Лекция 8

Непрекъснати случайни величини

<u>Деф:</u> Нека $(\Omega, \mathcal{A}, \mathbb{P})$ е вероятностно пространство. Тогава $X : \Omega \to \mathbb{R}$ е непрекъсната сл. величина ако приема <u>неизброимо</u> много стойности.

Примери:

а) X е с възм. ст-ти $\frac{k}{n}$ $0 \le k \le n$ $\mathbb{P}(X = \frac{k}{n}) = p_k$;

б) T е температура $T \in (1^\circ, 5^\circ)$

<u>Деф:</u> X е абсолютно непрекъсната (непрекъсната) ако $\exists f_X : \mathbb{R} \to [0, \infty)$, такава че:

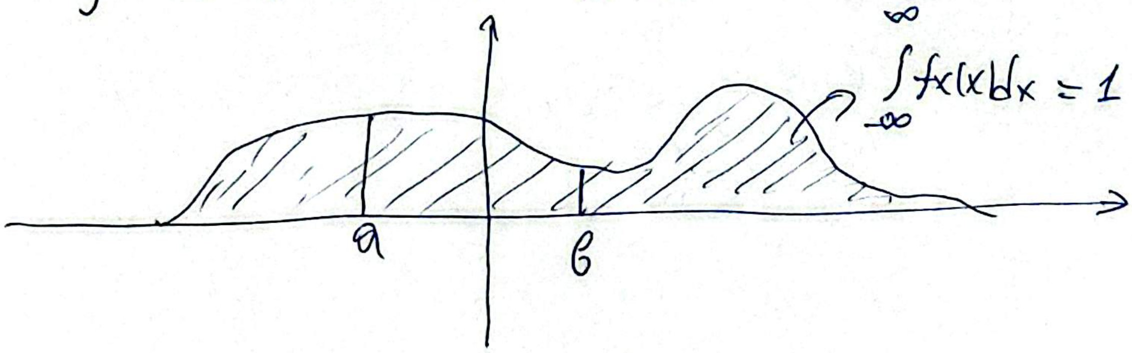
а) $f_X(x) \ge 0$, $\forall x \in \mathbb{R}$

б) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

в) $\forall a < b \Rightarrow \mathbb{P}(X \in (a, b)) = \int_a^b f_X(x)\,dx$

a и b може да са $\pm\infty$

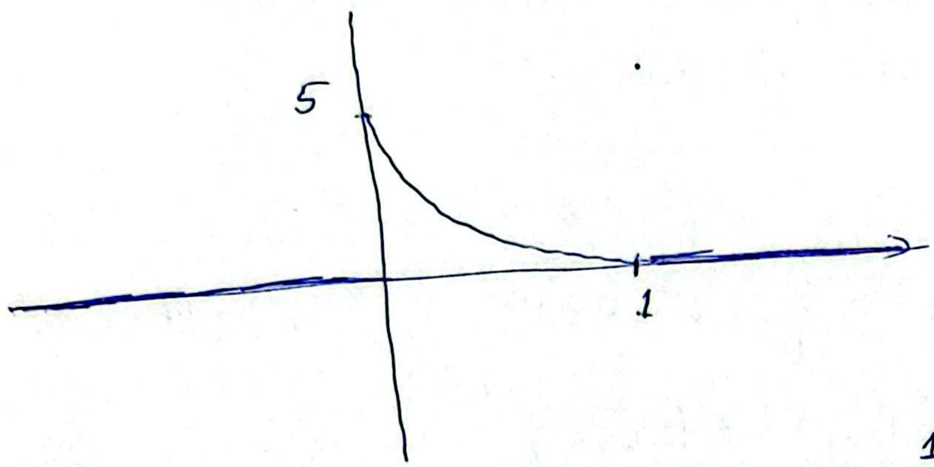
f_X се нарича плътност на X



Пример:

X са разходите на застрахователна фирма

$X = 100000Y$, където Y е н.с.в и $f_Y(y) = \begin{cases} 5(1-y)^4, & x \in (0,1) \\ 0, & \text{иначе} \end{cases}$



$$\mathbb{P}(X \geq 10000) = \mathbb{P}\left(Y \geq \frac{1}{10}\right) = 5 \cdot \int_{\frac{1}{10}}^{1} (1-y)^4 dy = \left(\frac{9}{10}\right)^5 \approx 0{,}59$$

Твърдение: Нека X е непр. сл. в. и $c \in \mathbb{R}$. Тогава

$\mathbb{P}(X=c) = 0$ и следователно за $a < b$

$$\mathbb{P}(X \in (a,b)) = \mathbb{P}(a < X < b) = \mathbb{P}(a < X \leq b) = \mathbb{P}(a \leq X \leq b) =$$
$$= \mathbb{P}(a \leq X < b).$$

Д-во:
$$\{X = c\} \subseteq \bigcap_{n \geq 1} \left\{X \in \left(c - \frac{1}{n}; c + \frac{1}{n}\right)\right\}$$

$$\mathbb{P}(X=c) \leq \mathbb{P}\left(X \in \left(c-\frac{1}{n}, c+\frac{1}{n}\right)\right) = \int_{c-\frac{1}{n}}^{c+\frac{1}{n}} f_X(x)dx$$

$$\mathbb{P}(X=c) \leq \lim_{n \to \infty} \int_{c-\frac{1}{n}}^{c+\frac{1}{n}} f_X(x)dx = 0$$

Деф:) 'ф-я на разпределение'

X е сл. вел. $F_X(x) = \mathbb{P}(X<x)$

Ако X е непрекъсната, то $F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$, $\forall x \in \mathbb{R}$

Ако f_X е непрекъсната в x_0 $\left.\frac{dF_X}{dx}\right|_{x=x_0} = f_X(x_0)$

$F_X' = f_X$

Смяна на променливите

X е н.с.в. $Y = g(X)$ Кога Y е н.с.в?

$$\begin{array}{ccc} X & & Y \\ \downarrow & & \downarrow \\ f_X & & f_Y \end{array}$$

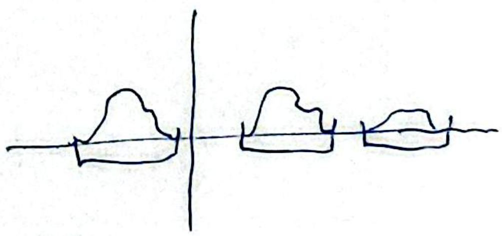
Ако $g(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ $Y = g(X) \in \{0,1\}$ не е н.с.в

Теорема:) Нека X е н.с.в. и $g: \mathbb{R} \to \mathbb{R}$, която е строго монотонно растяща или намаляваща. Тогава $Y = g(X)$, то

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left|(g^{-1}(y))'\right| , y \in \mathbb{R}$$

и Y е н.с.в. с плътност f_Y.

Д-во: Нека $Df_X = \{x \in \mathbb{R} : f_X(x) > 0\}$



Достатъчно е g да е монотонна само в Df_X

$$(1)\ \mathbb{P}(Y \in (a,b)) = \mathbb{P}(g(X) \in (a,b)) \overset{g\uparrow}{=} \mathbb{P}(X \in (g^{-1}(a), g^{-1}(b)))$$

$$\overset{g\downarrow}{=} \mathbb{P}(X \in (g^{-1}(b), g^{-1}(a)))$$

НЕКА g Е МОН. РАСТЯЩА

$$= \int_{g^{-1}(a)}^{g^{-1}(b)} f_X(w)\,dw \overset{w = g^{-1}(v)}{=\!=\!=} \int_a^b f_X(g^{-1}(v)) \cdot (g^{-1}(v))'\,dv$$

КОГАТО g Е МОН. НАМАЛЯВАЩА ЗНАКЪТ Е МИНУС, ТОГАВ:

$$(1) = \int_a^b f_X(g^{-1}(v)) \cdot |(g^{-1}(v))'| \cdot dv$$

$$\Rightarrow f_X(g^{-1}(v)) \cdot |(g^{-1}(v))'| \text{ е } f_Y(v) \quad \blacksquare$$

МАТЕМАТИЧЕСКО ОЧАКВАНЕ И ДИСПЕРСИЯ.

