Algorithms and Data Structures

COMP261
Fast Fourier Transform 1

Yi Mei

yi.mei@ecs.vuw.ac.nz

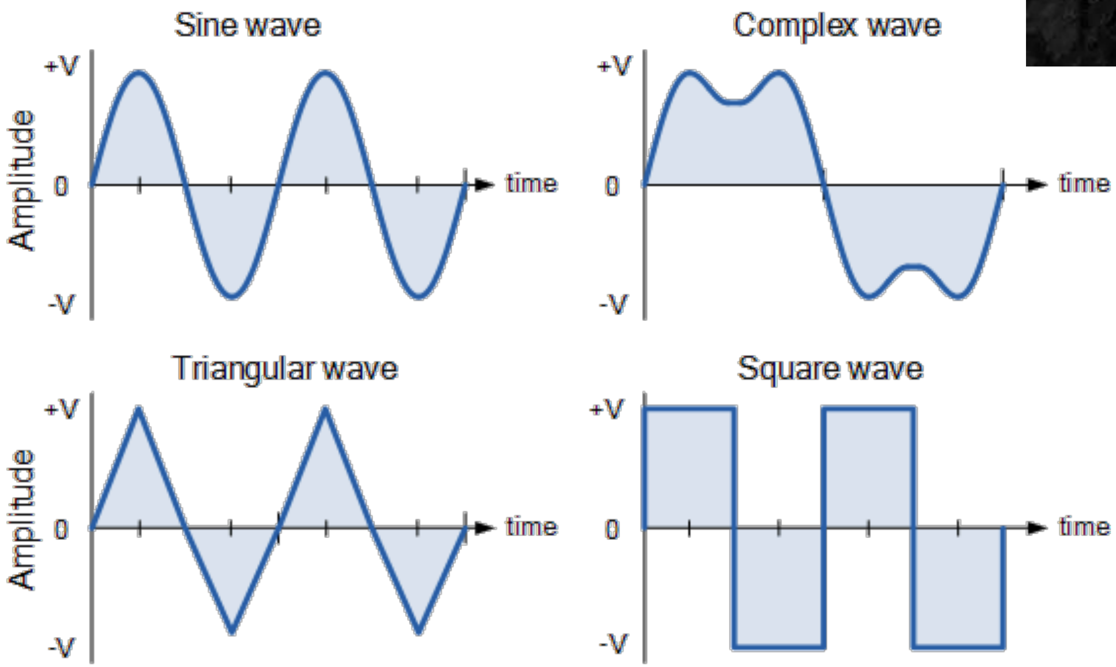
Outline

- Waveform (Signal) processing
- Fourier Series (Periodic waveform)
  - Real numbers
  - Complex numbers
- Fourier Transform (General waveform)
- Discrete Fourier Transforms

Waveform Processing

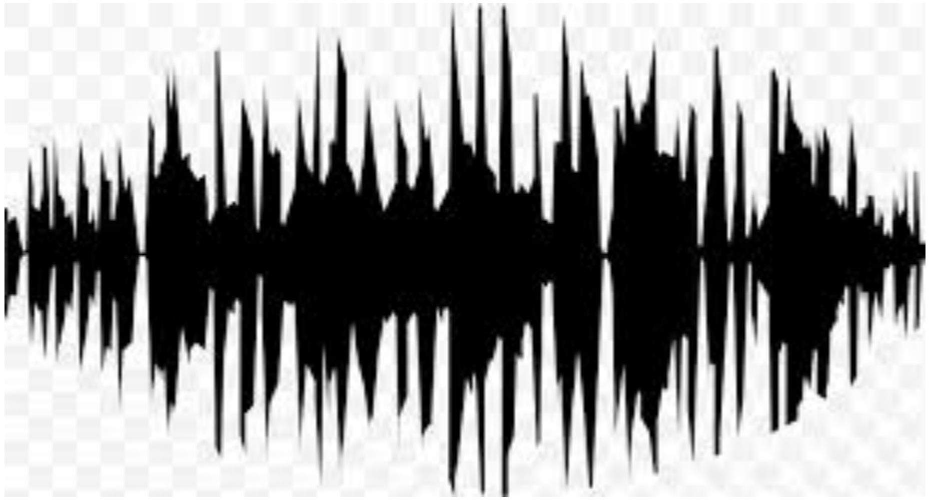
- Virtually everything in the world can be described via a waveform
  - Sound waves
  - Electromagnetic fields
  - Stock price series

