1.(a)

$$f_X = \begin{cases} \frac{1}{4} & \text{if } w = 1 \text{ or } 2 \text{ or } 3 \\ \frac{5}{12} & \text{if } w = 4 \text{ or } 5 \text{ or } 6 \text{ or } 7 \text{ or } 8 \\ \frac{1}{3} & \text{if } w = 9 \text{ or } 10 \text{ or } 11 \text{ or } 12 \\ 0 & \text{otherwise} \end{cases}$$

(b) $F_X(2.5) = \frac{1}{4} + \frac{5}{12} = \frac{8}{12} = \frac{2}{3}$

(c) $\lim_{x \to 2} F_X(x) = \frac{1}{4}$

(d) $\sum_{x \in X} x f_X(x) = 1 \times \frac{1}{4} + 2 \times \frac{5}{12} + 3 \times \frac{1}{3} = \frac{1}{4} + \frac{10}{12} + 1 = \frac{25}{12}$

(e) $\sum_{w \in \Omega} X(w) = 1 \times 3 + 2 \times 5 + 3 \times 4 = 3 + 10 + 12 = 25$

$$\frac{\sum_{w \in \Omega} X(w)}{n(\Omega)} = \frac{25}{12}$$

This is the same as $\sum_{x \in X} x f_X(x)$ in (d)

(f) $\sigma_X^2 = E((X - E(X))^2) = E((X - \frac{25}{12})^2) = E(X^2 - \frac{25}{6}X + \frac{25^2}{12^2})$

$= E(X^2) - \frac{25}{6}E(X) + \frac{25^2}{12^2}$

$= \frac{3 \cdot 1^2 + 5 \cdot 2^2 + 4 \cdot 3^2}{12} - \frac{25}{6} \cdot \frac{25}{12} + \frac{25^2}{12^2} = 0.576$

(g) X and Y are related. If Y=2, then X=2 or 3

$P(X=2 \mid Y=2) = \frac{1}{3}$ and $P(X=3 \mid Y=2) = \frac{2}{3}$

$P(X=2 \mid Y=2) = \frac{P(X=2 \cap Y=2)}{P(Y=2)} = \frac{\frac{2}{12}}{\frac{1}{2}} = \frac{4}{12} = \frac{1}{3} \neq P(Y=2)$, so

X and Y are not independent, so they are related

(h) $E(X|Y=1) = \sum_{x \in X} x f_{X|Y}(x|y=1) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} + 3 \cdot 0 = \frac{3}{2}$

(i) $Var(X|Y=1) = E((X-E(X|Y=1))^2|Y=1)$

$= E((X-\frac{3}{2})^2|Y=1)$

$= E((X^2-3X+\frac{9}{4})|Y=1)$

$= E(X^2|Y=1) - 3E(X|Y=1) + \frac{9}{4}$

$= \frac{3 \cdot 1^2 + 3 \cdot 2^2}{6} - 3 \cdot \frac{3}{2} + \frac{9}{4}$

$= \frac{15}{6} - \frac{9}{2} + \frac{9}{4}$

$= \frac{1}{4}$

2.(a) $h_X(64.4) = \frac{\frac{1}{8}}{65.4-62.2} = \frac{0.125}{3.2} = 0.039$

(b) $\int_{62.2}^{70} h(x)dx = F_X(70) - F_X(62.2) = \frac{8}{8} - \frac{5}{8} = \frac{3}{8}$

3.(a) $L(\theta|x) = \prod_{i=1}^{10} \frac{x_i^2}{2\theta^3} e^{-\frac{x_i}{\theta}} = \frac{\prod_{i=1}^{10} x_i^2}{2^{10}\theta^{30}} e^{-\frac{\sum_{i=1}^{10} x_i}{\theta}}$

$l(\theta|x) = \sum_{i=1}^{10} (2\log(x_i) - \log(2) - 3\log(\theta) - \frac{x_i}{\theta})$

$= 2\sum_{i=1}^{10} \log(x_i) - 10\log(2) - 30\log(\theta) - \frac{\sum_{i=1}^{10} x_i}{\theta}$

$l'(\theta|x) = \frac{-30}{\theta} + \frac{\sum_{i=1}^{10} x_i}{\theta^2} = 0$

$-30 = -\frac{\sum_{i=1}^{10} x_i}{\theta}$

$\theta = \frac{\sum_{i=1}^{10} x_i}{30}$

$l''(\theta|x) = \frac{30}{\theta^2} - \frac{2\sum_{i=1}^{10} x_i}{\theta^3} = \frac{30\theta - 2\sum_{i=1}^{10} x_i}{\theta^3} < 0$

$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{10} x_i}{30} \approx 0.45$ (As shown in below picture)

STAC58A1.R

yulunwu

2022-02-07

```
#3 (a)
x = c(1.56,2.54,1.08,2.45,0.39,0.4,2.56,1.24,1.03,0.33)
MLE_thetahat = sum(x)/30 # claculate the value of thetahat
round(MLE_thetahat,2)

## [1] 0.45
```

$= \hat{\theta}_{MLE}$

```
# 3(b)
# Likelihood function of theta
L = function(theta) {exp(-sum(x)/theta)*prod(x^2)/(2^10*theta^30)}
relative_L = L(0.5)/L(0.45) # relative likelihood = L(theta0=0.5)/~~MLE of theta~~
relative_L # print result
```

L(MLE of θ)

```
## [1] 0.8667214
```

$= P_{\theta_0}(X)$

(b) $P_{\theta_0}(x) = \dfrac{L(\theta_0|x)}{L(\hat{\theta}_{MLE}(x)|x)} = \dfrac{L(0.5|x)}{L(0.45|x)} = 0.8667214$ (as shown in above picture)

4. Since the distribution is Uniform$[\theta-0.5, \theta+0.5]$, the width of interval is 1.

Due to the property of Uniform distribution

$\theta - 0.5 \le \min(x) \le \max(x) \le \theta + 0.5$

$\because \min(x) = 1.5$, $\max(x) = 2$

$\therefore \theta - 0.5 \le 1.5 < 2 \le \theta + 0.5$

$\therefore \theta \le 1.5 + 0.5 \qquad 2 - 0.5 \le \theta$

$\Leftrightarrow \theta \le 2 \qquad \Leftrightarrow \theta \ge 1.5$

$\therefore \theta \in [1.5, 2]$

$$L(\theta|x) = \begin{cases} \dfrac{1}{\theta+0.5-\theta+0.5} = 1 & \theta \in [1.5, 2] \\ 0 & \text{otherwise} \end{cases}$$

$= I_{[1.5,2]}(\theta)$

STAC58A1.R

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```
#3 (a)
x = c(1.56,2.54,1.08,2.45,0.39,0.4,2.56,1.24,1.03,0.33)
MLE_thetahat = sum(x)/30 # claculate the value of thetahat
round(MLE_thetahat,2)
```

```
## [1] 0.45
```

```
# 3(b)
# Likelihood function of theta
L = function(theta) {exp(-sum(x)/theta)*prod(x^2)/(2^10*theta^30)}
relative_L = L(0.5)/L(0.45) # relative likelihood = L(theta0=0.5)/MLE_of_theta
relative_L # print result
```

```
## [1] 0.8667214
```

```
set.seed(2022)
# 5(a)
# function of generate sample with n = 10, mean = 6, sd = 2
sample=function(){
  x = rnorm(10, mean = 6, sd = 2)
  return(x)
}
x_a = sample()
Xa_bar = sum(x_a)/10 # Calculate X_bar
# Function to calculate r_i
R = function(x_i) {(x_i-Xa_bar)/sqrt(sum(x_a^2)-10*Xa_bar^2)}
sum_r = 0
# Loop through each x_i and calculate sum(r_i^3)
for (xa_i in x_a) {
  sum_r = sum_r + R(xa_i)^3
}
sk = sum_r/10 # Calculate sk
sk # print skew
```

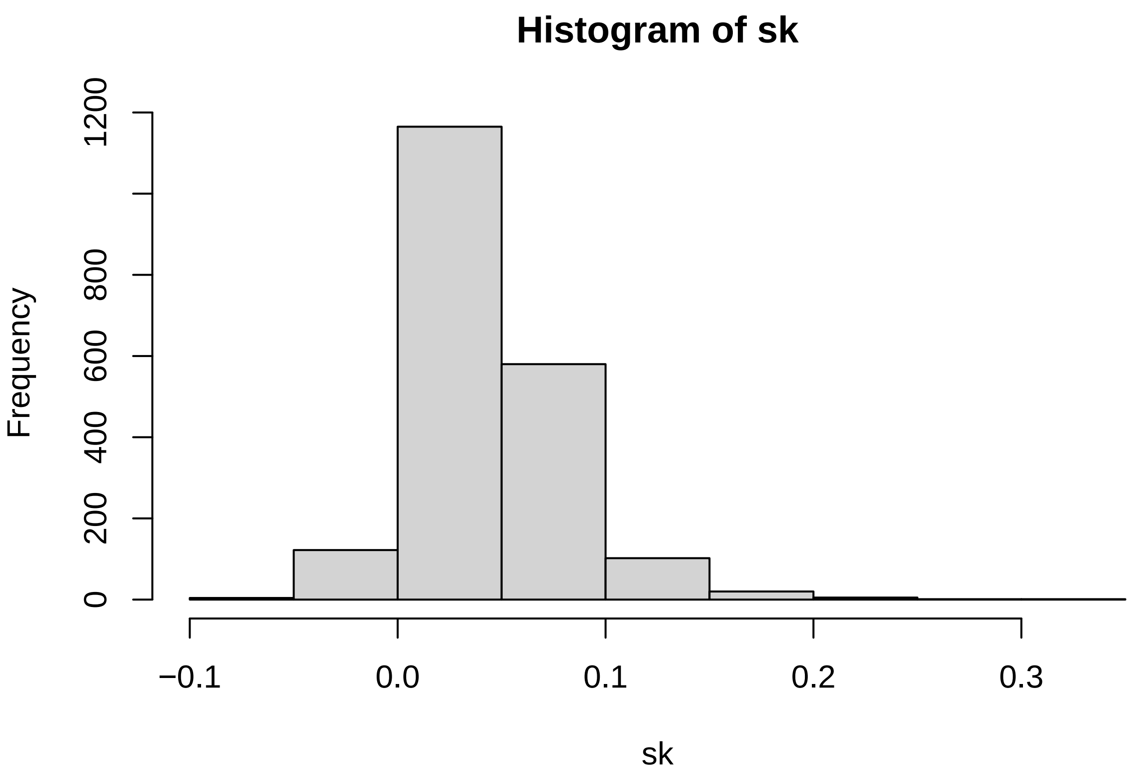
```
## [1] -0.01703587
```

```
# 5(b)
x_b = replicate(20,sample())# Generate 20 samples with n = 10, mean = 6, sd = 2
Xb_bar = sum(x_b)/10 # Calculate X_bar
sum_r = rep(0,20) # Initialize a array with length 20 with 0s
# Loop through each x[i,] and calculate sum(r_i^3)
for (i in c(1,2,3,4,5,6,7,8,9,10)) {
  sum_r = sum_r + R(x_b[i,])^3
}
sk = sum_r/10 # Calculate sk
```

```r
sk # print sk for the samples(total 20 sks)
```

```
##  [1]  0.026071125  0.033790646  0.027333826  0.007414895  0.070909859
##  [6]  0.066912588  0.087210386  0.154884602  0.108175937  0.040450849
## [11]  0.005249296  0.054470654  0.013292367  0.062075552 -0.004579188
## [16]  0.021162518  0.018829799  0.023203855  0.038953700 -0.004315430
```

```r
# 5(c)
# Generate 2000 samples with n = 10, mean = 6, sd = 2
x_c = replicate(2000,sample())
Xc_bar = sum(x_b)/10 # Calculate X_bar
sum_r = rep(0,2000) # Initialize a array with length 2000 with 0s
# Loop through each x[i,] and calculate sum(r_i^3)
for (i in c(1,2,3,4,5,6,7,8,9,10)) {
  sum_r = sum_r + R(x_c[i,])^3
}
sk = sum_r/10 # Calculate sk
hist(sk)
```




```r
# The histogram looks like skew more to the right, and the sk values are
# between -0.1 and 0.4.
# 5(d)
x = c(6.3,8.2,11.37,6.77,9.5,10.65,11.44,14.63,6.38,10.33)
X_bar = sum(x)/10
# Loop through each x_i and calculate sum(r_i^3)
sum_r = 0
for (x_i in x) {
  sum_r = sum_r + R(x_i)^3
}
skew = sum_r/10 # Calculate sk
```

```r
skew # print sk
```

```
## [1] 0.5808942
```

```r
# How many of those 2000 values have absolute value greater than or equal to
# the absolute value calculated for this sample (x)?
n_extreme = sum(abs(sk)>=skew)
n_extreme
```

```
## [1] 0
```

```r
# Calculate the propotion that of those 2000 samples having skewness statistic
# as extreme as or more extreme than the value calculated for this sample (x)
propotion = n_extreme/2000
propotion
```

```
## [1] 0
```

```r
# Conclusion: This sample x is not generated from Normal distribution with
# mean = 6 and sd = 2, because there is not sample in 2000 samples have
# skewness morw extreme than this sample.
```