

SOME EXAMPLES OF NONCOMMUTATIVE PROJECTIVE CALABI-YAU SCHEMES OBTAINED FROM NONCOMMUTATIVE SEGRE PRODUCTS AND VERONESE SUBRINGS

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ABSTRACT. In this article, we construct new noncommutative projective Calabi-Yau schemes by using noncommutative Segre products and noncommutative Veronese subalgebras. The first examples are constructed by Kanazawa ([11]).

1. INTRODUCTION

Calabi-Yau varieties are very rich objects. They play an important role in mathematics and physics. In noncommutative geometry, (skew) Calabi-Yau algebras are often treated as noncommutative analogues. They have a deep relation with quiver algebras ([7], [23]). Actually, many known Calabi-Yau algebras are constructed by them. They are also used to characterize Artin-Schelter regular algebras ([17], [18]). In particular, a connected graded algebra A over a field k is Artin-Schelter regular if and only if A is skew Calabi-Yau.

On the other hand, a triangulated subcategory of the derived category of a cubic fourfold in \mathbb{P}^5 which is obtained by some orthogonal decompositions has the 2-shift functor [2] as the Serre functor. Moreover, its structure of Hochschild (co)homology is the same as its of a projective K3 surface ([12]). However, some of such categories are not obtained as the derived category of coherent sheaves of a projective K3 surface. They are called noncommutative K3 surfaces.

Artin and Zhang constructed the framework of noncommutative projective schemes which are defined from noncommutative graded algebras in [3]. In the framework, we can think of Artin-Schelter algebras as noncommutative analogues of projective spaces, which are called quantum projective spaces. Our objective is producing examples of noncommutative projective Calabi-Yau varieties which are not obtained from commutative ones. As the definition of noncommutative projective Calabi-Yau schemes, we adopt the definition by Kanazawa ([11]). His definition is a direct generalization of the definition of (commutative) Calabi-Yau varieties to noncommutative projective schemes. He also constructed the first examples of noncommutative projective Calabi-Yau schemes as hypersurfaces of quantum projective spaces there.

In this paper, we construct new examples of noncommutative projective Calabi-Yau schemes by using noncommutative Segre products and Veronese subrings. We detail that below. Many examples of K3 surfaces are known in algebraic geometry. Among them, it is well-known some of them are divisors of Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, Reid gave the complete list of K3 surfaces which are hypersurfaces in weighted projective spaces ([16], [10]). In this paper, we construct noncommutative analogues of the 2 types of examples of K3 surfaces. Although the methods by Kanazawa are efficient in our cases, we also need different approaches to proceed our study. In order to construct the former case, we perform more detailed analysis about noncommutative projective schemes of \mathbb{Z}^2 -graded algebras which are studied by Rompay ([24]). A different approach to noncommutative Segre products is also studied in [8]. For the latter case, we need to treat quotients of weighted quantum polynomial rings. However, they are not generated in degree 1 in general. So, we take some Veronese subrings of them and consider the noncommutative projective schemes of them (about a different

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approach, see [21]). We bridge between properties of quotients of weighted quantum polynomial ring and those of thier Veronese subrings by using Čech complexe for noncommutative algebras (this is used in the former case). Those mentioned above are treated in Section 3.

We also show some of them are not isomorphic to commutative projective Calabi-Yau varieties and the first examples above in Section 4. To be more precise, we compare the dimensions of moduli spaces of point modules. We focus on noncommutative projective Calabi-Yau schemes of dimension 2 and 3. In particular, calculating the dimensions of moduli of point modules of Veronese subrings by hand is difficult because the number of generators and relations of them is large. To calculate them, we use noncommutative Veronese maps (cf.[1]). We also use a computer algebra system and give the source code in Section 5.

2. PRELIMINARIES

Notation 2.1. In this article, k means an algebraically closed field of characteristic 0. We suppose \mathbb{N} contains 0.

Definition 2.2 ([3, Section 2], [27, Section 0]). For any connected graded right Noetherian k -algebra $A = \bigoplus_{i=0}^{\infty} A_i$, we denote the category of graded right A -modules (resp. finitely generated graded right A -modules) by $\text{Gr}(A)$ (resp. $\text{gr}(A)$). We denote the shift functor by $(-)(1) : \text{Gr}(A) \rightarrow \text{Gr}(A)$, $M \mapsto M(1) := \bigoplus M(1)_i := \bigoplus M_{i+1}$. When we write $M, N \in \text{Gr}(A)$, $\text{Hom}_A(M, N) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{Gr}(A)}(M, N(n))$.

We also denote the subcategory of torsion modules in $\text{Gr}(A)$ (resp. $\text{gr}(A)$) by $\text{Tor}(A)$ (resp. $\text{tor}(A)$). We denote the quotient category $\text{Gr}(A)/\text{Tor}(A)$ (resp. $\text{gr}(A)/\text{tor}(A)$) by $\text{QGr}(A)$ (resp. $\text{qgr}(A)$) and the canonical projection by $\pi : \text{Gr}(A) \rightarrow \text{QGr}(A)$. Let $\mathcal{A} := \pi(A)$. The (general) projective scheme of A is defined as $\text{Proj}(A) := (\text{Qgr} A, \mathcal{A}, s)$. We also define the (Noetherian) projective scheme $\text{proj}(A) := (\text{qgr} A, \mathcal{A}, s)$. Let $X := \text{proj}(A)$. The global section of any object \mathcal{N} is $H^0(X, \mathcal{N}) = \text{Hom}_{\text{qgr}(A)}(\mathcal{A}, \mathcal{N})$. The cohomology is $H^i(X, \mathcal{N}) := \text{Ext}_{\text{qgr}(A)}^i(\mathcal{A}, \mathcal{N})$ for $i > 0$.

Definition 2.3 ([26, Section 4], [22, Section 4]). Let A be a connected graded k -algebra and m_A be $A_{\geq 1}$. Let M be a right graded A -module.

Then, we denote $\lim_{n \rightarrow \infty} \text{Hom}_A(A/A_{\geq n}, M)$ by $\Gamma_{m_A}(M)$. By using this, we define a functor $\Gamma_{m_A} : \text{Gr}(A) \rightarrow \text{Gr}(A)$ (we call Γ_{m_A} torsion functor). We denote the derived functor of Γ_{m_A} by $R\Gamma_{m_A}$ and $H^i R\Gamma_{m_A}$ by $H_{m_A}^i$.

Definition 2.4 ([26, Definition 3.3, 4.1], [22, Definition 6.1, 6.2]). Let A be a right and left Noetherian connected graded k -algebra, and A°, A^e be the opposite algebra, and the enveloping algebra of A , respectively. Let R be an object of $\text{D}^b(A^e)$. Then, R is called a dualizing complex of A if (1) R has finite injective dimension over A and A° . (2) The cohomology of R is finitely generated as both A and A° -modules. (3) The natural morphism $A \rightarrow \text{RHom}_A(R, R)$ and $A \rightarrow \text{RHom}_{A^\circ}(R, R)$ are isomorphisms $\text{D}^b(A^e)$.

