Лекция 11

Нека $(X_i)_{i=1}^{\infty}$ са i.i.d. Нека $\mu = \mathbb{E}X_1$ и $\mathbb{E}|X_1| < \infty$ и

да означим $S_n = \sum_{j=1}^{n} X_j$, $n \geq 1$.

От УЗГЧ имаме, че $\frac{S_n}{n} \xrightarrow[n\to\infty]{п.с} \mu$

$E_n = \frac{S_n}{n} - \mu \xrightarrow[n\to\infty]{п.с} 0$ Колко бързо?

Какво можем да кажем за E_n?

Теорема: 'ЦГТ'

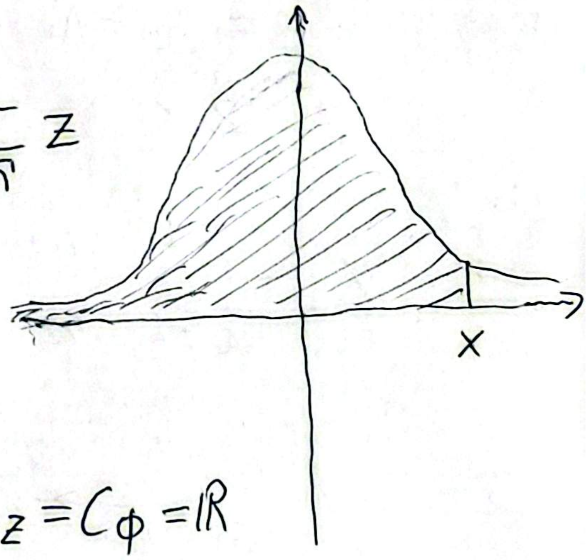
Нека $(X_i)_{i=1}^{\infty}$ са i.i.d със $\mu = \mathbb{E}X_1$ и $DX = \sigma^2$.

Нека $Z_n := \frac{\sqrt{n}}{\sigma} E_n = \frac{\sqrt{n}}{\sigma}\left(\frac{S_n}{n} - \mu\right) = \frac{S_n - n\mu}{\sigma\sqrt{n}}$.

Тогава $Z_n \xrightarrow[n\to\infty]{d} Z \sim N(0,1)$.

Коментари: $E_n = \frac{S_n}{n} - \mu = \frac{\sigma}{\sqrt{n}} Z_n \approx \frac{\sigma}{\sqrt{n}} Z$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$$



Z, $\Phi(x)$ е непрекъсната и $C_Z = C_\Phi = \mathbb{R}$

$\forall b \in C_\Phi = \mathbb{R}$: $\mathbb{P}(Z_n < b) \xrightarrow[n\to\infty]{} \mathbb{P}(Z < b) = \Phi(b)$

$a < b$ $\quad \mathbb{P}(Z_n < b) - \mathbb{P}(Z_n < a) = \mathbb{P}(Z_n \in (a,b))$

$\qquad\qquad \Phi(b) \quad - \quad \Phi(a) \quad = \mathbb{P}(Z \in (a,b))$

Пример: $S_n = \sum_{j=1}^{n} X_j$, $\mu = \mathbb{E} X_1$, $\sigma^2 = DX$

$$\mathbb{P}(a < Z_n < b) = \mathbb{P}\left(a < \frac{S_n - n\mu}{\sigma\sqrt{n}} < b\right) =$$

$$= \mathbb{P}(a\sigma\sqrt{n} + n\mu < S_n < b\sigma\sqrt{n} + n.\mu)$$

$$\Big\downarrow n \to \infty$$

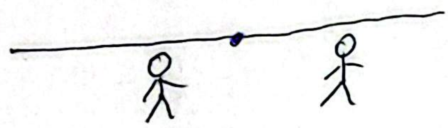
$$\frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{y^2}{2}} dy = \phi(b) - \phi(a)$$

Пример: $\mu = 0$, $\frac{S_n}{\sigma\sqrt{n}} \xrightarrow[n\to\infty]{d} Z$

$$\mathbb{P}(S_n > 0) = \mathbb{P}\left(\frac{S_n}{\sigma\sqrt{n}} > 0\right) \xrightarrow[n\to\infty]{} \mathbb{P}(Z > 0) = \frac{1}{2}$$

