Лекция 7

Е. Хипергеометрично разпределение $\quad X \sim HG(N, M, n)$

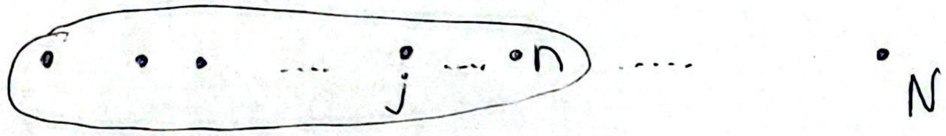
Имаме N елемента, от които M са маркирани. Правим извадка с размер n от тях. X изброява броя маркирани в тази извадка.

Твърдение:

а) $$\mathbb{P}(X=k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$$

б) $$\mathbb{E}X = n \cdot \frac{M}{N}, \quad \mathbb{D}X = \frac{n.M}{N} \cdot \frac{N-M}{N} \cdot \frac{N-n}{N-1}$$

Д-во: б)



$$X_j = \begin{cases} 1 & \text{ако на } j\text{-та позиция има маркиран} \\ 0 & \text{ако на } j\text{-та позиция не е маркиран} \end{cases}$$

$$X = \sum_{j=1}^{n} X_j \Rightarrow \mathbb{E}X = \sum_{j=1}^{n} \mathbb{E}X_j$$

$$X_j \sim Ber(p_j) \quad \Rightarrow \mathbb{E}X = \sum_{j=1}^{n} p_j$$

$$p_j = \frac{\binom{N-1}{M-1}}{\binom{N}{M}} = \frac{M}{N} = \mathbb{P}(X_j = 1)$$

$$\Rightarrow \mathbb{E}X = \sum_{j=1}^{n} p_j = n \cdot \frac{M}{N}$$

Съвместни дискретни разпределения

Деф: Нека X и Y са две дискретни случайни величини в едно вероятностно пространство. Тогава под съвместно разпределение на X и Y разбираме таблицата:

$Y \backslash X$	x_1	x_2	$\dots$	x_n	$Y \downarrow$
y_1	p_{11}	p_{21}	$\dots$	p_{n1}	$\sum_i p_{i1}$
y_2	p_{12}	p_{22}	$\dots$	p_{n2}	$\sum_i p_{i2}$
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\dots$	$\sum_i \dots$
y_k	p_{1k}	p_{2k}	$\dots$	p_{nk}	$\sum_i p_{ik}$
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\vdots$
y_n	p_{1n}	p_{2n}	$\dots$	p_{nn}	$\sum_i p_{in}$
$X \rightarrow$	$\sum_j p_{1j}$	$\sum_j p_{2j}$	$\dots$	$\sum_j p_{nj}$	← маргинално разпределение

$$p_{ij} = \mathbb{P}(X = x_i \cap Y = y_j) = \mathbb{P}(X = x_i ; Y = y_j)$$

$$\sum_{i,j} p_{ij} = 1$$

Пример:

