0. 误差 = 精确值 − 近似值

向量范数 $X=(x_1, x_2, \dots x_n)^T$ L_p范数 $\|X\|_p \triangleq (\sum_i |x_i|^p)^{\frac{1}{p}}$

$\|X\|_1 = \sum_i |x_i|$ $\|X\|_\infty = \max_i \{|x_i|\}$ $\|X\|_2 = \sqrt{\sum_i x_i^2} = \sqrt{X^H X}$

矩阵范数 $\|A\|^{n\times m} \triangleq \sup_{\|X\|=1} \|AX\|$ $\|A\|_E \triangleq (\sum_{i,j} |a_{ij}|^2)^{\frac{1}{2}}$

$\|A\|_1 = \max_j \sum_i |a_{ij}|$ 列和最大 $\|A\|_\infty = \max_i \sum_j |a_{ij}|$ 行和最大

$\|A\| = \sqrt{\rho(A^T A)}$ 相容性 $\|AX\| \le \|A\|\|X\|$

对于相容的范数 $\rho(A) \le \|A\|$ $\rho(A) = \max_i \{|\lambda_i|\}$ 最大谱半径

1. 插值 Φ上有函数 $\varphi(x)$ $\varphi(x_i) = f(x_i)$

多项式插值 $L(x) = \sum l_i(x) f(x_i)$ $l_i(x) = \prod_j \frac{x-x_j}{x_i-x_j}$

$N(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \cdots f[x_0, \dots x_n](x-x_0)\cdots(x-x_{n-1})$

$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n) = f[x_0, x_1, \dots, x_n, x](x-x_0)\cdots(x-x_n)$

差商 $f[x_0, x_1, \dots x_k] = \frac{f[x_1, \dots x_k] - f[x_0, \dots x_{k-1}]}{x_k - x_0} = \sum_i \frac{f(x_i)}{\prod_j (x_i - x_j)} = \frac{f^{(k)}(\xi)}{k!}$

Hermite 插值 x_{n_i} $f(x_n)$ $f'(x_n)$ x_{m_i} $f(x_m)$

$f(x_n)$ $f'(x_n) \Rightarrow f(x_n)$ $f(x_n+\varepsilon)$ 用 $L(x)$ 或 $N(x)$ 外理

其公式 $n_i = \prod_j \frac{x-x_{n_j}}{x_{n_i}-x_{n_j}}$ $m_i = \prod_j \frac{x-x_{m_j}}{x_{m_i}-x_{m_j}}$ $l = n^2 m$ $g = (x-x_n) n^2 m$

$h = [1-(2n'|_{x_n} + m'|_{x_n})(x-x_n)] n^2 m$ $H(x) = \sum l f(x_m) + \sum h f(x_n) + \sum g f'(x_n)$

分段(多项式)插值. 三次样条函数

k次样条: $S(x) = a_0 + a_1 x + \cdots a_k x^k + b_1 \max(0, x-x_1)^k + \cdots b_{n-1} \max(0, x-x_{n-1})^k$

2.1 数值微分

向前 $f[x_0, x_0+h]$ $R = -\frac{h}{2} f''(\xi) = O(h)$ 向后 $f[x_0-h, x_0]$ $R = \frac{h}{2} f''(\xi) = O(h)$

中心差商 $f[x_0-h, x_0+h]$ $R = -\frac{h^2}{6} f'''(\xi) = O(h^2)$

插值法 $f'(x) = \sum l_i'(x) f(x_i)$ $R = \frac{d}{dx}\left[\frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x-x_i)\right]$

2.2 数值积分 $I(f)=\int_a^b f(x)dx \quad I_n(f)=\sum_0^n \alpha_i f(x_i) \quad E_n(f)\triangleq I(f)-I_n(f)$

代数精度，m阶：$I(x^k)=I_n(x^k)\quad k\le m \quad I(x^{m+1})\ne I(x^{m+1})$

插值型数值积分 ≥n阶精度 <2n阶 用 $f(x)=x^0\ x^1 \cdots x^n$ 来定 α_i 即可 $E_n=\int_a^b \frac{f^{(n+1)}(\xi)}{(n+1)!}\ f[x_0,x_1\cdots x_n,x]\ \pi(x-x_i)\,dx$

梯形积分 $(b-a)[\frac{1}{2}\quad \frac{1}{2}]^T$ 1阶 $E_1=-\frac{f''(\xi)}{12}(b-a)^3$

Simpson $(b-a)[\frac{1}{6}\quad \frac{4}{6}\quad \frac{1}{6}]^T$ 3阶 $E_2=-\frac{f^{(4)}(\xi)}{2880}(b-a)^5$

n奇 $E_n(f)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\int_a^b \pi(x-x_i)dx$ n阶

n偶时，由等距造成的结果 $E_n(f)=\frac{f^{(n+2)}(\xi)}{(n+2)!}\int_a^b (x-c)\pi(x-x_i)dx$

c任取 n+1阶精度

复化数值积分

复化梯形 $h=\frac{b-a}{n}\quad T_n(f)\quad h[\frac{1}{2}\ \frac{1}{2}]^T\cdot\begin{bmatrix}1&1&1&1&1&1&0\\0&1&1&1&1&1&1\end{bmatrix}^T_{(n+1)\times 2}=h\begin{bmatrix}\frac{1}{2}\\1\\ \vdots\\1\\ \frac{1}{2}\end{bmatrix}_{(n+1)\times 1}$

$E_n=-\frac{h^3}{12}\sum_1^n f''(\xi_i)=-\frac{h^3}{12}nf''(\xi)=-\frac{(b-a)^3}{12n^2}f''(\xi)$