Пример: $S_n = Д_n - Л_n =$ – ДЪРПАНЕ НА ВЪЖЕ

$$= \sum_{j=1}^{n} X_j$$



$$X_j = \begin{cases} 1 & 1/2 \\ -1 & 1/2 \end{cases}, \quad \mu = 0 = \mathbb{E} X_1$$

$$D X_1 = \mathbb{E} X_1^2 = 1$$

$$\Rightarrow \frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{d} Z \sim N(0,1)$$

$$\mathbb{P}(S_n > 0) = \mathbb{P}\left(\frac{S_n}{\sqrt{n}} > 0\right) \approx \mathbb{P}(Z > 0) = \frac{1}{2}$$

Функции на моментите

Деф: Нека X е сл. вел. Тогава X има ф-я на момента $M_X(t)$, ако $M_X(t) = \mathbb{E} e^{tX}$ за $|t| < a$.

$\to \mathbb{E}e^{tX} = \sum_i e^{tx_i} \cdot \mathbb{P}(X = x_i)$ или

$\to \mathbb{E}e^{tX} = \int_{-\infty}^{\infty} e^{tx} f_X(x) . dx$

Пример: $X \sim U(0,1)$ $\quad M_X(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x) dx = \int_0^1 e^{tx} dx =$

$$= \frac{e^t - 1}{t}$$

M_X съществува в $t \in \mathbb{R} \setminus \{0\}$. Но $\lim_{t \to 0} \frac{e^t - 1}{t} = 1 = M_X(0)$.

M_X съществува в $|t| < \infty$

Пример: $X \sim Exp(1)$ $\quad M_X(t) = \int_0^{\infty} e^{tx} . e^{-x} dx = \int_0^{\infty} e^{-x(1-t)} dx =$

$= \frac{1}{1-t}$, за $t < t$ $\quad$ M_X съществува за $|t| < t$

Деф: 'моменти от ред k' Нека X е сл. вел. Тогава $\mathbb{E}X^k$ е момент от ред k.

$$\to \mathbb{E}X^k = \sum_i x_i^k \mathbb{P}(X = x_i)$$

$$\to \mathbb{E}X^k = \int_{-\infty}^{\infty} x^k . f_X(x) dx$$

Деф: 'Абсолютни моменти от ред k, централни моменти от ред k'

Нека X е сл. вел. Тогава $\mathbb{E}|X|^k$ е абсолютен момент от ред k.

$$\rightarrow \mathbb{E}|X|^k = \sum_i |x_i| \cdot \mathbb{P}(X=x_i)$$

$$\rightarrow \mathbb{E}|X|^k = \int_{-\infty}^{\infty} |x|^k \cdot f_X(x)dx \qquad , a$$

$\mathbb{E}|X-\mathbb{E}X|^k$ е абсолютен централен момент от ред k.

За $k=2$ $\quad DX = \mathbb{E}(X-\mathbb{E}X)^2$.

Свойства на ф-ите на моментите:

а) $M_X(0) = \mathbb{E}e^{0X} = \mathbb{E}1 = 1$

б) $M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot \mathbb{E}X^k$,

Д-во: $M_X(t) = \mathbb{E}e^{tX} = \mathbb{E} \sum_{k=0}^{\infty} \frac{t^k X^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot \mathbb{E}X^k$

в) $M_X^{(k)}(0) = \mathbb{E}X^k$

г) Ако $X \perp\!\!\!\perp Y$, то $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

д) Ако $\lim_{n\to\infty} M_{X_n}(t) = M_X(t)$, $|t|<\varepsilon \Rightarrow X_n \xrightarrow[n\to\infty]{d} X$

е) $M_X(t) = M_Y(t)$, $|t|<a$, то $X \stackrel{d}{=} Y$

ж) Ако $Y = aX + b$, то $M_Y(t) = e^{bt} \cdot M_X(at)$

Твърдение: Нека $X \sim N(\mu, \sigma^2)$. Тогава $M_X(t) = e^{\mu t} \cdot M_Z(\sigma t)$, където $Z \sim N(0,1)$ и $M_X(t) = e^{\mu t + \frac{\sigma^2}{2} t}$, т.е. $M_Z(t) = e^{\frac{t^2}{2}}$.

