Calculus (Integration)

Evaluation of definite and improper integrals:

Definite Integration can be used to find areas, volumes, central points and many useful things.

7.7.1 *Definite integral as the limit of a sum*

Let f be a continuous function defined on close interval $[a, b]$. Assume that all the values taken by the function are non negative, so the graph of the function is a curve above the x-axis.

The definite integral $\int_a^b f(x)\,dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis. To evaluate this area, consider the region PRSQP between this curve, x-axis and the ordinates $x = a$ and $x = b$ (Fig 7.2).

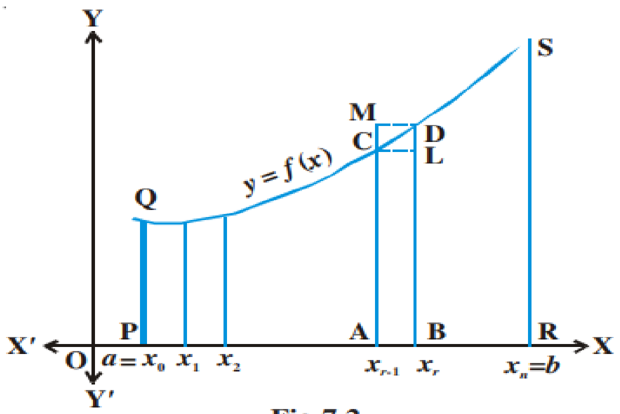

Fig 7.2

Divide the interval $[a, b]$ into n equal subintervals denoted by $[x_0, x_1]$, $[x_1, x_2]$,...., $[x_{r-1}, x_r]$, ..., $[x_{n-1}, x_n]$, where $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ... , $x_r = a + rh$ and $x_n = b = a + nh$ or $n = \frac{b-a}{h}$. We note that as $n \to \infty$, $h \to 0$.

The region PRSQP under consideration is the sum of n subregions, where each subregion is defined on subintervals $[x_{r-1}, x_r]$, $r = 1, 2, 3, \ldots, n$.

From Fig 7.2, we have

area of the rectangle (ABLC) < area of the region (ABDCA) < area of the rectangle (ABDM) ... (1)

Evidently as $x_r - x_{r-1} \to 0$, i.e., $h \to 0$ all the three areas shown in (1) become nearly equal to each other. Now we form the following sums.

$$s_n = h\,[f(x_0) + \ldots + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r) \qquad \ldots (2)$$

and

$$S_n = h[f(x_1) + f(x_2) + \ldots + f(x_n)] = h\sum_{r=1}^{n} f(x_r) \qquad \ldots (3)$$

Here, s_n and S_n denote the sum of areas of all lower rectangles and upper rectangles raised over subintervals $[x_{r-1}, x_r]$ for $r = 1, 2, 3, \ldots, n$, respectively.

In view of the inequality (1) for an arbitrary subinterval $[x_{r-1}, x_r]$, we have

$$s_n < \text{area of the region PRSQP} < S_n \qquad \ldots (4)$$

As $n \to \infty$ strips become narrower and narrower, it is assumed that the limiting values of (2) and (3) are the same in both cases and the common limiting value is the required area under the curve.

Symbolically, we write

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n = \text{area of the region PRSQP} = \int_a^b f(x)dx \qquad \ldots (5)$$

It follows that this area is also the limiting value of any area which is between that of the rectangles below the curve and that of the rectangles above the curve. For the sake of convenience, we shall take rectangles with height equal to that of the curve at the left hand edge of each subinterval. Thus, we rewrite (5) as

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + \ldots + f(a+(n-1)h]$$

or
$$\int_a^b f(x)dx = (b-a) \lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) + \ldots + f(a+(n-1)h] \qquad \ldots (6)$$

where
$$h = \frac{b-a}{n} \to 0 \text{ as } n \to \infty$$

The above expression (6) is known as the definition of definite integral as the *limit of sum*.

Remark The value of the definite integral of a function over any particular interval depends on the function and the interval, but not on the variable of integration that we choose to represent the independent variable. If the independent variable is denoted by t or u instead of x, we simply write the integral as $\int_a^b f(t)\,dt$ or $\int_a^b f(u)\,du$ instead of $\int_a^b f(x)\,dx$. Hence, the variable of integration is called a *dummy variable*.