复化Simpson $h=\frac{b-a}{n}$ ~~n=2m~~ 等分 $S_n\ \frac{2h}{6}[1\ 4\ 1]^T\cdot\begin{bmatrix}1&0&1&0&1&\cdots&1&0&0\\0&1&0&1&0&\cdots&0&1&0\\0&0&1&0&1&\cdots&1&0&1\end{bmatrix}^T_{(n+1)\times 3}=\frac{h}{3}\begin{bmatrix}1\\4\\2\\4\\ \vdots\\4\\1\end{bmatrix}_{(n+1)\times 1}$

$E_n=-\cancel{\ \ }-\frac{(2h)^5}{2880}mf^{(4)}(\xi)=-\frac{(b-a)^5}{180n^4}f^{(4)}(\xi)$

Romberg 积分（外推算法 Richardson）

$T_n=R_{1,1}\quad S_{2n}=R_{2,2}\quad T_{2n}=R_{2,1}\quad R_{k,j}$ 共 $n2^{k-1}$ 个分段

$R_{k,j+1}=\frac{4^jR_{k,j}-R_{k-1,j}}{4^j-1}\quad |R_{k,k}-R_{k+1,k+1}|<\varepsilon$

$R_{k,k}\quad O(h^{2k})\quad h_k=\frac{h}{2^{k-1}}$

重积分 $\begin{bmatrix}\frac{1}{2}\\ \vdots\\ \frac{1}{2}\end{bmatrix}\cdot[\frac{1}{2}\ 1\ 1\ \cdots\ 1\ \frac{1}{2}]$ 对应相乘 $\begin{bmatrix} f(x_i,y_j)\end{bmatrix}$

或 $\frac{1}{3}\begin{bmatrix}1\\4\\ \vdots\\1\end{bmatrix}\cdot\frac{1}{3}[1\ 4\ 2\ 4\ 2\ 4\ \cdots 4\ 1]$

Gauss积分 Legendre 多项式 $L_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\quad [-1,1]$

在 $(-1,1)$ 上 n个互异实根 若 $P(x)$ 小于n次 $\int_{-1}^1 L_nP=0$

$\int_{-1}^1 L_iL_j\,dx=0\quad i\ne j$

取 $\frac{d^n}{dx^n}[(x-a)^n(x-b)^n]$ 的根插值积分，代数精度 2n-1

3. 拟合 $\varphi(x)$ 逼近 y $\quad Q=(\varphi(x_1),\varphi(x_2)\cdots\varphi(x_m))^T\quad Y=(y_1,y_2\cdots y_m)^T\quad R=\|Q-Y\|_2^2$

线性拟合 $P(x)=a+bx$ 均方误差 $Q(a,b)=\sum(a+bx_i-y_i)^2=\left[\begin{pmatrix}x&1\\x&1\\ \vdots&\vdots\\x&1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}-\begin{pmatrix}y\\y\\ \vdots\\y\end{pmatrix}\right]^T[\cdots]$

$\begin{pmatrix}x&1\\x&1\\ \vdots&\vdots\\x&1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}\approx\begin{pmatrix}y\\y\\ \vdots\\y\end{pmatrix}\quad \begin{pmatrix}a\\b\end{pmatrix}=\left[\begin{pmatrix}x&x&\cdots&x\\1&1&\cdots&1\end{pmatrix}\begin{pmatrix}x&1\\x&1\\ \vdots&\vdots\\x&1\end{pmatrix}\right]^{-1}\begin{pmatrix}x&x&\cdots&x\\1&1&\cdots&1\end{pmatrix}\begin{pmatrix}y\\y\\ \vdots\\y\end{pmatrix}$ 可用cramer法则写出来

线性方程组的最小二乘解 $Ax\approx b$ $x=(A^TA)^{-1}b$ $x=A^{-1}b$ 广义逆

$A=P\begin{pmatrix}\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots \sigma_n & \\ & & & 0 \ddots\end{pmatrix}Q \Rightarrow x=Q^{-1}\begin{pmatrix}\sigma_1^{-1} & & & \\ & \sigma_2^{-1} & & \\ & & \ddots \sigma_n^{-1} & \\ & & & 0 \ddots\end{pmatrix}P^{-1}b$ PQ正交

4. 非线性方程求根

对分法

不动点法 $f(x)=0 \to \varphi(x)=x$ $x\in[a,b]$, $\varphi(x)\in[a,b]$, $|\varphi'(x)|\le L<1$ $|x^*-x_k|\le\frac{L^k}{1-L}|x_1-x_0|$

Newton 迭代法 $0=f(x)\approx f(x_0)+f'(x_0)(x-x_0)$ $x\approx x_0-\frac{f(x_0)}{f'(x_0)}$

$\varphi(x)=x-\frac{f(x)}{f'(x)}$ $\varphi'(x)=\frac{f(x)f''(x)}{(f'(x))^2}$

$\varphi(x)-\alpha=\varphi(\alpha+x-\alpha)-\alpha=\varphi(\alpha)-\alpha+\varphi'(\alpha)(x-\alpha)+\varphi''(\xi)(x-\alpha)^2=\varphi'(\xi)(x-\alpha)^2$

二阶方法，当 α 重根时 $x_{k+1}=x_k-p\frac{f(x_k)}{f'(x_k)}$ 仍是二阶方法

弦截法 $x_{k+1}=x_k-\frac{f(x_k)}{f[x_k,x_{k-1}]}=x_k-\frac{f(x_k)(x_k-x_{k-1})}{f(x_k)-f(x_{k-1})}$ 1.618 阶

$0\approx\varphi(x_{k+1})=f(x_{k+1})-\frac{f''(\xi)}{2!}(x_{k+1}-x_{k-1})(x_{k+1}-x_k)$ ←插值的 R_2

$f(x_{k+1})=f(\alpha)+f'(\eta)(x_{k+1}-\alpha)$

$\Rightarrow \underbrace{x_{k+1}-\alpha}_{o(|x_{k-1}-\alpha|^{r^2})}=\frac{f''(\xi)}{2f'(\eta)}\underbrace{(x_{k+1}-x_{k-1})}_{o(|x_{k-1}-\alpha|^{1})}\underbrace{(x_{k+1}-x_k)}_{o(|x_{k-1}-\alpha|^{r})}$ $r^2=r+1$

