SOME EXAMPLES OF NONCOMMUTATIVE PROJECTIVE CALABI-YAU SURFACES OBTAINED FROM NONCOMMUTATIVE SEGRE PRODUCTS AND VERONESE SUBRINGS

YUKI MIZUNO

ABSTRACT. In this article, we construct new noncommutative projective Calabi-Yau schemes by using noncommutative Segre products and noncommutative Veronese subalgebras. The first examples are constructed by Kanazawa ([11]).

1. Introduction

Calabi-Yau (CY) varieties are very rich objects. They play an important role in mathematics and physics. In noncommutative geometry, (skew) Calabi-Yau algebras are often treated as noncommutative analogues. They have a deep relation with quiver algebras ([8], [4]). Actually, many known Calabi-Yau algebras are constructed by them. They are also used to characterize Artin-Schelter (AS) regular algebras ([17], [18]). In particular, a connected graded algebra A over a field k is AS-regular if and only if A is skew Calabi-Yau.

On the other hand, a triangulated subcategory of the derived category of a cubic fourfold in $\mathbb{P}^5$ which is obtained by some orthogonal decompositions has the 2-shift functor [2] as the Serre functor. Moreover, its structure of Hochschild (co)homology is the same as its of a projective K3 surface ([12]). However, some of such categories are not obtained as the derived category of coherent sheaves of a projective K3 surface. They are called noncommutative K3 surfaces.

Artin and Zhang constructed the framework of noncommutative projective schemes which are defined from noncommutative graded algebras in [3]. In the framework, we can think of AS algebras as noncommutative analogues of projective spaces, which are called quantum projective spaces. Our objective is producing examples of noncommutative projective Calabi-Yau varieties which are not obtained from commutative ones. As the definition of noncommutative projective Calabi-Yau schemes, we adopt the definiton by Kanazawa ([11]). His definition is a direct generalization of the definition of (commutative) Calabi-Yau varieties to noncommutative projective schemes. He also consturcted the first exmaples of noncommutative projective Calabi-Yau schemes as hypersurfaces of quantum projective spaces there.

In this paper, we construct new examples of noncommutative projective Calabi-Yau schemes by using noncomuutaitve Segre products and Veronese subrings. We also show some of them are not isomorphic to commutative projective Calabi-Yau varieties and the first examples above. To be more precise, many examples of K3 surfaces are known in algebraic geometry. Among them, it is well-known some of them are divisors of Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, Reid gave the complete list of K3 surfaces which are hypersurfaces in weighted projective spaces ([16], [10]). In this paper, we construct noncommutative analogues of the 2 types of examples of K3 surfaces. Although the methods by Kanazawa are efficient in our cases, we also need different approaches to proceed our study. In order to construct the former case, we perform more detailed analysis about noncommutaitve projective schemes of $\mathbb{Z}^2$-graded algebras which are studied by Rompay ([23]). For the latter case, we need to treat quotients of weighted quantum polynomial rings. However, they are not generated in degree 1 in general. So, we take some Veronese subrings of them and consider the

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noncommutative projective schemes of them (about a different approach, see [21]). We bridge between properties of quotients of weighted quantum polynomial ring and those of thier Veronese subrings by using noncommutaive Čech complexe (this is used in the former case).

2. Preliminaries

Notation 2.1. In this article, k means an algebraically closed field of characteristic 0. We suppose $\mathbb{N}$ contains 0.

Definition 2.2 ([3, Section 2], [26, Section 0])**.** For any connected graded right Notherian k-algebra $A = \bigoplus_{i=0}^{\infty} A_i$, we denote the category of graded right A-modules (resp. finitely generated graded right A-modules) by $\mathrm{Gr}(A)$ (resp.$\mathrm{gr}(A)$). We denote the shift functor by $(-)(1) : \mathrm{Gr}(A) \to \mathrm{Gr}(A), M \mapsto M(1) := \oplus M(1)_i := \oplus M_{i+1}$. When we write $M, N \in \mathrm{Gr}(A)$, $\mathrm{Hom}_A(M,N) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr(A)}}(M, N(n))$.

We also denote the subcategory of torsion modules in $\mathrm{Gr}(A)$ (resp.$\mathrm{gr}(A)$) by $\mathrm{Tor}(A)$ (resp.$\mathrm{tor}(A)$). We denote the quotient category $\mathrm{Gr}(A)/\mathrm{Tor}(A)$ (resp.$\mathrm{gr}(A)/\mathrm{tor}(A)$) by $\mathrm{QGr(A)}$ (resp.$\mathrm{qgr(A)}$) and the canonical projection by $\pi : \mathrm{Gr}(A) \to \mathrm{QGr}(A)$. Let $\mathcal{A} := \pi(A)$. The (general) projective scheme of A is defined as $\mathrm{Proj}(A) := (\mathrm{Qgr}A, \mathcal{A}, s)$. We also define the (Noetherian) projective scheme $\mathrm{proj}(A) := (\mathrm{qgr}A, \mathcal{A}, s)$. Let $X := \mathrm{proj}(A)$. The global section of any object $\mathcal{N}$ is $\mathrm{H}^0(X, \mathcal{N}) = \mathrm{Hom}_{\mathrm{qgr}(A)}(\mathcal{A}, \mathcal{N})$. The cohomology is $\mathrm{H}^i(X, \mathcal{N}) := \mathrm{Ext}^i_{\mathrm{qgr}(A)}(\mathcal{A}, \mathcal{N})$ for $i > 0$.

Definition 2.3 ([25, Section4], [22, Section 4])**.** Let A be a connected graded k-algebra and m_A be $A_{\geq 1}$. Let M be a right graded A-module.

Then, we denote $\lim_{n\to\infty} \mathrm{Hom}_A(A/A_{\geq n}, M)$ by $\Gamma_{m_A}(M)$. By using this, we define a functor $\Gamma_{m_A} : \mathrm{Gr}(A) \to \mathrm{Gr}(A)$(we call Γ_{m_A} torsion functor). We denote the derived functor of Γ_{m_A} by $R\Gamma_{m_A}$ and $H^i R\Gamma_{m_A}$ by $H^i_{m_A}$.

Definition 2.4 ([25, Definition 3.3, 4.1], [22, Definition 6.1, 6.2])**.** Let A be a right and left Noetherian connected graded k-algebra, and A°, A^e be the opposite algebra, and the envelopping algebra of A, respectively. Let R be an object of $\mathrm{D^b}(A^e)$. Then, R is called a dualizing complex of A if (1) R has finite injective dimension over A and A°. (2) The cohomology of R is finitely generated as both A and A°-modules. (3) The natural morphism $A \to \mathrm{RHom}_A(R,R)$ and $A \to \mathrm{RHom}_{A^\circ}(R,R)$ are isomorhisms $\mathrm{D^b}(A^e)$.

