k Y) \to \mathrm{Mod}^{[p,q]}(A\otimes\mathscr{O}_Y)$$

has the left adjoint $\mathscr{S}$ (see [BCFHR14, Proposition 3.1]).

3.1.1. *Monomorphisms.*

Lemma 3.3. $\Gamma^{\mathrm{fil}}_{[p,q]}|_{\mathcal{F}\mathcal{C}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X)} : \mathcal{F}\mathcal{C}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X) \to \mathcal{F}\mathcal{M}\mathrm{od}^{[p,q]}_{(\alpha_1,\cdots,\alpha_s)}(A)$ *is monomorphism if* $q \gg p \gg 0$.

Proof. First, let $S := \{\mathscr{F} \in \mathrm{Coh}(X) \mid \text{the HN-type of } \mathscr{F} \text{ is } (\alpha_1,\cdots\alpha_s)\}$. Then, the sets $S_i = \{\mathrm{gr}_i(\mathscr{F}) \mid \mathscr{F}\in S\}(i\in\{1,\cdots,s\})$ are bounded because the stack of HN-filtration of type $(\alpha_1,\cdots,\alpha_s)$ is a quotient of a relative flag scheme by an algebrac group (see [Yos09, Lemma 2.5]). So, for $p\gg 0$, any sheaf in $\cup_i S_i$ is p-regular. So, for $p\gg 0$, $\Gamma^{\mathrm{fil}}_{[p,q]}|_{\mathcal{F}\mathcal{C}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X)}$ is well-defined.

In order to prove $\Gamma^{\mathrm{fil}}_{[p,q]}|_{\mathcal{F}\mathcal{C}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X)}$ is a monomorphism, we will prove this is fully faithful because monomorphisms between algebraic stacks are the same as fully faithful functors ([Sta, Lemma 98.8.4]).

If $\mathscr{F},\mathscr{G}$ are objects of $\mathcal{F}\mathcal{C}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X)$, we have an isomorphism

$$\mathrm{Hom}_{\mathscr{O}_{X\times Y}}(\mathscr{F},\mathscr{G}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{gr},A\otimes\mathscr{O}_Y}(\Gamma_{[p,q]}(\mathscr{F}),\Gamma_{[p,q]}(\mathscr{G})) \quad q \gg p \gg 0$$

because the induced morphism $\Gamma_{[p,q]} : \mathcal{U}_\alpha \to \mathcal{M}\mathrm{od}^{[p,q]}_\alpha(A)$ is a monomorphism for $q \gg p \gg 0$([BCFHR14, Proposition 3.2]), where $\mathcal{U}_\alpha$ is an open substack of $\mathcal{C}\mathrm{oh}_\alpha(X)$ of finite type. So, $\Gamma^{\mathrm{fil}}_{[p,q]}$ is faithful.

Next we show that $\Gamma^{\mathrm{fil}}_{[p,q]}$ is full. For this, we take integers p,q such that $\Gamma_{[p,q]} : \mathcal{U}_{\alpha_i} \to \mathcal{M}\mathrm{od}^{[p,q]}_{\alpha_i}(A)$ are monomorphism. Let $\psi : \Gamma^{\mathrm{fil}}_{[p,q]}(\mathscr{F}) \to \Gamma^{\mathrm{fil}}_{[p,q]}(\mathscr{G})$ be a morphism in $\mathcal{F}\mathcal{M}\mathrm{od}^{[p,q]}(A)$. From the above isomorphism, there exists a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ such that $\Gamma_{[p,q]}(\varphi) = \psi$. Then, it is sufficient to see that φ is a morphism in $\mathcal{F}\mathcal{C}\mathrm{oh}(X)^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}$. We may assume that $s = 2$. And, we consider the following diagrams and the corresponce by $\Gamma_{[p,q]}$ and $\mathscr{S}$.

$$\left(\begin{tikzcd} \mathscr{F}_2 \arrow[r, "\varphi"] & \mathscr{G}_2 \\ \mathscr{F}_1 \arrow[u, hook, "i_1"] \arrow[r, "\varphi'"] & \mathscr{G}_1 \arrow[u, hook, "i_2"'] \end{tikzcd}\right) \overset{\Gamma_{[p,q]}}{\mapsto} \left(\begin{tikzcd} \Gamma_{[p,q]}\mathscr{F}_2 \arrow[r, "\psi"] & \Gamma[p,q]\mathscr{G}_2 \\ \Gamma_{[p,q]}\mathscr{F}_1 \arrow[u, hook, "\Gamma_{[p,q]}i_1"] \arrow[ur, phantom, "\circlearrowleft"] \arrow[r, "\psi|_{\Gamma_{[p,q]}\mathscr{F}_1}"] & \Gamma_{[p,q]}\mathscr{G}_1 \arrow[u, hook, "\Gamma_{[p,q]}i_2"'] \end{tikzcd}\right) \overset{\mathscr{S}}{\mapsto} \left(\begin{tikzcd} \mathscr{S}\Gamma_{[p,q]}\mathscr{F}_2 \arrow[r, "\mathscr{S}\psi"] & \mathscr{S}\Gamma_{[p,q]}\mathscr{G}_2 \\ \mathscr{S}\Gamma_{[p,q]}\mathscr{F}_1 \arrow[u, "\mathscr{S}\Gamma_{[p,q]}i_1"] \arrow[ur, phantom, "\circlearrowleft"] \arrow[r, "\mathscr{S}\psi|_{\Gamma_{[p,q]}\mathscr{F}_1}"] & \mathscr{S}\Gamma_{[p,q]}\mathscr{G}_1 \arrow[u, "\mathscr{S}\Gamma_{[p,q]}i_2"'] \end{tikzcd}\right)$$

, where φ and φ' are the morphisms corresponding to ψ and $\psi|_{\Gamma_{[p,q]}\mathscr{F}_1}$ respectively. And, the middle and the right squares are commutative. A diagram chase and the monomorphisity of $\Gamma_{[p,q]}$ yield the commutativity of the left diagram above because that $\Gamma_{[p,q]}$ is fully faithful and $\mathscr{S}$ is a left adjoint to $\Gamma_{[p,q]}$ is equivalnt to that $\mathscr{S}\circ\Gamma_{[p,q]} \simeq \mathrm{id}$ ([Sta, Lemma 4.24.4]).