插值原理：x_{k+1} 是 x_k x_{k-1} 插值函数的根

非线性方程组 Newton 法 $J(x^{(k)})\Delta x^{(k)}=-F(x^{(k)})$

5. 解线性方程组的直接法

主对角 $o(n)$ 上下三角 $o(n^2)$ $\begin{pmatrix}+- & \frac{n^2}{2}+o(n) \\ \times\div & \frac{n^2}{2}+o(n)\end{pmatrix}$

Gauss 消元法：(A,b) 化为上三角再回代 $\begin{pmatrix}+- & \frac{n^3}{3}+O(n^2) \\ \times\div & \frac{n^3}{3}+O(n^2)\end{pmatrix}$ Gauss-Jordan 化为 $\begin{pmatrix}\ddots & \\ & \ddots\end{pmatrix}$ 代回时不慢

选主元，再加查找 $\frac{1}{2}n^2+o(n)$ (+-)

乘积分解 $A=LU$ Dolittle 单位下三角 Courant 单位上三角 $A=LDL^T$ $A=\overset{正交}{Q}\overset{上三角}{R}$

条件数 $\mathrm{cond}_p(A)=\|A\|_p\|A^{-1}\|_p$ cond(A)大时病态 det A很小时，A总是病态的

6.解线性方程组的迭代法 $AX=b$

Jacobi迭代 $DX=-(A-D)X+b$ $\quad Y=D^{-1}[B-(A-D)X]$ $\quad C=D^{-1}$

$y=-D^{-1}(A-D)x+D^{-1}b$ $\quad y_i=\frac{1}{a_{ii}}(b_i-\sum_{j\neq i}a_{ij}x_j)$

Gauss-Seidel迭代 $Dy=-Ly-Ux+b$ $\quad Y=D^{-1}(b-LY-UX)$ $\quad C=(D+L)^{-1}$

$y=(D+L)^{-1}(b-UX)$ $\quad y_i=\frac{1}{a_{ii}}(b_i-\sum_{j<i}a_{ij}y_j-\sum_{j>i}a_{ij}x_j)$

一般迭代过程 核是计算 $x=A^{-1}b$ 取 C $(\propto A^{-1})$ $\quad A=C^{-1}-P$

$(C^{-1}-P)x=b$ $\quad x=CPx+Cb=(I-CA)x+Cb$

$y=(I-CA)x+Cb$ $\quad \cancel{(Ay-b)=(I-AC)(Ax-b)}$ $\quad y=x-C(Ax-b)$

$x^*-y=(I-CA)(x^*-x)$

收敛 $\Leftrightarrow$ $\rho(I-CA)<1$

$C=D^{-1},(D+L)^{-1}$ A行或列对角占优时，收敛 $\quad C=(D+L)^{-1}$ A对称正定时收敛

松弛迭代 $C=(\frac{D}{\omega}+L)^{-1}$ $\quad Y=(1-\omega)X+\omega D^{-1}(-LY-UX+b)$

$I-CA=(\frac{D}{\omega}+L)^{-1}(\frac{D}{\omega}+L-A)=(\frac{D}{\omega}+L)^{-1}(\frac{1-\omega}{\omega}D-U)$

$=\begin{pmatrix}\frac{\omega}{a_{11}} & & \\ * & \frac{\omega}{a_{22}} & \\ & & \ddots\end{pmatrix}\begin{pmatrix}\frac{1-\omega}{\omega}a_{11} & & * \\ & \frac{1-\omega}{\omega}a_{22} & \\ & & \ddots\end{pmatrix}$ $\quad Det=(1-\omega)^n<1$ ，$0<\omega<2$

0~1 亚松弛 $\quad$ 1~2 超松弛

逆矩阵 $A\cdot A^{-1}=I$ 解 $Ax_j=e_j$ n个方程 Gauss消元 LU分解 QR分解

迭代 $AX=I$ $\quad Y=(I-CA)X+CI=X-C(AX-I)$ $\quad C=D^{-1}$ $(D+L)^{-1}$ X

7.计算矩阵特征值和特征向量

幂法 随机产生 X_0 $\quad Y^{(k)}=\frac{X^{(k)}}{\|X^{(k)}\|_\infty}$ $\quad X^{(k+1)}=AY^{(k)}$

按模最大特征值 $\lambda_1=\max_i\{X_i^{(k+1)}\}$ $\quad v_1=Y^{(k)}$ $\quad (\lambda_1^{(k)}=\frac{X^{(k)H}AX^{(k)}}{X^{(k)H}X^{(k)}})$

若 $X^{(2k)}$ $X^{(2k+1)}$ 收敛到互为相反的向量 $\lambda_1=-\max_i\{|X_i^{(k+1)}|\}$

$v_1=Y(k)$

若 $X^{(2k)}$ $X^{(2k+1)}$ 收敛到两个不同的向量 $Z=AX^{(k+1)}$

$\lambda_1=\max_i\{\sqrt{\frac{Z_i}{y_i^{(k)}}}\}$ $\quad \lambda_2=-\lambda_1$ $\quad v_1=Z+\lambda_1X^{(k+1)}$

$v_2=Z+\lambda_2X^{(k+1)}$

$A=P\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n\end{pmatrix}P^{-1}$ $\quad |\lambda_1|>|\lambda_2|>\cdots$ $\quad v_1=P\begin{pmatrix}1\\0\\ \vdots\end{pmatrix}$

$\frac{A^k}{\lambda_1^k}\to P\begin{pmatrix}1 & & \\ & 0 & \\ & & \ddots\end{pmatrix}P^{-1}=P\begin{pmatrix}1\\0\\ \vdots\end{pmatrix}\cdot(1\,0\,0\,0\cdots)\cdot P^{-1}$