Moreover, R is called balanced if $R\Gamma_{m_A}(R) \simeq A'$ and $R\Gamma_{m_{A^\circ}}(R) \simeq A'$, where A' is Matlis dual of A .

3. CALABI-YAU CONDITIONS

Definition 3.1 ([11, Section 2.2]). Let A be a connected right Noetherian graded k -algebra. Then, $\text{proj}(A)$ is a Calabi-Yau n scheme if its global dimension is n and the Serre functor of the derived category $\text{D}^b(\text{qgr}(A))$ is the n -shift functor $[n]$.

3.1. Segre products. In commutative algebraic geometry, when let X be the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^1$ into \mathbb{P}^6 , smooth hypersurfaces $S \subset X$ of type $(3, 2)$ provide Calabi-Yau surfaces. In this section, we construct noncommutative analogues of these examples. And, we prove that some of those are not obtained from twisted homogeneous coordinates of the examples in Section 4.

Let C be a connected \mathbb{N}^2 -graded k -algebra. We denote the category of bigraded right C -modules (resp. finitely generated bigraded right C -modules) by $\text{BiGr}(C)$ (resp. $\text{bigr}(C)$). In the same way as in

Definition 2.2, we define \mathbb{Z}^2 -graded torsion right C -modules. Let M be a \mathbb{Z}^2 -graded right C -module. If $M_{(\geq s, \geq s)} := \bigoplus_{i \geq s, j \geq s} M_{ij} = 0$ for $s \gg 0$, then we say M is a torsion C -module. We denote the category of \mathbb{Z}^2 -graded torsion C -modules by $\text{Tor}(C)$. We also define $\text{tor}(C)$ to be the intersection of $\text{bigr}(C)$ and $\text{Tor}(C)$. So, we can construct the quotient category $\text{QBiGr}(C) := \text{BiGr}(C)/\text{Tor}(C)$ (resp. $\text{qbigr}(C) := \text{bigr}(C)/\text{tor}(C)$) (cf. [24, Section 2]).

Moreover, we define $C_{++} := C_{(>0, >0)}$ and the torsion functor $\Gamma_{C_{++}}$ for a \mathbb{N}^2 -graded k -algebra C to be the map which sends M to $\{m \in M \mid (C_{++})^n m = 0 \text{ for some } m \in \mathbb{N}\}$.

We also denote the maximal ideal of C by m_C and define the notion of dualizing complexes of C in the same way as in Section 2.

Remark 3.2. In [17], the authors treat \mathbb{Z}^n -graded algebras. They define torsion functors of \mathbb{Z}^n -graded modules by using total degrees of homogeneous elements of multigraded modules. So, their definition and ours are different.

In this section, we prove the following theorem.

Theorem 3.3. *Let n, m be positive integers such that $n \geq m$. Let A be $k\langle x_0, \dots, x_n \rangle / (x_j x_i - q_{ji} x_i x_j)_{i,j}$ and let B be $k\langle y_0, \dots, y_m \rangle / (y_j y_i - q'_{ji} y_i y_j)_{i,j}$. Let $f := \sum_{i=0}^m x_i^{n+1} y_i^{m+1} + \sum_{j=0}^m y_j^{m+1} \sum_{i=m+1}^n x_i^{n+1}$ be a bihomogeneous polynomial of degree $(n+1, m+1)$ in $C := A \otimes_k B$. We assume that (1) $q_{ii} = 1, q_{ij} q_{ij} = 1$ and $q_{ij}^{n+1} = 1$, (2) $q'_{ii} = 1, q'_{ij} q'_{ji} = 1$ and $q'_{ij}^{m+1} = 1$.*

Then, $\text{qbigr}(C/(f))$ is a Calabi-Yau $(n+m-1)$ -category if $\prod_{i=0}^n q_{ij}$ and $\prod_{i=0}^m q'_{ij}$ is independent of j .

Note that f is a normal element because of the choice of $\{q_{ij}\}, \{q'_{ij}\}$. To prove the theorem, we need to show some lemmas.

Lemma 3.4. *Let A, B be as in Theorem 3.3. Then, $\mathbb{R}\Gamma_{C_{++}}(C)^*$ is isomorphic to $A_\phi(-n-1) \otimes_k B_\psi(-m-1)[n+m+1]$ in $\text{D}(\text{qbigr}(C))$ where ϕ (resp. ψ) is the graded automorphism of A (resp. B) which maps $x_j \mapsto \prod q_{ji} x_j$ (resp. $y_j \mapsto \prod q'_{ji} y_j$).*

Proof. First, let I_1, I_2 be the ideals generated by m_A, m_B respectively. Then, we have $C_{++} = I_1 \cap I_2$ and $m_C = I_1 + I_2$ and have the following long exact sequence

$$\dots \rightarrow H_{C_{++}}^{i-1}(C) \rightarrow H_{m_C}^i(C) \rightarrow H_{I_1}^i(C) \oplus H_{I_2}^i(C) \rightarrow H_{C_{++}}^i(C) \rightarrow H_{m_C}^{i+1}(C) \rightarrow \dots$$

by the Mayer-Vietris sequence. Note that we can use the Mayer-Vietris sequence in our case (see Remark 3.6 below).

On the other hand, $\mathbb{R}\Gamma_{m_C}(C)^*$ is the balanced dualizing complex of C (see [17, Proof of Lemma 3.5]). Then, we have

$$H_{m_C}^i(C)^* \simeq \begin{cases} 0 & i \neq n+m+2 \\ H_{m_A}^{n+1}(A)^* \otimes H_{m_B}^{m+1}(B)^* \simeq A_\phi(-n-1) \otimes_k B_\psi(-m-1) & i = n+m+2 \end{cases}$$

from [11, Proposition 2.4].

Moreover, we have $H_{I_1}^i(C)$ and $H_{I_2}^i(C)$ are torsion modules for C_{++} (about the commutative case, see [13, Proposition 2.1.6]).

Thus, we have isomorphisms

$$\mathbb{R}\Gamma_{C_{++}}(C)^* \simeq \mathbb{R}\Gamma_{m_C}(C)^*[-1] \simeq A_\phi(-n-1) \otimes_k B_\psi(-m-1)[n+m+1]$$

in $\text{D}(\text{qbigr}(C))$. □

Remark 3.5. In our situation, the balanced dualizing complex R_C of C is not suitable to prove the theorem because we treat the torsion category of the torsion modules for C_{++} to define $\text{qbigr}(C)$.