□

3.1.2. *Étale morphisms.* Here, we prove that the morphism of algebraic stacks

$$\Gamma^{\mathrm{fil}}_{[p,q]} : \mathcal{F}\mathcal{C}\mathrm{oh}_{(\alpha_1,\cdots,\alpha_s)}(X) \to \mathcal{F}\mathcal{M}\mathrm{od}^{[p,q]}_{(\alpha_1,\cdots,\alpha_s)}(A)$$

is étale. To prove this, we need to consider the deformation theories of $\mathcal{F}\mathcal{C}\mathrm{oh}_{(\alpha_1,\cdots,\alpha_s)}(X)$ and $\mathcal{F}\mathcal{M}\mathrm{od}^{[p,q]}_{(\alpha_1,\cdots,\alpha_s)}(A)$ and compare them.

First, we consider $\mathcal{F}\mathcal{M}\mathrm{od}^{[p,q]}_{(\alpha_1,\cdots,\alpha_s)}(A)$. $0 = V_0 \subset V_1 \subset \cdots \subset V_s = V$ is a filtration of $[p,q]$-graded k-vector spaces such that $\dim_k(V_i)_j = \alpha_i(j)$, where $V_i := \oplus_{i=p}^q (V_i)_j$. Then,

$$L := \bigoplus_{n=0}^\infty L^n := \bigoplus_{n=0}^\infty \mathrm{Hom}_{k-\mathrm{gr}}(A^{\otimes n}, \mathrm{End}_k(V)) = \mathrm{Hom}_{k-\mathrm{gr}}(A^{\otimes n}\otimes V, V)$$

has a dgla structure by the Hochschild differential d and the Gerstenhaber bracket $[\cdot,\cdot]$ (in detail, see [BCFHR14] or [BN18]). Note that when let $G := \mathrm{GL}_{\mathrm{gr}}(V)$ (called the gauge group), we have an action of G on L called the gauge action. However, in our situation, we need to consider the filtration of V. So, we consider the following graded vector spaces similar to L.

$$L_- := \bigoplus_{n=0}^\infty L^n_- := \bigoplus_{n=0}^\infty \mathrm{Hom}_{k-\mathrm{gr}}(A^{\otimes n}, \mathrm{End}_{k,-}(V)) = \bigoplus_{n=0}^\infty \mathrm{Hom}_{k-\mathrm{gr},-}(A^{\otimes n}\otimes V, V)$$

(for the above notation, see subsection 2.2). Note that L_- is a sub graded k-vector space of L. We can easily see Hochschild differential and the Gerstenhaber bracket of L is closed in L_-. So, $(L_-.d,[\cdot,\cdot])$ is also a dgla. We also consider the parabolic subgroup P of G consisting of elements preserving the filtration of V. This also induces an action on L_-. And, the Maurer-Cartan locus $\mathrm{MC}(L_-) = \{\mu \in \mathrm{Hom}_{k,-}(V,V) \mid \mu \text{ induces a filtered graded } A\text{-module structure }\}$. So, we have

$$[\mathrm{MC}(L_-)/P] \simeq \mathcal{F}\mathcal{M}\mathrm{od}^{[p,q]}_{(\alpha_1,\cdots,\alpha_s)}(A) \tag{2}$$

Moreover, since L_-, is a dgla, we get a differential graded structure on L_- by using Theorem 2.4 and Example 2.5. Because the Lie algebra of P is $\mathrm{Hom}_{k-\mathrm{gr},-}(V,V)$, we can describe the tangent and obstruction spaces of a filtered graded A-module (V,μ) by describing the deformation theory of $[\mathrm{MC}(L_-)/P]$. I.e.,

$$\text{The infinitesimal deforamtion of } (V,\mu) \text{ is classified by } H^1((L_-, d^\mu, [\cdot,\cdot])),$$

$$\text{The obstructions of the deformation of } (V,\mu) \text{ are contained in } H^2((L_-, d^\mu, [\cdot,\cdot])).$$

, where $(L_-, d^\mu, [\cdot,\cdot])$ is a dgla whose differential d^μ is defined by $[\mu,\cdot]$ (for detail, see [BCFHR14] or [BN18]). Although the right-hand sides of the above equalities are equal to the Hochschild cohomology, we need a lemma to associate them with $\mathrm{Ext}_{k-\mathrm{gr},-}(-,-)$ functor (see Remark 3.5).

Definition 3.4. (for the case of ungraded filtered modules, see [Luc71], [Kre14], [NVO79]) Let P be a filtered graded A-module. Then, P is filtered projective if $\mathrm{gr}_i(P)$ is projective object in the category of graded A-modules for any i.

Let M be a filtered graded A-module. Then, a filtered graded projective resolution of M is a sequence

$$\cdots \to P^1 \to P^0 \to M \to 0$$

such that the induced sequences

$$\cdots \to \mathrm{gr}_i(P^1) \to \mathrm{gr}_i(P^0) \to \mathrm{gr}_i(M) \to 0$$

are projective resolutions of $\mathrm{gr}_i(M)$ in the category of graded A-module.

Remark 3.5. We have graded versions of Definition 2.6, 2.8, 2.9 as above. And, we can define a graded version $\mathrm{Ext}^i_{A-\mathrm{gr},-}(-,-)$ of the filtered Ext functor $\mathrm{Ext}^i_{A-gr,-}(-,-)$ in Definition 2.10 and have a graded version of Theorem 2.12.

Lemma 3.6. *For filtered graded A-modules M,N, we can calculate $\mathrm{Ext}_-(M,N)$ by filtered graded projective resolutions of M. I.e, we have*

$$\mathrm{Ext}^i_{A-\mathrm{gr},-}(M,N) = H^i(\mathrm{Hom}_{A-\mathrm{gr},-}(P^\bullet,N))$$

, where $P^\bullet \to M \to 0$ is any filtered graded projective resolution of M.

