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【床関数】

xの床関数$\lfloor x \rfloor$とはx以下の最大の整数を表す。

$x \in [k, k+1) \quad (k \in \mathbb{Z}) \implies \lfloor x \rfloor = k$ を満たす。

例) $\lfloor 1.5 \rfloor = 1, \lfloor -4.225 \rfloor = -5$

解答

$$u = \left(\sqrt{2}\right)^x$$

$$\mathrm{d}u = \left(\sqrt{2}\right)^x \cdot \left(\frac{\ln(2)}{2}\right) \mathrm{d}x$$

$$\mathrm{d}x = \frac{2}{\ln(2)} \cdot \frac{1}{u} \mathrm{d}u$$

$$\begin{aligned}
\int_0^3 \left\lfloor \left(\sqrt{2}\right)^x \right\rfloor \mathrm{d}x &= \frac{2}{\ln(2)} \int_1^{2\sqrt{2}} \frac{\lfloor u \rfloor}{u} \mathrm{d}u \\
&= \frac{2}{\ln(2)} \left(\int_1^2 \frac{1}{u} \mathrm{d}u + \int_2^{2\sqrt{2}} \frac{2}{u} \mathrm{d}u \right) \\
&= \frac{2}{\ln(2)} \left(\Big[\ln(u) \Big]_1^2 + \Big[2\ln(u) \Big]_2^{2\sqrt{2}} \right) \\
&= \frac{2}{\ln(2)} \left(\ln(2) + \ln(2) \right) \\
&= 4
\end{aligned}$$

結論

$$\int_0^3 \left\lfloor \left(\sqrt{2}\right)^x \right\rfloor \mathrm{d}x = 4$$