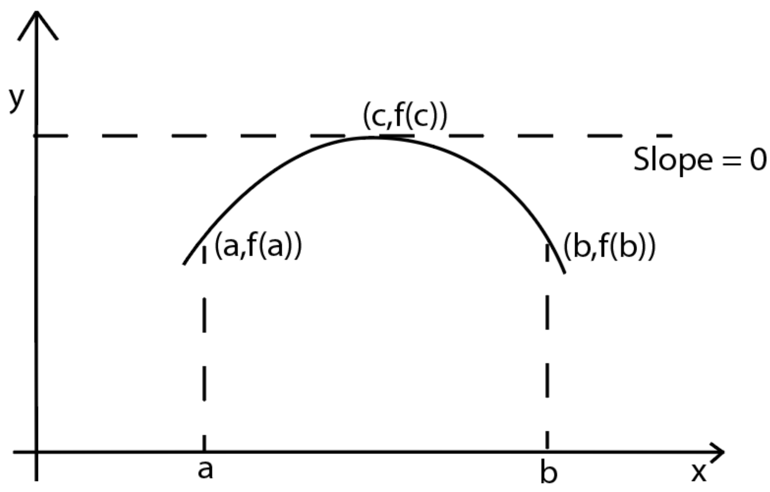
Calculus (Differentiation)

Rolle's Theorem

- Basic Requirements for applying Rolle's theorem, For $f(x) \rightarrow \ Real \rightarrow [a, b]$
  - $a \leq x \leq b$ (f is continuous)
  - $a < x < b$ (f is differentiable)
  - $f(a) = f(b)$
- Then we can say, There exist a $c \in (a.b)$ such that $f'(x) = 0$ i.e whose tangent is || to the x-axis.



Example for Rolle's theorem

- $f(x) = sin\,x$ in the range $[0, \pi]$
  - $f(0) = 0$
  - $f(\pi) = 0$
  - $f'(x) = cos\,x$
  - For what value of x, $f'(x)$ is 0.
  - $cos\frac{\pi}{2} = 0$
  - Therefore,

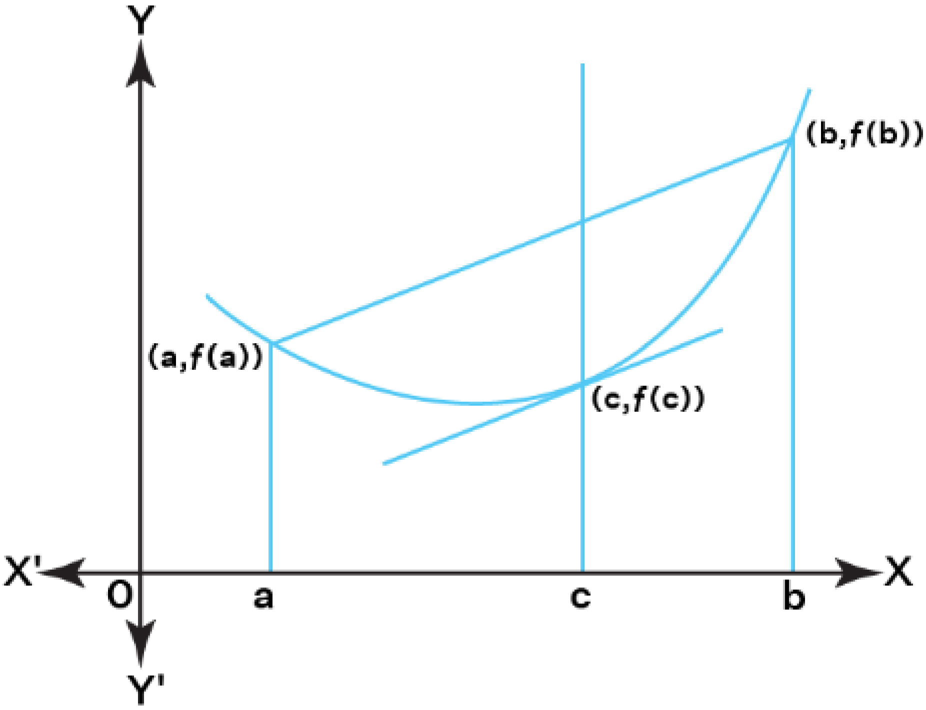
$$c = \frac{\pi}{2}$$

  - The tangent of the slope at this point is 0.