Proof. First, we construct a spectral sequence which is convergent to $H^i(\mathrm{Hom}_{A-\mathrm{gr},-}(P^\bullet,N))$. This is similar to the proof of [DLP85, Proposition 1.3]. We get a natural filtration of a chain complex $C = \mathrm{Hom}_{A-\mathrm{gr},-}(P^\bullet,N)$:

$$\begin{tikzcd}
C{=}F_0C: & 0 \arrow[r] & \mathrm{Hom}_{A-\mathrm{gr},-}(P^0,N) \arrow[r] & \mathrm{Hom}_{A-\mathrm{gr},-}(P^1,N) \arrow[r] & \mathrm{Hom}_{A-\mathrm{gr},-}(P^2,N) \arrow[r] & \cdots \\
F_1C: & 0 \arrow[r] & \mathrm{Hom}_{A-\mathrm{gr},-1}(P^0,N) \arrow[r] \arrow[u, hook] & \mathrm{Hom}_{A-\mathrm{gr},-1}(P^1,N) \arrow[r] \arrow[u, hook] & \mathrm{Hom}_{A-\mathrm{gr},-1}(P^2,N) \arrow[r] \arrow[u, hook] & \cdots \\
F_2C: & 0 \arrow[r] & \mathrm{Hom}_{A-\mathrm{gr},-2}(P^0,N) \arrow[r] \arrow[u, hook] & \mathrm{Hom}_{A-\mathrm{gr},-2}(P^1,N) \arrow[r] \arrow[u, hook] & \mathrm{Hom}_{A-\mathrm{gr},-2}(P^2,N) \arrow[r] \arrow[u, hook] & \cdots \\
 & & \vdots \arrow[u, hook] & \vdots \arrow[u, hook] & \vdots \arrow[u, hook] &
\end{tikzcd}$$

,where $\mathrm{Hom}_{A-\mathrm{gr},-i}(P^j,N) = \{f \in \mathrm{Hom}_{A-\mathrm{gr}}(P^j,N) \mid f((P^j)_k) \subset N_{k-i}\} (i \in \mathbb{N})$. Although filtrations in Section 2.2 are ascending, note that the above filtration is descending. So, we have a spectral sequence by [Wei94, Theorem 5.5.1] as follows :

$$H^{p+q}(\mathrm{Hom}_{\mathrm{gr},-}(P^\bullet,N)) \Leftarrow E_1^{pq} := \begin{cases} H^{p+q}(F_pC/F_{p+1}C) & p \geq 0 \\ 0 & p < 0 \end{cases}.$$

Because P^i is filtered graded projective A-module, by [NVO79, I, Lemma 6.4], we have $F_pC/F_{p+1}C \simeq \oplus_i \mathrm{Hom}_{A-\mathrm{gr}}(\mathrm{gr}_i(P^\bullet), \mathrm{gr}_{i-p}(N))$ if $p \geq 0$. So, we have

$$H^{p+q}(F_pC/F_{p+1}C) = \oplus_i \mathrm{Ext}^{p+q}_{A-\mathrm{gr}}(\mathrm{gr}_i(M), \mathrm{gr}_{i-p}(N)) \quad \text{if } p \geq 0.$$

We have a spectral sequence converging to $H^{p+q}(\mathrm{Hom}_{A-\mathrm{gr},-}(P^\bullet, N))$ and $\mathrm{Ext}_{A-\mathrm{gr},-}(M,N)$ (Theorem 2.12, Remark 3.5). Thus, we have $H^i(\mathrm{Hom}_{A-\mathrm{gr},-}(P^\bullet, N)) \simeq \mathrm{Ext}^i_{A-\mathrm{gr},-}(M,N)$. $\square$

We can describe the tangent space and the obstruction space of filtered graded A-module M from the above discussion and Lemma 3.6.

Proposition 3.7. *For any filtered graded A-module M, then*

The infinitesimal deforamtion of M is classified by $\mathrm{Ext}^1_{A-\mathrm{gr},-}(M,M)$,

The obstructions of the deformation of M are contained in $\mathrm{Ext}^2_{A-\mathrm{gr},-}(M,M)$.

Proof. If let M be a filtered graded A-module, we have the 'Tensor-Hom adoint' between the category of the filtered graded R-R modules and that of the filtered graded R-k modules (see [NVO79, Lemma 8.1]). In our situation, we have

$$\mathrm{Hom}_{A-A-\mathrm{gr},-}(B(A,A), \mathrm{Hom}_{k,-}(M,M)) \simeq \mathrm{Hom}_{A-k-\mathrm{gr}}(B(A,A) \otimes_A M, M),$$

where $B(A,A)$ is the Bar resolution of A and $B(A,A) \otimes_A M \to M \to 0$ is a filtered graded projective resolution of M. And note that A and M are thought as $A - A$ (filtered) graded bimodules by their (filtered) graded A-module structures. For any Hochschild cochain, a cochain complex which is quasi-isomorphism to it can be constructed by using Bar resolutions. We also describe this fact in our situation in the following:

$$\mathrm{Hom}_{k-\mathrm{gr},-}(A^{\otimes \bullet}, \mathrm{Hom}_{k,-}(M,M)) \simeq \mathrm{Hom}_{A-A-\mathrm{gr},-}(B(A,A), \mathrm{Hom}_{k,-}(M,M)).$$

Thus, from Lemma 3.6 and the above formulas, we have the conclusion (For the non filtered graded case, the proof are in [Wei94, Lemma 9.1.9]).

$\square$

Next, we compare the deformation theories of filtered sheaves on X and filtered graded A-modules. We use the method of [CFK01] where them of sheaves on X and graded A-modules are compared.

Proposition 3.8. *([Wal95]) For any filtered coherent sheaf $\mathscr{F}$, we can describe the deformation theory of F as follows:*

The infinitesimal deformation of $\mathscr{F}$ is classified by $\mathrm{Ext}^1_-(\mathscr{F}, \mathscr{F})$,

The obstructions of the deformation of $\mathscr{F}$ are contained $= \mathrm{Ext}^2_-(\mathscr{F}, \mathscr{F})$.