X Е Н.С.В. И $\int_{-\infty}^{\infty} |x| \cdot f_X(x)\,dx < \infty$, ТО $\mathbb{E}X = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$

$(\mathbb{E}X = \sum_i x_i p_i)$

$$\mathbb{D}X = \mathbb{E}(X - \mathbb{E}X)^2 = \int_{-\infty}^{\infty} (x - \mathbb{E}X)^2 f_X(x)\,dx$$

$$\left(\mathbb{D}X = \sum_i (x_i - \mathbb{E}X)^2 p_i\right)$$

ЗА $g(X)$, ТО $\mathbb{E}g(X) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\,dx$ (ДОКАЗВА СЕ С ТЕОРЕМАТА)

Свойства:

$$\mathbb{E}c = c$$

$$\mathbb{E}(aX+b) = a\mathbb{E}X + b$$

$$\mathbb{E}(aX+bY) = a\mathbb{E}X + a\mathbb{E}Y$$

$$X \perp\!\!\!\perp Y \quad \mathbb{E}XY = \mathbb{E}X . \mathbb{E}Y$$

$$Dc = 0$$

$$DaX = a^2 DX$$

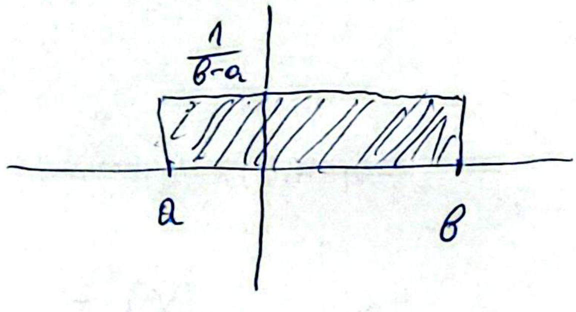
$$X \perp\!\!\!\perp Y \quad D(X+Y) = DX + DY$$

$$D(X+c) = DX$$

Видове непрекъснати случайни величини

A.) Равномерно разпределение, $X \sim U(a,b)$, ако $a<b$ и

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in (a,b) \\ 0 & \text{иначе} \end{cases}$$



$$\int_a^b \frac{1}{b-a}\,dx = 1$$

$$F_X(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & x \in (a,b] \\ 1, & x > b \end{cases}$$

$$\mathbb{E}X = \int_a^b x \cdot \frac{1}{b-a} = \frac{x^2}{2} \cdot \frac{1}{b-a}\Big|_a^b = \frac{a+b}{2}$$

$$\mathbb{E}X^2 = \int_a^b x^2 \cdot \frac{1}{b-a} = \frac{x^3}{3} \cdot \frac{1}{b-a}\Big|_a^b = \frac{a^2+ab+b^2}{3}$$

$$DX = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4} = \frac{(a-b)^2}{12}$$

Трансформация: $Y \sim U(0,1)$, $X \sim U(a,b)$

$$Y = \frac{X-a}{b-a} = g(X), \quad g(x) = \frac{x-a}{b-a} \quad g\uparrow$$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot |(g^{-1}(y))'|$$

$$g^{-1}(y) = x \cdot (b-a) + a, \quad (g^{-1}(y))' = b-a$$

$$\Rightarrow f_Y(y) = f_X(a+(b-a)y) \cdot (b-a) = \begin{cases} \frac{b-a}{b-a} = 1, & y \in (0,1) \\ 0 & \text{иначе} \end{cases}$$