Lagrange's Mean Value Theorem

- For $f : [a,b] \to R$
  - f is continuous on [a,b]
  - f is differentiable on [a,b]
  - There exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

  - Or tangent drawn at c is || to the line connecting a & b.
  - 

Example For Lagrange's theorem

- $f(x) = x^2 - 2x + 3$ in [0,3]
  - $f(3) = 6$ and $f(0) = 3$
  - $$f'(x) = 2x - 2$$
  - $$f'(c) = \frac{f(3) - f(0)}{3 - 0} = 1$$
  - $$2x - 2 = 1$$
  - $$x = \frac{3}{2}$$

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#question Verify Rolle's Theorem for function $f(x) = x^2 - 5x + 6$ in range [2,3]

- We know f(x) is continuous in $[2, 3]$ because it is a polynomial.
- f'(x) exists in $[0, 1]$

$$f'(x) = 2x - 5$$

- $$\begin{aligned} f(2) &= 2^2 - 5 \cdot 2 + 6 \\ &= 0 \end{aligned} \qquad \begin{aligned} f(3) &= 3^2 - 5 \cdot 3 + 6 \\ &= 0 \end{aligned}$$
  $$f(2) = f(3)$$
- Therefore there exists c where f'(c) = 0.
- $$\begin{gathered} f'(c) = 0 \\ 2c - 5 = 0 \\ c = \tfrac{5}{2} = 2.5 \end{gathered}$$
- At c = 2.5, the slope of the equation will be 0.

#question Verify LMVT for $f(x) = 2x - x^2$ in $[0, 1]$

- We know that f(x) is continuous in $[0, 1]$
- f'(x) exists in $[0, 1]$

$$f'(x) = 2 - 2x$$

- There exists $c \in (0, 1)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- $$f'(c) = \tfrac{f(b)-f(a)}{b-a}$$
  $$2 - 2c = \tfrac{f(1)-f(0)}{1-0}$$
  $$2(1 - c) = \tfrac{1}{1}$$
  $$1 - c = \tfrac{1}{2}$$
  $$\Rightarrow c = \tfrac{1}{2}$$
- c lies between $[0, 1]$.

#question Verify LMVT in $f(x) = x^2 + 2x + 3$ in the range $4 \leq x \leq 6$

- $$f(4) = 16 + 8 + 3 = 27$$

- $$f(6) = 36 + 12 + 3 = 51$$
- $$f'(c) = 2c + 2 = \frac{51 - 24}{6 - 4} \Rightarrow 2(c+1) = \frac{24}{2} \Rightarrow c = 5$$

Cauchy Mean Value Theorem

- $f(x)$ & $g(x)$ are continuous
- $f(x)$ & $g(x)$ are differentiable
- $g'(c) \neq 0$
- Therefore,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

> If $g(x) = x$ then the resultant condition gives us LMVT.

#question Verify CMVT for functions $\begin{matrix} f(x) = e^x \\ g(x) = e^{-x} \end{matrix}$ in range $[a, b]$

- $$f'(x) = e^x \quad g'(x) = -e^{-x}$$
- $$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{e^b-e^a}{e^{-b}-e^{-a}} = \frac{e^c}{e^{-c}}$$

$$\frac{e^b-e^a}{\frac{e^a-e^b}{e^{a+b}}} = -e^{2c}$$

$$e^{a+b} = e^{2c}$$

$$2c = a + b$$

$$c = \frac{a+b}{2}$$

- C lies between (a, b).

