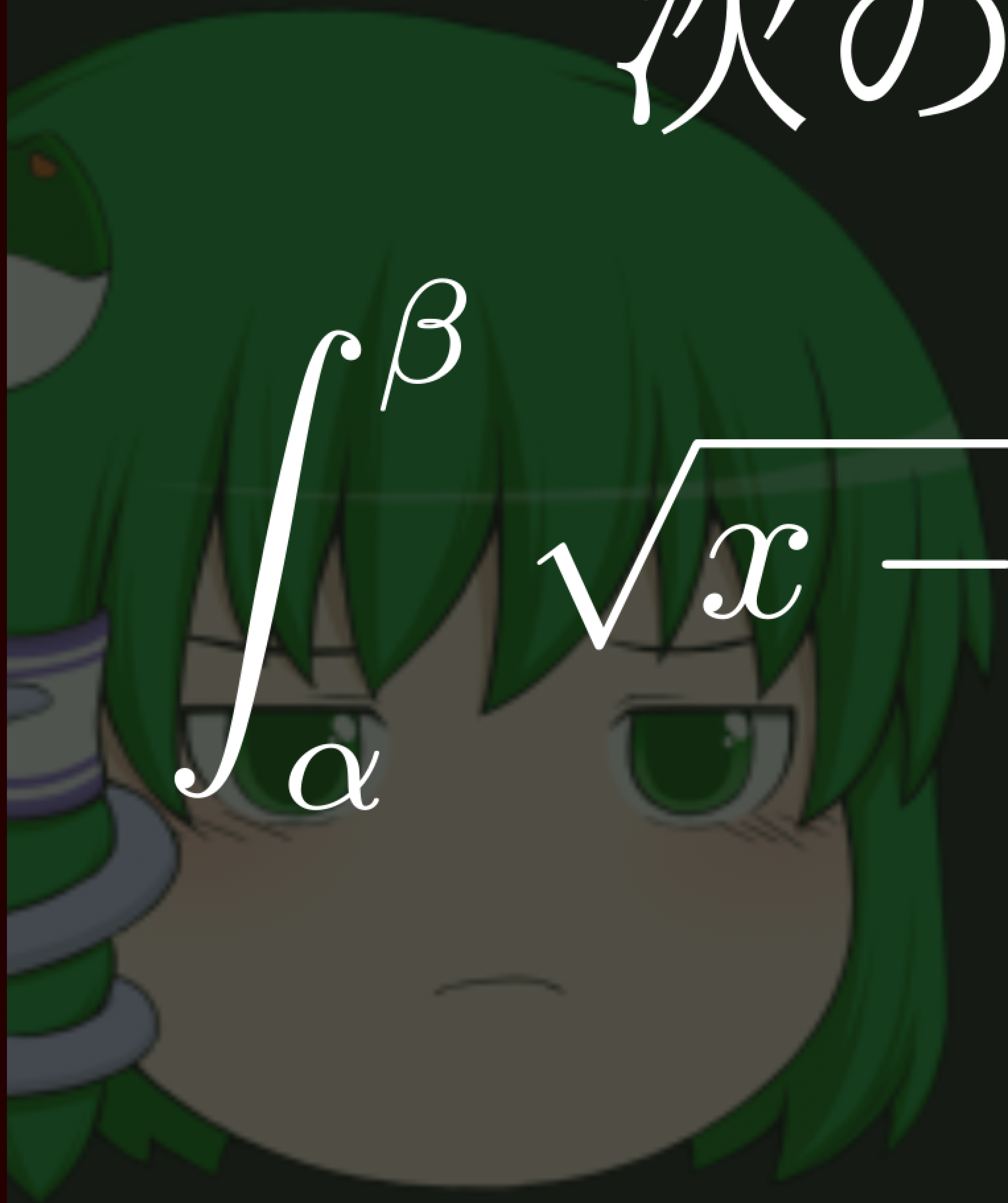


次の積分を計算せよ。

$$\int_{\alpha}^{\beta} \sqrt{x-\alpha}\sqrt{\beta-x}\,\mathrm{d}x \quad (\alpha < \beta)$$

次の積分を計算せよ。

$$\int_{\alpha}^{\beta} \sqrt{x-\alpha}\sqrt{\beta-x}\,\mathrm{d}x \quad (\alpha<\beta)$$

解答

$$\begin{aligned}
&\int_{\alpha}^{\beta} \sqrt{x-\alpha}\sqrt{\beta-x}\,\mathrm{d}x \\
&= \int_{\alpha}^{\beta} \sqrt{(x-\alpha)(\beta-x)}\,\mathrm{d}x \\
&= \int_{\alpha}^{\beta} \sqrt{\left(\frac{\beta-\alpha}{2}\right)^2 - \left(x-\frac{\beta+\alpha}{2}\right)^2}\,\mathrm{d}x
\end{aligned}$$

文字の置換

$$x-\frac{\beta+\alpha}{2} = \left(\frac{\beta-\alpha}{2}\right)\sin(\theta)$$

$$\begin{aligned}
&\int_{\alpha}^{\beta} \sqrt{x-\alpha}\sqrt{\beta-x}\,\mathrm{d}x \\
&= \left(\frac{\beta-\alpha}{2}\right)^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^2\,\mathrm{d}\theta \\
&= \left(\frac{\beta-\alpha}{2}\right)^2 \int_{0}^{\frac{\pi}{2}} \mathrm{d}\theta \\
&= \left(\frac{\beta-\alpha}{2}\right)^2 \cdot \frac{\pi}{2} \\
&= \frac{\pi}{8}(\beta-\alpha)^2
\end{aligned}$$

結論

$$\int_{\alpha}^{\beta} \sqrt{x-\alpha}\sqrt{\beta-x}\,\mathrm{d}x = \frac{\pi}{8}(\beta-\alpha)^2$$

