MECEE 4602: Introduction to Robotics, Fall 2019

Midterm Review

1 Transforms

1.1 2D Transforms

Consider three 2D coordinate frames: A, B and C.

- the transform ${}^AT_B$ from frame A to frame B consists of a rotation of 90° followed by a translation of 2 units along the y axis.
- the transform ${}^BT_C$ from frame B to frame C consists of a rotation of 60° followed by a translation of 2 units along the x axis.

a) Compute the homogeneous transform ${}^AT_C$ from frame A to frame C.

$$
\begin{aligned}
{}^AT_B &= \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 0 & -1 & -2 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] &(1)\\
{}^BT_C &= \left[\begin{array}{cc|c} 0.5 & -0.87 & 0 \\ 0.87 & 0.5 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 0.5 & -0.87 & 1 \\ 0.87 & 0.5 & 1.74 \\ \hline 0 & 0 & 1 \end{array}\right] &(2)\\
{}^AT_C &= {}^AT_B\ {}^BT_C = \left[\begin{array}{cc|c} -0.87 & -0.5 & -3.74 \\ 0.5 & -0.87 & 1 \\ \hline 0 & 0 & 1 \end{array}\right] &(3)
\end{aligned}
$$

b) A point p is expressed in the C coordinate frame as ${}^Cp = [1, 1]^T$. What are the coordinates of ${}^Ap$, the same point as seen from frame A?

$$
{}^Ap = {}^A T_C\ {}^Cp = \left[\begin{array}{cc|c} -0.87 & -0.5 & -3.74 \\ 0.5 & -0.87 & 1 \\ \hline 0 & 0 & 1 \end{array}\right]\left[\begin{array}{c} 1 \\ 1 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} -5.11 \\ 0.63 \\ \hline 1 \end{array}\right] \tag{4}
$$

c) Compute ${}^CT_A$, the homogeneous transform from frame C to frame A.

$$
{}^CT_A = {}^A T_C^{-1} \tag{5}
$$

Recall the formula for inverting transform matrices:

$$
\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{t} \\ 0 & 1 \end{bmatrix} \tag{6}
$$

Thus, we have:

$$
{}^CT_A = \left[\begin{array}{cc|c} -0.87 & 0.5 & -3.75 \\ -0.5 & -0.87 & -1 \\ \hline 0 & 0 & 1 \end{array}\right] \tag{7}
$$

d) A point q is expressed in the A coordinate frame as ${}^{A}q = [2, 0]^T$. What are the coordinates of ${}^{C}q$, the same point as seen from frame C?

$$ {}^{C}p = {}^{C}T_A \, {}^{A}p = \left[\begin{array}{cc|c} -0.87 & 0.5 & -3.75 \\ -0.5 & -0.87 & -1 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} 2 \\ 0 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} -5.49 \\ -2 \\ \hline 1 \end{array}\right] \tag{8} $$

1.2 3D Rotation matrices

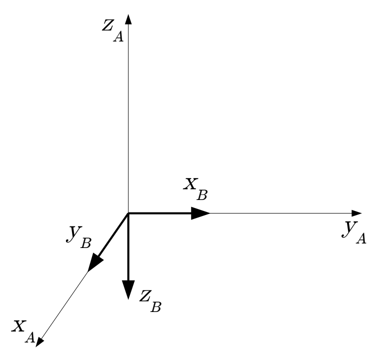
a) Which of the following do **not** represent a 3D Rotation matrix, and why?

$$ \boldsymbol{R}_1 = \begin{bmatrix} 0.707 & -0.707 & 0 \\ 0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{R}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9} $$

$$ \boldsymbol{R}_3 = \begin{bmatrix} 0.8 & 0 & -0.6 \\ 0 & 0.8 & 0 \\ 0.6 & 0 & 0.8 \end{bmatrix} \qquad \boldsymbol{R}_4 = \begin{bmatrix} 0.8 & 0 & 0.6 \\ 0 & 1.0 & 0 \\ 0.6 & 0 & 0.8 \end{bmatrix} \tag{10} $$

Answer: $\boldsymbol{R}_3$ is not a rotation matrix because it is not orthonormal - the norm of column 2 is not 1. $\boldsymbol{R}_4$ is not a rotation matrix because it is also not orthonormal - the dot product of columns 1 and 3 is not 0.

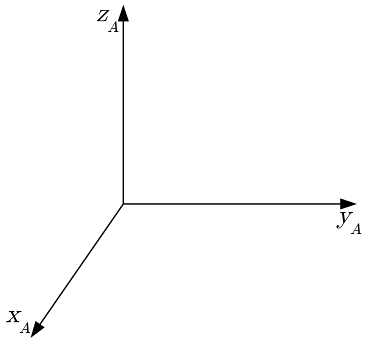
b) The axes of two coordinate frames A and B are shown below. What is the rotation matrix from frame A to frame B?



Answer: Recall that the new axes obtained after applying a rotation can be found as columns of the rotation matrix:

$$ \boldsymbol{R} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{11} $$

c) The axes of a coordinate frame A are shown below:

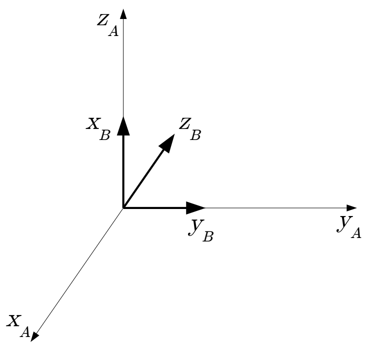


The rotation matrix from frame A to a new frame B is

$$ {}^{A}\boldsymbol{R}_{B} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{12} $$

Draw the axes of frame B superimposed over frame A.