Remark 3.6. To use the Mayer-Vietris sequence, we need to prove the following equivalences

$$\begin{aligned} & \lim_{n \rightarrow \infty} \operatorname{Ext}_C^i(C/(I_1^n + I_2^n), -) \text{ and } H_{I_1 + I_2}^i(-), \\ & \lim_{n \rightarrow \infty} \operatorname{Ext}_C^i(C/I_1^n \oplus C/I_2^n, -) \text{ and } H_{I_1}^i(-) \oplus H_{I_2}^i(-), \\ & \lim_{n \rightarrow \infty} \operatorname{Ext}_C^i(C/(I_1^n \cap I_2^n), -) \text{ and } H_{I_1 \cap I_2}^i(-). \end{aligned}$$

In the commutative ring theory, these equivalences are proved by using cofinality and the Artin-Rees Lemma (cf.[6, Chapter A1D], [4, Chapter 3]). In general, an ideal of a noncommutative ring does not satisfy the Artin-Rees Lemma. However I_1, I_2 satisfy the Artin-Rees property in the sense of [14, Chapter 4.2], because I_1, I_2 are generated by normal elements in our case. Thanks to this fact, we prove the above equivalences in the same way as in the case of commutative rings.

Lemma 3.7. $\operatorname{gl.dim}(\operatorname{qgr}(C/(f))) = n + m - 1$

Proof. We consider a bigraded (commutative) algebra $D := k[s_0, \dots, s_n, t_0, \dots, t_m]/(f)$ and the projective spectrum $\operatorname{biproj}(D)$ in the sense of [9, Section 1]. Then, an object in $\operatorname{qbigr}(C/(f))$ can be thought as an object in the category of modules over the sheaf \mathcal{A} of algebras, where \mathcal{A} is the sheaf associated to algebras $(k[x_0, \dots, x_n, y_0, \dots, y_m]/(f)_{x_i y_j})_{(0,0)}$ for each open affine scheme $D_+(s_i t_j) \simeq \operatorname{Spec}((D_{s_i t_j})_{(0,0)})$. Hence, it is enough to prove that the global dimension of $(k[x_0, \dots, x_n, y_0, \dots, y_m]/(f)_{x_i y_j})_{(0,0)} = n + m - 1$.

In our case, we can proceed the rest of the proof in the same way as in [11, Section 2.3] with an exception. We give its sketch and mention the exception. We consider the condition $i = j$ or $i > m \cdots (*)$.

We assume that $(*)$ holds. For simplicity, we prove it when $i = j = 0$. Let $S_i := s_i/s_0, T_i := t_i/t_0, X_i := x_i/x_0, Y_i := y_i/y_0$. Then we consider the $k[S_1, \dots, S_n, T_1, \dots, T_m]/(1 + \sum_{i=1}^m S_i T_i + (1 + \sum_{i=1}^m T_i \sum_{i=m+1}^n S_i)$ -algebra $k[X_1, \dots, X_n, Y_1, \dots, Y_m]/(1 + \sum_{i=1}^m X_i^{n+1} Y_i^{m+1} + (1 + \sum_{i=1}^m Y_i^{m+1} \sum_{i=m+1}^n X_i^{n+1}))$, where the module structure is given by the identification $S_i = X_i^{n+1}, T_i = Y_i^{m+1}$. We denote the former by E and the latter by F . It is enough to prove that the global dimension of localization F_m of F at any maximal ideal $m := (S_1 - a_1, \dots, S_n - a_n, T_1 - b_1, \dots, T_m - b_m)$ of E with $f(1, a_1, \dots, a_n, 1, b_1, \dots, b_m)$ is $n + m - 1$.

If any a_i, b_i is not 0, then F/m is a twisted group ring and hence semisimple. Moreover, $S_1 - a_1, \dots, S_n - a_n, T_1 - b_1, \dots, T_m - b_m$ is a regular sequence. This induces the claim. On the other hand, we assume that one of $\{a_1, \dots, a_n, b_1, \dots, b_m\}$ is 0. For example, we assume $a_1 = 0$. Then, we consider E/X_1 . We repeat taking quotients and reduce to considering the global dimensions of the algebras $k\langle X \rangle/(X^{n+1} + 1)$ and $k\langle Y \rangle/(Y^{m+1} + 1)$, which are 0.

We assume $(*)$ does not hold. For example, let $i = 0, j = 1$. We also consider m and F_m . When all a_i, b_i are not 0, the proof is the same. When some a_i, b_i are 0, an exception can occur. We suppose each term of $f(1, a_1, \dots, a_n, b_0, 1, \dots, b_m)$ is $0 \cdots (**)$. After repeating taking quotients, E becomes an algebra of the form $R = k\langle t_0, \dots, t_k \rangle/(t_j t_i - p_{ji} t_j t_i, t_0^l)$, which can not be applied the above proof. Because $\operatorname{gl.dim}(R) = \operatorname{gl.dim}(k\langle t_1, \dots, t_k \rangle/(t_j t_i - p_{ji} t_j t_i)) = k$, we can complete the proof in this exceptional case. If not $(**)$, the proof is the same as that of the first case. \square

Proof of Theorem 3.3. First, we calculate $\mathbb{R}\Gamma_{C/(f)++}(C/(f))^*$. For this, we consider the distinguished triangle in $\operatorname{D}(\operatorname{qbigr}(C/(f)))$

$$\mathbb{R}\Gamma_{C++}(C(-n-1, -m-1)) \xrightarrow{\times f} \mathbb{R}\Gamma_{C++}(C) \longrightarrow \mathbb{R}\Gamma_{C/(f)++}(C/(f))$$

from the exact sequence

$$0 \longrightarrow C(-n-1, -m-1) \xrightarrow{\times f} C \longrightarrow C/(f) \longrightarrow 0.$$

Now, we have the isomorphism in $\operatorname{D}(\operatorname{qbigr}(C/(f)))$

$$\mathbb{R}\Gamma_{C++}(C)^* \simeq A_\phi(-n-1) \otimes_k B_\psi(-m-1)[n+m+1] \simeq A \otimes_k B[n+m+1]$$

from Lemma 3.4 and the choice of $\{q_{ij}\}$ and $\{q'_{ij}\}$. Hence, we have $\mathbb{R}\Gamma_{C/(f)_{++}}(C/(f))^* \simeq C/(f)[n+m]$. In addition, we have the local duality for the torsion functor $\Gamma_{C/(f)_{++}}$ and the Serre duality in $D^b(\text{qbigr}(C/(f)))$ from Lemma 3.8 (we prove this below). Thus, $\mathbb{R}\Gamma_{C/(f)_{++}}(C/(f))^*[-1]$ induces the Serre functor in $D^b(\text{qbigr}(C/(f)))$. Since $\mathbb{R}\Gamma_{C/(f)_{++}}(C/(f))^* \simeq C/(f)[n+m]$, we complete the proof. \square

Lemma 3.8 (Local Duality and Serre Duality). *Let $D := C$ or $C/(f)$. We have the following.*

(1) *Let $M \in D^b(\text{bigr}(D))$. Then, we have*

$$\mathbb{R}\Gamma_{D_{++}}(M) \simeq \mathbb{R}\text{Hom}_D(M, \mathbb{R}\Gamma_{D_{++}}(D)')$$

in $D^b(\text{bigr}(D))$.