Moreover, R is called balanced if $R\Gamma_{m_A}(R) \simeq A'$ and $R\Gamma_{m_{A^\circ}}(R) \simeq A'$, where A' is Matlis dual of A.

3. Calabi-Yau conditions

Definition 3.1 ([11, Section 2.2])**.** Let A be a connected right Noetherian graded k-algebra. Then, $\mathrm{proj}(A)$ is a Calabi-Yau n scheme if its global dimension is n and the Serre functor of the derived category $\mathrm{D^b}(\mathrm{qgr}(A))$ is the n-shift functor $[n]$.

3.1. Segre products.

In commutative algebraic geometry, when let X be the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^1$ into $\mathbb{P}^6$, smooth hypersurfaces $S \subset X$ of typr $(3,2)$ provide Calabi-Yau surfaces. In this section, we construct noncommutative analogues of these examples. And, we prove that some of those are not obtained from twisted homogeneous coordinates of the examples in Section 4.

Let C be a connected $\mathbb{N}^2$-graded k-algebra. We denote the category of bigraded right C-modules (resp. finitely generated bigraded right C-modules) by $\mathrm{BiGr}(C)$ (resp.$\mathrm{bigr}(C)$). In the same way as in Definition 2.2, we define $\mathbb{Z}^2$-graded torsion right C-modules. Let M be a $\mathbb{Z}^2$-graded right C-module. If $M_{(\geq s, \geq s)} := \bigoplus_{i\geq s, j\geq s} M_{ij} = 0$ for $s \gg 0$, then we say M is a torsion C-module. We denote the category of $\mathbb{Z}^2$-graded torsion C-modules by $\mathrm{Tor}(C)$. We also define $\mathrm{tor}(C)$ to be the intersection of $\mathrm{bigr}(C)$ and $\mathrm{Tor}(C)$. So, we can construct the quotient category $\mathrm{QBiGr}(C) := \mathrm{BiGr}(C)/\mathrm{Tor}(C)$ (resp.$\mathrm{qbigr}(C) := \mathrm{bigr}(C)/\mathrm{tor}(C)$) (cf. [23, Section 2]).

Moreover, we define $C_{++} := C_{(>0,>0)}$ and the torsion functor $\Gamma_{C_{++}}$ for a $\mathbb{N}^2$-graded k-algebra C to be the map which sends M to $\{m \in M | (C_{++})^n m = 0 \text{ for some } m \in \mathbb{N}\}$.

We also denote the maximal ideal of C by m_C and define the notion of dualizing complexs of C in the same way as in Section 2.

Remark 3.2. In [17], the authors treat $\mathbb{Z}^n$-graded algebras. They define torsion functors of $\mathbb{Z}^n$-graded modules by using total degrees of homogeneous elements of multigraded modules. So, their definition and ours are different.

In this section, we prove the following theorem.

Theorem 3.3. *Let n, m be positive integers such that $n \geq m$. Let A be $k\langle x_0, \cdots, x_n\rangle/(x_jx_i - q_{ji}x_ix_j)_{i,j}$ and let B be $k\langle y_0, \cdots, y_m\rangle/(y_jy_i - q'_{ji}y_iy_j)_{i,j}$. Let $f := \sum_{i=0}^m x_i^{n+1}y_i^{m+1} + \sum_{j=0}^m y_j^{m+1}\sum_{i=m+1}^n x_i^{n+1}$ be a bihomogeneous polynomial of degree $(n+1, m+1)$ in $C := A \otimes_k B$. We assume that (1) $q_{ii} = 1, q_{ij}q_{ij} = 1$ and $q_{ij}^{n+1} = 1$, (2) $q'_{ii} = 1, q'_{ij}q'_{ji} = 1$ and $q_{ij}'^{m+1} = 1$.*

Then, $\mathrm{qbigr}(C/(f))$ *is a Calabi-Yau $(n + m - 1)$-category if $\prod_{i=0}^n q_{ij}$ and $\prod_{i=0}^m q'_{ij}$ is independent of j.*

Note that f is a normal element because of the choice of $\{q_{ij}\}, \{q'_{ij}\}$. To prove the theorem, we need to show some lemmas.

Lemma 3.4. *Let A, B be as in Theorem 3.3. Then, $\mathbb{R}\Gamma_{C_{++}}(C)^*$ is isomorphic to $A_\phi(-n-1) \otimes_k B_\psi(-m-1)[n+m+1]$ in $\mathrm{D}(\mathrm{qbigr}(C))$ where ϕ(resp. ψ) is the graded automorphism of A(resp. B) which maps $x_j \mapsto \prod q_{ji}x_j$ (resp. $y_j \mapsto \prod q_{ji}y_j$).*

Proof. First, let I_1, I_2 be the ideals generated by m_A, m_B respectively. Then, we have $C_{++} = I_1 \cap I_2$ and $m_C = I_1 + I_2$ and have the following long exact sequence

$$\cdots \to H^{i-1}_{C_{++}}(C) \to H^i_{m_C}(C) \to H^i_{I_1}(C) \oplus H^i_{I_2}(C) \to H^i_{C_{++}}(C) \to H^{i+1}_{m_C}(C) \to \cdots$$

by the Mayer-Vietris sequence. Note that we can use the Mayer-Vietris sequence in our case (see Remark 3.6 below).

On the other hand, $\mathbb{R}\Gamma_{m_C}(C)^*$ is the balanced dualizing complex of C (see, [17, Proof of Lemma 3.5]). Then, we have

$$H^i_{m_C}(C)^* \simeq \begin{cases} 0 & i \neq n+m+2 \\ H^{n+1}_{m_A}(A)^* \otimes H^{m+1}_{m_B}(B)^* \simeq A_\phi(-n-1) \otimes_k B_\psi(-m-1) & i = n+m+2 \end{cases}$$

from [11, Proposition 2.4].

Moreover, we have $H^i_{I_1}(C)$ and $H^i_{I_2}(C)$ are torsion modules for C_{++}(about the commutative case, see [13, Proposition 2.1.6]).

Thus, we have isomorphisms

$$\mathbb{R}\Gamma_{C_{++}}(C)^* \simeq \mathbb{R}\Gamma_{m_C}(C)^*[-1] \simeq A_\phi(-n-1) \otimes_k B_\psi(-m-1)[n+m+1]$$

in $\mathrm{D}(\mathrm{qbigr}(C))$. $\square$

Remark 3.5. In our situation, the balanced dualizing complex R_C of C is not suitable to prove the theorem because we treat the torsion category of the torsion modules for C_{++} to define $\mathrm{qbigr}(C)$.