Answer: Recall that the new axes obtained after applying a rotation can be found as columns of the rotation matrix.



1.3 3D Transforms

Consider three 3D coordinate frames: A, B and C.

- the transform ${}^{A}T_{B}$ from frame A to frame B consists of a rotation of 45° around the z axis followed by a translation of 2 units along the x axis.
- the transform ${}^{B}T_{C}$ from frame B to frame C consists of a rotation of 45° around the y axis followed by a translation of 2 units along the x axis.

a) Compute the homogeneous transform ${}^{A}T_{C}$ from frame A to frame C.

$$ {}^{A}T_{B} = \left[\begin{array}{ccc|c} 0.7 & -0.7 & 0 & 0 \\ 0.7 & 0.7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 0.7 & -0.7 & 0 & 1.4 \\ 0.7 & 0.7 & 0 & 1.4 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{13} $$

$$
{}^{B}T_{C} = \left[\begin{array}{ccc|c} 0.7 & 0 & 0.7 & 0 \\ 0 & 1 & 0 & 0 \\ -0.7 & 0 & 0.7 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 0.7 & 0 & 0.7 & 1.4 \\ 0 & 1 & 0 & 0 \\ -0.7 & 0 & 0.7 & -1.4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{14}
$$

$$
{}^{A}T_{C} = {}^{A}T_{B}\ {}^{B}T_{C} = \left[\begin{array}{ccc|c} 0.5 & -0.7 & 0.5 & 2.38 \\ 0.5 & 0.7 & 0.5 & 2.38 \\ -0.7 & 0 & 0.7 & -1.4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{15}
$$

b) A point p is expressed in the C coordinate frame as ${}^{C}p = [0, 0, 1]^{T}$. What are the coordinates of ${}^{A}p$, the same point as seen from frame A?

$$
{}^{A}p = {}^{A}T_{C}\ {}^{C}p = \left[\begin{array}{ccc|c} 0.5 & -0.7 & 0.5 & 2.38 \\ 0.5 & 0.7 & 0.5 & 2.38 \\ -0.7 & 0 & 0.7 & -1.4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} 2.88 \\ 2.88 \\ -0.7 \\ \hline 1 \end{array}\right] \tag{16}
$$

c) Compute ${}^{C}T_{A}$, the homogeneous transform from frame C to frame A.

$$
{}^{C}T_{A} = {}^{A}T_{C}^{-1} \tag{17}
$$

Recall the formula for inverting transform matrices:

$$
\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{R}^{T} & -\boldsymbol{R}^{T}\boldsymbol{t} \\ 0 & 1 \end{bmatrix} \tag{18}
$$

Thus, we have:

$$
{}^{C}T_{A} = \left[\begin{array}{ccc|c} 0.5 & 0.5 & -0.7 & -3.36 \\ -0.7 & 0.7 & 0 & 0 \\ 0.5 & 0.5 & 0.7 & -1.4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{19}
$$

d) A point q is expressed in the A coordinate frame as ${}^{A}q = [2, 0, 0]^{T}$. What are the coordinates of ${}^{C}q$, the same point as seen from frame C?

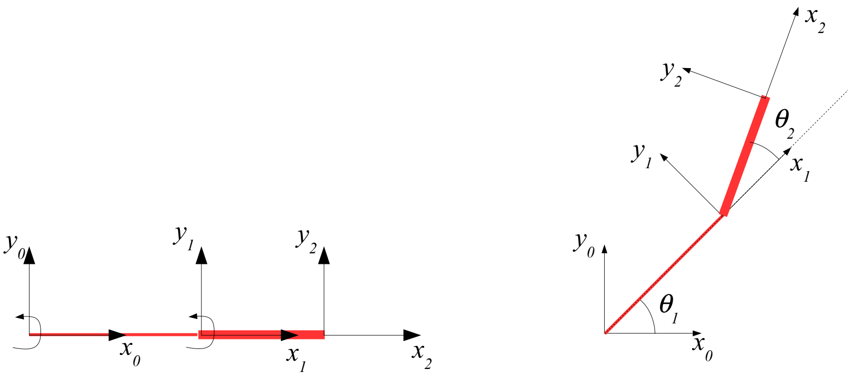
$$
{}^{C}p = {}^{C}T_{A}\ {}^{A}p = \left[\begin{array}{ccc|c} 0.5 & 0.5 & -0.7 & -3.36 \\ -0.7 & 0.7 & 0 & 0 \\ 0.5 & 0.5 & 0.7 & -1.4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} 2 \\ 0 \\ 0 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} -2.36 \\ -1.4 \\ -0.4 \\ \hline 1 \end{array}\right] \tag{20}
$$

2 Planar 2-Link Robot

i	θ	d	a	α
1	q_1	0	0.5	0
2	q_2	0	0.3	0

2.1 Sketch

In an XY coordinate frame, draw a sketch of this robot showing the spatial relationships between the coordinate frames associated with each link.



