Calculating the intersections of two ellipses in a plane

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In this paper we explore a method of determining the intersections (if any) of two arbitrary ellipses in a plane.[1] The ellipses are assumed in the form :

$$A(x,y)=a_0+a_1x+a_2y+a_3x^2+a_4xy+y^2=0$$

$$B(x,y)=b_0+b_1x+b_2y+b_3x^2+b_4xy+y^2=0$$

Because they are both ellipses, they will have discriminates:

$$a_4^2-4a_3<0$$

$$b_4^2-4b_3<0$$

Subtracting B from A results in:

$$D(x,y)=d_0+d_1x+d_2y+d_3x^2+d_4xy=0$$

where $d_i=a_i-b_i$

Define the variable w as:

$$w=y+(a_2+a_4^2)/2$$

And substituting for y in the equation A results in

$$w^2+(a_0+a_1x+a_3x^2)-(a_2+a_4x)^2/4=0$$

Expanding and collection terms for *x*

$$C(x,w)=w^2+(c_0+c_1x+c_2x^2=0)$$

where

$$c_0=a_0-a_2^2/4$$
$$c_1=a_1-a_2a_4/2$$
$$c_2=a_3-a_4^2/4$$

1 This presentation uses the approach used by David Eberly in his package of geometric tools.

Substituting for y in the equation D results in

$$E(x,w) = (d_2+d_4 x)w + (e_0 + e_1 x + e_2 x^2) = 0$$

where

$$e_0 = d_0 - d_2^2/2$$
$$e_1 = d_1 - (a_2 d_4 + a_4 d_2)/2$$
$$e_2 = d_3 - a_4 d_4/2$$

Substituting w from equation E into equation C and collecting terms results in the fourth order polynomial

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4$$

where

$$f_0 = c_0 d_2^2 + e_0^2$$
$$f_1 = c_1 d_2^2 + 2(c_0 d_2 d_4 + e_0 e_1)$$
$$f_2 = c_2 d_2^2 + c_0 d_4^2 + e_1^2 + 2(c_1 d_2 d_4 + e_0 e_2)$$
$$f_3 = c_1 d_4^2 + 2(c_2 d_2 d_4 + e_1 e_2)$$
$$f_4 = c_2 d_4^2 + e_2^2$$

Calculate the roots of the F is through the well-known process of determining the depressed version of F through substituting $w = x - f_3/4$ to eliminate the f_3 term and redefining F in terms of two quadratics

$$F'(w) = (w^2 + p x + q)(x^2 + r x + s) = 0$$

This procedure is described in the Wikipedia entry for “Quartic equation” in the section “Alternative method.”

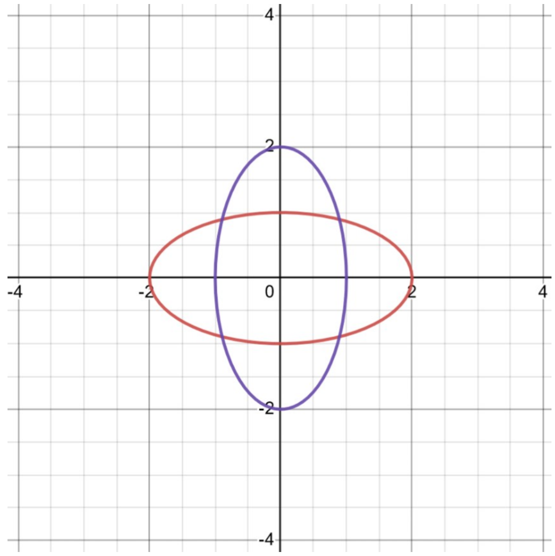
The determination of y is through the use of $E(x,w) = (d_2+d_4 x)w + (e_0 + e_1 x + e_2 x^2) = 0$ unless $d_2+d_4 x$ is zero, in which case $C(x,w) = w^2 + (c_0 + c_1 x + c_2 x^2 = 0)$ is used.

Sample calculations.

In **case 1** the two ellipses are

$$\frac{x^2}{4} + y^2 - 1 = 0 \quad \text{and} \quad x^2 + \frac{y^2}{4} - 1 = 0$$

These two ellipses are centered at $(0,0)$ and each is the other rotated by 90º. In the image below, the first ellipse is in red; the second in magenta.