Remark 3.6. To use the Mayer-Vietris sequence, we need to prove the following equivalences

$$\lim_{n\to\infty} \mathrm{Ext}^i_C(C/({I_1}^n + {I_2}^n), -) \text{ and } H^i_{I_1+I_2}(-),$$
$$\lim_{n\to\infty} \mathrm{Ext}^i_C(C/I_1^n \oplus C/I_2^n, -) \text{ and } H^i_{I_1}(-) \oplus H^i_{I_2}(-),$$
$$\lim_{n\to\infty} \mathrm{Ext}^i_C(C/(I_1^n \cap I_2^n), -) \text{ and } H^i_{I_1 \cap I_2}(-).$$

In the commutative ring theory, these equivalences are proved by using cofinality and the Artin-Rees Lemma (cf.[7, Chapter A1D], [5, Chapter 3]). In general, an ideal of a noncommutative ring does not satisfy the Artin-Rees Lemma. However I_1, I_2 satisfy the Artin-Rees property in the sense of [14, Chapter 4.2], because I_1, I_2 are generated by normal elements in our case. Thanks to this fact, we prove the above equivalences in the same way as in the case of commutative rings.

Lemma 3.7. gl.dim(qgr($C/(f)$)) $= n+m-1$

Proof. We consider a bigraded (commutative) algebra $D := k[s_0, \cdots, s_n, t_0, \cdots, t_m]/(f))$ and the projective spectrum biproj(D) in the sense of [9, Section 1]. Then, an object in qbigr($C/(f)$) can be thought as an object in the category of modules over the sheaf $\mathscr{A}$ of algebras, where $\mathcal{A}$ is the sheaf associated to algebras $(k[x_0, \cdots, x_n, y_0, \cdots, y_m]/(f)_{x_i y_j})_{(0,0)}$ for each open affine scheme $\mathrm{D}_+(s_i t_j) \simeq \mathrm{Spec}((D_{s_i t_j})_{(0,0)})$. Hence, it is enough to prove that the global dimension of $(k[x_0, \cdots, x_n, y_0, \cdots, y_m]/(f)_{x_i y_j})_{(0,0)} = n+m-1$.

In our case, we can proceed the rest of the proof in the same way as in [11, Section 2.3] with an exception. We give its sketch and mention the exception. We consider the condition $i = j$ or $i > m \cdots (*)$.

We asuume that $(*)$ holds. For simplicity, we prove it when $i = j = 0$. Let $S_i := s_i/s_0, T_i := t_i/t_0, X_i := x_i/x_0, Y_i := y_i/y_0$. Then we consider the $k[S_1, \cdots, S_n, T_1, \cdots, T_m]/(1 + \sum_{i=1} S_i T_i + (1+\sum_{i=1} T_i \sum_{i=m+1} S_i)$-algebra $k[X_1, \cdots, X_n, Y_1, \cdots, Y_m]/(1+\sum_{i=1} X_i^{n+1} Y_i^{m+1} + (1+ \sum_{i=1} Y_i^{m+1} \sum_{i=m+1} X_i^{n+1})$, where the module structure is given by the identification $S_i = X_i^{n+1}, T_i = Y_i^{m+1}$. We denote the former by E and the latter by F. It is enough to prove that the global dimension of localization F_m of F at any maximal ideal $m := (S_1 - a_1, \cdots, S_n - a_n, T_1 - b_1, \cdots, T_m - b_m)$ of E with $f(1, a_1, \cdots, a_n, 1, b_1, \cdots, b_m)$ is $n+m-1$.

If any a_i, b_i is not 0, then F/m is a twisted group ring and hence semisimple. Moreover, $S_1 - a_1, \cdots, S_n - a_n, T_1 - b_1, \cdots, T_m - b_m$ is a regular seqnence. This induces the claim. On the other hand, we assume that one of $\{a_1, \cdots, a_n, b_1, \cdots, b_m\}$ is 0. For example, we assume $a_1 = 0$. Then, we consider E/X_1. We repeat taking quotients and reduce to considering the global dimensions of the algebras $k\langle X\rangle/(X^{n+1}+1)$ and $k\langle Y\rangle/(Y^{m+1}+1)$, which are 0.

We asseume $(*)$ does not hold. For example, let $i = 0, j = 1$ and we also consider m and F_m. When all a_i, b_i are not 0, the proof is the same. When some a_i, b_i are 0, an exception can occur. We suppose each term of $f(1, a_1, \cdots, a_n, b_0, 1, \cdots, b_m)$ is $0 \cdots (**)$. After repeating taking quotients, E becomes an algebra of the form $R = k\langle t_0, \cdots, t_k\rangle/(t_j t_i - p_{ji} t_j t_i, t_0^l)$. However, $\mathrm{gl.dim}(R) = \mathrm{gl.dim}(k\langle t_1, \cdots, t_k\rangle/(t_j t_i - p_{ji} t_j t_i))$. If not $(**)$, the proof is the same as that of the first case. $\square$

Proof of Theorem 3.3. First, we calculate $\mathbb{R}\Gamma_{C/(f)_{++}}(C/(f))^*$. For this, we consider the distinguished triangle in D(qbigr)($C/(f)$)

$$\mathbb{R}\Gamma_{C_{++}}(C(-n-1, -m-1)) \xrightarrow{\times f} \mathbb{R}\Gamma_{C_{++}}(C) \longrightarrow \mathbb{R}\Gamma_{C/(f)_{++}}(C/(f))$$

from the exact sequence

$$0 \longrightarrow C(-n-1, -m-1) \xrightarrow{\times f} C \longrightarrow C/(f) \longrightarrow 0.$$

Now, we have the isomorphism in D(qbigr)($C/(f)$)

$$\mathbb{R}\Gamma_{C_{++}}(C)^* \simeq A_\phi(-n-1) \otimes_k B_\psi(-m-1)[n+m+1] \simeq A \otimes_k B[n+m+1]$$

from Lemma 3.4 and the choice of $\{q_{ij}\}$ and $\{q'_{ij}\}$. Hence, we have $\mathbb{R}\Gamma_{C/(f)_{++}}(C/(f))^* \simeq C/(f)[n+m]$. In addition, we have the local duality for the torsion functor $\Gamma_{C/(f)_{++}}$ and the Serre duality in $\mathrm{D^b}$(qbigr($C/(f)$)) from Lemma 3.8 (we prove this below). Thus, $\mathbb{R}\Gamma_{C/(f)_{++}}(C/(f))^*[-1]$ induces the Serre functor in $\mathrm{D^b}$(qbigr($C/(f)$)). Since $\mathbb{R}\Gamma_{C/(f)_{++}}(C/(f))^* \simeq C/(f)[n+m]$, we complete the proof. $\square$

Lemma 3.8 (Local Duality and Serre Duality)**.** *Let $D := C$ or $C/(f)$. We have the following.*

(1) Let $M \in \mathrm{D}^b(\mathrm{bigr}(D))$. Then, we have

$$\mathrm{R}\Gamma_{D_{++}}(M) \simeq \mathrm{RHom}_D(M, \mathrm{R}\Gamma_{D_{++}}(D)')$$

in $\mathrm{D}^b(\mathrm{bigr}(D))$.