2.2 Forward Kinematics

Compute the transform matrix ${}^b\boldsymbol{T}_{ee}$ from the base of the robot to its end-effector.

$$
\begin{aligned}
{}^b\boldsymbol{T}_{ee} &= \prod_i \boldsymbol{T}_{ROT(\theta_i,z)} \cdot \boldsymbol{T}_{TRANS(d_i,z)} \cdot \boldsymbol{T}_{TRANS(a_i,x)} \cdot \boldsymbol{T}_{ROT(\alpha_i,x)} \\
&= \boldsymbol{T}_{ROT(\theta_1,z)} \cdot \boldsymbol{T}_{TRANS(0.5,x)} \cdot \boldsymbol{T}_{ROT(\theta_2,z)} \cdot \boldsymbol{T}_{TRANS(0.3,x)} \cdot \\
&= \left[\begin{array}{cc|c} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{cc|c} 1 & 0 & 0.5 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{cc|c} C_2 & -S_2 & 0 \\ S_2 & C_2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{cc|c} 1 & 0 & 0.3 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{cc|c} C_1C_2 - S_1S_2 & -C_1S_2 - S_1C_2 & 0.3C_1C_2 - 0.3S_1S_2 + 0.5C_1 \\ S_1C_2 + C_1S_2 & -S_1S_2 + C_1C_2 & 0.3S_1C_2 + 0.3C_1S_2 + 0.5S_1 \\ \hline 0 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{cc|c} C_{12} & -S_{12} & 0.3C_{12} + 0.5C_1 \\ S_{12} & C_{12} & 0.3S_{12} + 0.5S_1 \\ \hline 0 & 0 & 1 \end{array}\right]
\end{aligned} \tag{21}
$$

2.3 Inverse Kinematics

Assume we require the end-effector to be at position $[a, b]^T$, and we do not care about end-effector orientation. Show how to compute the values for all the robot joints such that the end-effector achieves the desired position. Be sure to list all possible solutions. If the number of solutions depends on certain characteristics of a and b, explain how and why.

From the translation component of ${}^b\boldsymbol{T}_{ee}$ we obtain:

$$
\begin{aligned}
0.3C_{12} + 0.5C_1 &= a && (22)\\
0.3S_{12} + 0.5S_1 &= b && (23)
\end{aligned}
$$

If we square each equation and add them up, we obtain:

$$
\begin{aligned}
0.09C_{12}^2 + 0.3C_{12}C_1 + 0.25C_1^2 &+ && (24)\\
+0.09S_{12}^2 + 0.3S_{12}S_1 + 0.25S_1^2 &= a^2 + b^2 && (25)
\end{aligned}
$$

Using the fact that $S^2 + C^2 = 1$ we then obtain:

$$
\begin{aligned}
0.3C_2 + 0.34 &= a^2 + b^2 && (26)\\
C_2 &= \frac{a^2 + b^2 - 0.34}{0.3} && (27)
\end{aligned}
$$

Case 1:

$$
\begin{aligned}
\frac{a^2 + b^2 - 0.34}{0.3} &> 1 && (28)\\
a^2 + b^2 &> 0.64 && (29)\\
a^2 + b^2 &> (0.5 + 0.3)^2 && (30)
\end{aligned}
$$

There are no solutions. The point we are asking for is beyond the reach of the robot, given it's link lengths.

Case 2:

$$
\begin{aligned}
\frac{a^2 + b^2 - 0.34}{0.3} &= 1 && (31)\\
a^2 + b^2 &= (0.5 + 0.3)^2 && (32)\\
C_2 &= 1 && (33)\\
\theta_2 &= 0 && (34)
\end{aligned}
$$

From (22) and (23):

$$
\begin{aligned}
0.8C_1 &= a && (35)\\
0.8S_1 &= b && (36)\\
\theta_1 &= \mathtt{atan2}(b, a) && (37)
\end{aligned}
$$

The arm is fully stretched out in an effort to reach the desired point which is at the edge of its workspace. There is a single solution to the IK problem.

Case 3:

$$
\begin{aligned}
-1 < \quad & \tfrac{a^2+b^2-0.34}{0.3} & < 1 \qquad (38)\\
(0.5-0.3)^2 < \quad & a^2+b^2 & < (0.5+0.3)^2 \qquad (39)\\
\theta_2 \quad & = & \pm\texttt{acos}\left(\frac{a^2+b^2-0.34}{0.3}\right) \qquad (40)
\end{aligned}
$$

For each possible value of θ_2, we can plug that value back into (22) and (23) and compute the values of S_1 and C_1. These in turn identify a single value for θ_1. There are two solutions to the IK problem, with different configurations for the robot's "elbow".

Case 4:

$$
\begin{aligned}
\frac{a^2+b^2-0.34}{0.3} &= -1 \qquad (41)\\
a^2+b^2 &= (0.5-0.3)^2 \qquad (42)\\
C_2 &= -1 \qquad (43)\\
\theta_2 = \pi & \qquad (44)
\end{aligned}
$$

From (22) and (23):

$$
\begin{aligned}
0.2C_1 &= a \qquad (45)\\
0.2S_1 &= b \qquad (46)\\
\theta_1 &= \texttt{atan2}(b,a) \qquad (47)
\end{aligned}
$$

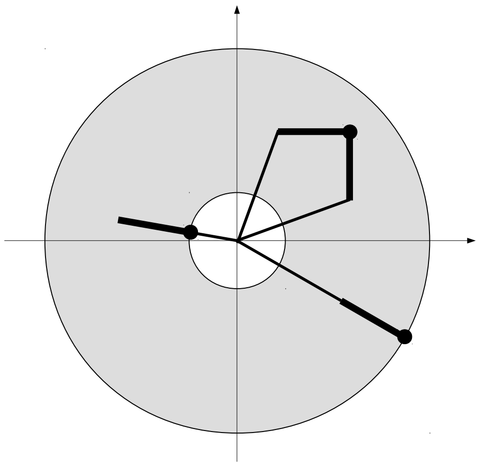
The arm is fully bent in on itself in an effort to reach close to the origin, on the inner edge of the workspace, and there is a single solution to the IK problem.

Case 5:

$$
\begin{aligned}
\frac{a^2+b^2-0.34}{0.3} &< -1 \qquad (48)\\
a^2+b^2 &< (0.5-0.3)^2 \qquad (49)
\end{aligned}
$$

There are no solutions. The point we are asking for is too close to the origin, and the second link is too short relative to the first to reach there.