$$\Rightarrow X = (b-a)Y + a \sim U(0,1)$$

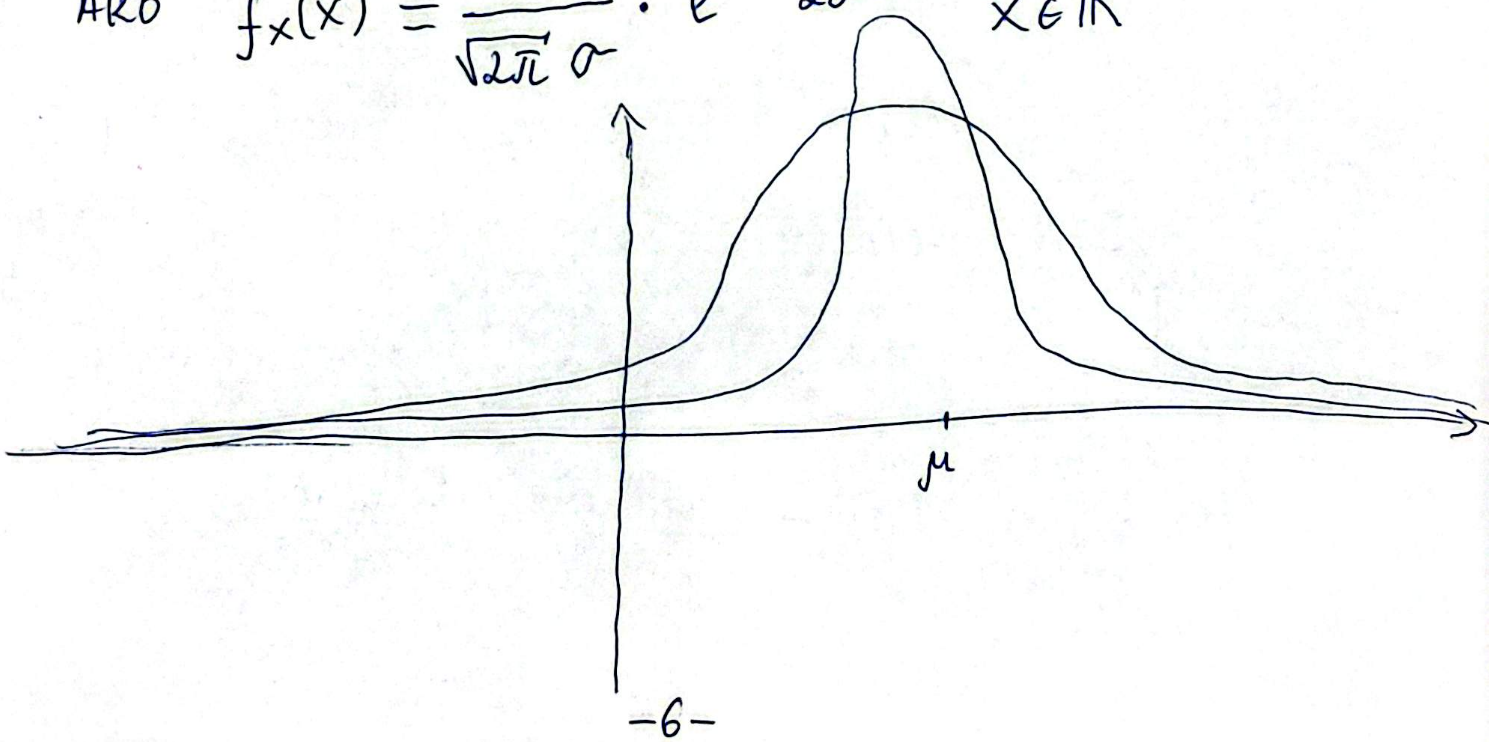
$$\mathbb{E}X = a + (b-a)\mathbb{E}Y = a + (b-a) \cdot \int_0^1 x\,dx = a + \frac{b-a}{2} = \frac{a+b}{2}$$

