HW3: Fourier Transform Proofs

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1 Proof of Inverse Fourier Transform

We want to prove the following inverse Fourier transform formula:

$$\psi(x) = \int_{\mathbb{R}^K} \Psi(f) e^{j2\pi f^T x} df. \tag{1}$$

From the lecture notes (page 10), we use the result:

$$\int_{-\infty}^{\infty} e^{j2\pi f_0 x} dx = \delta(f_0), \tag{2}$$

where $\delta(f_0)$ is the Dirac delta function.

Taking the Fourier transform definition:

$$\Psi(f) = \int_{\mathbb{R}^K} \psi(x) e^{-j2\pi f^T x} dx, \tag{3}$$

we substitute this into our equation:

$$\int_{\mathbb{R}^K} \Psi(f) e^{j2\pi f^T x} df = \int_{\mathbb{R}^K} \left(\int_{\mathbb{R}^K} \psi(y) e^{-j2\pi f^T y} dy \right) e^{j2\pi f^T x} df.$$

Interchanging the order of integration:

$$\int_{\mathbb{R}^K} \psi(y) \left(\int_{\mathbb{R}^K} e^{-j2\pi f^T y} e^{j2\pi f^T x} df \right) dy. \tag{4}$$

Using the result $\int e^{j2\pi f(x-y)} df = \delta(x-y)$, we obtain:

$$\psi(x) = \int_{\mathbb{R}^K} \psi(y) \delta(x-y) dy = \psi(x), \tag{5}$$

which completes the proof.

2 Proof of Convolution Theorem

We aim to prove:

$$\phi(x) = \psi(x) * h(x) \leftrightarrow \Phi(f) = \Psi(f)H(f). \tag{6}$$

By definition, the convolution is:

$$\phi(x) = (\psi * h)(x) = \int_{\mathbb{R}^K} \psi(y)h(x-y)dy. \tag{7}$$

Taking the Fourier transform:

$$\begin{aligned}\Phi(f) &= \int_{\mathbb{R}^K} \phi(x)e^{-j2\pi f^T x}dx \\ &= \int_{\mathbb{R}^K} \left(\int_{\mathbb{R}^K} \psi(y)h(x-y)dy\right) e^{-j2\pi f^T x}dx.\end{aligned}$$

Interchanging the order of integration:

$$\int_{\mathbb{R}^K} \psi(y) \left(\int_{\mathbb{R}^K} h(x-y)e^{-j2\pi f^T x}dx\right) dy. \tag{8}$$

Using the substitution $u = x - y$, we get:

$$\int_{\mathbb{R}^K} \psi(y)e^{-j2\pi f^T y} \left(\int_{\mathbb{R}^K} h(u)e^{-j2\pi f^T u}du\right) dy. \tag{9}$$

Recognizing the Fourier transforms, we obtain:

$$\Phi(f) = \Psi(f)H(f), \tag{10}$$

which completes the proof.