In the form for the calculation, these are

$$A(x,y) = -1 + 0.25x^2 + y^2 = 0$$

$$B(x,y) = -4 \quad +4\ x^2 + y^2 = 0$$

The coefficients of the factors in $F(x)$ are

$$f_0 = 9$$
$$f_1 = 0$$
$$f_2 = -22.5$$
$$f_3 = 0$$
$$f_4 = 14.0625$$

so that F becomes

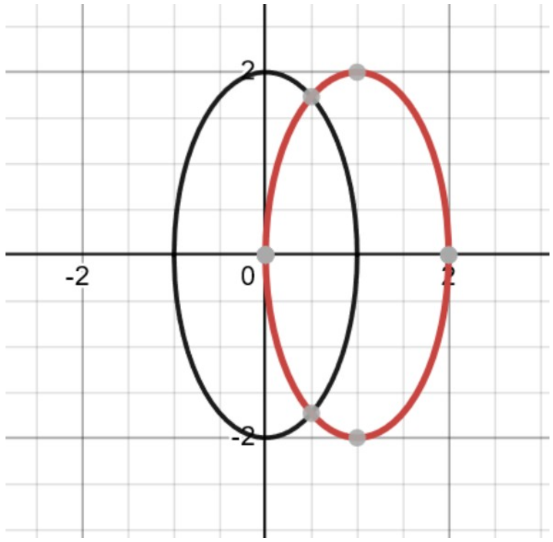
$$\frac{(225x^4 - 360x^2 + 144)}{16} = 0$$

This 4 roots of this quadratic are $x = \pm 2/\sqrt{(5)},\ y = \pm 2/\sqrt{(5)}$. The quantity $2/\sqrt{(5)}$ is approximately 0.894427191.

For **Case 2** the ellipses are

$$x^2 + \frac{y^2}{4} - 1 = 0 \quad \text{and} \quad x^2 + \frac{y^2}{4} - 2x = 0$$

The second ellipse is now centered at $(1,0)$ so that there are two intersection points, both at x = ½. The in image below, the first ellipse is in black, the second in red.



In the form for the calculation, these ellipses are

$$A(x,y) = -4 \quad +4\,x^2 + y^2 = 0$$

$$B(x,y) \;=\; -\,8\,x + 4\,x^2 + y^2 = 0$$

The coefficients of the factors in $F(x)$ are

$$f_0 = 16$$
$$f_1 = -64$$
$$f_2 = 16$$
$$f_3 = 0$$
$$f_4 = 0$$

so that F becomes

$$64\,x^2 - 64\,x + 16 = 0$$

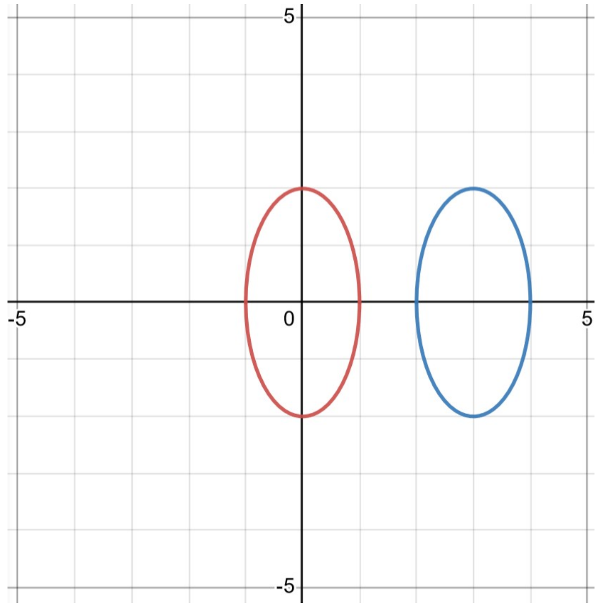
the root of this quadratic is ½.

Using the definitions for w, $y = \sqrt{(3)}$.

For **Case 3** the ellipses are

$$x^2 + \frac{y^2}{4} - 1 = 0 \quad \text{and} \quad x^2 + \frac{y^2}{4} - 2\,x = 0$$

The second ellipse is now centered at $(3,0)$ so the calculation should show that there is no intersection. The first ellipse is in red; the second in green.