Duc de Broglie



Waveform Processing

- Two representations

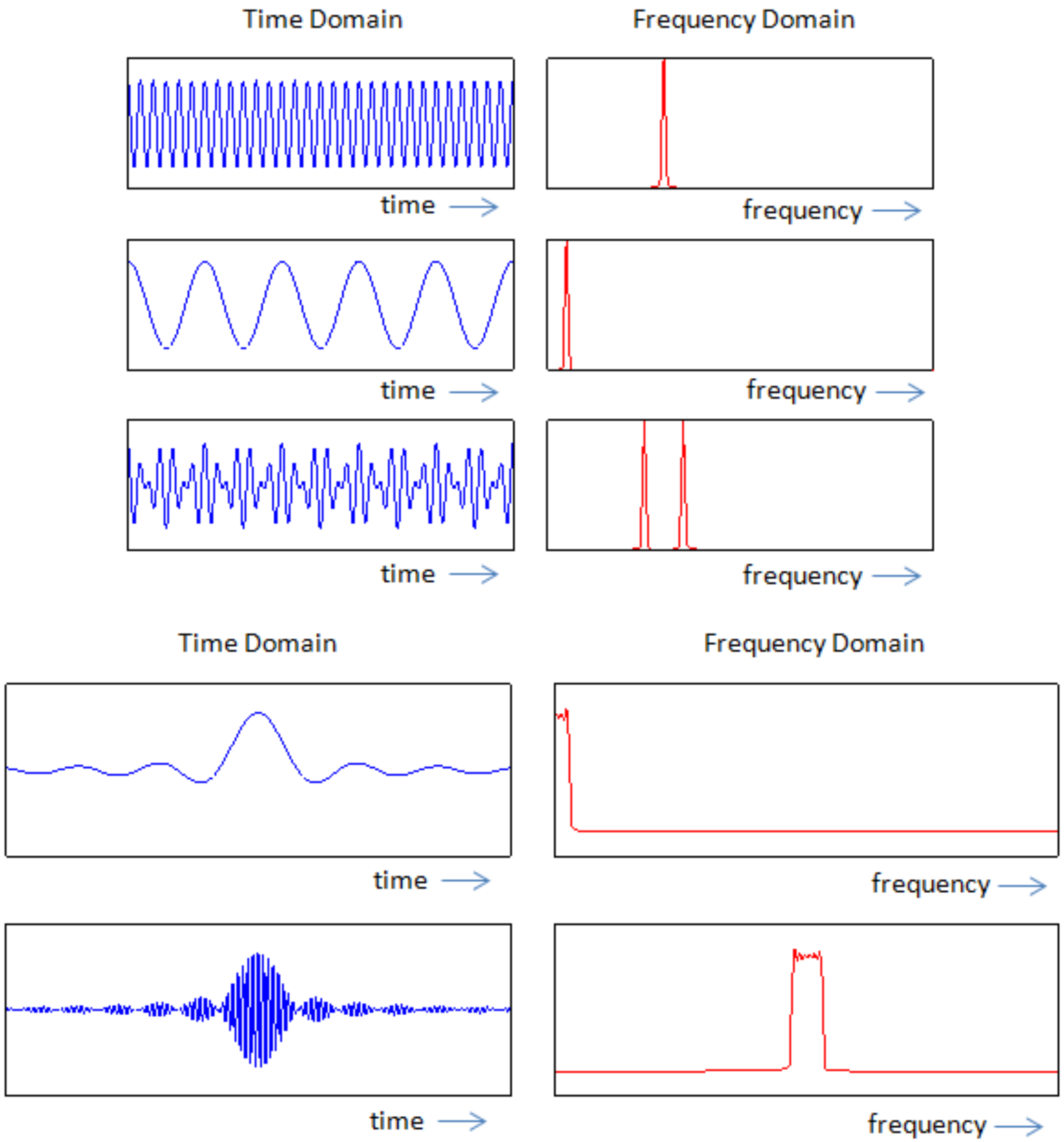


Time domain

Frequency domain

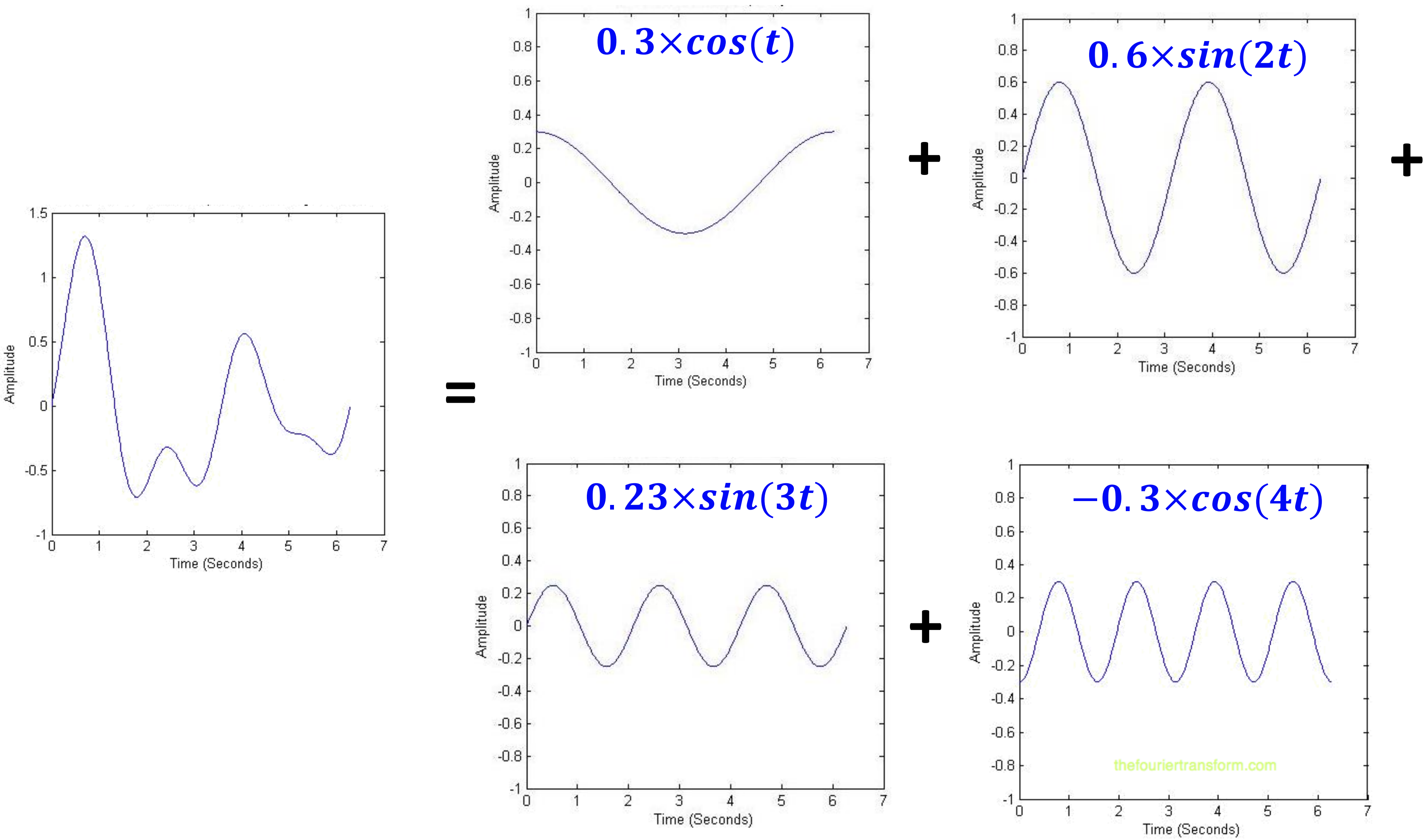
Waveform Processing

- Time domain = Frequency domain
  - **Fourier Transform**



Fourier Transform

- **Any** waveform can be decomposed into a number of sinusoids (sine/cosine functions)
正弦曲线



Fourier Transform

- Something like:

$$x(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\boldsymbol{m} \times t \times \frac{2\pi}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\boldsymbol{n} \times t \times \frac{2\pi}{T}\right)$$

- Need to calculate the coefficients
  - a_0
  - $a_1, \ldots,$ for $\cos()$
  - $b_1, \ldots,$ for $\sin()$

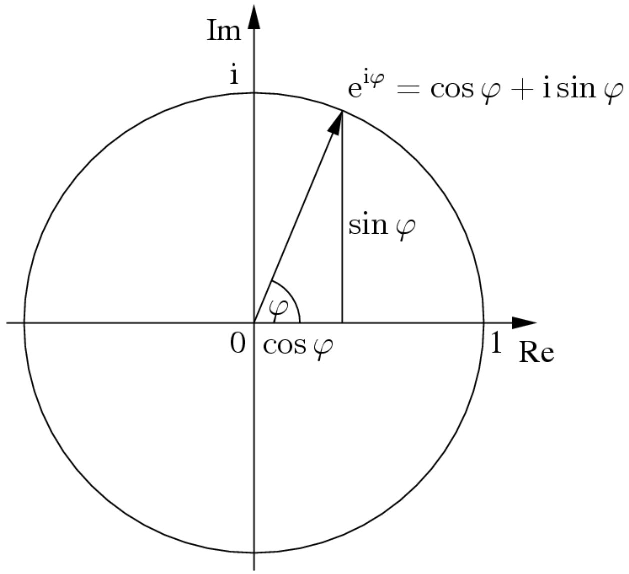
  - How about the period T? How do we know the period of the time series? If it is not periodic?

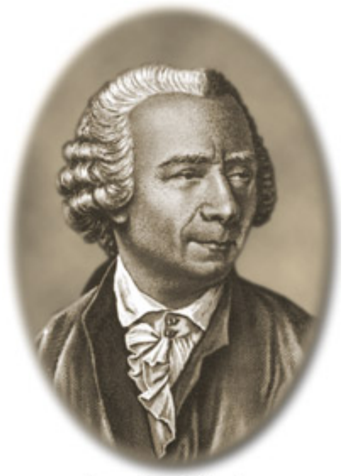
Fourier Transform in Computers

- All data are discrete (stored under certain time intervals) (按一定时间间隔储存)
  - $t \rightarrow n$ becomes discrete rather than continuous
- Discrete Fourier Transform (DFT)
  - From time domain to frequency domain
- Inverse DFT (IDFT)
  - From frequency domain to time domain

x. 时间信号

- To help calculation, we use complex numbers
  - $x = x.Re + i * x.Im$ (Re = real part, Im = imaginary part)
  - $i = \sqrt{-1}$ $(i^2 = -1)$
- Euler's formula
  - $\boldsymbol{e^{it} = \cos t + i * \sin t}$