$P\begin{pmatrix}1 & & \\ & 0 & \\ & & \ddots\end{pmatrix}P^{-1}\cdot X_0=P\begin{pmatrix}1\\0\\ \vdots\end{pmatrix}\cdot[(1\,0\,0\,0\cdots)P^{-1}\cdot X_0]=cv_1$ 单位化 $\Rightarrow$ $\frac{v_1}{\|v_1\|}$

反幂法 用 A^{-1} 取代 A，求按模最小特征值

No.
Date

Jacobi 方法 则 $P^TAP = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ A对称 则 $*=0$

$$\begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} = \begin{pmatrix} a-tb & \\ & a+tb \end{pmatrix}$$

$\cot 2\theta = \frac{c-a}{2b} = s$ $t^2+2st-1=0$ 的 按模小的根 (选择能使得 $\omega>0$)

$\cos = \frac{1}{\sqrt{1+t^2}}$ $\sin = \frac{t}{\sqrt{1+t^2}}$

QR 法 Givens 旋转 $Q(p,q,\theta) = \begin{pmatrix} \ddots & & & \\ & \cos & \sin & \\ & -\sin & \cos & \\ & & & \ddots \end{pmatrix}$

Householder 反射 ~~阵~~ $H = I - 2vv^T$ $\|v\|=1$

$A_k = \underset{正交}{Q_k}\underset{上三角}{R_k}$ $A_{k+1} = R_kQ_k = Q_{k+1}R_{k+1}$ ~~若收敛~~

~~$A^k = Q_1Q_2\cdots Q_kR_kR_{k-1}\cdots R_1$~~ $Q_k^TQ_{k-1}^T\cdots Q_1^TAQ_1Q_2\cdots Q_k = A_{k+1}$

$R_k \to \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $Q_k \to I$

$A=QR$: 对A作正交行变换 $R=PA$ $A=P^TR$

幂法推广 X_0随机生成 $n\times 2$ $[x_{01}\ x_{02}]$ $Y_k = AX_k$

把Y_k标准正交化 $Y_k = (\alpha_{1k}\ \alpha_{2k})\begin{pmatrix} x & x \\ 0 & x \end{pmatrix} \leftarrow R_{k+1}$ $\alpha_1 \perp \alpha_2$

$X_{k+1} = (\alpha_{1k}\ \alpha_{2k})$ $AX_k = X_{k+1}R_{k+1}$

收敛 $AX_\infty = X_\infty\begin{pmatrix} \lambda_1 & * \\ & \lambda_2 \end{pmatrix}$ $x_\infty = [\alpha_1\ \alpha_2]$

Shift 法 先估 $\lambda = a+bi$ 考虑 $A-\underset{s}{(a+bi)}I$ 的最小特征值

$$Y_k = (A - S_{k-1}I)^{-1}X_k \qquad X_k = \frac{Y_k}{\|Y_k\|}$$

$$S_k = S_{k-1} + \frac{1}{Y_{k-1}^TY_k}$$

8.常微分方程初值问题 $\begin{cases} y'(x) = f(x,y) \\ y(a) = y_0 \end{cases}$

Euler 方法 用导数近似差商 $y(x_{n+1}) = y(x_n) + hy[x_n, x_{n+1}]$

向前Euler公式 $y_{n+1} = y_n + hf(x_n, y_n)$

向后Euler公式 $y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$

Picard 迭代格式 $y_{n+1}^{(0)} = y_n + hf(x_n, y_n)$

$y_{n+1}^{(k+1)} = y_n + hf(x_{n+1}, y_{n+1}^{(k)})$

中心 $y_{n+1} = y_{n-1} + hf(x_n, y_n)$

局部截断误差 $y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2}f'(\xi)$ $= \cancel{y_{n+1} + \frac{h^2}{2}f'(\xi)}$

$T_{n+1} = O(h^2)$ $T_{n+1} = O(h^{p+1})$ 称方法是 p 阶的

整体截断误差 $e_{n+1} = y(x_{n+1}) - y_{n+1}$ ~~Lipp~~ Lipschitz 条件 $|f(x,y) - f(x,\bar{y})| < L|y-\bar{y}|$

$e_{n+1} = y(x_n) - y_n + h[f(x_n, y(x_n)) - f(x_n, y_n)] + T_{n+1}$

$|e_{n+1}| \leq |e_n| + hL|e_n| + T$

$\Rightarrow \quad = (1+Lh)^{n+1}|e_0| + \frac{1-(1+Lh)^{n+1}}{1-(1+Lh)}T$

$< (1+Lh)^{n+1}(|e_0| + \frac{T}{Lh}) \leq e^{L(b-a)}(|e_0| + \frac{T}{Lh})$

基于数值积分的近似公式 $y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x,y)dx$

$f(x,y)$ 取 $f(x_n, y(x_n)) \to$ Euler 向前公式

$f(x,y)$ 取 $f(x_{n+1}, y(x_{n+1})) \to$ Euler 向后公式

梯形公式 $y_{n+1} = y_n + \frac{h}{2}[f(x_n,y_n) + f(x_{n+1}, y_{n+1})]$

(再预估 - 校正：先用显式公式算初始，再迭代一次隐式公式)

Runge-Kutta 方法

$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \cdots \frac{h^p}{p!}y^{(p)}(x) + T$ p阶精度

$p=1 \to$ Euler 向前 $p=2$

$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2}[f_x(x_n, y(x_n)) + f_y(x_n, y(x_n))f(x_n, y(x_n))] + T_{n+1}$ ①

$\Rightarrow \underset{y(x_{n+1})}{y_{n+1}} = \underset{y(x_n)}{y_n} + hf(x_n, y_n) + \frac{h^2}{2}[f_x(x_n,y_n) + f_y(x_n,y_n)f(x_n,y_n)]$ ②