$$DX = D(a+(b-a)Y) = D(b-a)Y = (b-a)^2 \cdot DY$$

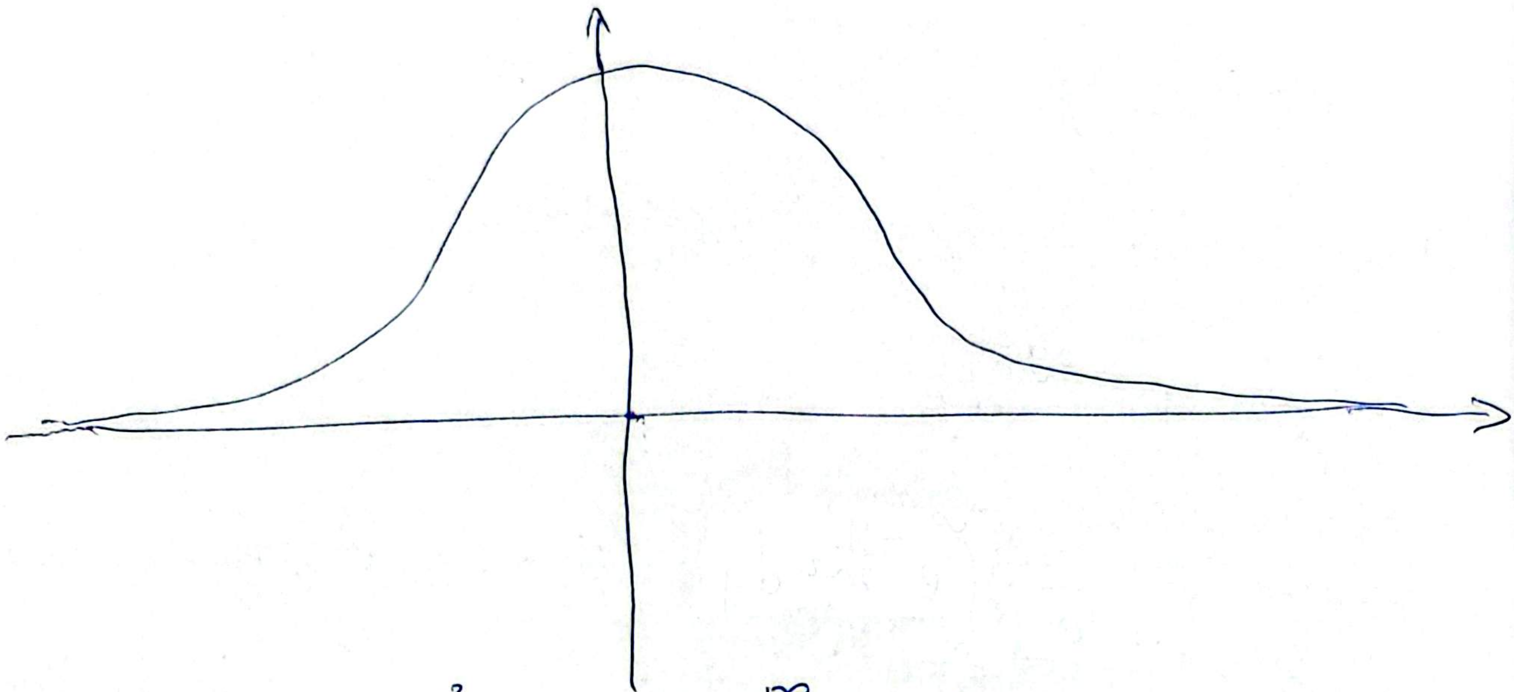
$$DY = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = \int_0^1 y^2 \cdot 1\,dy - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

5. Нормално разпределение (Гаусово) $X \sim N(\mu, \sigma^2)$ $\mu \in \mathbb{R}, \sigma^2 > 0$