(2) *Let $\mathcal{D} := \pi(D)$, $\mathcal{M} := \pi(M)$ and $\tilde{\mathcal{R}}_D := \pi(\mathbb{R}\Gamma_{D_{++}}(D)') \in D^b(\text{qbigr}(D))$. Then, we have*

$$\text{Ext}_{\text{qbigr}(D)}^i(\mathcal{D}, \mathcal{M})^* \simeq \text{Ext}_{\text{qbigr}(D)}^{-i}(\mathcal{M}, \tilde{\mathcal{R}}_D[-1]).$$

Proof. To prove (1), we want to apply [25, Theorem 0.4]. So, we show that the torsion class defined by D_{++} is quasi-compact, finite dimensional and stable (about the definition, see [25, Definition 3.4]). First, we prove that the torsion class is stable. D_{++} is generated by normal elements $\{x_i y_j\}$. So, D_{++} has Artin-Rees property in the sense of [14, Chapter 4.2]. Thanks to this property, we apply the proof of [6, Lemma A1.4]. This shows the stability of the torsion class. Let $l := \text{lcm}(n+1, m+1)$. Then, D_{++} and D_{++}^l define the same torsion class. We consider the latter in the rest of the proof. Note that D_{++}^l is generated by central elements $\{x_i^l y_j^l\}_{i,j}$ from the choice of $\{q_{ij}\}$ and $\{q'_{ij}\}$. Moreover, we have a surjective localization map $N \rightarrow N[(x_i^l y_j^l)^{-1}]$ for any $x_i^l y_j^l$ and any right injective D -module. Thus, we can calculate the local cohomology for D_{++}^l by using Čech complexes (cf. [6, Proof of Theorem A1.3], [15]). This shows that the torsion class is quasi-compact and finite. Hence, we can apply [25, Theorem 0.4]. Finally, we obtain the claim by taking dual. About (2), we can prove it in the same way as in [27, Theorem 4.2] by using (1). \square

As a corollary of Theorem 3.3, we construct examples of noncommutative projective Calabi-Yau schemes.

Definition 3.9. (1) The Segre product $A \circ B$ of A and B is the \mathbb{N} -graded k -algebra with $(A \circ B)_i = A_i \otimes_k B_i$.
 (2) Let $M \in \text{bigr}(C)$. We define a right graded $A \circ B$ -module M_Δ as the graded $A \circ B$ -module with $(M_\Delta)_i = M_{ii}$.

Lemma 3.10 ([24, Theorem 2.4]). *We have the following natural isomorphism*

$$\text{qbigr}(C) \longrightarrow \text{qgr}(A \circ B), \quad \pi(M) \longmapsto \pi(M_\Delta).$$

In addition, the functor defined by $-\otimes_{A \circ B} C$ is the inverse of this equivalence.

Remark 3.11. We similarly obtain an equivalence

$$\text{qbigr}(C/J) \simeq \text{qgr}(A \circ B/J_\Delta),$$

where $J := (f) \in \text{bigr}(C)$.

Combining Theorem 3.3 with Remark 3.11, we get the following.

Corollary 3.12. *Let $J := (f) \in \text{bigr}(C)$. Then, $\text{proj}(A \circ B/J_\Delta)$ is noncommutative projective Calabi-Yau scheme.*

3.2. Veronese subalgebras. Reid produced the list of all commutative weighted Calabi-Yau hypersurfaces of dimension 2 (for example, see [16], [10]).

In this section, we construct noncommutative Calabi-Yau schemes from Veronese subrings of noncommutative weighted projective hypersurfaces.

Let A be a connected graded k -algebra. Then the Veronese subring $B := A^{(r)}$ is the connected graded k -algebra $B = \bigoplus_i B_i := \bigoplus_i A_{rn}$.

We consider the (commutative) weighted polynomial ring $A = k[x_0, \dots, x_n]$ with $\deg(x_i) = a_i$. Then, $\text{Coh}(\text{Proj}(A))$ is not equivalent to $\text{qgr}(A)$, but to $\text{qgr}(A^{\text{nlcm}(a_0, \dots, a_n)})$. So, when we consider noncommutative projective schemes of quotient rings of quantum weighted polynomial rings, it seems that we should consider Veronese subrings of those rings.

Theorem 3.13. *Let $w = (a_0, \dots, a_n) \in \mathbb{Z}_{>0}^{n+1}$ and $d := \sum a_i$. We suppose that $a_i | d$ for any i . Let $C := k\langle x_0, \dots, x_n \rangle / (x_j x_i - q_{ji} x_i x_j)$ be a quantum weighted polynomial ring with $\deg(x_i) = a_i$ for $0 \leq i \leq n$. Let $f := \sum x_i^{h_i}$, where $h_i := d/a_i$. Let $D := C/(x_i^{n h_i} f)_i$. We assume that $q_{ii} = 1, q_{ij} q_{ji} = 1$ and $q_{ij}^{d/a_i} = q_{ij}^{d/a_j} = 1$.*

Then, $\text{proj}(D^{((n+1)d)})$ is a noncommutative projective Calabi-Yau $(n-1)$ scheme if $\prod_{i=0}^n q_{ij}$ is independent of $0 \leq j \leq n$.

Example 3.14. The types of weights of noncommutative Calabi-Yau surfaces which are well-formed are the following.

$$(a_0, a_1, a_2, a_3) = (1, 1, 1, 1), (1, 1, 1, 3), (1, 1, 2, 2), (1, 1, 2, 4), (1, 1, 2, 5), (1, 1, 4, 6), \\ (1, 2, 3, 6), (1, 3, 3, 4), (2, 3, 3, 4), (1, 2, 6, 9), (2, 3, 10, 15), (1, 6, 14, 21).$$

Note that (a_0, \dots, a_n) is well-formed if $\gcd(a_0, \dots, \hat{a}_i, \dots, a_n) = 1$ for any i (cf. [10]).

We use this list in Section 4.2.

Remark 3.15. $D^{((n+1)d)}$ is generated by elements of degree 1. And, f is a normal element from the choice of $\{q_{ij}\}$.