用 $y_{n+1} = y_n + h[c_1 f(x_n, \overset{y(x_n)}{y_n}) + c_2 f(x_n + ah, \overset{y(x_n)}{y_n} + bh f(x_n, \overset{y(x_n)}{y_n}))]$ 去逼近 ①②

②在 $x_n, y(x_n)$ 展开 $\Rightarrow \begin{cases} c_1 + c_2 = 1 \\ a = b = \frac{1}{2c_2} \end{cases}$

线性多步法 (Adams 公式 p=0)

用 q+1 低插值 $\varphi \approx y'$ $y_{n+1} = y_{n-p} + \int_{x_{n-p}}^{x_{n+1}} \varphi dx$ $x_n, x_{n-1} \ldots x_{n-q}$ 显示; $x_{n+1}, x_n \ldots x_{n-q+1}$ 隐示

$y_{n+1} = y_{n-p} + \sum C_i f(x_{n-i}, y_{n-i})$ 用 $f(t) = (t - x_{n})^{k=0,1,\ldots q}$ 计算 C_i

$T_{n+1} = \int_{x_{n-p}}^{x_{n+1}} R(x)dx = \int_{x_{n-p}}^{x_{n+1}} \frac{y^{(q+1)}(\eta)}{q!} \Pi(x - x_{n-i})dx = O(h^{q+1})$

常微分方程组及高阶常微分方程

$y \to \vec{Y}$ $f(x,y) \to \vec{F}(x, \vec{Y})$ $y_0 \to \vec{Y}_0$ $\begin{cases} \vec{Y}' = \vec{F}(x, \vec{Y}) \\ \vec{Y}(a) = \vec{Y}_0 \end{cases}$ 公式形式完全一致

高阶：$y_1 = y$ $y_2 = y'$ $y_3 = y''$ ... 化为常微分方程组

6章 算法流程 算法分析 数学原理

Jacobi 迭代 $y_i = \frac{1}{a_{ii}}(b_i - \sum_{j \neq i} a_{ij} x_j)$ 每轮 $n^2+O(n)$ 次 ± $n^2+O(n)$ 次 ×÷

$Ax=b$ $Dx = b-(A-D)x$ $D=\begin{pmatrix} a_{11} & & \\ & a_{22} & \\ & & \ddots & a_{nn}\end{pmatrix}$

$y = -D^{-1}(A-D)x + D^{-1}b = x - D^{-1}(Ax-b)$

D ╲

L 下三角(≠0)

U 上三角(≠0)

Gauss-Seidel $y_i = \frac{1}{a_{ii}}(b_i - \sum_{j<i} a_{ij} y_j - \sum_{j>i} a_{ij} x_j)$

$y = (D+L)^{-1}(b-Ux) = x-(D+L)^{-1}(Ax-b)$

Jacobi $Ay-b = (I-AD^{-1})(Ax-b)$

Gauss-Seidel $Ay-b = (I-A(D+L)^{-1})(Ax-b)$

$\|Ay-b\| \le \begin{cases}\|I-AD^{-1}\| \\ \|I-A(D+L)^{-1}\|\end{cases} \cdot \|Ax-b\|$

定理 $\lim B^n = 0 \iff \rho(B)<1$

$B = P\begin{pmatrix}\lambda_1 & * \\ & \ddots & \lambda_n\end{pmatrix}P^{-1}$ $B^n = P(\ddots)^n P^{-1}$ 约当标准型

$\rho < \|A\|$

谱半径 最大特征值 $\rho(A)$

一般迭代过程 $Ay-b = (I-AC)(Ax-b)$ $\rho(I-AC)<1$

$y = x - C(Ax-b)$ $C \approx A^{-1}$ 取? $\frac{A^T}{\|A\|^2}$ $\frac{A^T}{\|A \cdot A^T\|}$

$y = (I-CA)x + Cb$

$I-AC$ 相似 $I-CA$ $\rho(I-AC) = \rho(I-CA)$ P134

A行对角优势时 即 $|a_{ii}| > \sum_{j\neq i}|a_{ij}|$ $I-D^{-1}A$ 的特征值 λ

$(I-D^{-1}A)\alpha = \lambda\alpha$ ~~$(A+(\lambda-1)D)\alpha=0$~~ $\alpha = \begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix}$ 设 $|x_k|$ 最大

$\lambda x_i = x_i - \frac{1}{a_{ii}}\sum_{j=1}^{n} a_{ij}x_j = \frac{1}{a_{ii}}\sum_{j\neq i} a_{ij}x_j$

$|\lambda x_i| \le \left|\frac{1}{a_{ii}}\right| \sum_{j\neq i}^{n} |a_{ij}| \cdot |x_j|$ $\therefore |\lambda| \le \left|\frac{1}{a_{ii}}\right| \sum_{j\neq i}|a_{ij}| < 1$

A列对角优时　~~$\rho(I-D^{-1}A)$~~

取 $I-AD^{-1}$ 的 λ 和 α（左特征向量）

$\alpha(I-AD^{-1})=\lambda\alpha$　即 $(I-D^{-1}A^T)\alpha^T=\lambda\alpha^T$　证法同上

$A=\begin{pmatrix}1&a\\b&1\end{pmatrix}$　$I-(D+L)^{-1}A=(D+L)^{-1}U=\begin{pmatrix}0&-a\\0&ab\end{pmatrix}$　$\rho=|ab|$

A行对角优　证 $\rho(I-(D+L)^{-1}A)<1$

$(I-(D+L)^{-1}A)\alpha=\lambda\alpha$

$-(D+L)^{-1}U\alpha=\lambda\alpha$　$\alpha=\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$　$|x_i|$最大