X – брой 6-ци при хвърляне на 2 зара

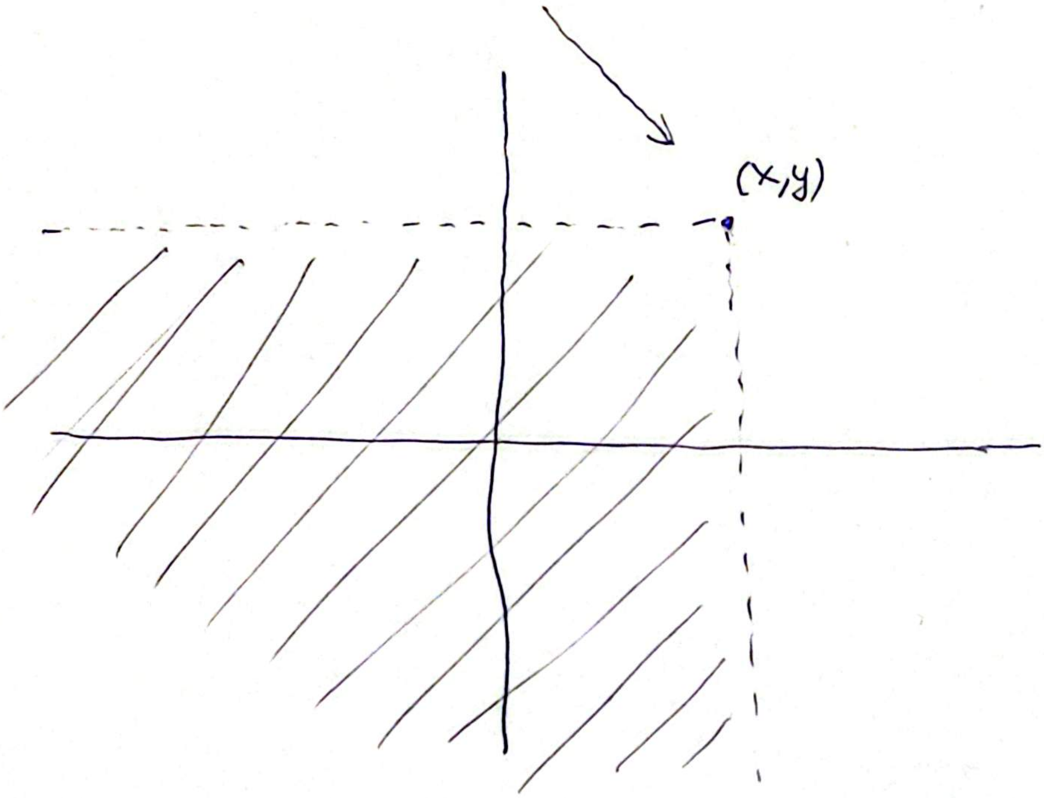
Y – брой 1-ци при хвърляне на 2 зара

$Y \backslash X$	0	1	2	Y
0	$\frac{16}{36}$	$\frac{8}{36}$	$\frac{1}{36}$	$\frac{25}{36}$
1	$\frac{8}{36}$	$\frac{2}{36}$	0	$\frac{10}{36}$
2	$1/36$	0	0	$\frac{1}{36}$
X	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$	

Съвместни функции на разпределение

<u>Деф:</u> Нека X и Y са две случайни величини. Тогава съвместна функция на разпределение на X и Y е

$$F_{X,Y}(x,y) = \mathbb{P}(X<x\,;Y<y) \quad \forall (x,y)\in\mathbb{R}^2$$



$$F_{X,Y}(x,\infty) = \mathbb{P}(X<x; Y<\infty) = \mathbb{P}(X<x) = F_X(x)$$

$$F_{X,Y}(\infty,y) = F_Y(y)$$

$$F_{X,Y}(\infty,\infty) = 1$$

$$F_{X,Y}(-\infty,y) = 0 = F_{X,Y}(x,-\infty)$$

Деф: 'независимост'

Две случайни величини X, Y във V са независими ако

$$F_{X,Y}(x,y) = \mathbb{P}(X<x).\mathbb{P}(Y<y) = F_X(x).F_Y(y)$$

$$\downarrow$$

$$\mathbb{P}(X<x; Y<y) \qquad \forall (x,y) \in \mathbb{R}^2$$

Ковариация и корелация

Деф: 'ковариация'

Нека X и Y са две случайни величини. Тогава

$Cov(X,Y) = \mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)]$ се

нарича ковариация

Пример: за 2 дискретни случайни величини

$$Cov(X,Y) = \sum_{i,j} p_{ij}(x_i-\mathbb{E}X)(y_j-\mathbb{E}Y)$$

Твърдение: $Cov(X,Y) = \mathbb{E}XY - \mathbb{E}X.\mathbb{E}Y$

Д-во: $\mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)] = \mathbb{E}[XY - X\mathbb{E}Y - Y\mathbb{E}X + \mathbb{E}X.\mathbb{E}Y] =$