Lemma 3.9. *(a) If $\mathscr{F}, \mathscr{G}$ are filtered coherent sheaves on X, then*

$$\mathrm{Ext}^i_-(\mathscr{F}, \mathscr{G}) = \varinjlim_p \mathrm{Ext}^i_{A-\mathrm{gr},-}(\Gamma^{\mathrm{fil}}_{\geq p}(\mathscr{F}), \Gamma^{\mathrm{fil}}_{\geq p}(\mathscr{G})).$$

(b) If M, N are finitely generated filtered graded A-module, then for $q \gg 0$, we have

$$\mathrm{Ext}^i_{A-\mathrm{gr},-}(M,N) = \mathrm{Ext}^i_{A-\mathrm{gr},-}(M_{\leq q}, N_{\leq q}).$$

Moreover, if let Y be a projective scheme and let $\mathscr{M}, \mathscr{N}$ be filtered graded $A \otimes \mathscr{O}_Y$-modules whose coponents are locally free sheaves on Y, then we can choose a q such that for any pair $(\mathscr{M}_y, \mathscr{N}_y)(y \in Y)$, the above equality holds.

Proof. (a) Let $\mathcal{S}$ be a category whose objects are finitely generated A-modules and morphisms between two objects M and N are

$$\mathrm{Hom}_{\mathcal{S}}(M,N) := \varinjlim_p \mathrm{Hom}_{A-\mathrm{gr}}(M_{\geq p}, N_{\geq p})$$

([CFK01, page.406-407]). Note that the category $\mathrm{Coh}(X)$ is equivalent to the category $\mathcal{S}$ (see [CFK01, Theorem 1.2.2] or [Ser55]). If let $\mathrm{Hom}_{\mathcal{S},-}(M,N)$ be the subset of $\mathrm{Hom}_{\mathcal{S}}(M,N)$ consisting of filtered preserving morphisms from M to N. Then, we have $\mathrm{Hom}_{\mathcal{S},-}(M,N) = \lim_{\substack{\rightarrow \\ p}} \mathrm{Hom}_{A-\mathrm{gr},-}(M_{\geq p}, N_{\geq p})$.

Let $0 \to \Gamma^{\mathrm{fil}}(G) \to I^{\bullet}$ be a filtered injective resolution of $\Gamma^{\mathrm{fil}}(G)$ and let $\mathscr{I}^i := \tilde{I}^i$. The natural functor from the $A-$graded modules to $\mathcal{S}$ is an exact functor([PS, Theorem 6.7]). So, a filtered graded resolution in the category of graded A-modules is one in $\mathcal{S}$. Any injective object in the category of the graded A-modules is injective in $\mathcal{S}$ from a direct calculation and the definition of injections in $\mathcal{S}$. It follows that $0 \to \Gamma^{\mathrm{fil}}(G) \to I^{\bullet}$ is also a filtered injective resolution in $\mathcal{S}$. So, $0 \to \mathscr{G} \to \mathscr{I}^{\bullet}$ is a filtered injective resolution of $\mathscr{G}$. Thus, we have

$$\begin{aligned}\mathrm{Ext}^i_{-}(\mathscr{F},\mathscr{G}) &= H^i(\mathrm{Hom}_{-}(\mathscr{F},\mathscr{I}^{\bullet})) = H^i(\mathrm{Hom}_{\mathcal{S},-}(\Gamma^{\mathrm{fil}}(\mathscr{F}), I^{\bullet}) \\ &= H^i(\lim_{\substack{\rightarrow \\ p}} \mathrm{Hom}_{A-\mathrm{gr},-}(\Gamma^{\mathrm{fil}}_{\geq p}(\mathscr{F}), I^{\bullet}_{\geq p})) = \lim_{\substack{\rightarrow \\ p}} H^i(\mathrm{Hom}_{A-\mathrm{gr},-}(\Gamma^{\mathrm{fil}}_{\geq p}(\mathscr{F}), I^{\bullet}_{\geq p}))\end{aligned}$$

In order to show (a), we have to prove the following claim.

Claim 3.10. *For $i, p \geq 0$,*

$$H^i(\mathrm{Hom}_{A-\mathrm{gr},-}(\Gamma^{\mathrm{fil}}_{\geq p}(\mathscr{F}), I^{\bullet}_{\geq p})) = \mathrm{Ext}^i_{A-\mathrm{gr},-}(\Gamma^{\mathrm{fil}}_{\geq p}(\mathscr{F}), \Gamma^{\mathrm{fil}}_{\geq p}(\mathscr{G}))$$

Proof of Claim 3.10. For simplicity, $\Gamma^{\mathrm{fil}}(\mathscr{F}) := M$ as filtered graded modules. First, we have a spectral sequence in the same way as in the proof of Lemma 3.6 :

$$H^{i+j}(\mathrm{Hom}_{A-\mathrm{gr},-}(M_{\geq p}, I^{\bullet}_{\geq p})) \Leftarrow E_1^{ij} := \begin{cases} H^{i+j}(F_iC_{\geq p}/F_{i+1}C_{\geq p}) & i \geq 0 \\ 0 & i < 0 \end{cases}$$

, where $F_iC_{\geq p} := \{0 \to \mathrm{Hom}_{A-\mathrm{gr},-i}(M_{\geq p}, I^0_{\geq p}) \to \mathrm{Hom}_{A-\mathrm{gr},-i}(M_{\geq p}, I^1_{\geq p}) \to \cdots\}$. Then, we see that

$$F_iC_{\geq p}/F_{i+1}C_{\geq p} = \oplus_k \mathrm{Hom}_{A-\mathrm{gr}}(\mathrm{gr}_k(M_{\geq p}), \mathrm{gr}_{k-i}(I^{\bullet}_{\geq p})) \quad \text{if } i \geq 0. \tag{3}$$