$\lambda(D+L)\alpha=-U\alpha$

$\lambda\sum_{j\le i}a_{ij}x_j=-\sum_{j>i}a_{ij}x_j$

$|\lambda|\le\dfrac{\sum_{j>i}|a_{ij}||x_i|}{a_{ii}x_i-\sum_{j<i}a_{ij}x_j}\le\dfrac{\sum_{j>i}|a_{ij}||x_i|}{a_{ii}x_i-\sum_{j<i}a_{ij}|x_i|}<1$

A正定对称时　$\rho(I-(D+L)^{-1}A)<1$　$U=L^T$　$D+L=A-U$

$-(D+L)^{-1}U\alpha=\lambda\alpha$

$\lambda(\alpha^TD\alpha+\alpha^TL\alpha)=-\alpha^TU\alpha$

$\lambda=\dfrac{-\alpha^TU\alpha}{\alpha^TD\alpha+\alpha^TL\alpha}=\dfrac{-\alpha^TU\alpha^T}{\alpha^TA\alpha-\alpha^TU\alpha}$　当 $Re(\alpha^TU\alpha)<0$ 时 $|\lambda|<1$

当 $Re(\alpha^TU\alpha)>0$ 时

$Re(\alpha^TL\alpha)$　也有 $|\lambda|<1$

$\|Ax^{(n)}-b\|\le\|I-AC\|^n\|Ax^{(0)}-b\|$

$n=\left|\dfrac{\log\varepsilon}{\log\|I-AC\|}\right|+O(1)$

A行/列对角优时　$\det A\neq0$

BusyBurn

计算方法 2014-11-27

松弛迭代 JOR SOR $\qquad Y = X - C(AX-b) = (I-CA)X + Cb$

$AY-b = (I-AC)(AX-b)$

JOR $\quad Y = X - \left(\frac{D}{\omega}\right)^{-1}(AX-b)$

SOR $\quad Y = X - \left(\frac{D}{\omega}+L\right)^{-1}(AX-b)$

松弛因子 ω $\qquad$ 定理：SOR收敛 $\Rightarrow 0<\omega<2$

$\Updownarrow$

$\rho\left(I - A\left(\frac{D}{\omega}+L\right)^{-1}\right) < 1$

$B = I - A\left(\frac{D}{\omega}+L\right)^{-1} = \left(\frac{D}{\omega}+L-A\right)\left(\frac{D}{\omega}+L\right)^{-1} = \left(\left(\frac{1}{\omega}-1\right)D-U\right)\left(\frac{D}{\omega}+L\right)^{-1}$

$= \begin{pmatrix} \frac{1-\omega}{\omega}a_{11} & & * \\ & \frac{1-\omega}{\omega}a_{22} & \\ & & \ddots \end{pmatrix}\begin{pmatrix} \frac{\omega}{a_{11}} & & \\ & \frac{\omega}{a_{22}} & \\ * & & \ddots \end{pmatrix}$ $\qquad \mathrm{Det}\,B = (1-\omega)^n$

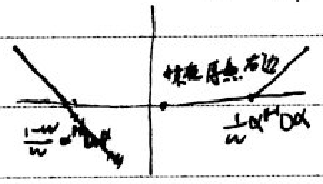
$\rho(B)<1 \Rightarrow \det B < 1 \Rightarrow |\omega-1|<1$

对角优：$0<\omega<1$ $\qquad$ A对称正定 SOR收敛 $\Leftrightarrow 0<\omega<2$

$\cancel{B\alpha=\lambda\alpha}\ \left[I-A\left(\frac{D}{\omega}+L\right)^{-1}\right]\alpha=\lambda\alpha \qquad \left(\frac{D}{\omega}+L-A\right)\alpha = \lambda\left(\frac{D}{\omega}+L\right)\alpha$ 两边左乘 α^H

$\lambda = \dfrac{\alpha^H\left(\frac{D}{\omega}+L-A\right)\alpha}{\alpha^H\left(\frac{D}{\omega}+L\right)\alpha} = \dfrac{\frac{1-\omega}{\omega}\alpha^H D\alpha - \alpha^H U\alpha}{\frac{1}{\omega}\alpha^H D\alpha + \alpha^H L\alpha}$ $\qquad \alpha^H U\alpha \quad \alpha^H L\alpha$ 共轭

$\alpha^H U\alpha + \alpha^H L\alpha + \alpha^H D\alpha = \alpha^H A\alpha > 0$



求 A^{-1} $\qquad$ 直接法：Gauss消元 $\quad$ LU分解 $\quad$ QR分解

迭代法：$AX=I \qquad Y=(I-CA)X+CI \qquad C=D^{-1} \quad (D+L)^{-1} \quad X$

$\uparrow$

矩阵迭代

$Y = X - C(AX-I)$

$AY-I = (I-AC)(AX-I)$

$I-AY = (I-AX)^2$

$Y = 2X - XAX$

No.
Date

特征值计算

幂法：最大特征值

反幂法：最小特征值

QR法：求所有特征值（最大的前k个）

实对称矩阵的 Jacobi 法：求实对称特征值

幂法：随机产生复向量 X_0 $\|X_0\|=1$ 　2> $X_k = \frac{AX_{k-1}}{\|AX_{k-1}\|}$ 　1> $\lambda_k = X_k^H A X_k$

$e_k = \|AX_k - \lambda X_k\|$

$A = P\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n\end{pmatrix}P^{-1}$ 　$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$

$P = (\alpha_1 \alpha_2 \dots \alpha_n)$

$A^k = P\begin{pmatrix}\lambda_1^k & \\ & \ddots\end{pmatrix}P^{-1}$ 　$\lim_{k\to\infty}\frac{A^k}{\lambda_1^k} = P\begin{pmatrix}1 & & \\ & 0 & \\ & & \ddots & 0\end{pmatrix}P^{-1}$ 　$X_k = P\begin{pmatrix}1\\0\\ \vdots\end{pmatrix}(1\,0\,0\,0\,0)P^{-1}X_0$ 　$X_k = \frac{\alpha_1}{\|\alpha_1\|}e^{i\theta}$