$= \mathbb{E}XY - \mathbb{E}[X.\mathbb{E}Y] - \mathbb{E}[Y\mathbb{E}X] + \mathbb{E}[\mathbb{E}X.\mathbb{E}Y] =$

$= \mathbb{E}XY - \mathbb{E}Y.\mathbb{E}X - \cancel{\mathbb{E}X.\mathbb{E}Y} + \cancel{\mathbb{E}X.\mathbb{E}Y}$ ☑

Следствие: $Cov(aX, bY) = \mathbb{E}aXbY - \mathbb{E}aX.\mathbb{E}bY = ab(\mathbb{E}XY - \mathbb{E}X.\mathbb{E}Y)$

$= ab.Cov(X,Y)$

— —

Твърдение: Ако $X \perp\!\!\!\perp Y$, то $Cov(X,Y)=0$

Д-во: $Cov(X,Y) = \mathbb{E}XY - \mathbb{E}X.\mathbb{E}Y \overset{X \perp\!\!\!\perp Y}{=} \mathbb{E}X\mathbb{E}Y - \mathbb{E}X.\mathbb{E}Y = 0$

Деф: 'корелация'

Нека X и Y са сл. вел. с $DX<\infty$ и $DY<\infty$.

Тогава $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{DX}.\sqrt{DY}}$ се нарича корелация.

Твърдение: Нека X и Y са сл. вел. такива, че $DX<\infty$ и $DY<\infty$.

Нека $\tilde{X} = \dfrac{X-\mathbb{E}X}{\sqrt{DX}}$ и $\tilde{Y} = \dfrac{Y-\mathbb{E}Y}{\sqrt{DY}}$. Тогава $\mathbb{E}\tilde{X} = \mathbb{E}\tilde{Y} = 0$ и $D\tilde{X} = D\tilde{Y} = 1$

и $\rho(X,Y) = \mathbb{E}\tilde{X}.\tilde{Y}$

Д-во: $\mathbb{E}\tilde{X} = \mathbb{E}\dfrac{X-\mathbb{E}X}{\sqrt{DX}} = \dfrac{1}{\sqrt{DX}}.\mathbb{E}(X-\mathbb{E}X) = 0$

$D\tilde{X} = D\dfrac{X-\mathbb{E}X}{\sqrt{DX}} = \dfrac{1}{DX}.D(X-\mathbb{E}X) = \dfrac{DX}{DX} = 1$, защото

$D(X-c) = DX$

$\rho(X,Y) = \dfrac{\mathbb{E}(X-\mathbb{E}X)(Y-\mathbb{E}Y)}{\sqrt{DX}\sqrt{DY}} = \mathbb{E}\left(\dfrac{X-\mathbb{E}X}{\sqrt{DX}} \cdot \dfrac{Y-\mathbb{E}Y}{\sqrt{DY}}\right) =$

$= \mathbb{E}\tilde{X}\tilde{Y}$

Теорема:) X и Y са сл. вел с $DX<\infty$ и $DY<\infty$. Тогава:

а) $|\rho(X,Y)| \le 1$

б) $|\rho(X,Y)| = 1 \iff Y = aX+b$ за някои $a,b \in \mathbb{R}$

Д-во:

а) $0 \le \mathbb{E}(\bar{X} \pm \bar{Y})^2 = \mathbb{E}(\bar{X}^2 \pm 2\bar{X}\bar{Y} + \bar{Y}^2) =$

$= \mathbb{E}\bar{X}^2 \pm 2\mathbb{E}\bar{X}\bar{Y} + \mathbb{E}\bar{Y}^2 \overset{\text{Тв.}}{=} D\bar{X} \pm 2\rho(X,Y) + D\bar{Y} =$

