Лекция 12

Статистика

Точкови оценки.

Постановка: X описва някакво св-во на някаква генерална съвкупност. $F_X(x) = \mathbb{P}(X<x)$, $f_X(x)$

Целта: $\vec{X} = (X_1, \dots, X_n)$, $(X_i)_{i=1}^{\infty}$ са i.i.d, $X_i \overset{d}{=} X$

На базата на тази извадка $\vec{X}$ искаме да определим /приближим F_X, f_X.

$$\vec{X} = (X_1, X_2, \dots, X_n) \xrightarrow{?!} f_X, F_X$$

Допускания: X принадлежи на някакъв клас от сл. вел.

$\theta = p$ Например $X \sim Ber(p)$, X, $F_X(x,p)$, $p \in (0,1)$.

$\theta = \lambda$ $X \sim Exp(\lambda)$, $\lambda > 0$, $F_X(x,\lambda) = 1 - e^{-\lambda x}$

$\theta = (\mu, \sigma^2)$ $X \sim N(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma^2 > 0$

$$X, F_X(x,\theta), f_X(x,\theta)$$

$X, F_X(x,\theta)$; $\vec{X} \Rightarrow \hat{\theta}$ ще бъде оценка за θ

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Деф.: 'точкова оценка'

X е от класа $F_X(x,\theta)$. Тогава

$\hat{\theta} = \hat{\theta}(\vec{X})$ е точкова оценка за θ.

А.) Метод на максималното правдоподобие (МПО)

$$X, \quad f_X(x,\theta), \quad \vec{X}$$

ф-я на максималното правдоподобие наричаме съвместната плътност на $\vec{X}$, т.е.

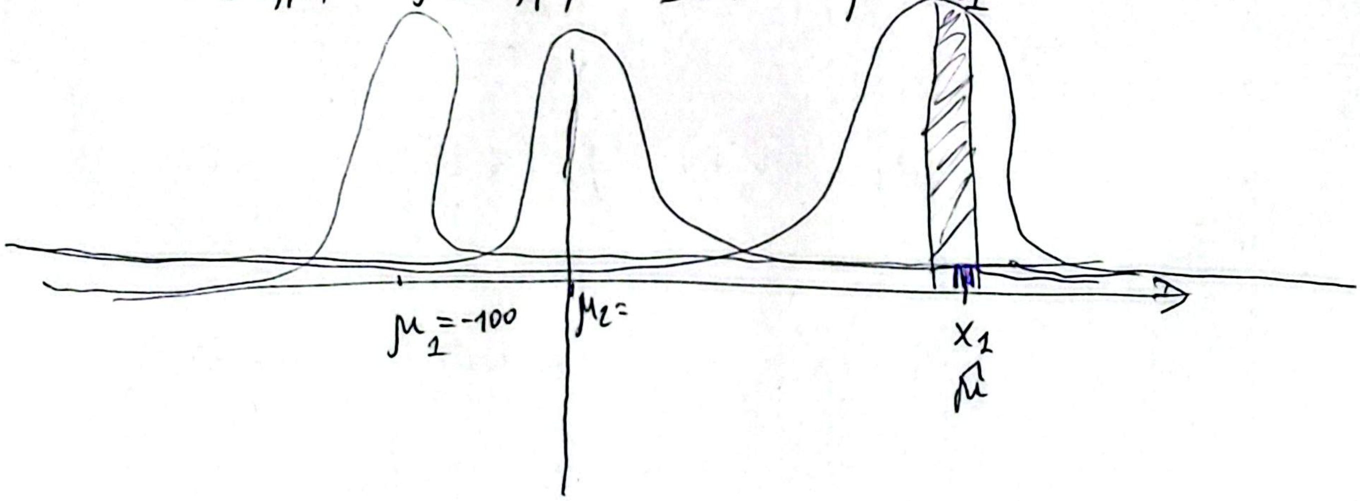
$$L(\vec{X},\theta) = \prod_{j=1}^{n} f_X(X_j,\theta)$$

$$\vec{x}, \quad L(\vec{x},\theta) = \prod_{j=1}^{n} f_X(x_j,\theta)$$

Пример: $X \sim N(\mu,1)$; $\vec{X} = X_1$, $X_1 = x_1$

$$L(\vec{X},\mu) = f_X(X_1,\mu)$$

$$L(x_1,\mu) = f_X(x_1,\mu) \xrightarrow{\text{избираме}} \hat{\mu} = x_1$$



Деф: Нека X е сл. вел с плътност $f_X(x,\theta)$.