Overall, the workspace of the robot is illustrated below:



2.4 Differential Kinematics

Compute the manipulator Jacobian with respect to end-effector position (and ignoring end-effector orientation). Find all the joint configurations where the Jacobian becomes singular; for each of them, briefly explain why the robot is in a singular configuration (for example, which direction of end-effector motion is impossible or which robot joint has no effect on end-effector position).

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.3C_{12} + 0.5C_1 \\ 0.3S_{12} + 0.5S_1 \end{bmatrix} \tag{50}$$

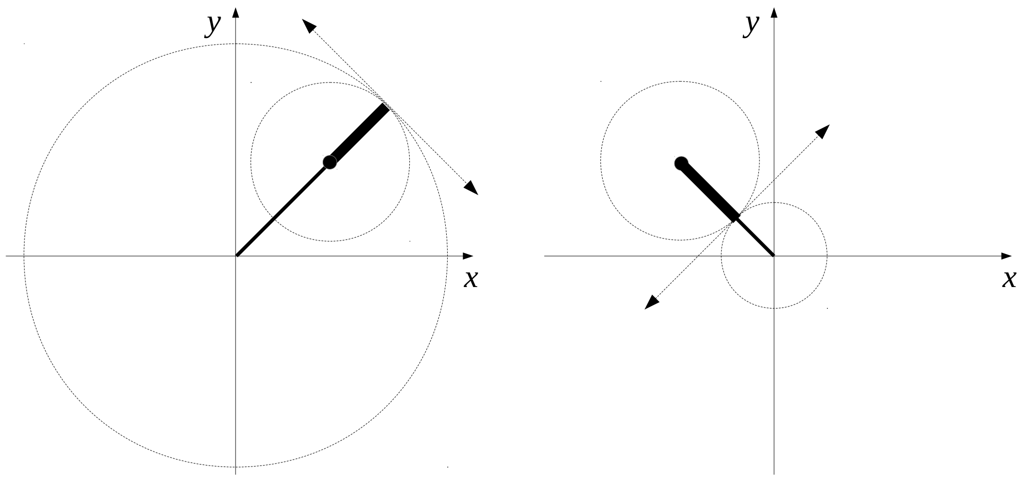
$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{bmatrix} = \begin{bmatrix} -0.3S_{12} - 0.5S_1 & -0.3S_{12} \\ 0.3C_{12} + 0.5C_1 & 0.3C_{12} \end{bmatrix} \tag{51}$$

$$\texttt{det}(\boldsymbol{J}) = -0.09S_{12}C_{12} - 0.15S_1C_{12} + 0.09S_{12}C_{12} + 0.15C_1S_{12} \tag{52}$$

$$= 0.15(C_1S_{12} - S_1C_{12}) = 0.15S_2 \tag{53}$$

The Jacobian is singular when $\sin\theta_2$ is 0, thus when $\theta_2 = 0$ or $\theta_2 = \pi$.

- when $\theta_2 = 0$, the robot is fully stretched out and both joints move the robot along the same line. In this configuration, this line is tangent to both the circle described by joint 1 and the smaller circle described by joint 2. Motion in any other direction is impossible.
- when $\theta_2 = \pi$, the robot is folded in on itself. Again, both joints move the robot along the same direction, and movement in any other direction is impossible.

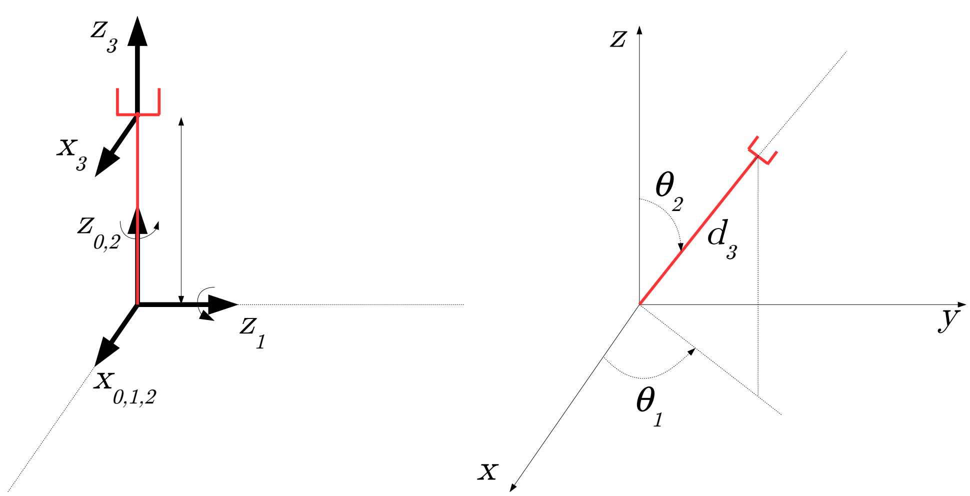


3 Spherical Robot

i	θ	d	a	α
1	q_1	0	0	$-90°$
2	q_2	0	0	$90°$
3	0	q_3	0	0

3.1 Sketch

In a right-handed XYZ coordinate frame, draw a sketch of this robot showing the spatial relationships between the coordinate frames associated with each link.