(2) Let $\mathcal{D} := \pi(D)$, $\mathcal{M} := \pi(M)$ *and* $\tilde{\mathcal{R}}_D := \pi(\mathrm{R}\Gamma_{D_{++}}(D)') \in \mathrm{D}^b(\mathrm{qbigr}(D))$. *Then, we have*

$$\mathrm{Hom}_{\mathrm{qbigr}(D)}(\mathcal{D}, \mathcal{M})^* \simeq \mathrm{Ext}^{i+1}_{\mathrm{qbigr}(D)}(\mathcal{M}, \tilde{\mathcal{R}}_D).$$

Proof. To prove (1), we want to apply [24, Theorem 0.4]. So, we show that the torsion class defined by D_{++} is quasi-compact, finite dimensional and stable (about the definition, see [24, Definition 3.4]). First, we prove that the torsion class is stable. D_{++} is generated by normal elements $\{x_i y_j\}$. So, D_{++} has Artin-Rees property in the sense of [14, Chapter 4.2]. Thanks to this property, we apply the proof of [7, Lemma A1.4]. This shows the stability of the torsion class. Let $l := \mathrm{lcm}(n+1, m+1)$. Then, D_{++} and D^l_{++} define the same torsion class. We consider the latter in the rest of the proof. Note that D^l_{++} is generated by central elements $\{x^l_i y^l_j\}_{i,j}$ from the choice of $\{q_{ij}\}$ and $\{q'_{ij}\}$. Moreover, we have a surjective localization map $N \to N[(x^l_i y^l_j)^{-1}]$ for any $x^l_i y^l_j$ and any right injective D-module. Thus, we can calculate the local cohomology for D^l_{++} by using Čech complexes (cf.[7, Proof of Theorem A1.3], [15]). This shows that the torsion class is quasi-compact and finite. Hence, we can apply [24, Theorem 0.4]. Finally, we obtaine the claim by taking dual. About (2), we can prove it in the same way as in [26, Theorem 4.2] by using (1). □

As a corollary of Theorem 3.3, we construct examples of noncommutative projective Calabi-Yau schemes.

Definition 3.9. (1) The Segre product $A \circ B$ of A and B is the $\mathbb{N}$-graded k-algebra with $(A \circ B)_i = A_i \otimes_k B_i$.
(2) Let $M \in \mathrm{bigr}(C)$. We define a right graded $A \circ B$-module M_Δ as the graded $A \circ B$-module with $(M_\Delta)_i = M_{ii}$.

Lemma 3.10 ([23, Theorem 2.4])**.** *We have the following natural isomorphism*

$$\mathrm{qbigr}(C) \longrightarrow \mathrm{qgr}(A \circ B), \qquad \pi(M) \longmapsto \pi(M_\Delta).$$

In addition, the functor defined by $- \otimes_{A \circ B} C$ *is the inverse of this equivalence.*

Remark 3.11. We similary obtaine an equivalence

$$\mathrm{qbigr}(C/J) \simeq \mathrm{qgr}(A \circ B/J_\Delta),$$

where $J := (f) \in \mathrm{bigr}(C)$.

Combining Theorem 3.3 with Remark 3.11, we get the following.

Corollary 3.12. *Let* $J := (f) \in \mathrm{bigr}(C)$. *Then,* $\mathrm{proj}(A \circ B/J_\Delta)$ *is noncommutative projective Calabi-Yau scheme.*

3.2. **Veronese subalgebras.** Reid produced the list of all commutative weighted Calabi-Yau hypersurfaces of dimension 2 (for example, see [16], [10]).

In this section, we construct noncommutative Calabi-Yau schemes from Veronese subrings of noncommutative weighted projective hypersurfaces.

Let A be a connected graded k-algebra. Then the Veronese subring $B := A^{(r)}$ is the connected graded k-algebra $B = \bigoplus_i B_i := \bigoplus_i A_{rn}$.

We consider the (commutative) weighted polynomial ring $A = k[x_0, \cdots, x_n]$ with $\deg(x_i) = a_i$. Then, $\mathrm{Coh}(\mathrm{Proj}(A))$ is not equivalent to $\mathrm{qgr}(A)$, but to $\mathrm{qgr}(A^{n\mathrm{lcm}(a_0, \cdots, a_n))})$. So, when we consider noncommutative projective schemes of quotient rings of quantum weighted polynomial rings, it seems that we should consider Veronese subrings of those rings.

Theorem 3.13. *Let* $w = (a_0, \cdots, a_n) \in \mathbb{Z}^{n+1}_{>0}$ *and* $d := \sum a_i$. *We suppose that* $a_i | d$ *for any* i. *Let* $C := k\langle x_0, \cdots, x_n\rangle/(x_j x_i - q_{ji} x_i x_j)$ *be a quantum weighted polynomial ring with* $\deg(x_i) = a_i$ *for* $0 \leq i \leq n$. *Let* $f := \sum x_i^{h_i}$, *where* $h_i := d/a_i$ *Let* $D := C/(x_i^{nh_i} f)_i$. *We assume that* $q_{ii} = 1, q_{ij} q_{ji} = 1$ *and* $q_{ij}^{d/a_i} = q_{ij}^{d/a_j} = 1$.

Then, $\mathrm{proj}(D^{((n+1)d))})$ *is a noncommutative projective Calabi-Yau (n-1) scheme if* $\prod_{i=0}^n q_{ij}$ *is independent of* $0 \leq j \leq n$.

Example 3.14. The types of weights of noncommutative CY surfaces which are well-formed are the following.

$$\begin{aligned}(a_0, a_1, a_2, a_3) = {} & (1,1,1,1), (1,1,1,3), (1,1,2,2), (1,1,2,4), (1,1,2,5), (1,1,4,6), \\ & (1,2,3,6), (1,3,3,4), (2,3,3,4), (1,2,6,9), (2,3,10,15), (1,6,14,21).\end{aligned}$$

Note that $(a_0, \cdots, a_n)$ is well-formed if $\gcd(a_0, \cdots, \hat{a}_i, \cdots, a_n) =$ for any i (cf. [10]).

We use this list in Section 4.2.

Remark 3.15. $D^{((n+1)d))}$ is generated by elements of degree 1. And, f is a normal element from the choice of $\{q_{ij}\}$.