Нека $\vec{X}$ са n независими наблюдения над X.

Тогава под МПО за θ разбираме $\hat{\theta}$:

$$L(\vec{X},\hat{\theta}) = \sup_{\theta\in\Theta} L(\vec{X},\theta)$$

практически $\vec{x}$ $\quad L(\vec{x},\hat{\theta}) = \sup_{\theta\in\Theta} L(\vec{x},\theta)$

Ако f_X е диференцируема по θ, то решаваме $\frac{\partial L}{\partial \theta} = 0 \Longrightarrow \hat{\theta}$

$$L(\vec{X},\theta) = \prod_{j=1}^{n} f_X(x_j,\theta)$$

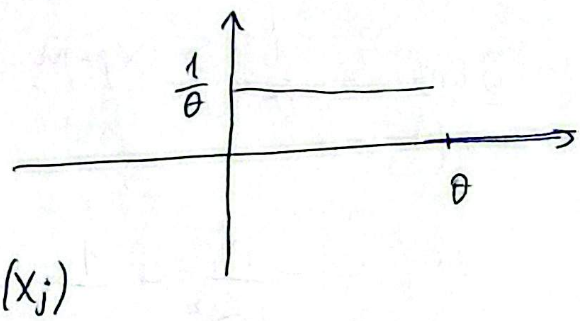
$$\frac{\partial \ln L}{\partial \theta} = 0$$

$$X \sim N(\mu,\sigma^2) \qquad \theta = (\mu,\sigma^2)$$

$$\left|\begin{array}{l} \frac{\partial \ln L}{\partial \mu} = 0 \\ \frac{\partial \ln L}{\partial \sigma^2} = 0 \end{array}\right. \Longrightarrow (\hat{\mu},\hat{\sigma}^2) = \hat{\theta}$$

⊕ $X \sim U(0,\theta)$, $\theta > 0$, $\quad f_X(x) = \frac{1}{\theta}\cdot 1_{[0,\theta]}(x)$

$\vec{X}$,



$$L(\vec{X},\theta) = \prod_{j=1}^{n} \frac{1}{\theta}\cdot 1_{[0,\theta]}(x) =$$

$$= \frac{1}{\theta^n}\cdot 1_{[0,\theta]}(x^*) \qquad x^* = \max_{1\le j\le n}(x_j)$$

$$= \begin{cases} \frac{1}{\theta^n}, & x^* \le \theta \\ 0, & x^* > 0 \end{cases}$$

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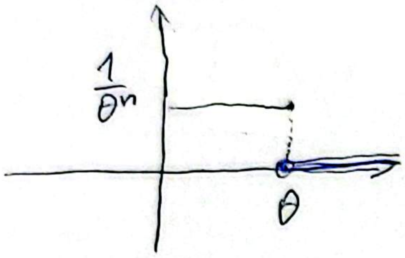
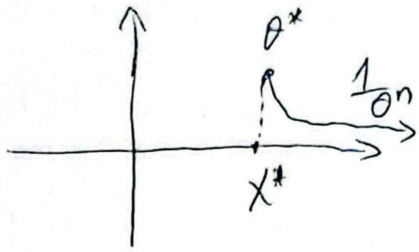


$\frac{1}{\theta^n}$ Е НАМАЛЯВАЩА ПО θ



$$L(\vec{X},\theta^*) = \sup_{\theta>0} L(\vec{X},\theta) \qquad \hat{\theta} = \max_{j\le n}(X_j)\,, \quad \hat{\theta}=\hat{\theta}(\vec{X})$$

ТВЪРДЕНИЕ: НЕКА $X \sim N(\mu,\sigma^2)$. ТОГАВА МПО ЗА μ Е

$$\hat{\mu} = \overline{X}_n^{(1)} = \frac{1}{n}\sum_{j=1}^{n} X_j$$

МПО ЗА σ^2 ЗАВИСИ ОТ ТОВА ДАЛИ ЗНАЕМ μ ИЛИ НЕ ЗНАЕМ μ, Т.Е.

а) $\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j-\mu)^2$ АКО ЗНАЕМ μ.

б) $\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j-\overline{X}_n^{(1)})^2$ АКО НЕ ЗНАЕМ μ.