3.2 Forward Kinematics

Compute the transform matrix ${}^{b}\boldsymbol{T}_{ee}$ from the base of the robot to its end-effector.

$$
\begin{aligned}
{}^{b}\boldsymbol{T}_{ee} &= \prod_i \boldsymbol{T}_{ROT(\theta_i,z)} \cdot \boldsymbol{T}_{TRANS(d_i,z)} \cdot \boldsymbol{T}_{TRANS(a_i,x)} \cdot \boldsymbol{T}_{ROT(\alpha_i,x)} && (54)\\
&= \left[\begin{array}{ccc|c} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{ccc|c} C_2 & -S_2 & 0 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \\
&\quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] && (55)\\
&= \left[\begin{array}{ccc|c} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{ccc|c} C_2 & 0 & S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] && (56)\\
&= \left[\begin{array}{ccc|c} C_1C_2 & -S_1 & C_1S_2 & 0 \\ S_1C_2 & C_1 & S_1S_2 & 0 \\ -S_2 & 0 & C_2 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] && (57)\\
&= \left[\begin{array}{ccc|c} C_1C_2 & -S_1 & C_1S_2 & C_1S_2q_3 \\ S_1C_2 & C_1 & S_1S_2 & S_1S_2q_3 \\ -S_2 & 0 & C_2 & C_2q_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] && (58)
\end{aligned}
$$

3.3 Inverse Kinematics

Assume we require the end-effector to be at position $[a, b, c]^T$, and we do not care about end-effector orientation. Show how to compute the values for all the robot joints such that the end-effector achieves the desired position. Be sure to list all possible solutions. If the number of solutions depends on certain characteristics of a, b and c, explain how and why.

$$
\begin{aligned}
C_1S_2q_3 &= a && (59)\\
S_1S_2q_3 &= b && (60)\\
C_2q_3 &= c && (61)
\end{aligned}
$$

which we must solve for all the joint values of our robot.

By squaring up all 3 equations and adding them together we obtain:

$$q_3^2 = a^2 + b^2 + c^2 \qquad (62)$$

Case 1: if $a^2 + b^2 + c^2 = 0$, or $a = b = c = 0$, then $q_3 = 0$. The rest of the equations are satisfied regardless of the values of θ_1 and θ_2. We thus have infinite solutions.

Case 2: if $a^2 + b^2 + c^2 > 0$:

$$q_3 = \pm\sqrt{a^2 + b^2 + c^2} \tag{63}$$

Note that there are two solutions for q_3. Intuitively, the robot can reach the same location by pointing towards it, and then extending by a positive q_3, or by pointing away from it, and then "extending" by a negative q_3.

By squaring up and adding (59) and (60) we obtain:

$$S_2^2 = (a^2 + b^2)/q_3^2 \tag{64}$$

$$S_2 = \pm\sqrt{(a^2 + b^2)/q_3^2} \tag{65}$$

Therefore, for each value of q_3 we have 2 possible values for S_2. For each of these, we compute:

$$C_2 = c/q_3 \tag{66}$$

$$\theta_2 = \mathtt{atan2}(S_2, C_2) \tag{67}$$

and finally

$$C_1 = a/S_2 q_3 \tag{68}$$

$$S_1 = b/S_2 q_3 \tag{69}$$

$$\theta_1 = \mathtt{atan2}(S_1, C_1) \tag{70}$$

and we are done.

Since for any query we have 2 possible values of q_3, and for each of those we have 2 possible values of S_2, we obtain a total of 4 possible solutions for each query.

3.4 Differential Kinematics

Compute the manipulator Jacobian with respect to end-effector position (and ignoring end-effector orientation). Find all the joint configurations where the Jacobian becomes singular; for each of them, briefly explain why the robot is in a singular configuration (for example, which direction of end-effector motion is impossible or which robot joint has no effect on end-effector position).

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} C_1 S_2 q_3 \\ S_1 S_2 q_3 \\ C_2 q_3 \end{bmatrix} \tag{71}$$

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{bmatrix} \tag{72}$$

$$= \begin{bmatrix} -S_1 S_2 q_3 & C_1 C_2 q_3 & C_1 S_2 \\ C_1 S_2 q_3 & S_1 C_2 q_3 & S_1 S_2 \\ 0 & -S_2 q_3 & C_2 \end{bmatrix} \tag{73}$$

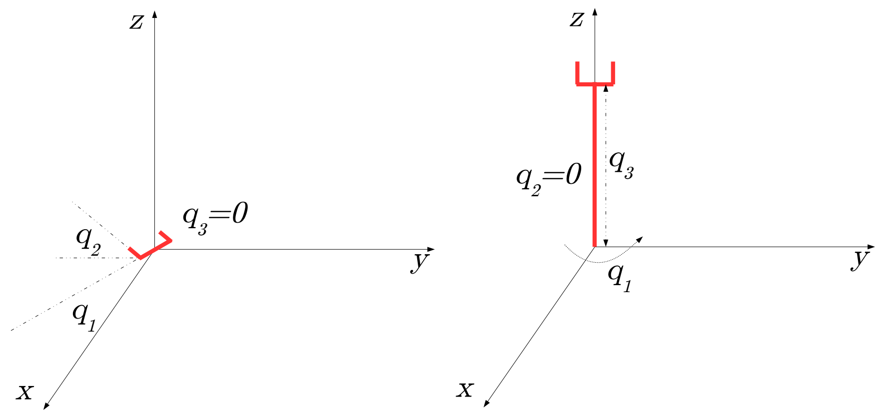
$$\mathtt{det}(\boldsymbol{J}) = S_2 q_3 \begin{vmatrix} -S_1 S_2 q_3 & C_1 S_2 \\ C_1 S_2 q_3 & S_1 S_2 \end{vmatrix} + C_2 \begin{vmatrix} -S_1 S_2 q_3 & C_1 C_2 q_3 \\ C_1 S_2 q_3 & S_1 C_2 q_3 \end{vmatrix} \tag{74}$$

$$= S_2 q_3^2(-S_1^2 S_2^2 - C_1^2 S_2^2) + C_2 q_3^2(-S_1^2 C_2 S_2 - C_1^2 S_2 C_2) \tag{75}$$

$$= -S_2^3 q_3^2 - C_2^2 S_2 q_3^2 = -S_2 q_3^2 \tag{76}$$

We notice there are two possible singular configurations: $q_3 = 0$ and $S_2 = 0$.