$A = P\begin{pmatrix}\lambda_1 & \\ & B\end{pmatrix}P^{-1}$ 　$|\lambda_1| > \rho(B)$ 　$\lim_{k\to+\infty} X_k^H A X_k = \lambda_1$

$A^k = P\begin{pmatrix}\lambda_1^k & \\ & B^k\end{pmatrix}P^{-1}$ 　$\lim_{k\to\infty}\frac{A^k}{\lambda_1^k} = P\begin{pmatrix}1 & & \\ & 0 & \\ & & \ddots & 0\end{pmatrix}P^{-1}$ 　步数 $o\left(\frac{\log \varepsilon}{\log \lambda_2/\lambda_1}\right)$

反幂法：幂法中A取 A^{-1}

QR：　X_0 随机生成 $n\times 2$ 　$[x_{01}\ x_{02}]$

$Y_k = AX_k$ 　把 Y_k 标准正交化 　$Y_k = (\alpha_{k1}\ \alpha_{k2})\begin{pmatrix}* & *\\ & *\end{pmatrix}$ 　$\alpha_1 \perp \alpha_2$

$\uparrow Q$ 　$\uparrow R_{k+1}$

$X_{k+1} = Q$ 　$AX_k = X_{k+1}R_{k+1}$

~~$A^k X_0 = X_k R_k R_{k-1} \cdots R_1$~~ 收敛：$AX_\infty = X_\infty\begin{pmatrix}\lambda_1 & *\\ & \lambda_2\end{pmatrix}$ 　$X_\infty = [\alpha_1\ \alpha_2]$ 　α_1是特征向量（α_2不是）

1> 若 A最大模特征值共轭复数 　考虑 $A + (a+bi)I$ 的最大模特征值

2> 若要求A的特征值距 $a+bi$ 最近的特征值 考虑 $A-(a+bi)I$ 的最小模特征值

↑
平移

shift法，先用1>估 $\lambda = a+bi$，再2>

↓ $s_k = a_k + b_k i$

反幂法 　$y_k = (A - s_{k-1}I)^{-1}X_{k-1}$

$X_k = y_k/\|y_k\|$

$s_k = s_{k-1} + \frac{1}{X_{k-1}^T \cdot y_k}$

No.
Date

Jacobi 法 $P^TAP=\begin{pmatrix}\lambda_1 & & * \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$ P正交 A对称则 $*=0$.

$$\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}a & b \\ b & c\end{pmatrix}\begin{pmatrix}\cos & +\sin \\ -\sin & \cos\end{pmatrix}=\begin{pmatrix}* & 0 \\ 0 & *\end{pmatrix}=\begin{pmatrix}a\cos^2+c\sin^2\theta-b\sin2\theta & \\ & a\cos^2\theta+c\sin^2\theta+b\sin2\theta\end{pmatrix}$$

$\tan2\theta=-\frac{2b}{a-c}$ $\cot2\theta=\frac{c-a}{2b}=s$ $t^2+2st-1=0$ 的 $|t|$小的根 $\cos=\frac{1}{\sqrt{1+t^2}}$ $\sin=\frac{t}{\sqrt{1+t^2}}$ $=\begin{pmatrix}a-tb & \\ & a+tb\end{pmatrix}$

这样能使得 $\cos\theta>0$

$$\begin{pmatrix}P^T & \\ & I\end{pmatrix}\begin{pmatrix}A_1 & A_2 \\ A_2^T & A_3\end{pmatrix}\begin{pmatrix}P & \\ & I\end{pmatrix}=\begin{pmatrix}P^TA_1P & P^TA_2 \\ A_2^TP & A_3\end{pmatrix}$$

QR $A_k=Q_kR_k$ (Q_k 正交, R_k 上三角)

$A_{k+1}=R_kQ_k=Q_{k+1}R_{k+1}$ 若收敛

$A^2=QR_1QR_1=\cancel{QA_2}\,Q_1Q_2R_2R_1$ $A^3=Q_1Q_2R_2R_1QR_1=Q_1Q_2R_2Q_2R_2R_1=Q_1Q_2Q_3R_3R_2R_1$

$A^k=Q_1\cdots Q_kR_k\cdots R_1$ 若 $\lim\prod Q_1\cdots Q_k=Q$

则 $\cancel{A=QR_{k+1}Q^T}$ $\lim_{k\to\infty}R_k=Q^TAQ$

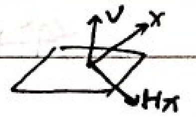
$Q_k^{-1}\cdots Q_1^{-1}AQ\cdots Q_k=A_{k+1}=R_kQ_k=Q_{k+1}R_{k+1}$

1. 对A的列向量作 Gram-Schmidt 正交化 得 $Q=A\cdot\begin{pmatrix}\ddots & * \\ & \ddots\end{pmatrix}$ $\leftarrow R^{-1}$
2. 对A~~的列向量~~作初等~~变换~~列变换 得 标准正交基 $Q=AR^{-1}$
3. 对A~~行向量~~作正交行变换 得 上三角阵 $R=PA$

正交行变换: ①Givens 旋转 ②Householder 反射

$\begin{pmatrix}\cos & -\sin \\ \sin & \cos\end{pmatrix}$ $H=I-2vv^T$

$\begin{pmatrix}1\\0\end{pmatrix}\to\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ $\|v\|=1$ $v\to -v$



$Hx=x-2\frac{v^Tx}{v^Tv}v$ x减 x在v投影的两倍

Givens $\begin{pmatrix}\cos & -\sin & \\ \sin & \cos & \\ & & I\end{pmatrix}\begin{pmatrix}a & b & * \\ c & d & \\ * & & *\end{pmatrix}\to\begin{pmatrix}* & * & \\ 0 & * & * \\ * & & *\end{pmatrix}$

$$\begin{pmatrix}\frac{\begin{pmatrix}a & c \\ -c & a\end{pmatrix}}{\sqrt{a^2+c^2}} & \\ & I\end{pmatrix}$$

