**BN5205 Assignment (Part 1)**

Due date: October $1^{st}$, 2018, 11:55pm

Electrical potential ($V$) propagates along the axon of neurons. Interestingly, the phenomenon can be described by a reaction-diffusion equation. If we assume the axon as a 1D structure (length $= 4cm$) with a coordinate $x$ along the axon, then

$$\sigma \frac{\partial^2 V}{\partial x^2} = c_m \frac{\partial V}{\partial t} + I_{ion} \tag{1}$$

where $t$ is time in $ms$, $\sigma$ is the conductivity of the axon, $c_m$ is the capacitance of the axon per unit area and $I_{ion}$ is an electrical current due to the opening and closing of the various ion channels on the neuronal membrane. $V$ is measured in $mV$. Note that $I_{ion}$ depends on $V$ because some ion channels are voltage and time dependent. The $I_{ion}$ term is given by the Hodgkin and Huxley model

$$I_{ion} = I_{Na} + I_K + I_{leak} + I_{stim} \tag{2}$$

where $I_{Na}$, $I_K$ and $I_{leak}$ are currents through the $Na^+$, $K^+$ and leak channels respectively, while $I_{stim}$ is a stimulus current. The equations for $I_{Na}$, $I_K$ and $I_{leak}$ are

$$\begin{aligned} I_{Na} &= G_{Na} * m^3 * h * (V - E_{Na}) \\ I_K &= G_K * n^4 * (V - E_K) \\ I_{leak} &= G_{leak} * (V - E_{leak}) \end{aligned} \tag{3}$$

where $G_{Na}$, $G_K$ and $G_{leak}$ are conductances and $E_{Na}$, $E_K$ and $E_{leak}$ are Nernst potential for $Na^+$, $K^+$ and leak channels respectively. The variables $m$, $h$, and $n$ are known as gating variables and they represent the ability of the ion channel to open and close depending of the value of the voltage $V$. They are given by

$$\begin{aligned} \frac{dm}{dt} &= \frac{m_\infty - m}{\tau_m} \\ \frac{dh}{dt} &= \frac{h_\infty - h}{\tau_h} \\ \frac{dn}{dt} &= \frac{n_\infty - n}{\tau_n} \end{aligned} \tag{4}$$

where

$$\begin{aligned} m_\infty &= \frac{\alpha_m}{\alpha_m + \beta_m}, \quad \tau_m = \frac{1}{\alpha_m + \beta_m} \\ h_\infty &= \frac{\alpha_h}{\alpha_h + \beta_h}, \quad \tau_h = \frac{1}{\alpha_h + \beta_h} \\ n_\infty &= \frac{\alpha_n}{\alpha_n + \beta_n}, \quad \tau_n = \frac{1}{\alpha_n + \beta_n} \end{aligned} \tag{5}$$

The expressions for $\alpha$ and $\beta$ for each gating variable is given by[1]

$$\begin{aligned} \alpha_m &= \frac{40 + V}{1 - e^{-0.1*(40+V)}}, \quad \beta_m = 0.108 * e^{-\frac{V}{18}} \\ \alpha_h &= 0.0027 * e^{-\frac{V}{20}}, \quad \beta_h = \frac{1}{1 + e^{-0.1*(35-V)}} \\ \alpha_n &= \frac{0.01 * (55 + V)}{1 - e^{-0.1*(55+V)}}, \quad \beta_n = 0.055 * e^{-\frac{V}{80}} \end{aligned} \tag{6}$$

[1] you may want to use `from math import exp` at the beginning of your code in order to be able to use the syntax `exp(x)` to calculate the exponential of `x`

The value of the constants for this model are given in the following table

| Symbol | Description | Value | Units |
|---|---|---|---|
| $G_{Na}$ | Maximal conductance ($Na^+$) | 120 | $mS/cm^2$ |
| $G_K$ | Maximal conductance ($K^+$) | 36 | $mS/cm^2$ |
| $G_{leak}$ | Maximal conductance ($leak$) | 0.3 | $mS/cm^2$ |
| $c_m$ | Cell capacitance | 1 | $\mu F/cm^2$ |
| $\sigma$ | Axon conductivity | 0.001 | $mS$[2] |
| $E_{Na}$ | Nernst potential ($Na^+$) | 50 | $mV$ |
| $E_K$ | Nernst potential ($K^+$) | $-77$ | $mV$ |
| $E_{leak}$ | Nernst potential ($leak$) | $-54.4$ | $mV$ |

We will consider a simulation time from $t = 0$ to $t = 25ms$. At $t = 0$, the value of $V$ is $-70mV$ throughout the axon, while the values of $m$, $h$ and $n$ is zero throughout the axon. Both ends of the axon are characterized by a "no flux" boundary condition. $I_{stim} = 0$ throughout the axon at all times except for the following: $I_{stim} = -25\mu A/cm^2$ at one end of the axon ($x <= 0.3cm$) during the time interval $0.5ms \leq t \leq 0.9ms$.

Using a spatial discretization step of $0.1cm$ and a suitable time step, use the Finite Difference FTCS method to solve Equation 1. Use the Forward Euler method for any ODE involved in the model.

You should submit to IVLE your python file (one file only) containing the necessary code. Besides solving Equation 1, your code is supposed to

- Plot $V$ versus time for the first and last node of the axon.
- Display (simply using "print") the value of conduction velocity between the beginning and end of the axon. Conduction velocity is defined as $D/\Delta T$, where $D$ is the length of the axon and $\Delta T = t_{peak-end} - t_{peak-start}$. $t_{peak-end}$ is the time where $V$ reaches its peak (highest) value at the last node of the axon, while $t_{peak-start}$ is the time where $V$ reaches its peak (highest) value at the first node of the axon.

[2]The units for conductivity are normally $mS$ per unit length. In standard electrophysiology theory, a surface-to-volume ratio term (in units of of *"per unit length"*) multiplies all the terms to the RHS of Equation 1 and the equation is still balanced. For simplicity, here we use a unit of $\sigma$ that will not require any additional terms to the RHS of Equation 1.