Important Properties of Definite Integrals

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b 1dx = b - a$

3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

4. Change of variable of integration is immaterial as long as the limits of integration remain the same, i.e.

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

5. If the limits are interchanged, i.e. the upper limit becomes the lower limit and vice versa, then

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

6. If f is a piecewise continuous function, then the integral is broken at the points of discontinuity or at the points where the definition of f changes,

i.e. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

7. $\int_{-a}^a f(x)dx = \int_0^a (f(x) + f(-x))dx$

$$= 0 if f(x) is odd$$

$$= 2\int_0^a f(x)dx if f(x) is even$$

8. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Another result that can be derived from this property is

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

In problems 1 through 9 find the area of the region R.

1. R is the triangle with vertices $(-4, 0)$, $(2, 0)$ and $(2, 6)$.
   Solution. From the corresponding graph (Figure 6.1) you see that the region in question is bellow the line $y = x + 4$ above the x axis, and extends from $x = -4$ to $x = 2$.

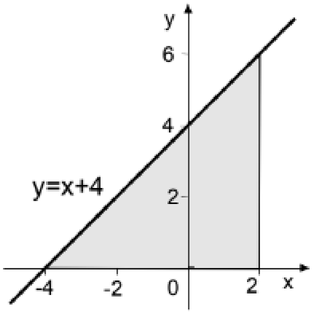

Figure 6.1.

   Hence,

$$A = \int_{-4}^{2} (x + 4)dx = \left(\frac{1}{2}x^2 + 4x\right)\Big|_{-4}^{2} = (2 + 8) - (8 - 16) = 18.$$

2. R is the region bounded by the curve $y = e^x$, the lines $x = 0$ and $x = \ln \frac{1}{2}$, and the x axis.
   Solution. Since $\ln \frac{1}{2} = \ln 1 - \ln 2 = -\ln 2 \simeq -0.7$, from the corresponding graph (Figure 6.2) you see that the region in question is bellow the line $y = e^x$ above the x axis, and extends from $x = \ln \frac{1}{2}$ to $x = 0$.

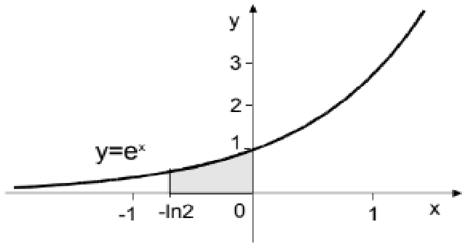

Figure 6.2.

Hence,

$$A = \int_{\ln\frac{1}{2}}^{0} e^x dx = e^x\Big|_{\ln\frac{1}{2}}^{0} = e^0 - e^{\ln\frac{1}{2}} = 1 - \frac{1}{2} = \frac{1}{2}.$$

3. R is the region in the first quadrant that lies below the curve $y = x^2 + 4$ and is bounded by this curve, the line $y = -x + 10$, and the coordinate axis.

Solution. First sketch the region as shown in Figure 6.3. Note that the curve $y = x^2 + 4$ and the line $y = -x + 10$ intersect in the first quadrant at the point $(2, 8)$, since $x = 2$ is the only positive solution of the equation $x^2 + 4 = -x + 10$, i.e. $x^2 + x - 6 = 0$. Also note that the line $y = -x + 10$ intersects the x axis at the point $(10, 0)$.

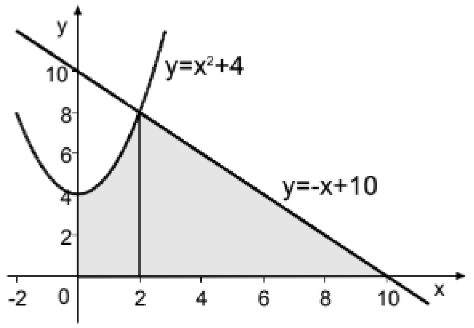

Figure 6.3.

Observe that to the left of $x = 2$, R is bounded above by the curve $y = x^2 + 4$, while to the right of $x = 2$, it is bounded by the line $y = -x + 10$. This suggests that you break R into two subregions, R_1 and R_2, as shown in Figure 6.3, and apply the integral formula for area to each subregion separately. In particular,

$$A_1 = \int_0^2 (x^2 + 4)dx = \left(\frac{1}{3}x^3 + 4x\right)\Big|_0^2 = \frac{8}{3} + 8 = \frac{32}{3}$$

and

$$A_2 = \int_2^{10} (-x + 10)dx = \left(-\frac{1}{2}x^2 + 10x\right)\Big|_2^{10} = -50 + 100 + 2 - 20 = 32.$$

Therefore,

$$A = A_1 + A_2 = \frac{32}{3} + 32 = \frac{128}{3}.$$

4. R is the region bounded by the curves $y = x^2 + 5$ and $y = -x^2$, the line $x = 3$, and the y axis.
Solution. Sketch the region as shown in Figure 6.4.