$= 2 \pm 2\rho(X,Y) \Rightarrow 1 \pm \rho(X,Y) \ge 0$ ■

б) "$\Leftarrow$" $Y = aX+b \Rightarrow \mathbb{E}Y = a\mathbb{E}X + b$

$DY = a^2 DX$

$$\rho(X,Y) = \frac{\mathbb{E}XY - \mathbb{E}X.\mathbb{E}Y}{\sqrt{DX}\sqrt{DY}} = \frac{\mathbb{E}X(aX+b) - \mathbb{E}X.\mathbb{E}(aX+b)}{\sqrt{DX}\,|a|.\sqrt{DX}} =$$

$$= \frac{a\mathbb{E}X^2 + \cancel{b\mathbb{E}X} - a(\mathbb{E}X)^2 - \cancel{b\mathbb{E}X}}{DX\,|a|} = \frac{a(\mathbb{E}X^2 - (\mathbb{E}X)^2)}{|a|\,DX} = \pm 1$$

"$\Rightarrow$" $|\rho(X,Y)| = 1$ 1сл.) Нека $\rho(X,Y) = 1$

$\mathbb{E}(\bar{X}-\bar{Y})^2 = \mathbb{E}\bar{X}^2 - 2\mathbb{E}\bar{X}\bar{Y} + \mathbb{E}\bar{Y}^2 = 1 - 2.1 + 1 = 0$

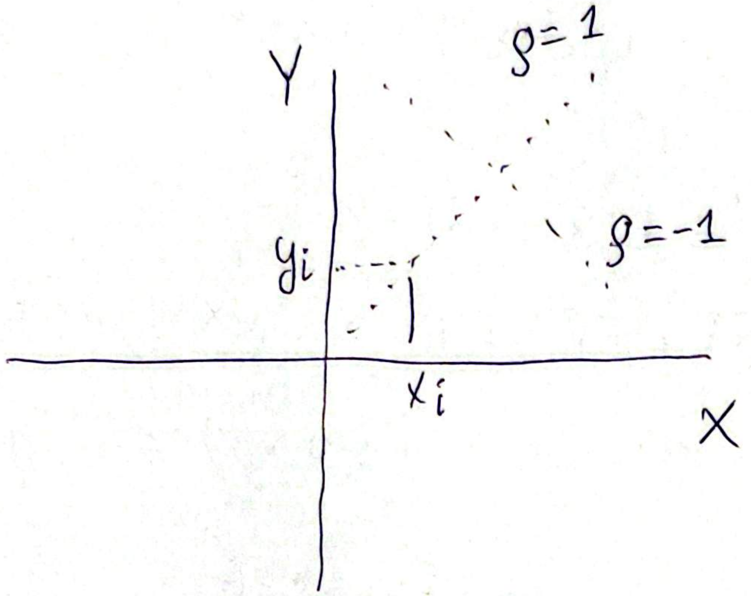
$\mathbb{E}(\bar{X}-\bar{Y})^2 = 0 = \sum_{i,j} p_{ij}(\bar{x}_i - \bar{y}_j)^2 \Rightarrow \bar{X} = \bar{Y}$