- $q_3 = 0$: if the only link of the robot has 0 length, then the robot can not really position the end-effector anywhere in space except at the origin.
- $S_2 = 0$: if $q_2 = 0$ or $q_2 = \pi$, the robot is pointing straight up or down, aligned with the rotation axis of joint 1; in this case, q_1 does not change the position of the end-effector, but just rotates it in place. The only infinitesimal movement that is possible is that generated by q_2.

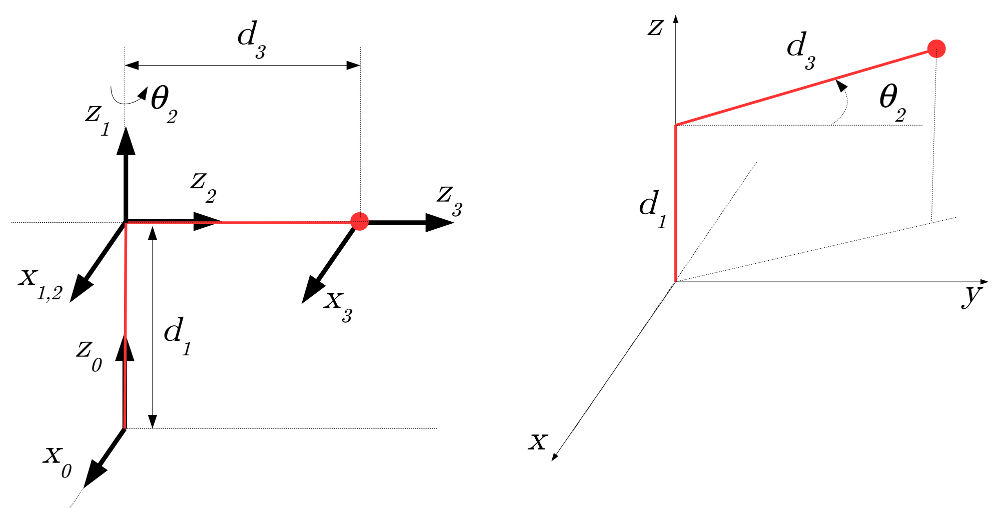


4 Exercise Robot 1

i	θ	d	a	α
1	0	q_1	0	0
2	q_2	0	0	$-90°$
3	0	q_3	0	0

4.1 Sketch

In a right-handed XYZ coordinate frame, draw a sketch of this robot showing the spatial relationships between the coordinate frames associated with each link.



4.2 Forward Kinematics

Compute the transform matrix ${}^b\boldsymbol{T}_{ee}$ from the base of the robot to its end-effector.

$$
\begin{aligned}
{}^bT_{ee} &= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} C_2 & -S_2 & 0 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{ccc|c} C_2 & -S_2 & 0 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & q_1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{ccc|c} C_2 & 0 & -S_2 & -S_2 q_3 \\ S_2 & 0 & C_2 & C_2 q_3 \\ 0 & -1 & 0 & q_1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad (77)
\end{aligned}
$$

4.3 Inverse Kinematics

Assume we require the end-effector to be at position $[a, b, c]^T$, and we do not care about end-effector orientation. Show how to compute the values for all the robot joints such that the end-effector achieves the desired position. Be sure to list all possible solutions. If the number of solutions depends on certain characteristics of a, b and c, explain how and why.

$$
\begin{aligned}
-S_2 q_3 &= a \qquad (78) \\
C_2 q_3 &= b \qquad (79) \\
q_1 &= c \qquad (80)
\end{aligned}
$$

The last equation directly gives us the value of $q_1 = c$.

Squaring up and adding the first two equations we obtain:

$$
\begin{aligned}
q_3^2 &= a^2 + b^2 \qquad (81) \\
q_3 &= \pm\sqrt{a^2 + b^2} \qquad (82)
\end{aligned}
$$

If $a = b = 0$, then $q_3 = 0$ and q_2 is undefined. We thus have an infinity of solutions in this case:

$$q_1 = c,\ q_2 = \text{undefined},\ q_3 = 0 \qquad (83)$$

If $a^2 + b^2 \neq 0$, then $q_3 = \pm\sqrt{a^2 + b^2}$ gives us 2 solutions for q_3. Plugging back into the first two equations, we obtain $S_2 = -\frac{a}{q_3}$ and $C_2 = \frac{b}{q_3}$, which uniquely identify q_2 as $\texttt{atan2}\left(-\frac{a}{q_3}, \frac{b}{q_3}\right)$. We thus have 2 total solutions in this case.

$$q_1 = c,\ q_2 = \texttt{atan2}\left(-\frac{a}{q_3}, \frac{b}{q_3}\right),\ q_3 = \pm\sqrt{a^2 + b^2} \qquad (84)$$

4.4 Differential Kinematics

Compute the manipulator Jacobian with respect to end-effector position (and ignoring end-effector orientation). Find all the joint configurations where the Jacobian becomes singular; for each of them, briefly explain why the robot is in a singular configuration (for example, which direction of end-effector motion is impossible or which robot joint has no effect on end-effector position).

$$\boldsymbol{J} = \begin{bmatrix} 0 & -C_2 q_3 & -S_2 \\ 0 & -S_2 q_3 & C_2 \\ 1 & 0 & 0 \end{bmatrix} \tag{85}$$

$$\mathtt{det}(\boldsymbol{J}) \quad = \quad -C_2^2 q_3 - S_2^2 q_3 = q_3 \tag{86}$$

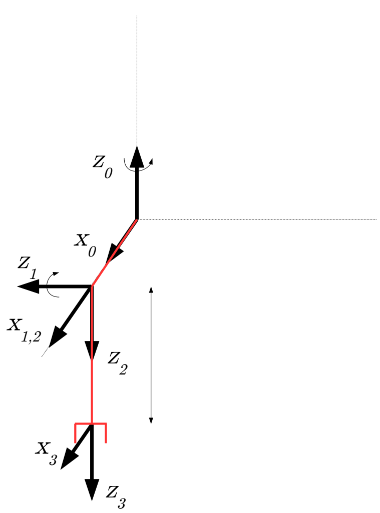
The robot is in a singular configuration, with $\mathtt{det}(\boldsymbol{J}) = 0$, if $q_3 = 0$. In this case, q_2 loses the ability to change the position of the end-effector.