How to prove (3) is similar to that of [Luc71, Chapitre V, Lemme 1.4.2]. First, note that each $I^i_{\geq p}$ is a direct sum of truncations of graded injective modules because each I^i is a direct sum of graded injective modules. When $0 = N_0 \subset N_1 \subset \cdots \subset N_t =: N$ and $0 = J_0 \subset J_1 \subset \cdots \subset J_t := J$ are filtered graded A-modules and J_{i+1}/J_i are injective objects (so each $J_{\bullet}$ is a direct sum of injective objects), then we have

$$\begin{aligned}\mathrm{Hom}_{A-\mathrm{gr},-}(N_{\geq p}, J_{\geq p}) &= \mathrm{Hom}_{A-\mathrm{gr},-}(N_{\geq p}, J) \\ &= \oplus_{k=0}\mathrm{Hom}_{A-\mathrm{gr}}((N/N_{t-k-1})_{\geq p}, J_{t-k}/J_{t-k-1}) \\ &= \oplus_{k=0}\mathrm{Hom}_{A-\mathrm{gr}}((N/N_{t-k-1})_{\geq p}, (J_{t-k}/J_{t-k-1})_{\geq p}).\end{aligned}$$

Remark 3.11. The second equal above can be obtained by the following correspondence.

$$\mathrm{Hom}_{A-\mathrm{gr},-}(N_{\geq p}, J) \rightleftarrows \oplus_{k=0}^{t-1}\mathrm{Hom}_{A-\mathrm{gr}}((N/N_{t-k-1})_{\geq p}, J_{t-k}/J_{t-k-1})$$

$$\psi \longmapsto \oplus_{k=0}^{t-1}(\mathrm{pr}_{t-k} \circ \psi)$$

$$\oplus_{k=0}^{t-1}(\varphi_{t-k} \circ \pi_{t-k-1}) \longleftarrow\!\shortmid \oplus_{k=0}^{t-1}\varphi_{t-k}$$

where $\mathrm{pr}_{t-k} : J = \oplus_{k=0}^{t-1} J_{t-k}/J_{t-k-1} \to J_{t-k}/J_{t-k-1}$ and $\pi_{t-k} : N \to N/N_{t-k}$ are natural projections.

So, if let $M := (M = M_s \supset M_{s-1} \supset \cdots \supset M_0 = 0)$ as a filtered module, we have

$$F_iC_{\geq p} = \mathrm{Hom}_{A-\mathrm{gr},-i}(M_{\geq p}, I^{\bullet}_{\geq p}) = \oplus_{k=0}\mathrm{Hom}_{A-\mathrm{gr}}(M_{\geq p}/(M_{s-i-k-1})_{\geq p}, \mathrm{gr}_{s-i-k}(I^{\bullet}_{\geq p})).$$

And, we have

$$F_iC_{\geq p}/F_{i+1}C_{\geq p} = \oplus_{k=0}\mathrm{Hom}_{A-\mathrm{gr}}((M/M_{s-k-1})_{\geq p}, \mathrm{gr}_{s-i-k}(I^\bullet_{\geq p}))/\mathrm{Hom}_{A-\mathrm{gr}}((M/M_{s-k})_{\geq p}, \mathrm{gr}_{s-i-k}(I^\bullet_{\geq p})).$$

Then, we apply the functor $\mathrm{Hom}_{A-\mathrm{gr}}(-, \mathrm{gr}_{s-i-k}(I^\bullet_{\geq p}))$ to the exact sequence

$$0 \to \mathrm{gr}_{s-k}(M_{\geq p}) \to (M/M_{s-k-1})_{\geq p} \to (M/M_{s-k})_{\geq p} \to 0.$$

Because $I^\bullet_{s-i-k}$ is an injective module and $\mathrm{Hom}_{A-\mathrm{gr}}(L_{\geq p}, \mathrm{gr}_{s-i-k}(I^\bullet_{\geq p})) = \mathrm{Hom}_{A-\mathrm{gr}}(L_{\geq p}, \mathrm{gr}_{s-i-k}(I^\bullet_{\geq p}))$ for a graded A-module L, we have an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{A-\mathrm{gr}}((M/M_{s-k})_{\geq p}, \mathrm{gr}_{s-i-k}(I^\bullet_{\geq p})) \to \mathrm{Hom}_{A-\mathrm{gr}}((M/M_{s-k-1})_{\geq p}, \mathrm{gr}_{s-i-k}(I^\bullet_{\geq p})) \\ &\to \mathrm{Hom}_{A-\mathrm{gr}}(\mathrm{gr}_{s-k}(M_{\geq p}), \mathrm{gr}_{s-i-k}(I^\bullet_{\geq p})) \to 0. \end{aligned}$$

Therefore, we have

$$F_iC_{\geq p}/F_{i+1}C_{\geq p} = \oplus_{k=0}\mathrm{Hom}_{\mathrm{gr}}(\mathrm{gr}_{s-k}(M_{\geq p}), \mathrm{gr}_{s-i-k}(I^\bullet_{\geq p})).$$

This is the desired equality.

Finally, because $\mathrm{Ext}^i_{A-\mathrm{gr}}(\mathrm{gr}_{s-k}(M_{\geq p}), \mathrm{gr}_{s-i-k}(I^\bullet_{\geq p})) = 0$ ([CFK01, Lemma 4.3.7]), $\mathrm{gr}_{s-i-k}(I^\bullet_{\geq p})$ is $\mathrm{Hom}_{A-\mathrm{gr}}((M_{s-k}/M_{s-k-1})_{\geq p}, -)-$acyclic. Thus, we have

$$H^{i+j}(F_iC_{\geq p}/F_{i+1}C_{\geq p}) = \oplus\mathrm{Ext}^{i+j}_{A-\mathrm{gr}}(\mathrm{gr}_k(M_{\geq p}), \mathrm{gr}_{k-i}(N_{\geq p})).$$

So, in the same way as in the proof of Lemma 3.6, we have the claim. □

Therefore, from Claim 3.10 and the above discussion, we have

$$\mathrm{Ext}^i_-(\mathscr{F}, \mathscr{G}) = \varinjlim_p \mathrm{Ext}^i_{A-\mathrm{gr},-}(\Gamma^{\mathrm{fil}}_{\geq p}(\mathscr{F}), \Gamma^{\mathrm{fil}}_{\geq p}(\mathscr{G})).$$