Lemma 3.16. *The balanced dualizing complex of $C/(f)$ is isomorphic to $C_\phi/(f)[n]$, where ϕ is a graded automorphism of C which maps $x_i \mapsto \prod q_{ij}x_j$.*

Proof. Let $C' := k\langle x'_0, \cdots, x'_n\rangle/(x'_jx'_i - q_{ji}x'_ix'_j)$ be a quantum polynomial ring with $\deg(x'_i) = 1$ for any i. Then, C' is Koszul. We can calculate the balanced dualizing complex $R_{C'}$ of C' by using its Koszul resolution $K^{'\cdot}$ and taking $\mathrm{Hom}_{(C')^e}(-, (C')^e)^{-1}$ by [22, Section 8,9]. If we take some shifts on the summands of each $K^{'i}$, this complex $K^\cdot$ is a graded free resolution of C as C^e-modules. Then, we take $\mathrm{Hom}_{C^e}(-, C^e)^{-1}$ and get the balanced dualizing complex R_C of C. So, R_C and $R_{C'}$ have the same forms except for shifts of degrees. Indeed, the difference of the shifts is $\sum_{i=0}^n a_i - (n+1)$. So, we get $R_C = C_\phi(-\sum_{i=0}^n a_i)[n+1] = C_\phi(-d)[n+1]$, where ϕ is the graded automorphism defined in the statement of this lemma. The rest of the proof is done in the same way as in the first half of the proof of Theorem 3.3. $\square$

Lemma 3.17. $\mathrm{gl.dim}(\mathrm{qgr}(D^{((n+1)d)})) = n - 1$

Proof. In order to use the method of the proof in Lemma 3.7, we need some reduction. First, $\mathrm{qgr}(D^{((n+1)d)})$ is thought as the category of modules over a sheaf $\mathcal{B}$ of algebras on $\mathrm{Proj}(k[t_0, \cdots, t_n]/(t_i^n \sum_{j=0}^n t_j)_i$, where $\mathcal{B}(D_+(t_i))$ is isomorphic to $\left(D^{((n+1)d)}_{x_i^{(n+1)h_i}}\right)_0$. This is because $\{\mathrm{Mod}((D^{((n+1)d)}_{x_i^{(n+1)h_i}})_0)\}_i$ is an open cover of $\mathrm{qgr}(D^{((n+1)d)})$. About open covers of noncommutative projective schemes and open subspaces, see [20, Section 3.7, 5.4]. So, it is enough to prove that $\mathrm{gl.dim}\left(D^{((n+1)d)}_{x_i^{(n+1)h_i}}\right)_0 = n - 1$. Then, we have isomorphisms

$$\left(D^{((n+1)d)}_{x_i^{(n+1)h_i}}\right)_0 \simeq \left(D_{x_i^{(n+1)h_i}}\right)_0 \simeq (D_{x_i})_0 \simeq (C/(f)_{x_i})_0.$$

So, it is enough to prove the global dimension of $((C/f)_{x_i})_0$ is $n-1$. Moreover, $((C/f)_{x_i})_0$ is an algebra over $(k[t_0, \cdots, t_n]/(\sum_{j=0}^n t_j)_{t_i})_0$. Thus, we can prove it in the same way as in the latter half of the proof of Lemma 3.7. $\square$

Proof of Theorem 3.13. First, we have an isomorphism $\mathrm{proj}(C/(f)) \simeq \mathrm{proj}(D)$. From Lemma 3.16 and the choice of $\{q_{ij}\}$, $\mathrm{proj}(C/(f))$ is a noncommutative projective Calabi-Yau $(n-1)$ schemes. And, so does $\mathrm{proj}(D)$. This means the balanced dualizing complex R_D of D is $\mathbb{R}\Gamma_{m_D}(D)' \simeq D[3]$. We compute the balanced dualizing complex $R_{D^{((n+1)d)}}$ of $D^{((n+1)d)}$, which is isomorphic to $\mathbb{R}\Gamma_{m_{D^{(4d)}}}(D^{((n+1)d)})'$. Let $n_D \subset D$ be the ideal of D generated by $\{x_0^{(n+1)h_0}, \cdots, x_n^{(n+1)h_n}\}$. Then, m_D and n_D (resp. $n_D \cap D^{((n+1)d)}$ and $m_{D^{((n+1)d)}}$) define the same local cohomology of D (resp. $D^{((n+1)d)}$). In addition, $x_0^{(n+1)h_0}, \cdots, x_n^{(n+1)h_n}$ are central elements of D. Thus, the Čech complex $C(x_0^{(n+1)h_0}, \cdots, x_n^{(n+1)h_n}; D)$ (resp. $C(x_0^{(n+1)h_0}, \cdots, x_n^{(n+1)h_n}; D^{((n+1)d)})$) associated by $\{x_0^{(n+1)h_0}, \cdots, x_n^{(n+1)h_n}\}$ computes the local cohomology of D (resp. $D^{((n+1)d)}$) for n_D (resp. $n_D \cap D^{((n+1)d)}$) (for details, see the proof of 3.8). Since $C(x_0^{(n+1)h_0}, \cdots, x_n^{(n+1)h_n}; D)^{((n+1)d)}$ is isomorphic to $C(x_0^{(n+1)h_0}, \cdots, x_n^{(n+1)h_n}; D^{((n+1)d)})$ as chain complexes of graded $D^{((n+1)d)}$-modules, we have $H^i_{n_D}(D)^{((n+1)d)} \simeq H^i_{n_D \cap D^{((n+1)d)}}(D^{((n+1)d)})$.

Hence, we have $H^i_{m_D}(D)^{((n+1)d)} \simeq H^i_{m_{D^{((n+1)d)}}}(D^{((n+1)d)})$ and $R_{D^{((n+1)d)}} \simeq D^{((n+1)d)}[n]$. This shows that $\mathrm{proj}(D^{((n+1)d)})$ is a noncommutative projective Calabi-Yau $(n-1)$ schemes. □

4. Point modules

In this section, we calculate point modules of noncommutative Calabi-Yau varieties obtained in Section 3 and show some of them give new examples of noncommutative Calabi-Yau varieties different from commutative ones and the examples obtained in [11].

Definition 4.1 ([2, Definition 3.8], [19, Definition 3.1], [6])**.** Let A be a connected graded k-algebra which is generated in degree 1. Let M be a graded right A-module. We say M is a point module if M is cyclic, generated in degree 0 and $\dim_k(M_i) = 1$ for all $i \geq 0$.

We mention some basic facts which are needed in this section. For details, see [19, Section 3], [11, Section 3], etc.