Ако $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x \in \mathbb{R}$



$Z \sim N(0,1)$ – СТАНДАРТНО НОРМАЛНО РАЗПРЕДЕЛЕНИЕ



$$f_Z(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \qquad \int_{-\infty}^{\infty} f_Z(x)\,dx = 1$$

$$F_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \int_{-\infty}^{x} e^{-\frac{(y-\mu)^2}{2\sigma^2}}\,dy$$

$$F_Z = \Phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{\frac{-y^2}{2}}\,dy$$

Трансформация: $X \sim N(\mu, \sigma^2) \qquad Y = \frac{X-\mu}{\sigma} \overset{?}{\sim} N(0,1)$

$$Y = g(X), \quad g(x) = \frac{x-\mu}{\sigma}, \quad g'(x) = \frac{1}{\sigma} \quad g\uparrow$$

$$g^{-1}(y) = \sigma y + \mu \qquad (g^{-1}(y))' = \sigma$$

$$f_Y = f_X(\sigma y + \mu) \cdot \sigma = \frac{1}{\sqrt{2\pi}\cancel{\sigma}} \cdot e^{-\frac{(\cancel{\sigma} y + \cancel{\mu} - \cancel{\mu})^2}{2\cancel{\sigma^2}}} \cdot \cancel{\sigma} =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-y^2}{2}} = \phi(y) \qquad X = \sigma Z + \mu$$

$\mathbb{E}X = \sigma\mathbb{E}Z + \mu$, $\mathbb{E}Z = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2}} dy = 0$ (НЕЧЕТНА ф-Я)

$\mathbb{E}X = \sigma \cdot 0 + \mu = \mu$

$DX = D(\sigma Z + \mu) = D\sigma Z = \sigma^2 DZ = \sigma^2 \mathbb{E}Z^2$

$\mathbb{E}Z^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2}} dy$

СМЯТАМЕ $\int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2}} dy$ ПО МЕТОДА НА ФАЙНМАН :

ЗНАЕМ, ЧЕ $1 = \frac{1}{\sqrt{2\pi}\sigma} \cdot \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy$, ЗАЩОТО Е ПЛЪТНОСТ.

$\Rightarrow \sqrt{2\pi}\sigma = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy$ ЗА ВСЯКО σ

ДИФЕРЕНЦИРАМЕ ПО σ: $\sqrt{2\pi} = \int_{-\infty}^{\infty} +\frac{y^2}{\cancel{2}} \cdot \frac{+\cancel{2}}{\sigma^3} \cdot e^{-\frac{y^2}{2\sigma^2}} dy$

$\Rightarrow \sqrt{2\pi}\sigma^3 = \int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2\sigma^2}} dy$

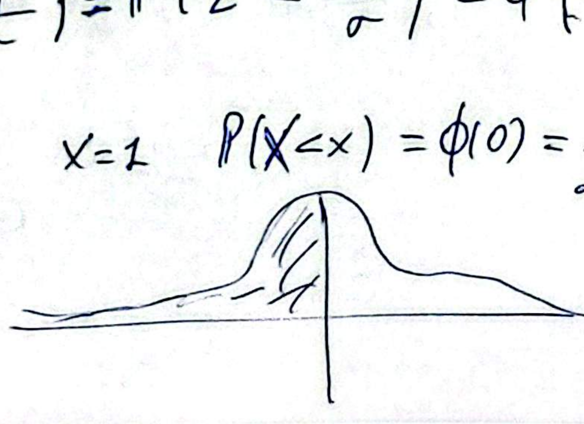
При $\sigma = 1$ ИМАМЕ $\sqrt{2\pi} = \int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2}} dy$

$\Rightarrow \mathbb{E}Z^2 = 1$ и $DZ = \sigma^2$

$F_X(x) = \mathbb{P}(X < x) = \mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) = \mathbb{P}\left(Z < \frac{x-\mu}{\sigma}\right) = \phi\left(\frac{x-\mu}{\sigma}\right)$

$\mu = \sigma = 1$ $\mathbb{P}(X < x) = \phi(x-1)$ $x = 1$ $\mathbb{P}(X < x) = \phi(0) = \frac{1}{2}$