Householder $(I-2vv^T)\begin{pmatrix}\alpha_1 & \\ x & \\ x & * \\ x & \\ x & \end{pmatrix}\to\begin{pmatrix}|\alpha_1| & \\ 0 & \\ 0 & * \\ 0 & \\ 0 & \end{pmatrix}$

$v=\underbrace{\begin{pmatrix}|\alpha_1|\\0\\0\\0\\0\end{pmatrix}-\alpha_1}_{\text{单位化}}$

计算方法 2014-12-18

常微分方程初值问题

$$\begin{cases} y' = f(x,y) \\ y(a)=y_0 \end{cases}$$

Euler方法

$h=\frac{b-a}{n}$

$\frac{y_{i+1}-y_i}{x_{i+1}-x_i} \approx y'(x_i) = f(x_i,y_i)$ 向前Euler公式$_1$

$\approx y'(x_{i+1}) = f(x_{i+1},y_{i+1})$ 向后Euler公式$_2$

$x_0=a$ $x_n=b$

$y_{i+1} = y_i + hf(x_i,y_i)_1$ 显示

$y_{i+1} = y_i + hf(x_{i+1},y_{i+1})_2$ 隐式

$\frac{y_{i+1}-y_{i-1}}{x_{i+1}-x_{i-1}} = y'(x_i) = f(x_i,y_i)$ 中心差商公式 $y_{i+1} = y_{i-1} + 2hf(x_i,y_i)_3$

$_2$ 迭代格式 $y_{i+1}^{(k+1)} = y_i + hf(x_{i+1}, y_{i+1}^{(k)})$ 改进公式

$y_1 = y(x_1)$? 误差: $y(x_1)-y_1$

y_0

x_0 x_1

$\frac{y(x_1)-y(x_0)}{h} = y'(x_0) + \frac{h}{2}y''(\xi)$ $O(h)$ $\frac{h^2}{2}y''(\xi)$

$y(x_1) = y_0 + hy'(x_0) + O(h^2) = y_1 + O(h^2)$

局部误差, 整体误差 假设初始是对的 局部误差的积累

① 局部误差 $y(x_1)-y_1 = \frac{f'(\xi)}{2}h^2$

② 局部误差 $y(x_0)-y(x_1) = y'(x_1)(x_0-x_1) + \frac{y''(\xi)}{2}(x_0-x_1)^2$

$y(x_1) = y_0 + \underbrace{hy'(x_1)}_{y_1} - \frac{y''(\xi)}{2}h^2$

$y(x_1)-y_1 = -\frac{y''(\xi)}{2}h^2$

③ $\frac{y(x_{i+1})-y(x_{i-1})}{2h} = y'(x_i) + \ldots$

$y(x_{i+1}) = y(x_i) + y'(x_i)h + \frac{y''(x_i)}{2}h^2 + \frac{y'''(\xi)}{6}h^3$

$y(x_{i-1}) = y(x_i) - y'(x_i)h + \frac{y''(x_i)}{2}h^2 - \frac{y'''(\xi)}{6}h^3$

$y(x_{i+1}) - y(x_{i-1}) = 2h\,y'(x_i) + \frac{y'''(\xi)}{3}h^3$

$y(x_2) - y_2 = \frac{y'''(\xi)}{3}h^3$

局部 $y(x_1)-y_1 = O(h^{p+1})$
$\leq Mh^{p+1}$

整体:

整体: $\begin{cases}\phi' = f(x,\phi) \\ \phi(x_i) = y_i\end{cases}$ $\quad \cancel{\phi(x_{i+1}) - y_{i+1} = O(h^{p+1})} \quad$ $\begin{cases}\varphi' = f(x,y) \\ \varphi(x_i) = y(x_i)\end{cases}$

$\cancel{y(x_{i+1}) \quad \phi(x_{i+1})}$ $|\phi - \varphi| = ?$ $\quad |\phi' - \varphi'| = |f(x,\phi) - f(x,\varphi)| \le L|\phi - \varphi|$

$\rightarrow |\phi(x_{i+1}) - \varphi(x_{i+1})| \le e^{Lh}|\phi(x_i) - \varphi(x_i)|$ $\quad \left(z' \le Lz \overset{积分}{\Longrightarrow} z(t) \le e^{Lt} z(0) \right)$

$|\phi(x_{i+1}) - y_{i+1}| \le M h^{p+1}$

$|y(x_{i+1}) - y_{i+1}| \le e^{Lh}|y(x_i) - y_i| + Mh^{p+1}$ 两边除 $e^{Lh(i+1)}$

$$\frac{|y(x_{i+1}) - y_{i+1}|}{e^{Lh(i+1)}} \le \frac{|y(x_i) - y_i|}{e^{Lhi}} + \frac{Mh^{p+1}}{e^{Lh(i+1)}} \qquad \sum i$$

$$\frac{|y(x_n) - y_n|}{e^{Lh\cdot n}} \le \sum 等比数列 \le \frac{Mh^{p+1}}{1-e^{-Lh}} \le \cancel{\frac{Mh^{p+1}}{Lh}} = \frac{M}{L}h^p$$

$$|y(x_n) - y_n| \le \cancel{\frac{M}{L} e^{(b-a)L} \cdot h^p} \qquad \frac{Me^{(b-a)L}}{1-e^{-Lh}} \cdot h^{p+1} \qquad O(h^p)$$

梯形④ $y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} y'(t)\,dt = y(x_i) + \int_{x_i}^{x_{i+1}} f(t, y(t))\,dt$ ← 基于数值积分的近似公式

$$\approx y(x_i) + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2} \cdot h \qquad O(h^3)?$$

改进⑤ $y_{i+1} = y_i + \dfrac{f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))}{2} \cdot h$