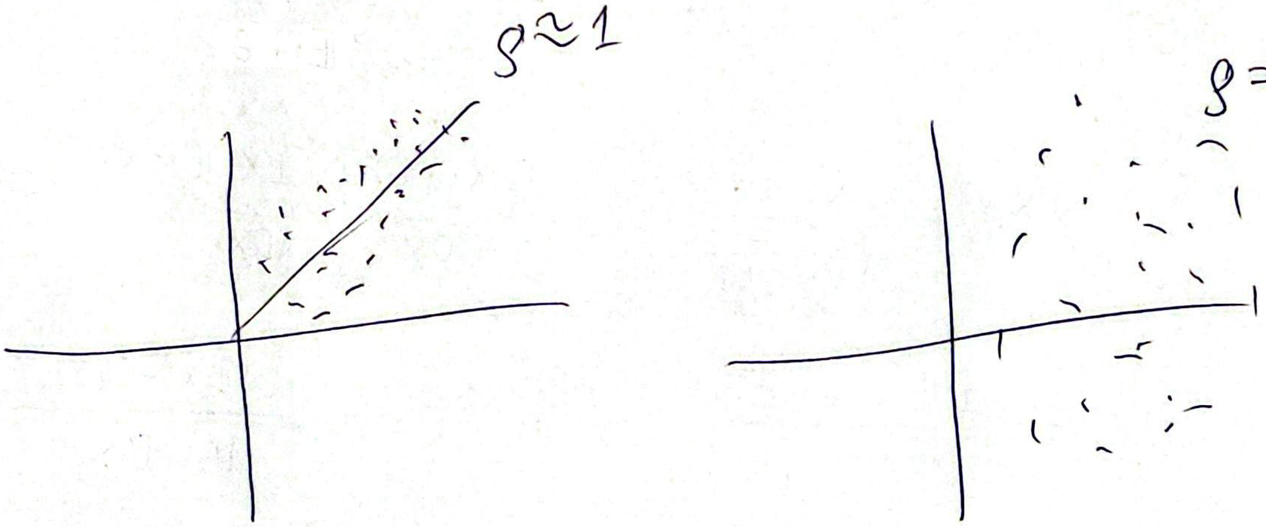
$$\bar{Y} = \frac{Y - \mathbb{E}Y}{\sqrt{DY}} = \frac{X - \mathbb{E}X}{\sqrt{DX}} \Rightarrow Y - \mathbb{E}Y = \frac{X}{\sqrt{DX}}.\sqrt{DY} - \frac{\mathbb{E}X\sqrt{DY}}{\sqrt{DX}}$$

$$\Rightarrow Y = X.\frac{\sqrt{DY}}{\sqrt{DX}} - \mathbb{E}X.\frac{\sqrt{DY}}{\sqrt{DX}} + \mathbb{E}Y$$ ■

2 сл.) Аналогично

$$Y = aX + b$$

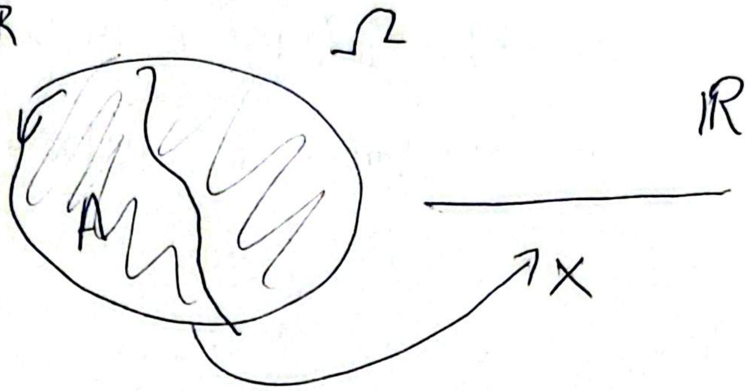




Условно математическо очакване

$$X, \mathbb{E}X; \quad \mathbb{E}(X-\mathbb{E}X)^2 = DX = \min_{a\in\mathbb{R}} \mathbb{E}(X-a)^2$$

Пример: $Y = 1_A = \begin{cases} 1 \\ 0 \end{cases}$



$$\mathbb{P}(A) = \mathbb{P}(Y=1) = p$$

$$\mathbb{P}(A^c) = 1-p = q$$

$$\min_{g} \mathbb{E}(X-g(Y))^2, \quad g(Y) = a\cdot 1_A + b\cdot 1_{A^c}$$

$$a = g(1) \quad b = g(0)$$

$$\min_{g} \mathbb{E}(X-g(Y))^2 = \min_{a,b\in\mathbb{R}} \underbrace{\mathbb{E}(X-a1_A-b1_{A^c})^2}_{f(a,b)} = \min_{a,b\in\mathbb{R}} f(a,b)$$