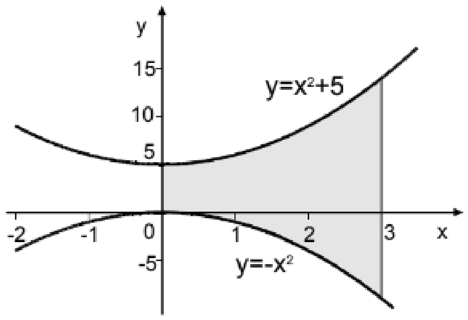

Figure 6.4.

Notice that the region in question is bounded above by the curve $y = x^2 + 5$ and below by the curve $y = -x^2$ and extends from $x = 0$ to $x = 3$. Hence,

$$A = \int_0^3 [(x^2+5)-(-x^2)]dx = \int_0^3 (2x^2+5)dx = \left(\frac{2}{3}x^3 + 5x\right)\Big|_0^3 = 18+15 = 33.$$

5. R is the region bounded by the curves $y = x^2 - 2x$ and $y = -x^2 + 4$.
Solution. First make a sketch of the region as shown in Figure 6.5 and find the points of intersection of the two curves by solving the equation

$$x^2 - 2x = -x^2 + 4 \qquad \text{i.e.} \qquad 2x^2 - 2x - 4 = 0$$

to get

$$x = -1 \qquad \text{and} \qquad x = 2.$$

The corresponding points $(-1, 3)$ and $(2, 0)$ are the points of intersection.

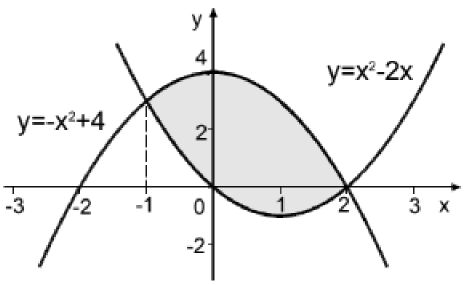

Figure 6.5.

Notice that for $-1 \le x \le 2$, the graph of $y = -x^2 + 4$ lies above that of $y = x^2 - 2x$. Hence,

$$\begin{aligned} A &= \int_{-1}^{2} [(-x^2+4) - (x^2-2x)]dx = \int_{-1}^{2} (-2x^2+2x+4)dx = \\ &= \left(-\frac{2}{3}x^3 + x^2 + 4x\right)\Big|_{-1}^{2} = -\frac{16}{3} + 4 + 8 - \frac{2}{3} - 1 + 4 = 9. \end{aligned}$$

6. R is the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$.
Solution. Sketch the region as shown in Figure 6.6. Find the points of intersection by solving the equations of the two curves simultaneously to get

$$x^2 = \sqrt{x} \qquad x^2 - \sqrt{x} = 0 \qquad \sqrt{x}(x^{\frac{3}{2}} - 1) = 0$$

$$x = 0 \qquad \text{and} \qquad x = 1.$$

The corresponding points $(0,0)$ and $(1,1)$ are the points of intersection.

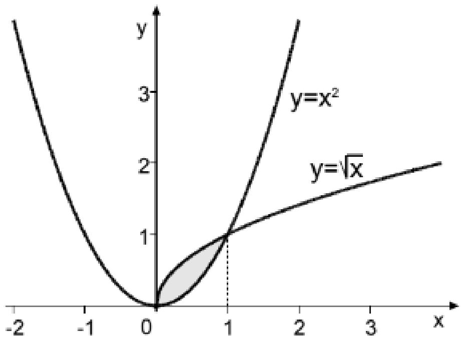

Figure 6.6.

Notice that for $0 \le x \le 1$, the graph of $y = \sqrt{x}$ lies above that of $y = x^2$. Hence,

$$A = \int_0^1 (\sqrt{x} - x^2)dx = \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

(a) R is the region to the right of the y axis that is bounded above by the curve $y = 4 - x^2$ and below the line $y = 3$.

(b) R is the region to the right of the y axis that lies below the line $y = 3$ and is bounded by the curve $y = 4 - x^2$, the line $y = 3$, and the coordinate axes.