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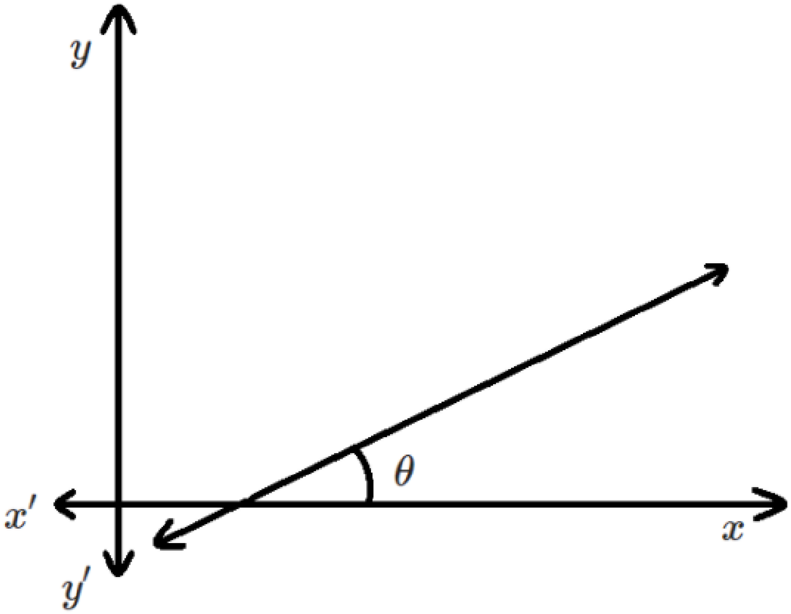
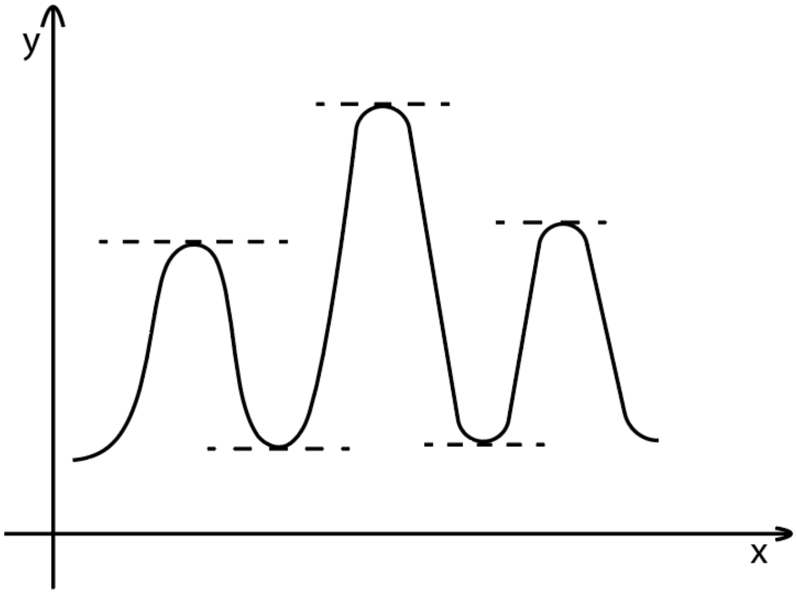
Maximum and Minimum



- The Slope/gradient of the sample line above is $\tan \theta$



- A function f(x) has maximum or minimum when the gradient is 0.
- $$\tan \theta = \frac{df}{dx} = 0$$
- When gradient is 0, the tangent from that point is parallel to the X axis.

> At a point, the function f(x) has maximum value when $\frac{d^2 f}{dx^2} < 0$

> At a point, the function f(x) has minimum value when $\frac{d^2 f}{dx^2} > 0$

#question Find the Critical points of the function $f(x) = x^3 - 9x^2 + 24x - 12$. Also find maximum or minimum value at those points.

- $$f(x) = x^3 - 9x^2 + 24x - 12$$
- $$f'(x) = 3x^2 - 18x + 24$$
- The function has maximum and minimum values at, $\frac{d\,f(x)}{dx} = 0$
- $$\begin{gathered} 3x^2 - 18x + 24 = 0 \\ x^2 - 6x + 8 = 0 \\ (x-4)(x-2) = 0 \\ \\ x = 2 \qquad\qquad OR \qquad\qquad x = 4 \end{gathered}$$
- The critical points are $x = 2$ and $x = 4$
- $$f''(x) = 6x - 18$$
- At $x = 2$,
  - $$\left|\frac{d^2 f}{dx^2}\right|_{x=2} = 6 \times 2 - 18 = -6$$
  - $\frac{d^2 f}{dx^2} < 0$ Therefore, Maximum value can be found from x=2.
- At $x = 4$,
  - $$\left|\frac{d^2 f}{dx^2}\right|_{x=4} = 6 \times 4 - 18 = 6$$
  - $\frac{d^2 f}{dx^2} > 0$ Therefore, Minimum value can be found from x=4.
- Maximum Value is $f(2) = 2^3 - 9 \times 2^2 + 24 \times 2 - 12 = 8$
- Maximum Value is $f(4) = 4^3 - 9 \times 4^2 + 24 \times 4 - 12 = 64 - 144 + 96 - 12 = 4$