(b) Next, it is sufficient to prove that if let $F^\bullet \to M \to 0$ be a filtered graded free resolution of M (about the exitence of filtered graded free resolution, see [Kre14, Proposition 5.6.5]) , then for $q \gg 0$

$$\mathrm{Ext}^i_{A-\mathrm{gr},-}(M_{\leq q}, N_{\leq q}) = H^i(\mathrm{Hom}_{A-\mathrm{gr},-}(F^\bullet_{\leq q}, N_{\leq q})).$$

First note that the truncated complex $F^\bullet_{\leq q} \to M_{\leq q} \to 0$ is a respolution of $M_{\leq q}$ (which is not free). In the same way as in the proof of 3.6, we can get a spectral sequence

$$H^{i+j}(\mathrm{Hom}_{A-\mathrm{gr},-}(P^\bullet_{\leq q}, N_{\leq q})) \Leftarrow E^{ij}_1 := \begin{cases} H^{i+j}(F_iC_{\leq q}/F_{i+1}C_{\leq q}) & i \geq 0 \\ 0 & i < 0 \end{cases}$$

, where $F_iC_{\leq q} := \{0 \to \mathrm{Hom}_{A-\mathrm{gr},-i}(P^0_{\leq q}, N_{\leq q}) \to \mathrm{Hom}_{A-\mathrm{gr},-i}(P^1_{\leq q}, N_{\leq q}) \to \cdots\}$. And, we have

$$H^{i+j}(F_iC_{\leq q}/F_{i+1}C_{\leq q}) = \oplus_k\mathrm{Ext}^{i+j}_{A-\mathrm{gr}}(\mathrm{gr}_k(M_{\leq q}), \mathrm{gr}_{k-i}(N_{\leq q})) \quad \text{if } i \geq 0$$

because the similar equality to 3 holds and $\mathrm{Ext}^i_{A-\mathrm{gr}}(M_{\leq q}, N_{\leq q})$ are calculated by using the truncated resolution $F^\bullet_{\leq q} \to M_{\leq q} \to 0$ ([CFK01, page 437]).

The latter claim is proved samely by using the fact any coherent sheaf on a projective scheme Y has a resolution by graded $A \otimes \mathscr{O}_Y$-modules $\mathscr{F}^\bullet$ which are of the form $\mathscr{F}^i = A \otimes \mathscr{E}^\bullet$ such that each $\mathscr{E}^i$ is locally free sheaves of finite rank (see also [CFK01, page 437]). □

Corollary 3.12. *Let $\mathscr{F}, \mathscr{G}$ be filtered coherent sheaves on X. Then, for $q \gg p \gg 0$, we have*

$$\mathrm{Ext}^i_-(\mathscr{F}, \mathscr{G}) = \mathrm{Ext}^i_{A-\mathrm{gr},-}(\Gamma^{\mathrm{fil}}_{[p,q]}(\mathscr{F}), \Gamma^{\mathrm{fil}}_{[p,q]}(\mathscr{G})).$$

Proof. From (a) of Lemma we have 3.9, $\mathrm{Ext}^i_-(\mathscr{F}, \mathscr{G}) = \mathrm{Ext}^i_{A-\mathrm{gr},-}(\Gamma^{\mathrm{fil}}_{\geq p}(\mathscr{F}), \Gamma^{\mathrm{fil}}_{\geq p}(\mathscr{G}))$ for $0 \ll p$. Then, we apply (b) of the lemma. □

Proposition 3.13. *$\Gamma^{\mathrm{fil}}_{[p,q]}$ is an étale morphism if $q \gg p \gg 0$*

Proof. This is followed from Proposition 3.7, Proposition 3.8 and Corollary 3.13. For the non-filtered case, see [BCFHR14, Proposition 3.3]. □

Finally, we can get the main result in this subsection by combining Lemma 3.3 and Proposition 3.13.

Corollary 3.14. *$\Gamma^{\mathrm{fil}}_{[p,q]}$ is an open immersion if $q \gg p \gg 0$*

3.2. **Explicit descriptions.** In this subsection, we describe the image of $\Gamma^{\mathrm{fil}}_{[p,q]}|_{\mathcal{F}\mathrm{Coh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X)}$.

Theorem 3.15. *For $q \gg p' \gg p \gg 0$, the functor $\Gamma^{\mathrm{fil}}_{[p,q]}$ induces an open immersion*

$$\Gamma^{\mathrm{fil}}_{[p,q]}|_{\mathcal{F}\mathrm{Coh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X)} : \mathcal{FC}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X) \to \mathcal{FM}\mathrm{od}^{[p,q],\mathrm{sfg}}_{(\alpha_1,\cdots,\alpha_s)}(A)$$

whose image is equal to the locus of whose truncations into the interval $[p',q]$ is HN-filtration. We denote this open substack of $\mathcal{FM}\mathrm{od}^{[p,q],\mathrm{sfg}}_{(\alpha_1,\cdots,\alpha_s)}(A)$ by $\mathcal{FM}\mathrm{od}^{[p,q],\mathrm{sfg},[p',q]-\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(A)$

Proof. For $p \gg 0$ and any sheaf $\mathscr{F}$ with HN-filtration $0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_s = \mathscr{F}$, all $\Gamma^{\mathrm{fil}}_{[p,q]}(\mathscr{F}_{i+1}/\mathscr{F}_i)$ are generated in degree p because $\mathscr{F}_{i+1}/\mathscr{F}_i$ is p-regular. It follows that $\Gamma^{\mathrm{fil}}_{[p,q]}|_{\mathcal{F}\mathrm{Coh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X)}$ is an open immersion from Corollary 3.14 for $q \gg p \gg 0$.

Next, we see the following : For $q \gg p' \gg p \gg 0$, any $M \in \mathcal{FM}\mathrm{od}^{[p,q],\mathrm{sfg}}_{(\alpha_1,\cdots,\alpha_s)}(A)$ such that $M_{\geq p'}$ is a HN-filtration is sent to a object of $\mathcal{FC}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X)$.