補足1

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^2 \, d\theta = \int_0^{\frac{\pi}{2}} d\theta$$

導出

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^2 \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\cos(\theta))^2 \, d\theta \quad \left((\cos(-\theta))^2 = (\cos(\theta))^2\right)$$

$$= 2\int_{\frac{\pi}{2}}^{0} \left(\cos\left(\frac{\pi}{2} - \phi\right)\right)^2 - d\phi \quad \left(\theta = \frac{\pi}{2} - \phi\right)$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin(\phi))^2 \, d\phi$$

$$\frac{1}{2}I + \frac{1}{2}I = \int_0^{\frac{\pi}{2}} (\cos(\theta))^2 \, d\theta + \int_0^{\frac{\pi}{2}} (\sin(\theta))^2 \, d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \left((\cos(\theta))^2 + (\sin(\theta))^2\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta$$

補足2（問題の解釈）

被積分関数 $\sqrt{x-\alpha}\sqrt{\beta-x}$ をyと置くと

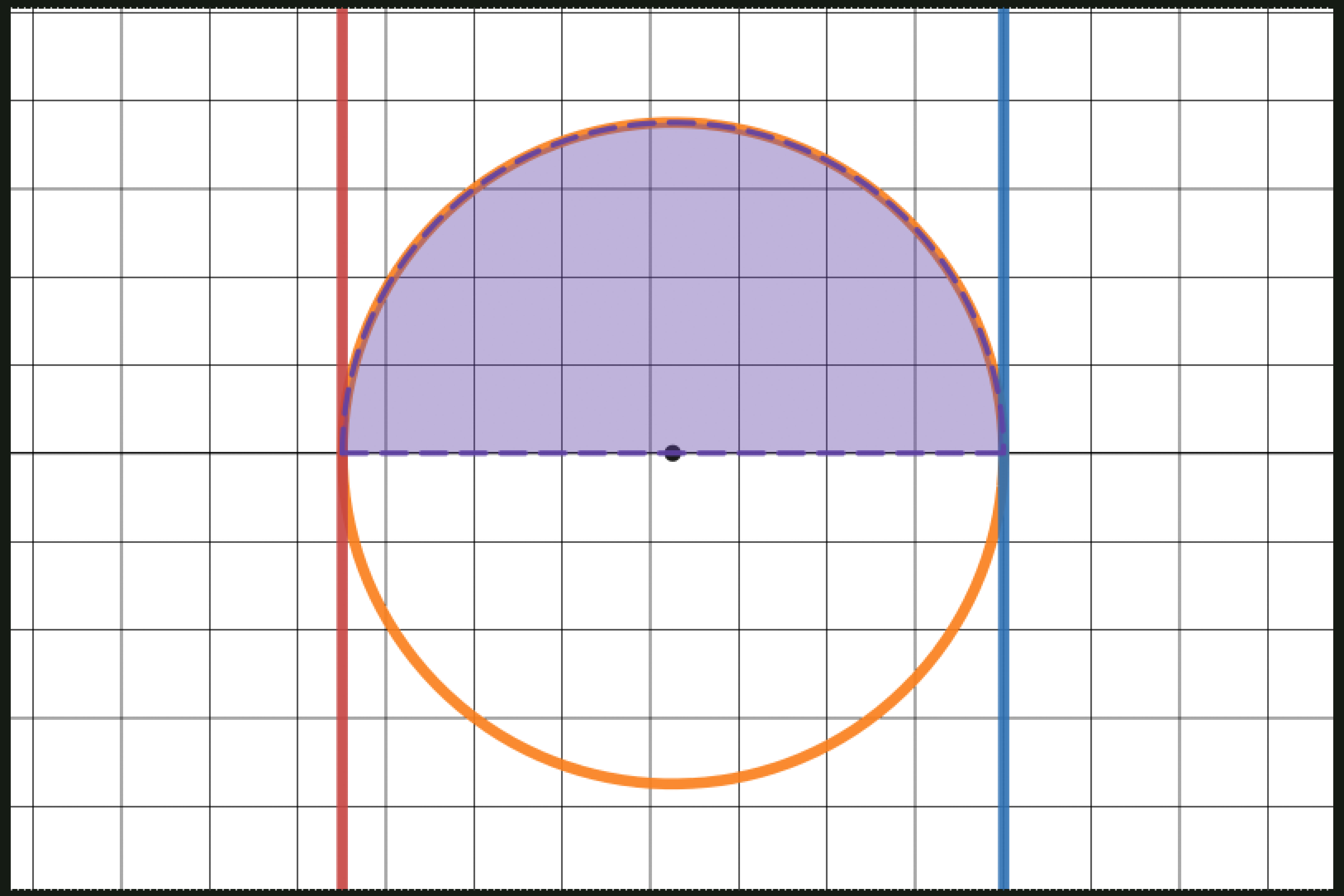
$$y=\sqrt{x-\alpha}\sqrt{\beta-x}$$
$$\Longrightarrow \left(x-\frac{\beta+\alpha}{2}\right)^2+y^2=\left(\frac{\beta-\alpha}{2}\right)^2$$

上記から、この問題の積分は

xy平面上における点$\left(\frac{\beta+\alpha}{2},0\right)$を中心とした

半径 $\frac{\beta-\alpha}{2}$ の円の半分の面積だと言える。

グラフ



$x=\alpha$

$x=\beta$

$$\left(x-\frac{\beta+\alpha}{2}\right)^2+y^2=\left(\frac{\beta-\alpha}{2}\right)^2$$

$$\int_\alpha^\beta \sqrt{x-\alpha}\sqrt{\beta-x}\,\mathrm{d}x$$