Let $A := k\langle x_0, \cdots, x_n\rangle$ be a free associative algebra with $n+1$-variable. Then, the moduli space $\mathcal{M}_A$ of point modules of A is isomorphic to $\prod_{i=0}^{\infty} \mathbb{P}^n$. Let $M := \bigoplus_i km_i$ be a point module of A. If $m_i x_j = \alpha_{i,j} m_{i+1}$, then we can describe the isomorphism between them as follows

$$\mathcal{M}_A \to \prod_{i=0}^{\infty} \mathbb{P}^n, \quad M \mapsto \{(\alpha_{i,0}, \cdots, \alpha_{i,n})\}_{i\in\mathbb{N}}.$$

Let $f := \sum a_x x \in A$ be a homogeneous element of degree d, where x means $x_{i_0}x_{i_1}\cdots x_{i_{d-1}}$. Then, the multilinearlization of f is an element $f^{\mathrm{mul}}$of the polynomial ring $k[y_{ij}]$ which is given by replacing x by $y_{0,i_0}y_{1,i_1}\cdots y_{d-1,i_{d-1}}$. Let $B := A/(f_1, \cdots, f_m)$, where f_i are homogeneous elements of degree d_i respecitvely . Then, the moduli spaces $\mathcal{M}_B$ of point modules of B is given by

$$\mathcal{M}_B = \{\{\alpha_i := (\alpha_{i,0}, \cdots, \alpha_{i,n})\}_{i\in\mathbb{N}} \in \mathcal{M}_A \mid f_k^{\mathrm{mul}}(\alpha_l, \cdots, \alpha_{l+d_k-1}) = 0, 1 \leq k \leq m, 0 \leq l\} \subset \mathcal{M}_A. \tag{4.0.1}$$

If $B = k\langle x_0, \cdots, x_n\rangle/(x_jx_i - q_{ji}x_ix_j)$ be a quantum polynomial ring with $n+1$-variables, then we have $\mathcal{M}_B \subset \mathbb{P}^n = \mathrm{Proj}(k[x_{0,0}, \cdots, x_{0,n}])$ and

$$\mathcal{M}_B \cap U_l = \bigcap_{i,j,k\neq l} \mathcal{Z}((q_{ij}q_{jk}q_{ki} - 1,)t_it_jt_k) \cap \bigcap_{i,j\neq l} \mathcal{Z}((q_{ij}q_{jl}q_{li} - 1)t_it_j) \subset U_i \tag{4.0.2}$$

on the basic open affines $U_i = \mathrm{Spec}(k[t_0, \cdots, \hat{t_i}, \cdots, t_n])$ ($\hat{t_i}$ means omit t_i). We say $\{q_{ij}\}$ is special if $q_{ij}q_{jk}q_{ki} = 1$ for any three relations $\{x_ix_j - q_{ij}x_jx_i, x_jx_k - q_{jk}x_kx_j, x_kx_i - q_{ki}x_ix_k\}$. Otherwise, we say $\{q_{ij}\}$ is general.

4.1. **Segre products.** Here, we compute the moduli space $\mathcal{M}_{A\circ B/J_\Delta}$ of point modules of $A \circ B/J_\Delta$, where $A \circ B/J_\Delta$ is $\mathbb{N}$-graded connected k-algebra defined in Corollary 3.12.

In the above, point moduleles of $\mathbb{N}$-graded algebras are defined. Similarly, we can defined point modules of $\mathbb{N}^2$-graded connected k-algebras. In particular, the moduli $\mathcal{M}_{A\otimes_k B}$ of point modules on $A\otimes_k B$ and the moduli $\mathcal{M}_{A\circ B}$ of point modules on $A \circ B$ are isomorphic to the fiber product $\mathcal{M}_A \times \mathcal{M}_B$ (see, Lemma 3.10 and [23, Corollary 2.10]). We also have an isomorphism between $\mathcal{M}_{A\otimes B/J}$ and $\mathcal{M}_{A\circ B/J_\Delta}$ from the commutativity of the follwing diagram

$$\begin{array}{ccc} \mathrm{qbigr}(A \otimes_k B) & \longrightarrow & \mathrm{qgr}(A \circ B) \\ \cup & \circlearrowright & \cup \\ \mathrm{qbigr}(A \otimes_k B/J) & \longrightarrow & \mathrm{qgr}(A \circ B/J_\Delta). \end{array}$$

In the following, we consider noncommutative CY projective 3 schemes obtained from $k\langle x_0, x_1, x_2, x_3, y_0, y_1\rangle/(x_jx_i - q_{ji}x_ix_j, y_ly_k - q'_{lk}y_ky_l, f)_{i,j,k,l}$. So, let $A \otimes B/J$ be the ring above in the rest of this subsection. Note that ones given by $k\langle x_0, x_1, x_2, y_0, y_1, y_2\rangle/(x_jx_i - q_{ji}x_ix_j, y_ly_k - q'_{lk}y_ky_l, f)_{i,j,k,l}$ become twists of some commutative CY variety.

Proposition 4.2. *If $\{q_{ij}\}$ is general, then $\dim(\mathcal{M}_{A\otimes_k B/J}) = 1$. Therefore, $\dim(\mathcal{M}_{A\circ B/J_\Delta}) = 1$ and $\mathrm{Proj}(A\circ B/J_\Delta)$ is not isomorphhic to any commutative projective Calavi-Yau 3-variety and the noncommutative projective Calabi-Yau 3-variety in* [11, theorem 1.1].

Proof. If $\{q_{ij}\}$ is general, then $\mathcal{M}_A$ is the 1-skelton $\mathcal{S}$ of $\mathbb{P}^3$ (cf. [11, proof pf Prop 3.4]). Let U_i, V_i be basic open affines of $\mathrm{Proj}(k[x_{0,0}, x_{0,1}, x_{0,2}, x_{0,3}])$ and $\mathrm{Proj}(k[y_{0,0}, y_{0,1}])$, respectively. If $i = j = 0$, then

$$\mathcal{M}_{A\otimes_k B/J} \cap (U_i \times V_j) = ((U_i \cap \mathcal{S}) \times \mathbb{A}^1) \cap \mathcal{Z}(\tilde{f}^{\mathrm{mul}}),$$

where $\tilde{f}^{\mathrm{mul}}$ is the polynomial obtained from the multilinearlization of f, eliminating $x_{1,0}, x_{1,1}, x_{1,2}, x_{1,3}$ and dehomogenezation. This argument holds for any i, j. On the other hand, the dimensions of projective commutative Calabi-Yau varieties are 3 and those in [11, theorem 1.1] are 3 or 0. Therefore, we complete the proof. □

Remark 4.3. If $A\otimes B/(f)$ gives a noncommutative projective Calabi-Yau 2 scheme , then the projective scheme is a twist of a commutative Calabi-Yau variety.

4.2. **Veronese subrings.** Let D be the connected $\mathbb{N}$-graded k-algebra defined in 3.13. In this subsection, we focus on noncommutative CY projective surfaces. So, we suppose $n = 3$. In the same way as in the above discussion, we compute $\mathcal{M}_{D^{(4d)}}$. First, we describe $D^{(4d)}$ as a quotient of a quantum polynomial ring in order to do this.