Дво: $X = \mu + \sigma Z \overset{\text{III)}}{\Longrightarrow} M_X(t) = e^{\mu t} \cdot M_Z(\sigma t)$

$$M_Z(s) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{sy} \cdot e^{\frac{-y^2}{2}} dy = \frac{e^{\frac{s^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sy - \frac{y^2}{2} - \frac{s^2}{2}} dy =$$

$$= \frac{e^{\frac{s^2}{2}}}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{-\frac{(y-s)^2}{2}} dy \overset{y-s=w}{=\!=} \frac{e^{\frac{s^2}{2}}}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{-\frac{w^2}{2}} dw =$$

$$= e^{\frac{s^2}{2}} \quad \blacksquare$$

Доказателство на Централна Гранична Теорема:

Цел: $Z_n = \dfrac{S_n - n\mu}{\sigma \sqrt{n}} \xrightarrow[n \to \infty]{d} Z \sim N(0,1)$

$$Z_n = \frac{1}{\sqrt{n}} \frac{S_n - n\mu}{\sigma} = \frac{1}{\sqrt{n}} \cdot \sum_{j=1}^{n} \frac{X_j - \mu}{\sigma} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} Y_j, \quad Y_j = \frac{X_j - \mu}{\sigma}.$$

Ще допуснем, че M_{X_1} е дефинирана за $|t| < \infty$.

$$M_{Y_j}(t) \overset{\text{III}}{=} e^{-\frac{\mu}{\sigma} \cdot t} \cdot M_{X_j}\left(\frac{t}{\sigma}\right) = e^{-\frac{\mu}{\sigma} t} \cdot M_{X_1}\left(\frac{t}{\sigma}\right) = M_{Y_1}(t)$$

$$M_{Z_n}(t) = M_{\frac{1}{\sqrt{n}} \sum_{j=1}^{n} Y_j}(t) = \mathbb{E}\, e^{t \cdot \frac{\sum_{j=1}^{n} Y_j}{\sqrt{n}}} = M_{\sum_{j=1}^{n} Y_j}\left(\frac{t}{\sqrt{n}}\right) =$$

$$\overset{\text{г)}}{=} \prod_{j=1}^{n} M_{Y_j}\left(\frac{t}{\sqrt{n}}\right) = \left(M_{Y_1}\left(\frac{t}{\sqrt{n}}\right)\right)^n$$

$$M_{Y_1}\left(\frac{t}{\sqrt{n}}\right) \overset{df}{=} \sum_{j=0}^{\infty} \frac{t^j}{(\sqrt{n})^j \cdot j!} \mathbb{E}Y_1^j = 1 + \cancel{\frac{t}{\sqrt{n}} \mathbb{E}Y_1} + \frac{t^2}{2n} \cdot \mathbb{E}Y_1^2 + o\left(\frac{t^2}{n}\right)$$

$$\mathbb{E}Y_1 = \mathbb{E}\, \frac{X_1 - \mu}{\sigma} = 0$$

$$DY_1 = \frac{1}{\sigma^2} D(X_1 - \mu) = \frac{DX_1}{\sigma^2} = 1$$

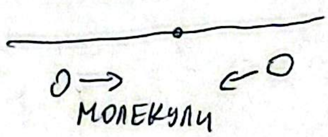
$$M_{Y_1}\left(\frac{t}{\sqrt{n}}\right) = 1 + \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)$$

$$\Rightarrow \quad M_{Z_n}(t) = \left(M_{Y_1}\left(\frac{t}{\sqrt{n}}\right)\right)^n = \left(1 + \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n =$$

$$= \left(1 + \frac{t^2}{2} \cdot \frac{1}{n} + o\left(\frac{t^2}{1} \cdot \frac{1}{n}\right)\right)^n \xrightarrow[n \to \infty]{} e^{\frac{t^2}{2}} = M_Z(t)$$

Примери:

⊕ Дърпане на въже $\quad X_j = \begin{cases} 1 & 1/2 \\ -1 & 1/2 \end{cases}$



$N = 10^6$ $\qquad S_N = \sum_{j=1}^{N} X_j = Д_N - Л_N$

$$\mathbb{P}(S_N > 1000) = \mathbb{P}(\text{да имаме движение})$$

$$\frac{S_n}{\sqrt{n}} = \frac{S_{10^6}}{1000} \approx Z \qquad \mathbb{P}\left(\frac{S_{10^6}}{1000} > 1\right) \approx \mathbb{P}(Z > 1) = 1 - \Phi(1)$$

$$\frac{S_n}{\sqrt{n}} \xrightarrow{п.с} \mu \qquad Z_n = \frac{S_n - n.\mu}{\sigma\sqrt{n}} = \frac{\sqrt{n}}{\sigma}\left(\frac{S_n}{n} - \mu\right) \xrightarrow[n\to\infty]{d} Z \sim N(0,1)$$

Колко бързо? За колко големи n?

