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解答

$$\begin{aligned}\lim_{x\to\pi}\frac{\sin(kx)}{x-\pi} &= \lim_{x\to\pi}\frac{\sin(kx)-\sin(k\pi)}{x-\pi}\\ &= \frac{\mathrm{d}(\sin(kx))}{\mathrm{d}x}(\pi)\\ &= k\cos(k\pi)\\ &= (-1)^k k\ (k\in\mathbb{Z})\end{aligned}$$

$$\begin{aligned}\lim_{x\to\pi}\sum_{n=0}^{N}\frac{\sin((2n+1)x)}{x-\pi} &= \sum_{n=0}^{N}\lim_{x\to\pi}\frac{\sin((2n+1)x)}{x-\pi}\\ &= \sum_{n=0}^{N}(-1)^{2n+1}(2n+1)\\ &= -\sum_{n=0}^{N}(2n+1)\\ &= -\left(2\sum_{n=0}^{N}n+\sum_{n=0}^{N}1\right)\\ &= -(N+1)(2N^2+N+1)\end{aligned}$$

結論

$$\lim_{x\to\pi}\sum_{n=0}^{N}\frac{\sin((2n+1)x)}{x-\pi} = -(N+1)(2N^2+N+1)$$