Let M be an A-module in $[p,q]$ of dimension $\alpha|_{[p,q]}$ and generated in degree p. Then, we have the following exact sequence

$$0 \to K \to A_{[0,q-p]} \otimes_k M_p \to M \to 0.$$

Then, if let $K' \subset A \otimes M_p$ be a submodule generated by K in $A \otimes M_p$, then $\mathscr{S}(M) \simeq \widetilde{A \otimes_k M_p/K'}$ where $\widetilde{A \otimes_k M_p/K'}$ is the associated sheaf of $A \otimes_k M_p/K'$ (see [Ser55], [BCFHR14, Theorem 3.10]). Note that K' is generated in degree $p+1$ if $p \gg 0$. So, any filtered graded A-module in $\mathcal{FM}\mathrm{od}^{[p,q],\mathrm{sfg}}_{(\alpha_1,\cdots,\alpha_s)}(A)$ is sent a filtered coherent sheaf on X by $\mathscr{S}$.

Next, we see that for $q \gg p' \gg p \gg 0$, any filtered graded A-module in $\mathcal{FM}\mathrm{od}^{[p,q],\mathrm{sfg}}_{(\alpha_1,\cdots,\alpha_s)}(A)$ is sent to a filtered coherent sheaf in $\mathcal{FC}\mathrm{oh}_{(\alpha_1,\cdots,\alpha_s)}(X)$. In the proof of [BCFHR14, Theorem 3.10], any finitely $[p,q]$-graded A-module M in degree p with dimension vector $\alpha|_{[p,q]}$ is sent to a p'-regular coherent sheaf with Hilbert polynomial α and $\Gamma_{[p,q]}(\mathscr{S}(M))_{\geq p'} \simeq M_{\geq p'}$ for $q > p' \gg p \gg 0$. The choice of p' is dependent only on p. So, for $q > p' \gg p \gg 0$, the similar claim holds. I.e, for any $(0 = M_0 \subset M_1 \subset \cdots \subset M_s = M) \in \mathcal{FM}\mathrm{od}^{[p,q],\mathrm{sfg}}_{(\alpha_1,\cdots,\alpha_s)}(A)$, we have that each $\mathscr{S}(M_{i+1})/\mathscr{S}(M_i)$ is p'-regular with Hilbert polynomial α_{i+1} and $\Gamma_{[p,q]}(\mathscr{S}(M_i))_{\geq p'} \simeq (M_i)|_{\geq p'}$.

On the other hand, for $q \gg p \gg 0$, any HN-filtration in $\mathcal{FC}\mathrm{oh}_{(\alpha_1,\cdots,\alpha_s)}(X)$ is sent to a HN-filtration in $\mathcal{FM}\mathrm{od}^{[p,q],\mathrm{sfg}}_{(\alpha_1,\cdots,\alpha_s)}(A)$. Actually, by [BCFHR14, Theorem 3.7], $\mathscr{F}_{i+1}/\mathscr{F}_i$ is semistable if and only if $\Gamma_{[p,q]}(\mathscr{F}_{i+1}/\mathscr{F}_i)$ is semistable with respect to $(h^0((\mathscr{F}_{i+1}/\mathscr{F}_i)(q)), -(h^0((\mathscr{F}_{i+1}/\mathscr{F}_i)(p)))$ for any HN-filtration $0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_s$ in $\mathcal{FC}\mathrm{oh}_{(\alpha_1,\cdots,\alpha_s)}(X)$. And, if let $\theta_p = \alpha(q)$ and $\theta_q = -\alpha(p)$ ($\alpha(t) \in \mathbb{Q}[t]$), then for sheaves $\mathscr{G}_1, \mathscr{G}_2$, $\mu_{(\theta_p,\theta_q)}(\Gamma_{[p,q]}\mathscr{G}_1) > \mu_{(\theta_p,\theta_q)}(\Gamma_{[p,q]}\mathscr{G}_2)$ if and only if $h^0(\mathscr{G}_1(p))/h^0(\mathscr{G}_1(q)) > h^0(\mathscr{G}_2(p))/h^0(\mathscr{G}_2(q))$ from the following calculations (a similar statement is mentioned in [Hos18, Lemma 5.2], where representations of quivers are treated and the slope treated there is different from ours):

$$\mu_{(\theta_p,\theta_q)}(\Gamma_{[p,q]}(\mathscr{G}_i)) = \frac{\alpha(q)h^0(\mathscr{G}_i(p)) - \alpha(p)h^0(\mathscr{G}_i(q))}{h^0(\mathscr{G}_i(p)) + h^0(\mathscr{G}_i(q))} = -\alpha(p) + \frac{(\alpha(p)+\alpha(q))h^0(\mathscr{G}_i(p)}{h^0(\mathscr{G}_i(p)) + h^0(\mathscr{G}_i(q))} \quad (i = 1,2).$$

$$\begin{aligned}\mu_{(\theta_p,\theta_q)}(\Gamma_{[p,q]}(\mathscr{G}_1)) - \mu_{(\theta_p,\theta_q)}(\Gamma_{[p,q]}(\mathscr{G}_2)) &= (\alpha(p)+\alpha(q))\left(\frac{h^0(\mathscr{G}_1(p))}{h^0(\mathscr{G}_1(p)) + h^0(\mathscr{G}_1(q))} - \frac{h^0(\mathscr{G}_2(p))}{h^0(\mathscr{G}_2(p)) + h^0(\mathscr{G}_2(q))}\right) \\ &= \frac{(\alpha(p)+\alpha(q))h^0(\mathscr{G}_1(q))h^0(\mathscr{G}_1(q))}{(h^0(\mathscr{G}_1(p)) + h^0(\mathscr{G}_1(q)))(h^0(\mathscr{G}_2(p)) + h^0(\mathscr{G}_2(q)))}\left(\frac{h^0(\mathscr{G}_1(p))}{h^0(\mathscr{G}_1)(q)} - \frac{h^0(\mathscr{G}_2(p))}{h^0(\mathscr{G}_2)(q)}\right)\end{aligned}$$