⊕ Избори, вероятност да гласуваме за партия b е p.

$$\frac{\sum_{j=1}^{N} X_j}{N} \xrightarrow[N\to\infty]{п.с} p \qquad X_j \sim Ber(p).$$

$$|E_N| = \left| \frac{\sum_{j=1}^{N} X_j}{N} - p \right| = \frac{\sigma}{\sqrt{N}} \cdot |Z_N| = \frac{\sqrt{p(1-p)}}{\sqrt{N}} \cdot |Z_N|$$

$$\mathbb{P}(|E_N| > \varepsilon) = \mathbb{P}\left(\frac{\sqrt{p(1-p)}}{\sqrt{N}} \cdot |Z_n| > \varepsilon\right) \le \mathbb{P}\left(\frac{1}{2\sqrt{N}} \cdot |Z_N| > \varepsilon\right)$$

$p(1-p) \le \frac{1}{4}$ за $p \in (0,1)$

$$\Rightarrow \mathbb{P}(|E_N| > \varepsilon) \le \mathbb{P}(|Z_N| > \varepsilon.2\sqrt{N}) \approx \mathbb{P}(|Z| > 2\varepsilon\sqrt{N}) =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{|y| > 2\varepsilon\sqrt{N}} e^{-\frac{y^2}{2}} dy \le \delta$$

$\varepsilon = 0{,}01$
$\delta = 0{,}01$ за удобство $\to \varepsilon = \frac{A}{2\sqrt{N}}$ $\quad \mathbb{P}\left(|E_N| > \frac{A}{2\sqrt{N}}\right) \le \frac{1}{\sqrt{2\pi}} \cdot \int_{|y|>A} e^{-\frac{y^2}{2}} dy \approx$

$$\approx \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{A^2}{2}}}{A} \le \delta$$

Неравенство на Бери - Есен:

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} < x \right) - \phi(x) \right| \leq \frac{0,4748}{\sqrt{n}\,\sigma^3} \cdot \mathbb{E}|X_1 - \mu|^3$$

Следствие: Нека $X_n \sim Bin(n,p)$. Тогава $\frac{X_n - np}{\sqrt{np(1-p)}} \xrightarrow[n\to\infty]{d} Z \sim N(0,1)$

Д-во: $X_n = \sum_{j=1}^{n} Y_j$, $Y_j \sim Ber(p)$, i.i.d.

$$\frac{X_n - np}{\sqrt{np(1-p)}} \xrightarrow[n\to\infty]{d} Z \quad , \quad np(1-p) = DX_n$$

$$\mathbb{P}(X_n \geq k_n) = \mathbb{P}\left(\frac{X_n - np}{\sqrt{np(1-p)}} \geq \frac{k_n - np}{\sqrt{np(1-p)}} \right) \approx$$

$$\approx \mathbb{P}\left(Z \geq \frac{k_n - np}{\sqrt{np(1-p)}} \right) = \mathbb{P}(Z \geq a)$$

Твърдение: Ако $X \sim \Gamma(\alpha,\beta)$ то $M_X(t) = \left(\frac{\beta}{\beta - t}\right)^{\alpha}$, $t < \beta$

Д-во: $$M_X(t) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot \int_0^{\infty} e^{tx} \cdot x^{\alpha-1} \cdot e^{-\beta x} \cdot dx =$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot \int_0^{\infty} x^{\alpha-1} \cdot e^{-x(\beta-t)} dx \underset{x(\beta-t)=y}{=} \frac{\beta^{\alpha}}{\Gamma(\alpha)(\beta-t)^{\alpha}} \cdot \int_0^{\infty} y^{\alpha-1} \cdot e^{-y} dy =$$

$$= \frac{\beta^{\alpha}}{(\beta-t)^{\alpha}}$$

$$\mathbb{E}X = M_X'(0) = \frac{\alpha\beta^\alpha}{(\beta-t)^{\alpha+1}}\Bigg|_{t=0} = \frac{\alpha}{\beta}$$

$$\mathbb{E}X^2 = M_X''(0) = \frac{\alpha(\alpha+1)\beta^\alpha}{(\beta-t)^{\alpha+2}} = \frac{\alpha(\alpha+1)}{\beta^2}$$