Lemma 4.4. *$D^{(4d)} \simeq k\langle\{x_{\boldsymbol{i}}\}_{\boldsymbol{i}\in W}\rangle/K$, where*

(1) $W = \{\boldsymbol{i} := (i_0, i_1, i_2, i_3) \in \mathbb{N}^4 \mid a_0 i_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 = 4d\}$,
(2) K is the two-sided ideal defined the elements in sets $S_1, S_2, S_3 \subset k\langle\{x_{\boldsymbol{i}}\}_{\boldsymbol{i}\in W}\rangle$.

$$S_0 = \{x_{\boldsymbol{j}}x_{\boldsymbol{i}} - q_{\boldsymbol{ji}}x_{\boldsymbol{i}}x_{\boldsymbol{j}} \mid \boldsymbol{i},\boldsymbol{j} \in W, q_{\boldsymbol{ji}} = \textstyle\prod_{0\le\alpha,\beta\le3} q_{\beta\alpha}^{j_\alpha i_\beta}\}$$
$$S_1 = \{x_{\boldsymbol{j}}x_{\boldsymbol{i}} - p_{\boldsymbol{ji}}^{\boldsymbol{lk}}x_{\boldsymbol{l}}x_{\boldsymbol{k}} \mid \boldsymbol{j} \ge \boldsymbol{i}, \boldsymbol{j}+\boldsymbol{i} = \boldsymbol{l}+\boldsymbol{k}, x_{\boldsymbol{l}}x_{\boldsymbol{k}} \in S_2', p_{\boldsymbol{ji}}^{\boldsymbol{lk}} = \textstyle\prod_{0\le\alpha<\beta\le3} q_{\beta\alpha}^{j_\alpha i_\beta} / \prod_{0\le\alpha<\beta\le3} q_{\beta\alpha}^{l_\alpha k_\beta}\}$$
$$S_1' = \{x_{\boldsymbol{lk}} \mid (i) : \boldsymbol{l} \ge \boldsymbol{k}, (ii) : x_{\boldsymbol{l}}x_{\boldsymbol{k}} \le x_{\boldsymbol{j}}x_{\boldsymbol{i}} \textit{ for any } (\boldsymbol{j},\boldsymbol{i}) \in W^2 \textit{ such that } \boldsymbol{j} \ge \boldsymbol{i} \textit{ and } \boldsymbol{l}+\boldsymbol{k} = \boldsymbol{j}+\boldsymbol{i}\}$$
$$S_2 = \left\{ x_{\boldsymbol{i}_0+\boldsymbol{h}} + x_{\boldsymbol{i}_1+\boldsymbol{h}} + x_{\boldsymbol{i}_2+\boldsymbol{h}} + x_{\boldsymbol{i}_3+\boldsymbol{h}} \;\middle|\; \begin{array}{c} \boldsymbol{i}_0 = (d/a_0, 0, 0, 0), \boldsymbol{i}_1 = (0, d/a_1, 0, 0), \\ \boldsymbol{i}_2 = (0, 0, d/a_2, 0), \boldsymbol{i}_3 = (0, 0, 0, d/a_3), \\ \boldsymbol{h} \in \{3\boldsymbol{i}_0, 3\boldsymbol{i}_1, 3\boldsymbol{i}_2, 3\boldsymbol{i}_3\} \end{array} \right\}.$$

Proof. We use ideas in [1, Section 4,5]. We consider the following map

$$\varphi : k\langle\{x_{\boldsymbol{i}}\}_{\boldsymbol{i}\in W}\rangle/(S_0) \to C^{(4d)}, \quad x_{\boldsymbol{i}} \mapsto x_0^{i_0}x_1^{i_1}x_2^{i_2}x_3^{i_3}.$$

Note that φ is a surjective homomorphism of graded k-algebra from the choice of elements of W and $\{q_{\boldsymbol{ji}}\}$. So, we want to know $\mathrm{Ker}(\varphi)$. Since C is quadric, $C^{(4d)}$ is also quadric. Hence, we have

$$\dim_k(k\langle\{x_{\boldsymbol{i}}\}_{\boldsymbol{i}\in W}\rangle/(S_1))_2 + \dim_k(\mathrm{Ker}(\varphi))_2 = \dim_k(C^{(4d)})_2.$$

Because $S_1 \subset \mathrm{Ker}(\varphi)_2$, S_1 is lineary independent and the number of S_1 is equal to $\dim_k(\mathrm{Ker}(\varphi))_2$, we have

$$\varphi : k\langle\{x_{\boldsymbol{i}}\}_{\boldsymbol{i}\in W}\rangle/(S_0, S_1) \xrightarrow{\simeq} C^{(4d)}.$$

The set which is obtained by pulling back $\{x_0^{3h_0}f, x_1^{3h_1}f, x_2^{3h_2}f, x_3^{3h_3}f\}$ along φ is S_2. Thus, we get the claim. □

Proposition 4.5. *For any $i = 0, 1, 2$, there exists a weight w in 3.14 such that $\dim(\mathcal{M}_{D^{(4d)}}) = i$.*

Proof. Taking Lemma 4.4 into account, we can compute $\mathcal{M}_{D^{(4d)}}$ in the same way as in the proof of Theorem 4.2 (or " 4.0.1 and 4.0.2 "). In this case, we compute $\mathcal{M}_{D^{(4d)}}$ directly by using computer algebra system because the number of elements of S_2 is large (see Section 5). For example, $\dim_k \mathcal{M}_{D^{(4d)}} = 0$ for the weight $(1, 1, 1, 3)$ and some $\{q_{ij}\}$ which is general. $\dim_k \mathcal{M}_{D^{(4d)}} = 1$ for the weight $w = (1, 2, 3, 6)$ and $\{q_{ij}\}$ which is general. □

5. Appendix

In this section, we give the source code for calculating the dimension of the moduli of point modules of $D^{(4d)}$ in Theorem 3.13, when the weight is $w = (1, 2, 3, 6)$, degree $d = 12$ and $Q := \{q_{ij}\}$ is given as follows

$$Q = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

We use SageMath. If you want to calculate the dimension of the moduli of point modules on $D^{(4d)}$ for different weights, degrees and $\{q_{ij}\}$, change the values of `ai, d, qij` in first few lines of the following source code.

