次の積分を計算せよ。

$$\int_0^1 \frac{\ln(1+x)}{x}\,\mathrm{d}x$$

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<u>解答</u>

$$\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{x}\,\mathrm{d}x &= \int_0^1 \left(\sum_{n=1}^{\infty} \frac{(-x)^{n-1}}{n}\right)\mathrm{d}x \\
&= -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 (-x)^{n-1}\,\mathrm{d}(-x) \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}
\end{aligned}$$

$$\begin{aligned}
-\sum_{n=1}^{2N} \frac{(-1)^n}{n^2} &= \sum_{n=1}^{2N} \frac{1}{n^2} - 2\sum_{n=1}^{N} \frac{1}{(2n)^2} \\
&= \sum_{n=1}^{2N} \frac{1}{n^2} - \frac{1}{2}\sum_{n=1}^{N} \frac{1}{n^2} \\
&\to \frac{\pi^2}{6} - \frac{1}{2}\cdot\frac{\pi^2}{6} \quad (n \to \infty)
\end{aligned}$$

$$\begin{aligned}
-\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} &= \frac{\pi^2}{12} \\
&= 0.82246703\ldots
\end{aligned}$$

<u>結論</u>

$$\int_0^1 \frac{\ln(1+x)}{x}\,\mathrm{d}x = \frac{\pi^2}{12}$$