Lemma 3.16. *The balanced dualizing complex of $C/(f)$ is isomorphic to $C_\phi/(f)[n]$, where ϕ is a graded automorphism of C which maps $x_i \mapsto \prod q_{ij} x_j$.*

Proof. Let $C' := k\langle x'_0, \dots, x'_n \rangle / (x'_j x'_i - q_{ji} x'_i x'_j)$ be a quantum polynomial ring with $\deg(x'_i) = 1$ for any i . Then, C' is Koszul. We can calculate the balanced dualizing complex $R_{C'}$ of C' by using its Koszul resolution K' and taking $\text{Hom}_{(C')^e}(-, (C')^e)^{-1}$ by [22, Section 8,9]. If we take some shifts on the summands of each K'^i , this complex K is a graded free resolution of C as C^e -modules. Then, we take $\text{Hom}_{C^e}(-, C^e)^{-1}$ and get the balanced dualizing complex R_C of C . So, R_C and $R_{C'}$ have the same forms except for shifts of degrees. Indeed, the difference of the shifts is $\sum_{i=0}^n a_i - (n+1)$. So, we get $R_C = C_\phi(-\sum_{i=0}^n a_i)[n+1] = C_\phi(-d)[n+1]$, where ϕ is the graded automorphism defined in the statement of this lemma. The rest of the proof is done in the same way as in the first half of the proof of Theorem 3.3. \square

Lemma 3.17. $\text{gl.dim}(\text{qgr}(D^{((n+1)d)})) = n - 1$

Proof. In order to use the method of the proof in Lemma 3.7, we need some reduction. First, $\text{qgr}(D^{((n+1)d)})$ is thought as the category of modules over a sheaf \mathcal{B} of algebras on $\text{Proj}(k[t_0, \dots, t_n]/(t_i^n \sum_{j=0}^n t_j)_i)$, where $\mathcal{B}(D_+(t_i))$ is isomorphic to $\left(D_{x_i^{(n+1)h_i}}^{((n+1)d)}\right)_0$. This is because $\{\text{Mod}((D_{x_i^{(n+1)h_i}}^{((n+1)d)})_0)\}_i$ is an open cover of $\text{qgr}(D^{((n+1)d)})$. About open covers of noncommutative projective schemes and open subspaces, see [20, Section 3.7, 5.4]. So, it is enough to prove that $\text{gl.dim} \left(D_{x_i^{(n+1)h_i}}^{((n+1)d)}\right)_0 = n - 1$. Then, we have isomorphisms

$$\left(D_{x_i^{(n+1)h_i}}^{((n+1)d)}\right)_0 \simeq \left(D_{x_i^{(n+1)h_i}}\right)_0 \simeq (D_{x_i})_0 \simeq (C/(f)_{x_i})_0.$$

So, it is enough to prove the global dimension of $((C/f)_{x_i})_0$ is $n - 1$. Moreover, $((C/f)_{x_i})_0$ is an algebra over $(k[t_0, \dots, t_n]/(\sum_{j=0}^n t_j)_{t_i})_0$. Thus, we can prove it in the same way as in the latter half of the proof of Lemma 3.7. \square

Proof of Theorem 3.13. First, we have an isomorphism $\text{proj}(C/(f)) \simeq \text{proj}(D)$. From Lemma 3.16 and the choice of $\{q_{ij}\}$, $\text{proj}(C/(f))$ is a noncommutative projective Calabi-Yau $(n - 1)$ schemes. And, so does $\text{proj}(D)$. This means the balanced dualizing complex R_D of D is $\mathbb{R}\Gamma_{m_D}(D)' \simeq D[3]$. We compute the balanced dualizing complex $R_{D^{((n+1)d)}}$ of $D^{((n+1)d)}$, which is isomorphic to $\mathbb{R}\Gamma_{m_{D^{(4d)}}}(D^{((n+1)d)})'$. Let $n_D \subset D$ be the ideal of D generated by $\{x_0^{(n+1)h_0}, \dots, x_n^{(n+1)h_n}\}$. Then, m_D and n_D (resp. $n_D \cap D^{((n+1)d)}$ and $m_{D^{((n+1)d)}}$) define the same local cohomology of D (resp. $D^{((n+1)d)}$). In addition, $x_0^{(n+1)h_0}, \dots, x_n^{(n+1)h_n}$ are central elements of D . Thus, the Čech complex $C(x_0^{(n+1)h_0}, \dots, x_n^{(n+1)h_n}; D)$ (resp. $C(x_0^{(n+1)h_0}, \dots, x_n^{(n+1)h_n}; D^{((n+1)d)})$) associated by $\{x_0^{(n+1)h_0}, \dots, x_n^{(n+1)h_n}\}$ computes the local cohomology of D (resp. $D^{((n+1)d)}$) for n_D (resp. $n_D \cap D^{((n+1)d)}$) (for details, see the proof of 3.8). Since $C(x_0^{(n+1)h_0}, \dots, x_n^{(n+1)h_n}; D)^{((n+1)d)}$ is isomorphic to $C(x_0^{(n+1)h_0}, \dots, x_n^{(n+1)h_n}; D^{((n+1)d)})$ as chain complexes of graded $D^{((n+1)d)}$ -modules, we have $H_{n_D}^i(D)^{((n+1)d)} \simeq H_{n_D \cap D^{((n+1)d)}}^i(D^{((n+1)d)})$. Hence, we have $H_{m_D}^i(D^{((n+1)d)}) \simeq H_{m_{D^{((n+1)d)}}}^i(D^{((n+1)d)})$ and $R_{D^{((n+1)d)}} \simeq D^{((n+1)d)}[n]$. This shows that $\text{proj}(D^{((n+1)d)})$ is a noncommutative projective Calabi-Yau $(n - 1)$ schemes. \square

4. POINT MODULES

In this section, we calculate point modules of noncommutative Calabi-Yau schemes obtained in Section 3 and show some of them give new examples of noncommutative Calabi-Yau schemes which are different from commutative ones and the examples obtained in [11].

Definition 4.1 ([2, Definition 3.8], [19, Definition 3.1], [5]). Let A be a connected graded k -algebra which is generated in degree 1. Let M be a graded right A -module. We say M is a point module if M is cyclic, generated in degree 0 and $\dim_k(M_i) = 1$ for all $i \geq 0$.

We mention some basic facts which are needed in this section. For details, see [19, Section 3], [11, Section 3], etc.