Д-ВО: $$L(\vec{X},\theta) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot \sigma^n \cdot e^{-\sum_{j=1}^{n}\frac{(X_j-\mu)^2}{2\sigma^2}}$$

$$\ln L(\vec{X},\theta) = n\ln\frac{1}{\sqrt{2\pi}} + \frac{n}{2}\ln\sigma - \sum_{j=1}^{n}\frac{(X_j-\mu)^2}{2\sigma^2}$$

$$\left|\begin{array}{l} \dfrac{\partial \ln L}{\partial \mu} = \dfrac{1}{\sigma^2}\sum_{j=1}^{n}(X_j-\mu) = 0 \\[2ex] \dfrac{\partial \ln L}{\partial \sigma^2} = -\dfrac{n}{2\sigma^2} + \dfrac{1}{2\sigma^4}\sum_{j=1}^{n}(X_j-\mu)^2 = 0 \end{array}\right.$$

$$\left|\begin{array}{ll} \dfrac{\partial \ln L}{\partial \mu} = 0 & \Rightarrow \hat{\mu} = \dfrac{1}{n}\sum_{j=1}^{n} X_j = X_n^{(1)} \\ \dfrac{\partial \ln L}{\partial \sigma^2} = 0 & \Rightarrow \hat{\sigma}^2 = \dfrac{1}{n}\sum_{j=1}^{n} (X_j - \mu)^2 \end{array}\right.$$

или

$$\sigma^2 = \frac{1}{n}\sum_{j=1}^{n} \left(X_j - X_n^{(1)}\right)^2$$

$\boxed{\text{Б.}}$ Метод на моментите (ММО)

$$\theta \in \mathbb{R}, \quad X, F_X(x,\theta) \qquad \mathbb{E}X = \mu^{(1)}(\theta); \quad \theta = \left(\mu^{(1)}\right)^{-1}(\mathbb{E}X)$$

$$\overline{X}_n^{(1)} = \frac{\sum_{j=1}^{n} X_j}{n} \xrightarrow[n\to\infty]{\text{п.с}} \mathbb{E}X$$

$$\mu^{(1)}(\hat{\theta}) = \overline{X}_n^{(1)}, \quad \hat{\theta} = \left(\mu^{(1)}\right)^{-1}\left(\overline{X}_n^{(1)}\right)$$

$$\downarrow n\to\infty \qquad \downarrow n\to\infty$$

$$\theta = \left(\mu^{(1)}\right)^{-1}(\mathbb{E}X)$$

Деф: 'ММО' $\quad \overline{X}_n^{(j)} := \dfrac{1}{n}\sum_{k=1}^{n} X_k^j, \quad j \geq 1$

Нека X е сл. вел. $F_X(x,\theta)$, $\theta = (\theta_1, \dots, \theta_s)$.

Тогава ММО оценка за θ или $\hat{\theta}$ намираме чрез решаването на с-мата:

$$\mu^{(j)}(\theta) = \overline{X}_n^{(j)}, \quad 1 \leq j \leq s \Rightarrow \hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_s)$$

където $\mu^{(j)}(\theta) = \mathbb{E}X^j, \quad 1 \leq j \leq s$

$$\hat{\theta}_j = \left(\mu^{(j)}\right)^{-1}\left(X_n^{(j)}\right)$$

Пример:

⊕ $X \sim U(0,\theta)$, $\vec{X}$, ММП $\hat{\theta} = \max_{j \le n} X_j$;

$$\mathbb{E}X = \mu^{(1)}(\theta) = \frac{\theta}{2} \Rightarrow \theta = 2\mathbb{E}X$$

$$\overline{X}_n^{(1)} = \frac{\theta}{2} \Rightarrow \underline{\hat{\theta} = 2\overline{X}_n^{(1)}} \text{ ММО}$$

⊕ $X \sim N(\mu,\sigma^2)$ $\mathbb{E}X = \mu$, $\mathbb{E}X^2 = DX + (\mathbb{E}X)^2 = \sigma^2 + \mu^2$

$\underline{\overline{X}_n^{(1)} = \hat{\mu}}$ ММО=ММП

$\overline{X}_n^{(2)} = \mu^2 + \sigma^2 \Rightarrow \hat{\sigma}^2 = \overline{X}_n^{(2)} - \left(\overline{X}_n^{(1)}\right)^2 = \frac{1}{n}\sum_{j=1}^{n}\left(X_j - \overline{X}_n^{(1)}\right)^2$

ММП=ММО

Свойства на точковите оценки:

а) неизместеност – казваме, че $\hat{\theta}$ е неизместена оценка за $\hat{\theta}$

$$\mathbb{E}\hat{\theta} = \theta \quad \left(\mathbb{E}\hat{\theta}_j = \theta_j,\ 1 \le j \le s,\ \theta = (\theta_1, \dots, \theta_s)\right)$$

$|\mathbb{E}\hat{\theta} - \theta|$ е системна грешка на $\hat{\theta}$.