In the form for the calculation, these ellipses are

$$A(x,y) = -4 \quad +4\,x^2 + y^2 = 0$$

$$B(x,y) \;=\; -\,8x + 4x^2 + y^2 = 0$$

The coefficients of the factors in $F(x)$ are

$$f_0 = 1296$$
$$f_1 = -1728$$
$$f_2 = 576$$
$$f_3 = 0$$
$$f_4 = 0$$

so that *F* becomes

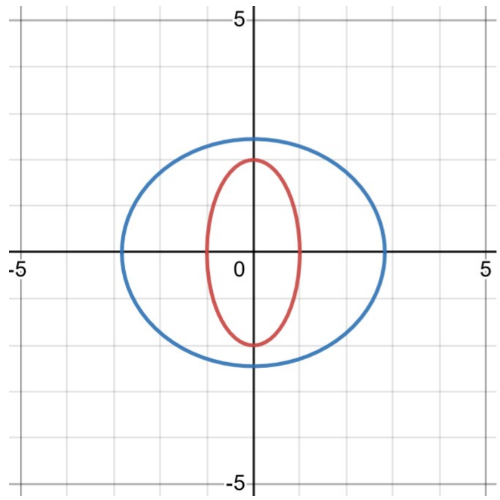
$$576x^2 - 1728x + 1296 = 0$$

The root of this quadratic is 1.5, but the calculation of y requires taking the square root of -5 thus indicating no intersection.

In **case 4** the two ellipses are

$$\frac{x^2}{4} + y^2 - 1 = 0 \quad \text{and} \quad 3x^2 + 4y^2 - 24 = 0$$

These two ellipses are centered at $(0,0)$ and the first is completely contained in the second.. In the image below, the first ellipse is in red; the second in blue.



In the form for the calculation, these are

$$A(x,y) = -4 \quad + 4\ x^2 + y^2 = 0$$

$$B(x,y) = -6 + 0.75\ x^2 + y^2 = 0$$

The coefficients of the factors in $F(x)$ are

$f_0 = 4$
$f_1 = 0$
$f_2 = 13$
$f_3 = 0$
$f_4 = 10.0625$

so that F becomes

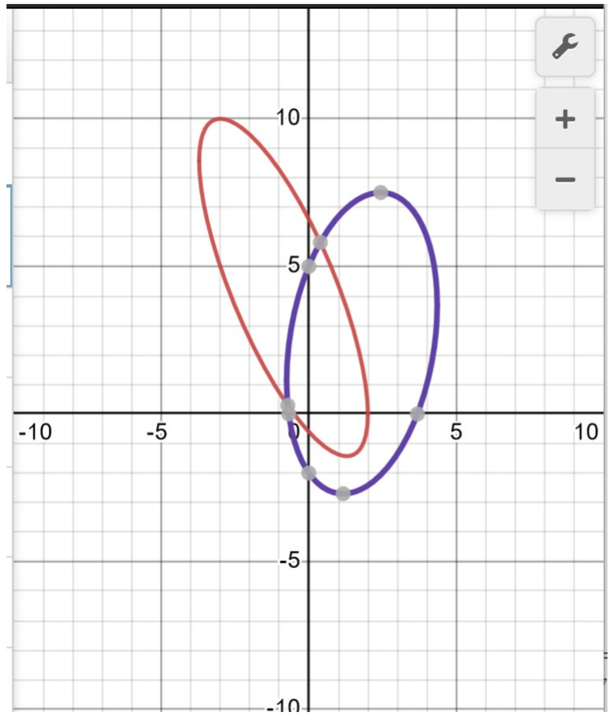
$$\frac{(164\,x^4 = 208\,x^2 + 64)}{16} = 0$$

This roots of this quadratic are both imaginary: $2^{1.5} / \sqrt{(13)}\, i$, so there is no intersection.

For **Case 5** the ellipses are

$$4\,x^2 + y^2 + 3\,xy - 6\,x - 6\,y - 4 = 0 \quad \text{and} \quad 4\,x^2 + y^2 - xy - 12\,x - 3\,y - 10 = 0$$

The first ellipse is shown in red; the second in magenta in the diagram below



From inspection of the diagram, the intersections are about $(-0.7, 0.3)$ and $(0.4, 5.8)$.

In the form for the calculation, these ellipses are

$$A(x,y) = -4 - 6x - 6y + 4x^2 + 3xy + y^2 = 0$$

$$B(x,y) = -10 - 12x + 3y + 4x^2 - xy + y^2 = 0$$

The coefficients of the factors in $F(x)$ are

$$\begin{aligned} f_0 &= -108 \\ f_1 &= 204 \\ f_2 &= 278 \\ f_3 &= -264 \\ f_4 &= 64 \end{aligned}$$

so that F becomes

$$64x^4 - 264x^3 + 278x^2 + 204x - 108 = 0$$

The roots of this equation (to 9 decimals) are -0.712845869 and 0.391064077 plus two roots with imaginary parts.

Calculating y as before gives the intersections as $(-0.712845869, 0.294447423)$ and $(0.391064077, 5.813283047)$.