Listing 1. A source code for computing dimensions of moduli of point modules

```
#SageMath version 9.4 using Python 3.9.5

#Input a weight w, a degree d and {q_ij}
a0 = 1; a1 = 2; a2 = 3; a3 = 6; d = 12
q00 = 1; q01 = 1; q02 = -1; q03 = 1; q10 = 1; q11 = 1; q12 = 1; q13 = -1
q20 = -1; q21 = 1; q22 = 1; q23 = 1; q30 = 1; q31 = -1; q32 = 1; q33 = 1
lc = lcm(lcm(a0,a1),lcm(a2,a3)); v0 =  (a0,a1,a2,a3) ; v1 = (d/a0,d/a1,d/a2,d/a3)

#Preparation for the next calculation
basis0 = [(s,t,u,v) for s in range(0,4*lc/a0+1) for t in range(0,4*lc/a1+1) for u in range(0,4*lc/a2+1) for v in range(0,4*lc/a3+1)]

#Calculating the basis of (D^(4d))_1
basis1=[]; N=0
for v in basis0:
    j = a0*v[0]+a1*v[1]+a2*v[2]+a3*v[3]
    if j == lc*4: N=N+1; basis1.append(v)

#Calculating the basis of (D^(4d))_2
basis2=[]; N = 0
for k in range(0,lc*4*2/a0+1):
    for l in range(0,lc*4*2/a1+1):
        for m in range(0,lc*4*2/a2+1):
            for n in range(0,lc*4*2/a3+1):
                j = a0*k+a1*l+a2*m+a3*n
                if j == lc*4*2: N=N+1; w = [k,l,m,n]; basis2.append(w)

#Dividing elements of (D^(4d))_2 into products of 2 elements of (D^(4d))_1
basis3=[]; N=0
for v in basis2:
    s = []
    for w in basis1:
        for u in basis1:
            if w >= u:
                if list(map(lambda a,b: a+b, w,u)) == v: s.append([w,u])
    basis3.append(s); N=N+len(s); print(N,s)

#Preparation for calculating three2
three1=[]
for v in basis1:
    for w in basis1:
        for u in basis1:
            if v > w and w > u:
                    if u[1] == v[0] and v[1] == w[0] and w[1] == u[0]: three1.append([v,w,u])

#Producing the variable assiciated with a list v.
def var1(v):
    return var('x%02d%02d%02d%02d' % (v[0],v[1],v[2],v[3]))

#Giving the polynomials defining the Calabi-Yau surface and dehomogenezation of them
def poly(i,v,d,y):
    e=4*lc-d; e = e/v0[i];y = list(y); ve = [0,0,0,0]; t0=1; t1=1; t2=1; t3=1
    for j in range(0,4):
        if j == i: ve[j] = e
        if j != i: ve[j] = 0
    w0 = [v[0],0,0,0]; w1 = [0,v[1],0,0]; w2 = [0,0,v[2],0]; w3 = [0,0,0,v[3]]
    w0 = list(map(lambda a,b: a+b, w0,ve)); w1 = list(map(lambda a,b: a+b, w1,ve))
```

```
    w2 = list(map(lambda a,b: a+b, w2,ve)); w3 = list(map(lambda a,b: a+b, w3,ve))
    if w0 != y: t0 = var1(w0)
    if w1 != y: t1 = var1(w1)
    if w2 != y: t2 = var1(w2)
    if w3 != y: t3 = var1(w3)
    f = t0+t1+t2+t3
    return f

#A formula for computing $q_{ij}$ in $S_0$
def mul1(v,w):
    c = q00^(v[0]*w[0])*q01^(v[1]*w[0])*q02^(v[2]*w[0])*q03^(v[3]*w[0])*q10^(v[0]*w[1])*q11^(v[1]*w[1])*q12^(v[2]*w[1])*q13^(v[3]*w[1])*q20^(v[0]*w[2])*q21^(v[1]*w[2])*q22^(v[2]*w[2])*q23^(v[3]*w[2])*q30^(v[0]*w[3])*q31^(v[1]*w[3])*q32^(v[2]*w[3])*q33^(v[3]*w[3])
    return c

#A formula for computing $p_{ji}^{lk}$ in $S_1$
def mul2(v,w):
    c = q00^(v[0]*w[0])*q10^(v[0]*w[1])*q11^(v[1]*w[1])*q20^(v[0]*w[2])*q21^(v[1]*w[2])*q22^(v[2]*w[2])*q30^(v[0]*w[3])*q31^(v[1]*w[3])*q32^(v[2]*w[3])*q33^(v[3]*w[3])
    return c

#Computing the set of 3 indexes of $W$ which are general
three2=[]; N=0
for v in three1:
    t = mul1(v[0],v[1])*mul1(v[1],v[2])*mul1(v[2],v[0])
    if  t != 1: N=N+1; three2.append(v)

#Calculating the dimension of the moduli of point modules on each basic open affines
S1=[]; S2=[]; S3=[]; g0=0; g1=0; g2=0; u0=0; u1=0; r=1; n=0
for y in basis1: #y is a basic open affine.
    s =[]
    #Producing the polynomial ring defineing the open affine y
    for v in basis1:
        if v != y: s.append(v)
    R1=['x%02d%02d%02d%02d' %(v[0],v[1],v[2],v[3]) for v in s]
    R=PolynomialRing(QQ,len(R1),R1); S1=[]; n=0
    #Giving $S1$ and the dehomogenezation of it
    for v in basis3:
        m = len(v); u0= 1/ mul2(v[0][0],v[0][1])
        if v[0][0] == y and v[0][1] == y: g0 = mul1(y,v[0][1])*u0
        if v[0][0] == y and v[0][1] != y: g0 = mul1(y,v[0][1])*u0*var1(v[0][1])
        if v[0][0] != y and v[0][1] == y: g0 = mul1(y,v[0][1])*u0*var1(v[0][0])
        if v[0][0] != y and v[0][1] != y: g0 = mul1(y,v[0][1])*u0*var1(v[0][0])*var1(v[0][1])
        for i in range(1,m):
            u1= 1/mul2(v[i][0],v[i][1])
            if v[i][0] == y and v[i][1] == y: g1 = mul1(y,v[i][1])*u1
            if v[i][0] == y and v[i][1] != y: g1 = mul1(y,v[i][1])*u1*var1(v[i][1])
            if v[i][0] != y and v[i][1] == y: g1 = mul1(y,v[i][1])*u1*var1(v[i][0])
            if v[i][0] != y and v[i][1] != y: g1 = mul1(y,v[i][1])*u1*var1(v[i][0])*var1(v[i][1])
            g = g1 - g0; S1.append(g); n=n+1
            if n % 500 == 0:
                J1 = R.ideal(S1); S1 = list(J1.groebner_basis('singular:slimgb')) #We calculate Groebner basis of J1 fruquently to reduce the calculation time of dimensions of ideals.
    J1 = R.ideal(S1); S1 = list(J1.groebner_basis('singular:slimgb'))
    #Giving $S2$ and the dehomogenezation of it
    S2=[poly(0,v1,d,y),poly(1,v1,d,y),poly(2,v1,d,y),poly(3,v1,d,y)]
    #Giving the set of hypersurfaces obtained from the set of general threes of $\{q_{ij}\}$
    g2=1; S3=[]
    for v in three2:
        if v[0] == y : g2 = var1(v[1])*var1(v[2])
        if v[1] == y : g2 = var1(v[0])*var1(v[2])
        if v[2] == y : g2 = var1(v[0])*var1(v[1])
        if v[0] != y and v[1] != y and v[2] != y: g2=var1(v[0])*var1(v[1])*var1(v[2])
        S3.append(f2)
    J3 = R.ideal(S3); S3 = list(J3.groebner_basis('singular:slimgb'))
    #Calculating what we want
    I = R.ideal(S1+S2+S3)
    print(y,I.dimension())
```

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Email address: `m7d5932a72xxgxo@fuji.waseda.jp`

Department of Mathematics, School of Science and Engineering, Waseda University, Ohkubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan