MECEE 4602: Introduction to Robotics, Fall 2019

Velocity Conversions

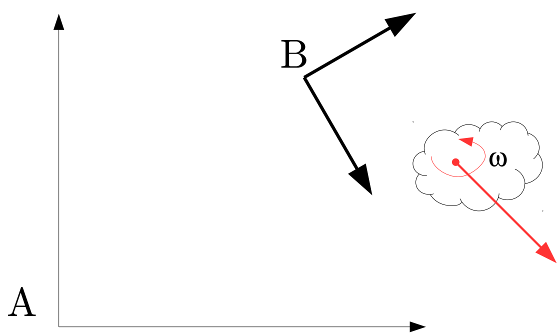
1 Velocity Conversions Between Frames

In this section, we focus on the following problem. Assume we know the velocity of a given point p, expressed relative to a frame B (${}^B\boldsymbol{v}_p$). We also have frame A, and we know the transform from A to B (${}^AT_B$). What is the velocity of the same point, expressed relative to frame A (${}^A\boldsymbol{v}_p$)?

$$\begin{array}{cc} {}^B\boldsymbol{v}_p & \text{known} \\ {}^AT_B & \text{known} \\ {}^A\boldsymbol{v}_p & ? \end{array}$$

1.1 2D Case

In this case, a velocity vector has 3 components: translation velocity components $\dot{x}$ and $\dot{y}$, and angular velocity ω (implied to be around the only possible axis of rotation in 2D, which is z). The transform between frames has the well-known rotation and translation components.



$$\boldsymbol{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega \end{bmatrix} \qquad {}^AT_B = \begin{bmatrix} {}^AR_B & {}^At_B \\ 0 & 1 \end{bmatrix} \tag{1}$$

In this case, velocity conversion between frames can be computed as:

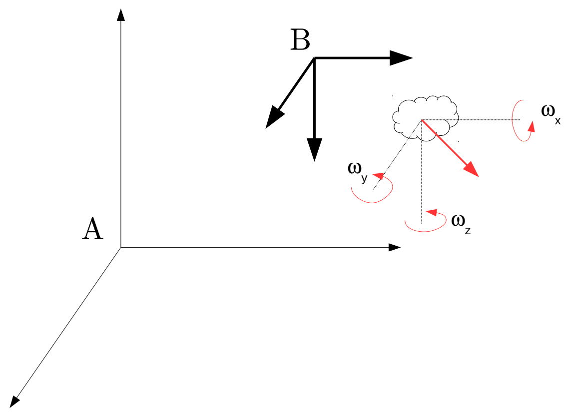
$${}^A\boldsymbol{v}_p = \begin{bmatrix} {}^AR_B & 0 \\ 0 & 1 \end{bmatrix} {}^B\boldsymbol{v}_p \tag{2}$$

We notice that:

- translational velocity components have simply been rotated to reflect the new axes
- angular velocity is unchanged (since the axis of rotation, which is z, has not changed, and in fact can not change as long as we remain in the same xy plane).

1.2 3D Case

In this case, a velocity vector has 3 components: translation velocity components along all the axes ($\dot{x}$, $\dot{y}$ and $\dot{z}$) as well as angular velocity around all the axes (ω_x, ω_y, ω_z).



$$\boldsymbol{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}t_{B} \\ 0 & 1 \end{bmatrix} \tag{3}$$

In this case, velocity conversion between frames can be computed as:

$${}^{A}\boldsymbol{v}_p = \begin{bmatrix} {}^{A}R_{B} & 0 \\ 0 & {}^{A}R_{B} \end{bmatrix} {}^{B}\boldsymbol{v}_p \tag{4}$$

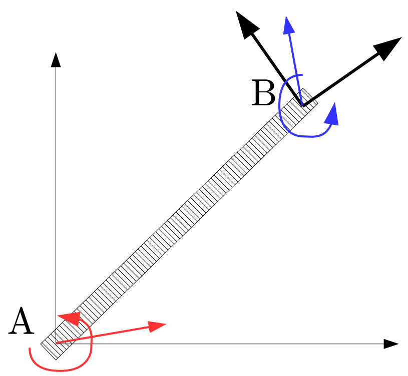
We notice that both the translational and angular components of the velocity simply get rotated to reflect the new axes.

2 Rigid Movement Transmission Between Frames

In this section, we are concerned with the following problem. Frames A and B are **rigidly connected to each other** - think of them as 2 points on the same rigid body. For a given local velocity of frame A, expressed in its own coordinate system (${}^{A}\boldsymbol{v}_A$), what is the resulting local velocity of frame B, also expressed in its own coordinate system (${}^{B}\boldsymbol{v}_B$)?

2.1 2D Case

In this case, a velocity vector has 3 components: translation velocity components $\dot{x}$ and $\dot{y}$, and angular velocity ω (implied to be around the only possible axis of rotation in 2D, which is z). The transform between frames has the well-known rotation and translation components.



$$\boldsymbol{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega \end{bmatrix} \qquad {}^{A}T_B = \begin{bmatrix} {}^{A}R_B & {}^{A}t_B \\ 0 & 1 \end{bmatrix} \tag{5}$$

In this case, the local velocity of frame B can be computed as:

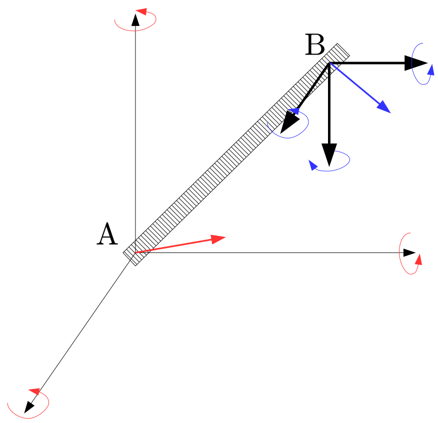
$${}^{B}\boldsymbol{v}_B = \begin{bmatrix} {}^{B}R_A & -{}^{B}R_A \cdot S({}^{A}t_B) \\ 0 & 1 \end{bmatrix} {}^{A}\boldsymbol{v}_A \tag{6}$$

where the function S is defined as:

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ -x \end{bmatrix} \tag{7}$$

2.2 3D Case

In this case, a velocity vector has 3 components: translation velocity components along all the axes ($\dot{x}$, $\dot{y}$ and $\dot{z}$) as well as angular velocity around all the axes (ω_x, ω_y, ω_z).



$$\boldsymbol{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}t_{B} \\ 0 & 1 \end{bmatrix} \tag{8}$$

In this case, the local velocity of frame B can be computed as:

$${}^{B}\boldsymbol{v}_{B} = \begin{bmatrix} {}^{B}R_{A} & -{}^{B}R_{A} \cdot S({}^{A}t_{B}) \\ 0 & {}^{B}R_{A} \end{bmatrix} {}^{A}\boldsymbol{v}_{A} \tag{9}$$

where the function S is defined as:

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \tag{10}$$

Note that the matrix obtained through the function S has the following property:

$$S(\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b} \tag{11}$$

where $\times$ denotes the cross-product of two vectors.