I

$$f(a,b) = \mathbb{E}\left(X^2 + a^2 1_A + b^2 1_{A^c} - 2a1_A X - 2bX1_{A^c}\right) = \qquad \boxed{\mathbb{P}(A) = \mathbb{E}1_A}$$

$$= \mathbb{E}X^2 + a^2\mathbb{P}(A) + b^2\mathbb{P}(A^c) - 2a\mathbb{E}X1_A - 2b\mathbb{E}X1_{A^c}$$

$$\left|\begin{array}{l} \dfrac{\partial f}{\partial a} = 0 = 2a\mathbb{P}(A) - 2\mathbb{E}X1_A \\ \dfrac{\partial f}{\partial b} = 0 = 2b\mathbb{P}(A^c) - 2\mathbb{E}X1_{A^c} \end{array}\right| \Rightarrow \begin{array}{l} a = \dfrac{\mathbb{E}X1_A}{\mathbb{P}(A)} = \dfrac{\mathbb{E}XY}{\mathbb{P}(Y=1)} \\ b = \dfrac{\mathbb{E}X1_{A^c}}{\mathbb{P}(A^c)} = \dfrac{\mathbb{E}X(1-Y)}{\mathbb{P}(Y=0)} \end{array}$$

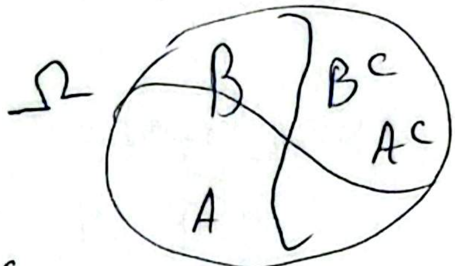
$$\Rightarrow g^*(Y) = \frac{\mathbb{E}X1_A}{\mathbb{P}(A)}\cdot 1_A + \frac{\mathbb{E}X1_{A^c}}{\mathbb{P}(A^c)}\cdot 1_{A^c} =: \mathbb{E}(X|Y)$$

Деф: 'УМО'

Нека X и Y са сл. вел. Тогава

$g^*(Y) = \mathbb{E}(X|Y)$ е тази случайна величина, която

минимизира $\min\limits_{g} \mathbb{E}(X - g(Y))^2 = \mathbb{E}(X - \mathbb{E}(X|Y))^2$



Пример: $X = 1_B$ $\quad Y = 1_A$

$$g^*(Y) = \frac{\mathbb{E}\, 1_A 1_B}{\mathbb{P}(A)} \cdot 1_A + \frac{\mathbb{E}\, 1_B \cdot 1_{A^c}}{\mathbb{P}(A^c)} \cdot 1_{A^c} =$$

$$= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \cdot 1_A + \frac{\mathbb{P}(A^c \cap B)}{\mathbb{P}(A^c)} \cdot 1_{A^c} = \mathbb{P}(B|A) 1_A + \mathbb{P}(B|A^c) \cdot 1_{A^c}$$

Пример:

Y	y_1	$\dots$	y_j	$\dots$
$\mathbb{P}$	p_1	$\dots$	p_j	$\dots$

$A_j = \{Y = y_j\}$

$$\mathbb{E}[X|Y] = \sum_j \frac{\mathbb{E} X 1_{A_j}}{\mathbb{P}(A_j)} \cdot 1_{A_j}$$

Ако X е дискретна, т.е: $X = \sum\limits_i x_i \cdot 1_{B_i}$

$$\mathbb{E} X 1_{A_j} = \mathbb{E} \sum_i x_i \cdot 1_{B_i} \cdot 1_{A_j} = \sum_i x_i \cdot \mathbb{E}\, 1_{B_i \cap A_j} = \sum_i x_i \cdot \mathbb{P}(A_j \cap B_i)$$

$$\Rightarrow \mathbb{E}[X|Y] = \sum_j \sum_i \frac{x_i\, \mathbb{P}(A_j \cap B_i)}{\mathbb{P}(A_j)} \cdot 1_{A_j} =$$

$$= \sum_j \sum_i x_i \cdot \mathbb{P}(B_i | A_j) \cdot 1_{A_j}$$

Деф: Нека X и Y са дискр. сл. вел. Тогава

$$\mathbb{E}[X|Y=y_j] = \sum_i x_i . \mathbb{P}(B_i|A_j) \qquad B_i = \{X=x_i\}$$
$$A_j = \{Y=y_j\}$$

$$\mathbb{E}[X|Y] = \sum_j \mathbb{E}[X|Y=y_j] . 1_{A_j}$$

Твърдение: Ако X и Y са дискретни сл. вел., то $\mathbb{E}[X|Y]$ е дискр. сл. вел. със следната таблица.