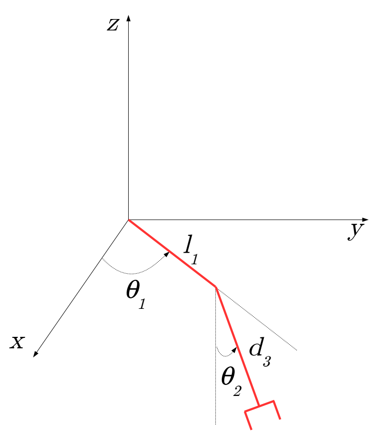
5 Exercise Robot 2

i	θ	d	a	α
1	q_1	0	l_1	90°
2	q_2	0	0	90°
3	0	q_3	0	0

5.1 Sketch

In a right-handed XYZ coordinate frame, draw a sketch of this robot showing the spatial relationships between the coordinate frames associated with each link.





5.2 Forward Kinematics

Compute the **translation part** of the transform matrix ${}^{b}\boldsymbol{T}_{ee}$ from the base of the robot to its end-effector.

Note that since we're asked to compute just the translation part of ${}^{b}T_{ee}$, we keep only the translation part of the last matrix in the transform chain. This simplifies our calculations.

$$
\begin{aligned}
{}^{b}T_{ee} &= \left[\begin{array}{ccc|c} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} C_2 & -S_2 & 0 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \\
&\quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} 0 \\ 0 \\ q_3 \\ \hline 1 \end{array}\right] \qquad (87) \\
&= \left[\begin{array}{ccc|c} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} C_2 & -S_2 & 0 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} 0 \\ -q_3 \\ 0 \\ \hline 1 \end{array}\right] \\
&= \left[\begin{array}{ccc|c} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} S_2 q_3 \\ -C_2 q_3 \\ 0 \\ \hline 1 \end{array}\right] \\
&= \left[\begin{array}{ccc|c} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} S_2 q_3 \\ 0 \\ -C_2 q_3 \\ \hline 1 \end{array}\right] \\
&= \left[\begin{array}{ccc|c} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} S_2 q_3 + l_1 \\ 0 \\ -C_2 q_3 \\ \hline 1 \end{array}\right] \\
&= \left[\begin{array}{c} C_1(S_2 q_3 + l_1) \\ S_1(S_2 q_3 + l_1) \\ -C_2 q_3 \\ \hline 1 \end{array}\right] \qquad (88)
\end{aligned}
$$

5.3 Differential Kinematics

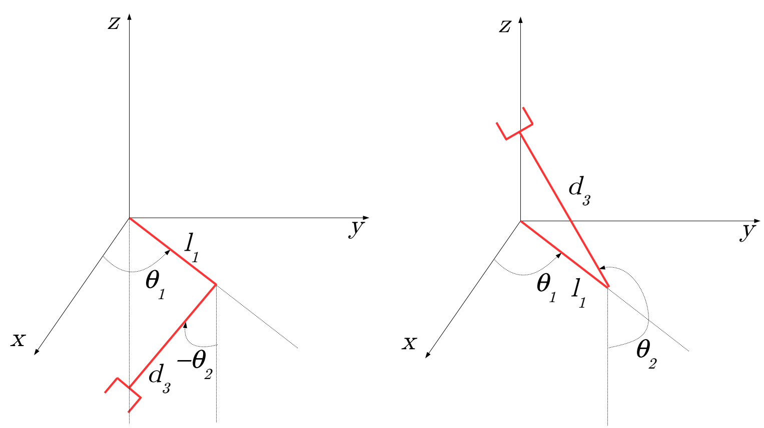
Compute the manipulator Jacobian with respect to end-effector position (and ignoring end-effector orientation). Find all the joint configurations where the Jacobian becomes singular; for each of them, briefly explain why the robot is in a singular configuration (for example, which direction of end-effector motion is impossible, which 2 joints produce the same end-effector motion, or which joint has no effect on end-effector position).

$$
\boldsymbol{J} = \begin{bmatrix} -S_1(S_2 q_3 + l_1) & C_1 C_2 q_3 & C_1 S_2 \\ C_1(S_2 q_3 + l_1) & S_1 C_2 q_3 & S_1 S_2 \\ 0 & S_2 q_3 & -C_2 \end{bmatrix} \qquad (89)
$$

$$
\begin{aligned}
\mathtt{det}(\boldsymbol{J}) &= -S_2q_3[-S_1^2S_2(S_2q_3+l_1) - C_1^2S_2(S_2q_3+l_1)] - \\
&\quad -C_2[-S_1^2C_2q_3(S_2q_3+l_1) - C_1^2C_2q_3(S_2q_3+l_1)] && (90)\\
&= S_2^2q_3(S_2q_3+l_1) + C_2^2q_3(S_2q_3+l_1) && (91)\\
&= q_3(S_2q_3+l_1) && (92)
\end{aligned}
$$

The robot is in a singular configuration ($\mathtt{det}(\boldsymbol{J}) = 0$) when:

- $q_3 = 0$. In this case, q_2 can no longer move the end-effector.
- $S_2q_3 + l_1 = 0$. In this case, the end-effector is positioned exactly along the (positive or negative) z_0 axis, thus exactly under or above the first joint. q_1 can no longer move the end-effector, and just rotates it in place.