ГЛЕДАТ СЕ В ТАБЛИЦИТЕ

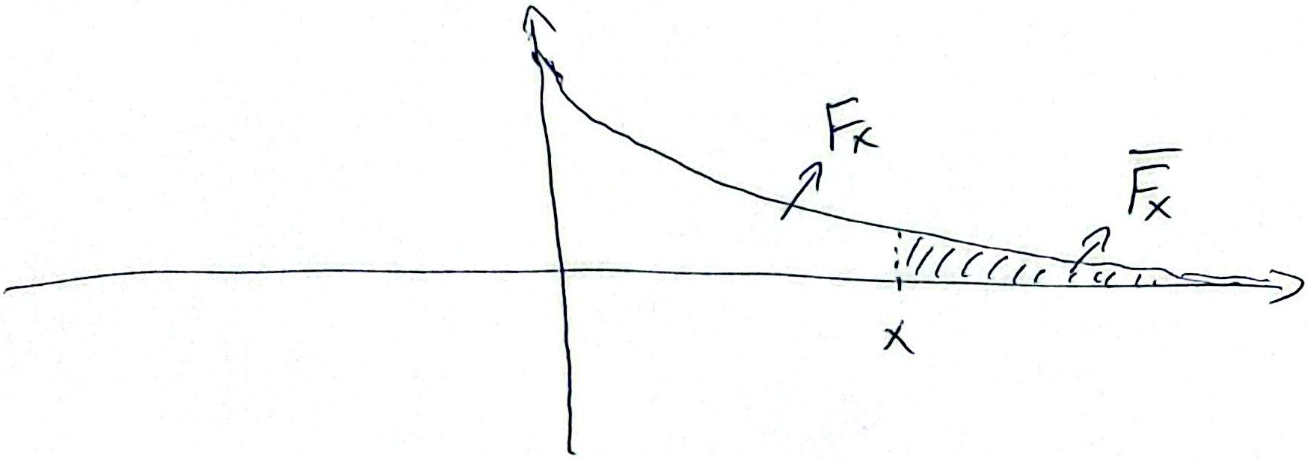


В.] Експоненциално разпределение. $X \sim Exp(\lambda)$, $\lambda > 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0 \\ 0 & \text{иначе} \end{cases}$$

$$F_X(x) = \int_0^x \lambda e^{-\lambda y} dy = 1 - e^{-\lambda x}$$

$$\overline{F}_X(x) = \mathbb{P}(X > x) = 1 - F_X(x) = e^{-\lambda x}$$ – опашка



Трансформация: $X \sim Exp(\lambda)$ $Y \sim Exp(1)$, $\lambda > 0$

$$Y = \lambda X, \quad \mathbb{P}(Y < x) = \mathbb{P}(\lambda X < x) = \mathbb{P}\left(X < \frac{x}{\lambda}\right) = 1 - e^{-\lambda \frac{x}{\lambda}} =$$

$$= 1 - e^{-x} = \mathbb{P}(Z < x) \quad Z \sim Exp(1)$$

$$\Rightarrow F_Y = F_Z \Rightarrow Y \sim Exp(1)$$

$$\mathbb{E}Y = \lambda \mathbb{E}X \xRightarrow{\mathbb{E}Y = 1} \mathbb{E}X = \frac{1}{\lambda}$$

$$DY = \lambda^2 DX \xRightarrow{DY = 1} DX = \frac{1}{\lambda^2}$$

Безпаметност: $$\mathbb{P}(X > x+y \mid X > y) = \frac{\mathbb{P}(X > x+y; X > y)}{\mathbb{P}(X > y)} = \frac{\mathbb{P}(X > x+y)}{\mathbb{P}(X > y)} =$$

$$= \frac{e^{-\lambda(x+y)}}{e^{-\lambda y}} = e^{-\lambda x} = \mathbb{P}(X > x)$$