$\mathbb{E}[X\|Y]$	$\mathbb{E}[X\|Y=y_1]$	...	$\mathbb{E}[X\|Y=y_j]$	...
	$p_1 = \mathbb{P}(A_1)$		$p_j = \mathbb{P}(A_j)$	

Свойства на УМО:

а) $\mathbb{E}[aX+bZ|Y] = a\mathbb{E}[X|Y] + b\,\mathbb{E}[Z|Y]$

д-во: $$\mathbb{E}[aX+bZ|Y] = \sum_j \frac{\mathbb{E}(aX+bZ).1_{A_j}}{\mathbb{P}(A_j)} . 1_{A_j} =$$

$$= \sum_j \frac{a\,\mathbb{E}X1_{A_j} + b\,\mathbb{E}Z1_{A_j}}{\mathbb{P}(A_j)} . 1_{A_j} = a\sum_j \frac{\mathbb{E}X1_{A_j}}{\mathbb{P}(A_j)} 1_{A_j} + b\sum_j \frac{\mathbb{E}Z1_{A_j}}{\mathbb{P}(A_j)} . 1_{A_j}$$

$$= a\,\mathbb{E}[X|Y] + b\,\mathbb{E}[Z|Y]$$

$\mathbb{E}f(X).g(Y) = \mathbb{E}f(X).\mathbb{E}\,g(Y)$

б) $X \perp\!\!\!\perp Y \Rightarrow \mathbb{E}[X|Y] = \mathbb{E}X$

д-во: $$\mathbb{E}[X|Y] = \sum_j \frac{\mathbb{E}X.1_{\{Y=y_j\}}}{\mathbb{P}(Y=y_j)} . 1_{\{Y=y_j\}} \overset{X\perp\!\!\!\perp Y}{=} \sum_j \frac{\mathbb{E}X.\mathbb{E}\,1_{\{Y=y_j\}}}{\mathbb{P}(Y=y_j)} . 1_{\{Y=y_j\}} =$$

$$= \sum_j \frac{\mathbb{E}X.\cancel{\mathbb{P}(Y=y_j)}}{\cancel{\mathbb{P}(Y=y_j)}} . 1_{A_j} = \mathbb{E}X . \sum_j 1_{A_j} = \mathbb{E}X$$

в) $X = f(Y)$, то $\mathbb{E}[X|Y] = X = f(Y)$

$$\mathbb{E}[X|Y] = \sum_j \frac{\mathbb{E}\, f(Y) \cdot 1_{A_j}}{\mathbb{P}(A_j)} \cdot 1_{A_j} = \sum_j f(y_j) \cdot \frac{\cancel{\mathbb{E}\, 1_{A_j}}}{\cancel{\mathbb{P}(A_j)}} \cdot 1_{A_j} = f(Y)$$

г) $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}X$

$$\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E} \sum_j \frac{\mathbb{E} X 1_{A_j}}{\mathbb{P}(A_j)} \cdot 1_{A_j} =$$

$$= \sum_j \frac{\mathbb{E} X 1_{A_j}}{\cancel{\mathbb{P}(A_j)}} \cdot \cancel{\mathbb{E} 1_{A_j}} = \sum_j \mathbb{E}(X \cdot 1_{A_j}) = \mathbb{E} \sum_j X 1_{A_j} =$$

$$= \mathbb{E} X \sum_j 1_{A_j} = \mathbb{E}X$$

Деф: 'Условно разпределение'

Нека X и Y са дискр. сл. вел. Тогава усл. разпр. на X при $Y = y_j$ (за всяка възможна ст-т на y_j)

$X \mid Y=y_j$	x_1	x_2	$\dots$	x_i
	$\mathbb{P}(X=x_1 \mid Y=y_j)$			$\mathbb{P}(X=x_i \mid Y=y_j)$