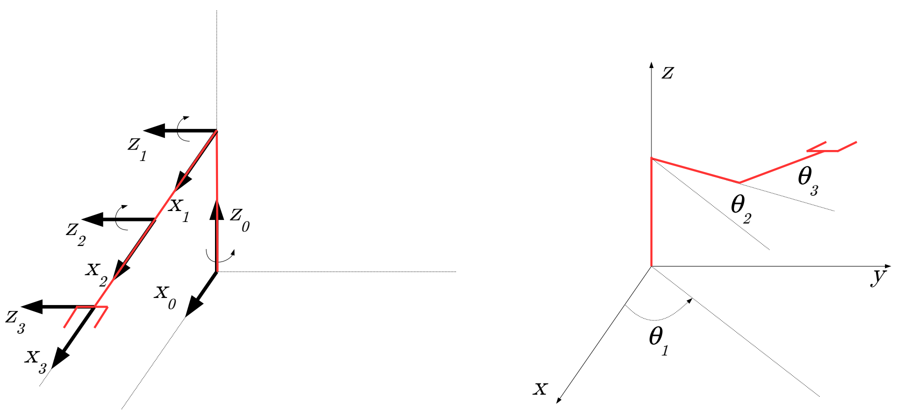


6 Exercise Robot 3

i	θ	d	a	α
1	q_1	1	0	90°
2	q_2	0	1	0
3	q_3	0	1	0

6.1 Sketch

In a right-handed XYZ coordinate frame, draw a sketch of this robot showing the spatial relationships between the coordinate frames associated with each link.



6.2 Forward Kinematics

Compute the **translation part** of the transform matrix ${}^b\boldsymbol{T}_{ee}$ from the base of the robot to its end-effector.

$$
{}^b\boldsymbol{T}_{ee} = \left[\begin{array}{ccc|c} & & & C_1(C_2 + C_{23}) \\ & \boldsymbol{R} & & S_1(C_2 + C_{23}) \\ & & & S_2 + S_{23} + 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{93}
$$

We recall that we are using the notation $S_{23} = S(q_2 + q_3)$.

6.3 Differential Kinematics

Compute the manipulator Jacobian with respect to end-effector position (and ignoring end-effector orientation). Find all the joint configurations where the Jacobian becomes singular; for each of them, briefly explain why the robot is in a singular configuration (for example, which direction of end-effector motion is impossible, which 2 joints produce the same end-effector motion, or which joint has no effect on end-effector position).

$$
\boldsymbol{J} = \begin{bmatrix} -S_1(C_2 + C_{23}) & -C_1(S_2 + S_{23}) & -C_1 S_{23} \\ C_1(C_2 + C_{23}) & -S_1(S_2 + S_{23}) & -S_1 S_{23} \\ 0 & C_2 + C_{23} & C_{23} \end{bmatrix} \tag{94}
$$

$$
\texttt{det}(\boldsymbol{J}) = -(C_2 + C_{23}) S_3 \tag{95}
$$

The robot is in a singular configuration ($\texttt{det}(\boldsymbol{J}) = 0$ when:

- $S_3 = 0$ ($q_3 = 0$ or $q_3 = \pi$). The last two links of the robot are either stretched out or folded on themselves. q_2 and q_3 move the robot along the same direction. Note that this is the same singularity of the planar 2-link robot. In fact, the last 2 links of this robot form a planar 2-link operating in the xz plane.
- $C_2 + C_{23} = 0$. The end-effector is positioned exactly along the (positive or negative) z_0 axis, thus exactly under or above the first joint. q_1 can no longer move the end-effector, and just rotates it in place.

7 Linear Algebra and Redundant Robots

Consider a 3-joint planar robot. The position Jacobian $\boldsymbol{J} \in \mathcal{R}^{2x3}$ relates velocities of the three joints to linear velocities of the end-effector in the plane. The SVD decomposition of the Jacobian yields the following matrices:

$$
\boldsymbol{s} = [1.41, 0.0] \quad \boldsymbol{U} = \begin{bmatrix} -0.70 & 0.70 \\ 0.70 & 0.70 \end{bmatrix} \quad \boldsymbol{V}^T = \begin{bmatrix} 0.70 & 0.0 & -0.70 \\ -0.63 & -0.44 & -0.63 \\ 0.31 & -0.89 & 0.31 \end{bmatrix} \tag{96}
$$

a) What is the rank of the Jacobian?
Answer: As a single singular value is non-zero, the Jacobian has rank 1.

b) What is the dimensionality of the nullspace of the Jacobian?
Answer: As the rank of the matrix (which is 1) plus the dimensionality of the nullspace have to sum up to the number of columns (which is 3), it follows that the nullspace has dimensionality 2.

c) What is the direction in Cartesian space that the robot can not generate any velocity in?
Answer: this direction is given by the column of the $\boldsymbol{U}$ matrix associated with a 0 singular value, or $[0.70, 0.70]^T$.

d) Write a joint velocity vector that is in the null space of the Jacobian.
Answer: any column of $\boldsymbol{V}$ (row of $\boldsymbol{V}^T$) corresponding to a zero singular value will be in the null space of the matrix. Thus, either of the last two rows of $\boldsymbol{V}^T$ (as well as linear combination of them) will be in the null space of $\boldsymbol{J}$. For example, the vector $[0.31, -0.89, 0.31]^T$ is an answer to this question.