Let $A := k\langle x_0, \dots, x_n \rangle$ be a free associative algebra with $n + 1$ -variable. Then, the moduli space \mathcal{M}_A of point modules of A is isomorphic to $\prod_{i=0}^{\infty} \mathbb{P}^n$. Let $M := \bigoplus_i km_i$ be a point module of A . If $m_i x_j = \alpha_{i,j} m_{i+1}$, then we can describe the isomorphism between them as follows

$$\mathcal{M}_A \rightarrow \prod_{i=0}^{\infty} \mathbb{P}^n, \quad M \mapsto \{(\alpha_{i,0}, \dots, \alpha_{i,n})\}_{i \in \mathbb{N}}.$$

Let $f := \sum a_x x \in A$ be a homogeneous element of degree d , where x means $x_{i_0} x_{i_1} \dots x_{i_{d-1}}$. Then, the multilinearization of f is an element f^{mul} of the polynomial ring $k[y_{ij}]$ which is given by replacing x by $y_{0,i_0} y_{1,i_1} \dots y_{d-1,i_{d-1}}$. Let $B := A/(f_1, \dots, f_m)$, where f_i are homogeneous elements of degree d_i respectively. Then, the moduli spaces \mathcal{M}_B of point modules of B is given by

$$\mathcal{M}_B = \{ \{ \alpha_i := (\alpha_{i,0}, \dots, \alpha_{i,n}) \}_{i \in \mathbb{N}} \in \mathcal{M}_A \mid f_k^{\text{mul}}(\alpha_l, \dots, \alpha_{l+d_k-1}) = 0, 1 \leq k \leq m, 0 \leq l \} \subset \mathcal{M}_A. \quad (4.0.1)$$

If $B = k\langle x_0, \dots, x_n \rangle / (x_j x_i - q_{ji} x_i x_j)$ be a quantum polynomial ring with $n + 1$ -variables, then we have $\mathcal{M}_B \subset \mathbb{P}^n = \text{Proj}(k[x_{0,0}, \dots, x_{0,n}])$ and

$$\mathcal{M}_B \cap U_l = \bigcap_{i,j,k \neq l} \mathcal{Z}((q_{ij} q_{jk} q_{ki} - 1) t_i t_j t_k) \cap \bigcap_{i,j \neq l} \mathcal{Z}((q_{ij} q_{jl} q_{li} - 1) t_i t_j) \subset U_l \quad (4.0.2)$$

on the standard open affines $U_l = \text{Spec}(k[t_0, \dots, \hat{t}_l, \dots, t_n])$ (\hat{t}_l means omit t_l). We say $\{q_{ij}\}$ is special if $q_{ij} q_{jk} q_{ki} = 1$ for any three relations $\{x_i x_j - q_{ij} x_j x_i, x_j x_k - q_{jk} x_k x_j, x_k x_i - q_{ki} x_i x_k\}$. Otherwise, we say $\{q_{ij}\}$ is general.

4.1. Segre products. Here, we compute the moduli space $\mathcal{M}_{A \circ B/J_\Delta}$ of point modules of $A \circ B/J_\Delta$, where $A \circ B/J_\Delta$ is \mathbb{N} -graded connected k -algebra defined in Corollary 3.12.

In the above, point modules of \mathbb{N} -graded algebras are defined. Similarly, we can define point modules of \mathbb{N}^2 -graded connected k -algebras. In particular, the moduli $\mathcal{M}_{A \otimes_k B}$ of point modules on $A \otimes_k B$ and the moduli $\mathcal{M}_{A \circ B}$ of point modules on $A \circ B$ are isomorphic to the fiber product $\mathcal{M}_A \times \mathcal{M}_B$ (see, Lemma 3.10 and [24, Corollary 2.10]). We also have an isomorphism between $\mathcal{M}_{A \otimes_k B/J}$ and $\mathcal{M}_{A \circ B/J_\Delta}$ from the commutativity of the following diagram

$$\begin{array}{ccc} \text{qbigr}(A \otimes_k B) & \longrightarrow & \text{qgr}(A \circ B) \\ \cup & \circ & \cup \\ \text{qbigr}(A \otimes_k B/J) & \longrightarrow & \text{qgr}(A \circ B/J_\Delta). \end{array}$$

In the following, we consider noncommutative Calabi-Yau projective 3 schemes obtained from $k\langle x_0, x_1, x_2, x_3, y_0, y_1 \rangle / (x_j x_i - q_{ji} x_i x_j, y_l y_k - q'_{lk} y_k y_l, f)_{i,j,k,l}$. So, let $A \otimes B/J$ be the ring above in the rest of this subsection. Note that ones given by $k\langle x_0, x_1, x_2, y_0, y_1, y_2 \rangle / (x_j x_i - q_{ji} x_i x_j, y_l y_k - q'_{lk} y_k y_l, f)_{i,j,k,l}$ become twists of some commutative Calabi-Yau variety.

Proposition 4.2. *If $\{q_{ij}\}$ is general, then $\dim(\mathcal{M}_{A \otimes_k B/J}) = 1$. Therefore, $\dim(\mathcal{M}_{A \circ B/J_\Delta}) = 1$ and $\text{Proj}(A \circ B/J_\Delta)$ is not isomorphic to any commutative projective Calabi-Yau 3-variety and the noncommutative projective Calabi-Yau 3 schemes in [11, theorem 1.1].*

Proof. If $\{q_{ij}\}$ is general, then \mathcal{M}_A is the 1-skelton \mathcal{S} of \mathbb{P}^3 (cf. [11, proof of Prop 3.4]). Let U_i, V_i be basic open affines of $\text{Proj}(k[x_{0,0}, x_{0,1}, x_{0,2}, x_{0,3}])$ and $\text{Proj}(k[y_{0,0}, y_{0,1}])$, respectively. If $i = j = 0$, then

$$\mathcal{M}_{A \otimes_k B/J} \cap (U_i \times V_j) = ((U_i \cap \mathcal{S}) \times \mathbb{A}^1) \cap \mathcal{Z}(\tilde{f}^{\text{mul}}),$$

where \tilde{f}^{mul} is the polynomial obtained from the multilinearization of f , eliminating $x_{1,0}, x_{1,1}, x_{1,2}, x_{1,3}$ and dehomogenization. This argument holds for any i, j . On the other hand, the dimensions of projective commutative Calabi-Yau varieties are 3 and those in [11, theorem 1.1] are 3 or 0. Therefore, we complete the proof. \square

Remark 4.3. If $A \otimes B/(f)$ gives a noncommutative projective Calabi-Yau 2 scheme, then the projective scheme is a twist of a commutative Calabi-Yau variety.

4.2. Veronese subrings. Let D be the connected \mathbb{N} -graded k -algebra defined in 3.13. In this subsection, we focus on noncommutative Calabi-Yau projective surfaces. So, we suppose $n = 3$. In the same way as in the above discussion, we compute $\mathcal{M}_{D^{(4d)}}$. In order to do this, we describe $D^{(4d)}$ as a quotient of a quantum polynomial ring by using noncommutative Veronese maps.