If let $\alpha(t) := \sum_{i=1}^{s} \alpha_i(t)$, this proves that $\Gamma_{[p,q]}(\mathscr{F}_{i+1}/\mathscr{F}_i)$ is semistable with respect to $(\alpha(q), -\alpha(p))$ and $\mu_{(\alpha(q),-\alpha(p))}(\Gamma_{[p,q]}(\mathscr{F}_{i+1}/\mathscr{F}_i)) > \mu_{(\alpha(q),-\alpha(p))}(\Gamma_{[p,q]}(\mathscr{F}_{i+2}/\mathscr{F}_{i+1}))$ (cf.[Hos18, Theorem 5.7]). In the same way, we can show that for $q \gg p' \gg p \gg 0$ and any HN-filtration $\mathscr{F} \in \mathcal{FC}\mathrm{oh}_{(\alpha_1,\cdots,\alpha_s)}(X)$, $\Gamma_{[p,q]}(\mathscr{F})|_{\geq p'}$ is also HN-filtration with respect to $(\alpha(q), -\alpha(p'))$.

Conversely, If let $M \in \mathcal{FM}\mathrm{od}^{[p,q],\mathrm{sfg}}_{(\alpha_1,\cdots,\alpha_s)}(A)$ such that $M|_{\geq p'}$ is HN-filtration with respect to $(\alpha(q), -\alpha(p'))$, for $q > p' \gg p \gg 0$, $\Gamma_{[p,q]}(\mathscr{S}(M))|_{\geq p'} = M_{\geq p'}$ as filtered graded A-modules (and the $\mathscr{S}(M_{i+1}/M_i)$ are p'-regular) from the above discussion. And, for $q \gg p' \gg p \gg 0$, such M is sent to a filtered sheaf in $\mathcal{FC}\mathrm{oh}_{(\alpha_1,\cdots,\alpha_s)}(X)$ such that the $\mathscr{S}(M_{i+1}/M_i)$ are semistable because $(\alpha(q), -\alpha(p'))$-stability is equivalent to $(\alpha_{i+1}(q), -\alpha_{i+1}(p'))$-stability. So, $\mathscr{S}(M)$ is a HN-filtration and we can assume the $\mathscr{S}(M_{i+1}/M_i)$ are p-regular. Thus, we have $\Gamma_{[p,q]}(\mathscr{S}(M)) \simeq M$ as filtered graded A-modules because we have the following commutative diagram (cf. [Laz17, Proposition 1.8.8])

$$\begin{tikzcd}
A\otimes_k K'_{p+1} \arrow[r] \arrow[d, "\simeq"] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes_k M_p \arrow[r] \arrow[d, "\simeq"] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[r] \arrow[d] & 0 & \text{(exact)} \\
\Gamma_{[p,q]}(K'_{p+1}\otimes\mathscr{O}_X(-p-1)) \arrow[r] & \Gamma_{[p,q]}(M_p\otimes\mathscr{O}_X(-p)) \arrow[r] & \Gamma_{[p,q]}(\mathscr{M}) \arrow[r] & 0 & \text{(exact)}
\end{tikzcd}$$

, where K' is a submodule of $A \otimes_k M_p$ defined above. □

3.3. **Derived enhancement.** Finally, we can define derived moduli stacks of Harder-Narasimhan filtrations. This is our aim in this paper.

Let $\alpha_1(t), \alpha_2(t), \cdots, \alpha_s(t) \in \mathbb{Q}[t]$ such that $\alpha_1(t) \succ \alpha_2(t) \succ \cdots \succ \alpha_s(t)$. We take integers p, p', q so that Theorem 3.15 holds for $\Gamma^{\mathrm{fil}}_{[p,q]}|_{\mathcal{FC}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X)} : \mathcal{FC}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X) \to \mathcal{FM}\mathrm{od}^{[p,q],\mathrm{sfg}}_{(\alpha_1,\cdots,\alpha_s)}(A)$. Let V be an open subscheme of $\mathrm{MC}(L_-)$ obtained by the pullback of $\mathcal{FM}\mathrm{od}^{[p,q],\mathrm{sfg},[p',q]-\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(A)$ from the following diagram:

$$\mathrm{MC}(L_-) \longrightarrow [\mathrm{MC}(L_-)/P] \xrightarrow{\simeq} \mathcal{FC}\mathrm{oh}^{[p,q]}_{(\alpha_1,\cdots\alpha_s)}(A) \hookleftarrow \mathcal{FC}\mathrm{oh}^{[p,q]\mathrm{sfg}}_{(\alpha_1,\cdots\alpha_s)}(A) \hookleftarrow \mathcal{FM}\mathrm{od}^{[p,q],\mathrm{sfg},[p',q]-\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(A) .$$

And we define a P-equivariant open subscheme U of L^1_- to be

$U := \{\lambda : A\otimes_k V \to V \in L^1_- \mid$ the truncation of λ to $[p', q]$ is a HN-filtration and V is strictly finitely generated$\}$.

Note that the restriction of U to $\mathrm{MC}(L_-)$ is the image of $\Gamma^{\mathrm{fil}}_{[p,q]}$ in Theorem 3.15.

Definition 3.16. Under the condition above, we the derived moduli stack $\mathcal{RFC}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X)$ of Harder-Narasimhan filtrations of type $(\alpha_1, \cdots, \alpha_s)$ on X is the restriction of the derived moduli stacks $[L_-/P]$ to $[U/P]$.

The derived moduli stacks $\mathcal{RFC}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X)$ is suiteble for the definition of derived moduli stacks of HN-filtrations. They have the following property from the discussion from the discussion so far.

Theorem 3.17. *(a)* $\pi_0\mathcal{RFC}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X) \simeq \mathcal{FC}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X)$.

(b) If $0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_s = \mathscr{F}$ *is a HN-filtration of type* $(\alpha_1, \cdots, \alpha_s)$ *on* X*, then*

$$H^i T^{\bullet}_{[\mathscr{F}]}\mathcal{RFC}\mathrm{oh}^{\mathrm{HN}}_{(\alpha_1,\cdots,\alpha_s)}(X) \simeq \mathrm{Ext}^i_-(\mathscr{F}, \mathscr{F}) \quad i \geq 0.$$

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