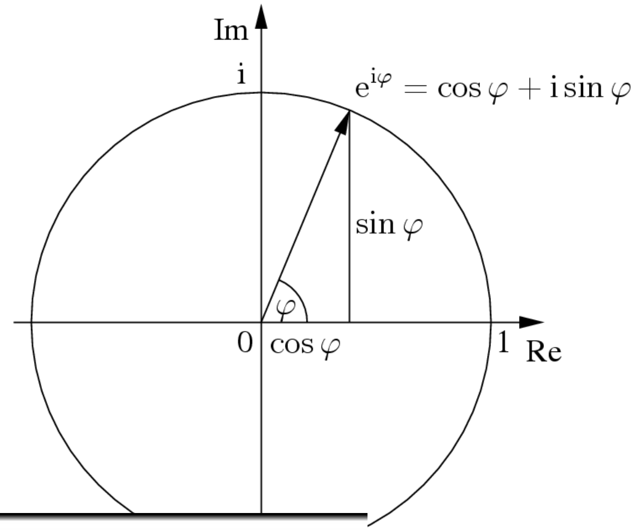


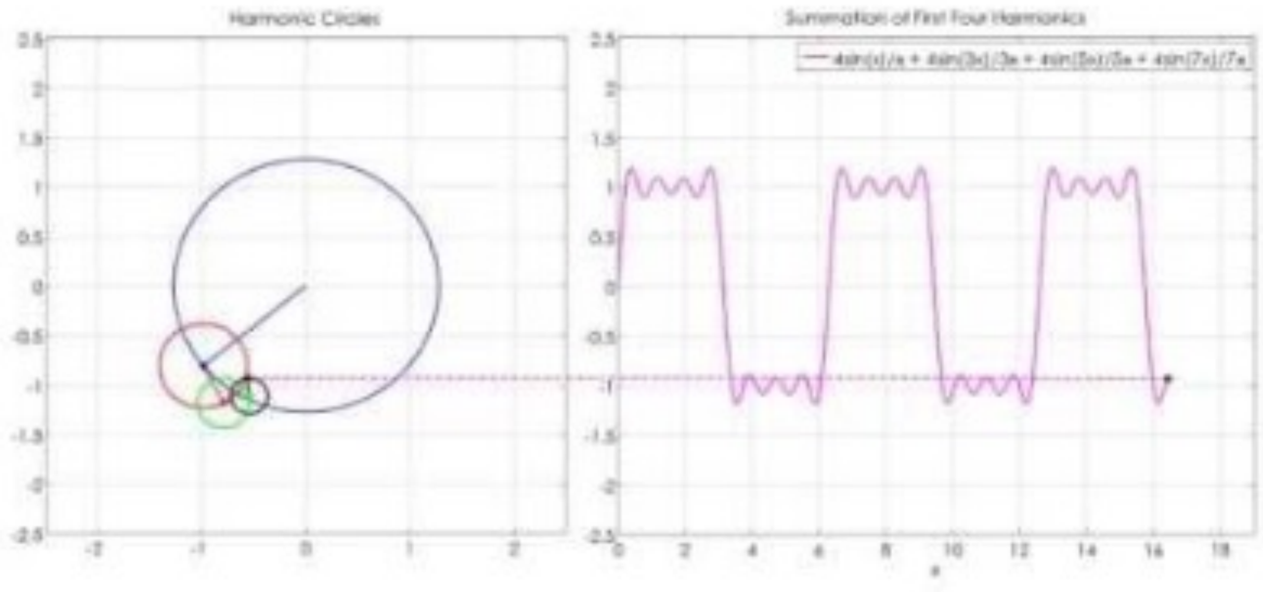

Leonhard Euler
(1707-1783)

Fourier Transform in Computers

- $\boldsymbol{e^{it} = \cos t + i * \sin t}$
- The angle is t
- The real part (x axis) is $\cos(t)$
- The imaginary part (y axis) is $\sin(t)$

- Increase t -> increase angle





https://hackaday.com/2015/09/17/visualizing-the-fourier-transform/

Complex Numbers

- Basic operations
  - $-21 = -21 + 0 * i$
  - $8i = 0 + 8 * i$
  - $i * i = i^2 = -1$
  - Addition:
    - $(a + b * i) + (c + d * i) = (a + c) + (b + d) * i$
  - Subtraction:
    - $(a + b * i) - (c + d * i) = (a - c) + (b - d) * i$
  - Multiplication
    - $(a + b * i) \; * \; (c + d * i) = (ac - bd) + (bc + ad) * i$
  - Division
    - $\frac{(a+b*i)}{(c+d*i)} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} * i$
  - Exponential
    - $e^{a+b*i} = e^a * e^{b*i} = e^a * (\cos b + i * \sin b) = e^a \cos b + e^a \sin b * i$

Discrete Fourier Transform

- Given a discrete sequence with N samples $[x(0), x(1), \ldots, x(N-1)]$

- Time -> Frequency (Discrete Fourier Transform)

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i*n*k*\frac{2\pi}{N}}, k = 0,1, \ldots, N-1$$

- Frequency -> Time (Inverse Discrete Fourier Transform)

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{i*n*k*\frac{2\pi}{N}}, n = 0, 1, \ldots, N-1$$

Example

- x = [3, 5, 2, 6, 4, 7, 1, 8]