Lemma 4.4. $D^{(4d)} \simeq k\langle \{x_i\}_{i \in W} \rangle / K$, where

- (1) $W = \{\mathbf{i} := (i_0, i_1, i_2, i_3) \in \mathbb{N}^4 \mid a_0 i_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 = 4d\}$,
- (2) K is the two-sided ideal defined the elements in sets $S_1, S_2, S_3 \subset k\langle \{x_i\}_{i \in W} \rangle$.

$$\begin{aligned} S_0 &= \{x_j x_i - q_{ji} x_i x_j \mid \mathbf{i}, \mathbf{j} \in W, q_{ji} = \prod_{0 \leq \alpha, \beta \leq 3} q_{\beta\alpha}^{j_\alpha i_\beta}\} \\ S_1 &= \{x_j x_i - p_{ji}^{lk} x_l x_k \mid \mathbf{j} \geq \mathbf{i}, \mathbf{j} + \mathbf{i} = \mathbf{l} + \mathbf{k}, x_l x_k \in S'_2, p_{ji}^{lk} = \prod_{0 \leq \alpha < \beta \leq 3} q_{\beta\alpha}^{j_\alpha i_\beta} / \prod_{0 \leq \alpha < \beta \leq 3} q_{\beta\alpha}^{l_\alpha k_\beta}\} \\ S'_1 &= \{x_l x_k \mid (i) : \mathbf{l} \geq \mathbf{k}, (ii) : x_l x_k \leq x_j x_i \text{ for any } (\mathbf{j}, \mathbf{i}) \in W^2 \text{ such that } \mathbf{j} \geq \mathbf{i} \text{ and } \mathbf{l} + \mathbf{k} = \mathbf{j} + \mathbf{i}\} \\ S_2 &= \left\{ x_{i_0+\mathbf{h}} + x_{i_1+\mathbf{h}} + x_{i_2+\mathbf{h}} + x_{i_3+\mathbf{h}} \mid \begin{array}{l} \mathbf{i}_0 = (d/a_0, 0, 0, 0), \mathbf{i}_1 = (0, d/a_1, 0, 0), \\ \mathbf{i}_2 = (0, 0, d/a_2, 0), \mathbf{i}_3 = (0, 0, 0, d/a_3), \\ \mathbf{h} \in \{3\mathbf{i}_0, 3\mathbf{i}_1, 3\mathbf{i}_2, 3\mathbf{i}_3\} \end{array} \right\}. \end{aligned}$$

Proof. We use ideas in [1, Section 4.5]. We consider the following map

$$\varphi : k\langle \{x_i\}_{i \in W} \rangle / (S_0) \rightarrow C^{(4d)}, \quad x_{\mathbf{i}} \mapsto x_0^{i_0} x_1^{i_1} x_2^{i_2} x_3^{i_3}.$$

Note that φ is a surjective homomorphism of graded k -algebra from the choice of elements of W and $\{q_{ij}\}$. So, we want to know $\text{Ker}(\varphi)$. Since C is quadric, $C^{(4d)}$ is also quadric. Hence, we have

$$\dim_k(k\langle\{x_i\}_{i \in W}\rangle/(S_1))_2 + \dim_k(\text{Ker}(\varphi))_2 = \dim_k(C^{(4d)})_2.$$

Because $S_1 \subset \text{Ker}(\varphi)_2$, S_1 is lineary independent and the number of S_1 is equal to $\dim_k(\text{Ker}(\varphi))_2$, we have

$$\varphi : k\langle\{x_i\}_{i \in W}\rangle/(S_0, S_1) \xrightarrow{\sim} C^{(4d)}.$$

The set which is obtained by pulling back $\{x_0^{3h_0}f, x_1^{3h_1}f, x_2^{3h_2}f, x_3^{3h_3}f\}$ along φ is S_2 . Thus, we get the claim. \square

Proposition 4.5. *For any $i = 0, 1, 2$, there exists a weight w in 3.14 such that $\dim(\mathcal{M}_{D^{(4d)}}) = i$.*

Proof. Taking Lemma 4.4 into account, we can compute $\mathcal{M}_{D^{(4d)}}$ in the same way as in the proof of Theorem 4.2 (or “4.0.1 and 4.0.2”). In this case, we compute $\mathcal{M}_{D^{(4d)}}$ directly by using computer algebra system because the number of elements of S_2 is large (see Section 5). For example, $\dim_k \mathcal{M}_{D^{(4d)}} = 0$ for the weight $(1, 1, 1, 3)$ and some $\{q_{ij}\}$ which is general. $\dim \mathcal{M}_{D^{(4d)}} = 1$ for the weight $w = (1, 2, 3, 6)$ and some $\{q_{ij}\}$ which is general. \square

5. APPENDIX

In this section, we give the source code for calculating the dimension of the moduli of point modules of $D^{(4d)}$ in Theorem 3.13, when the weight is $w = (1, 2, 3, 6)$, degree $d = 12$ and $Q := \{q_{ij}\}$ is given as follows

$$Q = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

We use SageMath. If you want to calculate the dimension of the moduli of point modules on $D^{(4d)}$ for different weights, degrees and $\{q_{ij}\}$, change the values of `ai`, `d`, `qi,j` in first few lines of the following source code.

LISTING 1. A source code for computing dimensions of moduli of point modules

```

1 #SageMath version 9.4 using Python 3.9.5
2
3 #Input a weight w, a degree d and {q_{ij}}
4 a0 = 1; a1 = 2; a2 = 3; a3 = 6; d = 12
5 q00 = 1; q01 = 1; q02 = -1; q03 = 1; q10 = 1; q11 = 1; q12 = 1; q13 = -1
6 q20 = -1; q21 = 1; q22 = 1; q23 = 1; q30 = 1; q31 = -1; q32 = 1; q33 = 1
7 lc = lcm(lcm(a0,a1),lcm(a2,a3)); v0 = (a0,a1,a2,a3); v1 = (d/a0,d/a1,d/a2,d/a3)
8
9 #Preparation for the next calculation
10 basis0 = [(s,t,u,v) for s in range(0,4*lc/a0+1) for t in range(0,4*lc/a1+1) for u in range(0,4*lc/
    a2+1) for v in range(0,4*lc/a3+1)]
11
12 #Calculating the basis of (D^{(4d)})_1
13 basis1=[]; N=0
14 for v in basis0:
15     j = a0*v[0]+a1*v[1]+a2*v[2]+a3*v[3]
16     if j == lc*4: N=N+1; basis1.append(v)
17
18 #Calculating the basis of (D^{(4d)})_2
19 basis2=[]; N = 0
20 for k in range(0,lc*4*2/a0+1):
21     for l in range(0,lc*4*2/a1+1):
22         for m in range(0,lc*4*2/a2+1):
23             for n in range(0,lc*4*2/a3+1):
24                 j = a0*k+a1*l+a2*m+a3*n
25                 if j == lc*4*2: N=N+1; w = [k,l,m,n]; basis2.append(w)
26
27 #Dividing elements of (D^{(4d)})_2 into products of 2 elements of (D^{(4d)})_1
28 basis3=[]; N=0
29 for v in basis2:
30     s = []
31     for w in basis1:
32         for u in basis1:

```

```

33         if w >= u:
34             if list(map(lambda a,b: a+b, w,u)) == v: s.append([w,u])
35         basis3.append(s); N=N+len(s); print(N,s)
36
37     #Preparation for calculating three2
38     three1=[]
39     for v in basis1:
40         for w in basis1:
41             for u in basis1:
42                 if v > w and w > u:
43                     if u[1] == v[0] and v[1] == w[0] and w[1] == u[0]: three1.append([v,w,u])
44
45     #Producing the variable associated with a list v.
46     def var1(v):
47         return var('x%02d%02d%02d%02d' % (v[0],v[1],v[2],v[3]))
48
49     #Giving the polynomials defining the Calabi-Yau surface and dehomogenization of them
50     def poly(i,v,d,y):
51         e=4*lc-d; e = e/v[0][i]; y = list(y); ve = [0,0,0,0]; t0=1; t1=1; t2=1; t3=1
52         for j in range(0,4):
53             if j == i: ve[j] = e
54             if j != i: ve[j] = 0
55         w0 = [v[0],0,0,0]; w1 = [0,v[1],0,0]; w2 = [0,0,v[2],0]; w3 = [0,0,0,v[3]]
56         w0 = list(map(lambda a,b: a+b, w0,ve)); w1 = list(map(lambda a,b: a+b, w1,ve))
57         w2 = list(map(lambda a,b: a+b, w2,ve)); w3 = list(map(lambda a,b: a+b, w3,ve))
58         if w0 != y: t0 = var1(w0)
59         if w1 != y: t1 = var1(w1)
60         if w2 != y: t2 = var1(w2)
61         if w3 != y: t3 = var1(w3)
62         f = t0+t1+t2+t3
63         return f
64
65     #A formula for computing  $q_{ij}$  in  $S_0$ 
66     def mul1(v,w):
67         c = q00^(v[0]*w[0])*q01^(v[1]*w[0])*q02^(v[2]*w[0])*q03^(v[3]*w[0])*q10^(v[0]*w[1])*q11^(v[1]*w[1])*q12^(v[2]*w[1])*q13^(v[3]*w[1])*q20^(v[0]*w[2])*q21^(v[1]*w[2])*q22^(v[2]*w[2])*q23^(v[3]*w[2])*q30^(v[0]*w[3])*q31^(v[1]*w[3])*q32^(v[2]*w[3])*q33^(v[3]*w[3])
68         return c
69
70     #A formula for computing  $p_{ji}^{lk}$  in  $S_1$ 
71     def mul2(v,w):
72         c = q00^(v[0]*w[0])*q10^(v[0]*w[1])*q11^(v[1]*w[1])*q20^(v[0]*w[2])*q21^(v[1]*w[2])*q22^(v[2]*w[2])*q30^(v[0]*w[3])*q31^(v[1]*w[3])*q32^(v[2]*w[3])*q33^(v[3]*w[3])
73         return c
74
75     #Computing the set of 3 indexes of  $W$  which are general
76     three2=[]; N=0
77     for v in three1:
78         t = mul1(v[0],v[1])*mul1(v[1],v[2])*mul1(v[2],v[0])
79         if t != 1: N=N+1; three2.append(v)
80
81     #Calculating the dimension of the moduli of point modules on each basic open affines
82     S1=[]; S2=[]; S3=[]; g0=0; g1=0; g2=0; u0=0; u1=0; r=1; n=0
83     for y in basis1: #y is a standard open affine set.
84         s=[]
85         #Producing the polynomial ring defining the open affine y
86         for v in basis1:
87             if v != y: s.append(v)
88         R1=['x%02d%02d%02d%02d' % (v[0],v[1],v[2],v[3]) for v in s]
89         R=PolynomialRing(QQ,len(R1),R1); S1=[]; n=0
90         #Giving  $S_1$  and the dehomogenization of it
91         for v in basis3:
92             m = len(v); u0= 1/ mul2(v[0][0],v[0][1])
93             if v[0][0] == y and v[0][1] == y: g0 = mul1(y,v[0][1])*u0
94             if v[0][0] == y and v[0][1] != y: g0 = mul1(y,v[0][1])*u0*var1(v[0][1])
95             if v[0][0] != y and v[0][1] == y: g0 = mul1(y,v[0][1])*u0*var1(v[0][0])
96             if v[0][0] != y and v[0][1] != y: g0 = mul1(y,v[0][1])*u0*var1(v[0][0])*var1(v[0][1])
97             for i in range(1,m):
98                 u1= 1/mul2(v[i][0],v[i][1])
99                 if v[i][0] == y and v[i][1] == y: g1 = mul1(y,v[i][1])*u1
100                 if v[i][0] == y and v[i][1] != y: g1 = mul1(y,v[i][1])*u1*var1(v[i][1])
101                 if v[i][0] != y and v[i][1] == y: g1 = mul1(y,v[i][1])*u1*var1(v[i][0])
102                 if v[i][0] != y and v[i][1] != y: g1 = mul1(y,v[i][1])*u1*var1(v[i][0])*var1(v[i][1])
103             g = g1 - g0; S1.append(g); n=n+1
104             if n % 500 == 0:
105                 J1 = R.ideal(S1); S1 = list(J1.groebner_basis('singular:slimgb')) #We calculate

```

Groebner basis of J_1 frequently to reduce the calculation time of dimensions of ideals.

```

106 J1 = R.ideal(S1); S1 = list(J1.groebner_basis('singular:slimgb'))
107 #Giving S2 and the dehomogenization of it
108 S2=[poly(0,v1,d,y),poly(1,v1,d,y),poly(2,v1,d,y),poly(3,v1,d,y)]
109 #Giving the set of hypersurfaces obtained from the set of general threes of {qij}
110 g2=1; S3=[]
111 for v in three2:
112     if v[0] == y : g2 = var1(v[1])*var1(v[2])
113     if v[1] == y : g2 = var1(v[0])*var1(v[2])
114     if v[2] == y : g2 = var1(v[0])*var1(v[1])
115     if v[0] != y and v[1] != y and v[2] != y: g2=var1(v[0])*var1(v[1])*var1(v[2])
116     S3.append(f2)
117 J3 = R.ideal(S3); S3 = list(J3.groebner_basis('singular:slimgb'))
118 #Calculating what we want
119 I = R.ideal(S1+S2+S3)
120 print(y,I.dimension())

```

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