⊕ $X \sim N(\mu,\sigma^2)$, $\hat{\mu} = \frac{1}{n}\sum_{j=1}^{n} X_j$ $\mathbb{E}\hat{\mu} = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}X_j = \frac{n\mu}{n} = \mu$

Ако знаете μ, $\mathbb{E}\hat{\sigma}^2 = \frac{1}{n}\mathbb{E}\sum_{j=1}^{n}(X_j-\mu)^2 = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}(X_j-\mu)^2 = \frac{n\sigma^2}{n} = \sigma^2$

Ако не знаете μ, то $\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}\left(X_j - \overline{X}_n^{(1)}\right)^2$

$\mathbb{E}\hat{\sigma}^2 = \left(\frac{n-1}{n}\right)\sigma^2 \neq \sigma^2$ изместена оценка

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$$S^2 = \frac{n}{n-1}\hat{\sigma}^2, \quad \mathbb{E}S^2 = \frac{n}{n-1}\cdot\frac{n-1}{n}\sigma^2 = \sigma^2$$

$$S^2 = \frac{1}{n-1}\cdot\sum_{j=1}^{n}\left(X_j - \overline{X}_n^{(1)}\right)^2$$

б) СЪСТОЯТЕЛНОСТ НА ТОЧКОВА ОЦЕНКА

<u>Деф:</u> 'СЪСТОЯТЕЛНОСТ'

Ако $\hat{\theta}(\vec{X}) \xrightarrow[n\to\infty]{\mathbb{P}} \theta$, то $\hat{\theta}$ е СЪСТОЯТЕЛНА ОЦЕНКА.

⊕ $\theta \in \mathbb{R}$ $\quad \mathbb{P}\left(|\hat{\theta}(\vec{X}) - \theta| > \varepsilon\right) \xrightarrow[n\to\infty]{} 0$

<u>Деф:</u> 'СИЛНА СЪСТОЯТЕЛНОСТ'

$\hat{\theta}(\vec{X}) \xrightarrow[n\to\infty]{\text{п.с}} \theta$, то $\hat{\theta}$ Е СИЛНО СЪСТОЯТЕЛНА.

⊕ $X \sim N(\mu, \sigma^2)$ $\quad \hat{\mu} = \overline{X}_n^{(1)} \xrightarrow[n\to\infty]{\text{п.с}} \mu$ (УЗГЧ)

$\hat{\mu}$ ОСВЕН НЕИЗМЕСТЕНА Е И СИЛНО СЪСТОЯТЕЛНА.

По общо $\hat{\theta}_k = \dfrac{\sum_{j=1}^{n} X_j^k}{n} \xrightarrow[n\to\infty]{\text{п.с}} \mathbb{E}X_1^k$

⊕ $X \sim U(0,\theta)$ $\quad \hat{\theta}_1 = \max\limits_{j\le n}(X_j)$, $\hat{\theta}_2 = 2\overline{X}_n^{(1)}$

$$\mathbb{E}\hat{\theta}_2 = 2\mathbb{E}\overline{X}_n^{(1)} = \frac{2}{n}\cdot\sum_{j=1}^{n}\mathbb{E}X_j = \frac{2}{n}\cdot\frac{n}{2}\cdot\theta = \theta$$

$\hat{\theta}_2 = 2\cdot\overline{X}_n^{(1)} \xrightarrow[n\to\infty]{\text{п.с}} 2\cdot\dfrac{\theta}{2} = \theta \implies$ СИЛНО СЪСТОЯТЕЛНА.

Нека $X^* = \max\limits_{j\le n}(X_j)$

$$F_{X^*}(x) = \mathbb{P}(X^* < x) = \mathbb{P}\left(\bigcap_{j=1}^{n}\{X_j < x\}\right) = \prod_{j=1}^{n}\mathbb{P}(X_j < x) = \mathbb{P}(X_1 < x)^n = \left(\frac{x}{\theta}\right)^n$$

$x \in (0,\theta)$

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$$f_{X^*}(x) = n \cdot \frac{x^{n-1}}{\theta^n} \quad , x \in (0, \theta)$$

$$\mathbb{E}X^* = \frac{n}{\theta^n} \int_0^\theta x \cdot x^{n-1} dx = \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} = \frac{n}{n+1}\theta = \mathbb{E}\hat{\theta}_1$$

$$\hat{\theta}_1 = X^* \xrightarrow[n\to\infty]{\text{п.с}} \theta$$

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