Proof: 3D DFT and IDFT can be computed by multi-pass 1D DFTs

We begin with the 3D DFT definition:

$$\Psi(f_x, f_y, f_t) = \sum_{x=0}^{w-1} \sum_{y=0}^{h-1} \sum_{t=0}^{n-1} \psi(x, y, t) \cdot e^{-j2\pi\left(\frac{f_x x}{w} + \frac{f_y y}{h} + \frac{f_t t}{n}\right)}$$

Because the exponential is separable, we can factor it:

$$= \sum_{x=0}^{w-1} \sum_{y=0}^{h-1} \sum_{t=0}^{n-1} \psi(x, y, t) \cdot e^{-j2\pi\frac{f_t t}{n}} \cdot e^{-j2\pi\frac{f_y y}{h}} \cdot e^{-j2\pi\frac{f_x x}{w}}$$

Rewriting the summation order by ppt:

$$= \sum_{x=0}^{w-1} e^{-j2\pi\frac{f_x x}{w}} \sum_{y=0}^{h-1} e^{-j2\pi\frac{f_y y}{h}} \sum_{t=0}^{n-1} \psi(x, y, t) \cdot e^{-j2\pi\frac{f_t t}{n}}$$

We now define intermediate terms:

$$\begin{aligned}
\psi_1(x, y, f_t) &= \sum_{t=0}^{n-1} \psi(x, y, t) \cdot e^{-j2\pi\frac{f_t t}{n}} \\
\psi_2(x, f_y, f_t) &= \sum_{y=0}^{h-1} \psi_1(x, y, f_t) \cdot e^{-j2\pi\frac{f_y y}{h}} \\
\Psi(f_x, f_y, f_t) &= \sum_{x=0}^{w-1} \psi_2(x, f_y, f_t) \cdot e^{-j2\pi\frac{f_x x}{w}}
\end{aligned}$$

Conclusion (DFT)

Thus, the 3D DFT can be computed as three passes of 1D DFT:

$$\psi(x, y, t) \xrightarrow{\text{1D DFT along } t} \psi_1(x, y, f_t) \xrightarrow{\text{1D DFT along } y} \psi_2(x, f_y, f_t) \xrightarrow{\text{1D DFT along } x} \Psi(f_x, f_y, f_t)$$

Now consider the 3D inverse DFT:

$$\psi(x, y, t) = \frac{1}{whn} \sum_{f_x=0}^{w-1} \sum_{f_y=0}^{h-1} \sum_{f_t=0}^{n-1} \Psi(f_x, f_y, f_t) \cdot e^{j2\pi\left(\frac{f_x x}{w} + \frac{f_y y}{h} + \frac{f_t t}{n}\right)}$$

This is also separable:

$$= \frac{1}{whn} \sum_{f_x=0}^{w-1} e^{j2\pi\frac{f_x x}{w}} \sum_{f_y=0}^{h-1} e^{j2\pi\frac{f_y y}{h}} \sum_{f_t=0}^{n-1} \Psi(f_x, f_y, f_t) \cdot e^{j2\pi\frac{f_t t}{n}}$$

Define intermediate terms for IDFT:

$$
\begin{aligned}
\Psi_1(f_x, f_y, t) &= \sum_{f_t=0}^{n-1} \Psi(f_x, f_y, f_t) \cdot e^{j2\pi \frac{f_t t}{n}} \\
\Psi_2(f_x, y, t) &= \sum_{f_y=0}^{h-1} \Psi_1(f_x, f_y, t) \cdot e^{j2\pi \frac{f_y y}{h}} \\
\psi(x, y, t) &= \frac{1}{whn} \sum_{f_x=0}^{w-1} \Psi_2(f_x, y, t) \cdot e^{j2\pi \frac{f_x x}{w}}
\end{aligned}
$$

Conclusion (IDFT)

3D IDFT can also be computed by three successive 1D IDFTs:

$$
\Psi(f_x, f_y, f_t) \xrightarrow{\text{1D IDFT along } f_t} \Psi_1(f_x, f_y, t) \xrightarrow{\text{1D IDFT along } f_y} \Psi_2(f_x, y, t) \xrightarrow{\text{1D IDFT along } f_x} \psi(x, y, t)
$$