#question Find Maximum and Minimum of $f(x) = x^3 - 3x^2 + 5$

- $$f(x) = x^3 - 3x^2 + 5$$
- $$f'(x) = 3x^2 - 6x$$
- For Critical points,

$$\begin{gathered} f'(x) = 3x^2 - 6x = 0 \\ x^2 - 2x = 0 \\ x(x-2) = 0 \\ x = 0 \qquad\qquad OR \qquad\qquad x = 2 \end{gathered}$$

- $$f''(x) = 6x - 6$$

- At x=0, $f''(0) = -6$ which is < 0. Therefore, Maximum value can be found from x=0.
- At x=2, $f''(2) = 6$ which is > 0. Therefore, Minimum value can be found from x=2.
- Maximum value = $f(0) = 5$
- Minimum value = $f(2) = 1$

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#question f'(c) = ?

$$(a)0 \quad (b)c \quad (c)a+b \quad (d)\frac{f(b)-f(a)}{b-a}$$

- Correct option is (d).

#question The value of c of Rolle's Theorem in $[-1,1]$ for function $f(x) = x^2 - x$ is _.

- $$\begin{aligned} f'(c) &= 2c - 1 = 0 \\ c &= \frac{1}{2} \end{aligned}$$

#question Verify Rolle's Theorem For $f(x) = x^2 - 5x + 6$ is $[2,3]$

- $f(x) \to$ Continuous
- $f'(x) \to$ exists
- $$\begin{aligned} f'(c) &= 2c - 5 = 0 \\ c &= \frac{5}{2} \end{aligned}$$
- c lies between (2,3) such that f'(c) is 0.

#question Find local maxima and minima points of $f(x) = x^3 - 9x^2 + 24x - 12$

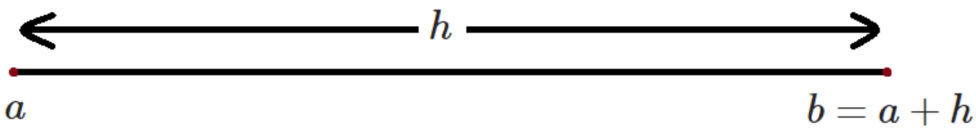
- $$\begin{gathered} f'(x) = 3x^2 - 18x + 24 = 0 \\ x^2 - 6x + 8 = 0 \\ (x-4)(x-2) = 0 \\ \begin{aligned} x &= 4 \\ f(4) &= 64 - 144 + 96 - 12 \\ &= 4 \end{aligned} \quad or \quad \begin{aligned} x &= 2 \\ f(2) &= 8 - 36 + 48 - 12 \\ &= 8 \end{aligned} \end{gathered}$$
- The Points are $(4,4)$ and $(2,8)$.

Taylor's Theorem

- $f(x)$ and it's First $(n-1)$ derivatives be continuous in $[a, a+h]$

- $f^n(x)$ exists for every value of x in $(a, a+h)$
- Then there exists, atleast one number θ $(0 < \theta < 1)$ such that,

$$f'(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^n(a+\theta h)$$

- This is called Taylor's Theorem with Lagrange's form remainder, the remainder R_n being $\frac{h^n}{n!}f^n(a+\theta h)$
- 

> Putting $a = 0$ and $h = x$, we get,
>
> - $$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(\theta x)$$
> - Which is known as Maclaurin's Theorem with Lagrange's form of remainder.

- Maclaurin's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots \infty$$

#question Using Maclaurin's Series, Expand $\tan x$ up to term containing x^3

- $$\begin{aligned} &Let\ f(x) = \tan x && f(0) = 0 \\ &f'(x) = sec^2 x && f'(0) = 1 \\ &f''(x) = 2\tan x \cdot \sec^2 x \\ &\quad = 2\tan x(1 + tan^2 x) && f''(0) = 0 \\ &f'''(x) = 2\sec^2 x + 6\tan^2 x \cdot \sec^2 x && f'''(0) = 2 \end{aligned}$$

- Now using Maclaurin's Series, we get,

$$\begin{aligned} f(x) = \tan x \quad &= 0 + x \cdot 1 + \tfrac{x^2}{2!} \cdot 0 + \tfrac{x^3}{3!} \cdot 2 \\ &= x + \tfrac{x^3}{6} \cdot 2 \\ &= x + \tfrac{x^3}{3} \end{aligned}$$