a.

$$y'' = -(x+1)y' + 2y + (1-x^2)e^{-x}$$

可以先轉換為一階系統

$$\begin{cases} y_1' = y_2 \\ y_2' = -(x+1)y_2 + 2y_1 + (1-x^2)e^{-x} \end{cases}$$

使用 Runge-Kutta method 來獲得y_1, y_2的值

$$y_1'' = p(x)y_1' + q(x)y_1 + r(x),\quad a \le x \le b,\quad y_1(a) = \alpha,\quad y_1'(a) = 0$$

$$y_2'' = p(x)y_2' + q(x)y_2,\quad a \le x \le b,\quad y_2(a) = 0,\quad y_2'(a) = 1.$$

The solutions $y_1(x)$ and $y_2(x)$ can be obtained from the Runge-Kutta method.

求得參數 c 和最終的 y

$$c = [\beta - y_1(b)]/y_2(b) \qquad y = y_1 + cy_2$$

b.

建構出陣列

$$[A]\{Y\} = \{F\}$$

where

$$[A] = \begin{bmatrix} 2+h^2q_1 & -1+0.5hp_1 & 0 & 0 \\ -1-0.5hp_2 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & -1+0.5hp_{n-1} \\ 0 & 0 & -1-0.5hp_n & 2+h^2q_n \end{bmatrix},\quad \{Y\} = \begin{Bmatrix} y_1 \\ \bullet \\ \bullet \\ y_n \end{Bmatrix},$$

$$\{F\} = \begin{Bmatrix} -h^2r_1 + (1+0.5hp_1)y_0 \\ -h^2r_2 \\ \bullet \\ -h^2r_n + (1-0.5hp_n)y_{n+1} \end{Bmatrix}.$$

使用 Thomas Algorithm 求解 Y（這個做法求解三對角矩陣比較快）

$$\begin{bmatrix} b_1 & c_1 & & & 0 \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ 0 & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix}.$$

三對角矩陣算法可分為如下兩步進行。第一步求解係數c'_i和d'_i：

$$c'_i = \begin{cases} \dfrac{c_i}{b_i} & ; \quad i = 1 \\ \dfrac{c_i}{b_i - a_i c'_{i-1}} & ; \quad i = 2, 3, \dots, n-1 \end{cases}$$

以及

$$d'_i = \begin{cases} \dfrac{d_i}{b_i} & ; \quad i = 1 \\ \dfrac{d_i - a_i d'_{i-1}}{b_i - a_i c'_{i-1}} & ; \quad i = 2, 3, \dots, n. \end{cases}$$

第二步通過回代得到最終結果：

$$x_n = d'_n$$
$$x_i = d'_i - c'_i x_{i+1} \qquad ; \; i = n-1, n-2, \dots, 1.$$

c.

Ex. $-\dfrac{d}{dx}[p(x)\dfrac{dy}{dx}] + q(x)y = f(x)$, $0 \le x \le l$, $y(0) = a$, $y(l) = b$

First, $y_1(x) = a(1 - \dfrac{x}{l}) + b(\dfrac{x}{l})$, $y(x) = y_1(x) + y_2(x)$, then $y_2(0) = y_2(l) = 0$

Second, $-\dfrac{d}{dx}[p(x)\dfrac{dy}{dx}] + q(x)y = f(x)$ becomes

$$-\frac{d}{dx}\{p(x)\frac{dy_2}{dx}\} + q(x)y_2 = f(x) + \frac{b-a}{l}\frac{dp}{dx} - q(x)[a(1-\frac{x}{l}) + b(\frac{x}{l})] \triangleq F(x)$$

$\dfrac{\partial I}{\partial c_i} = 0$ implies $[A]\{c\} = \{b\}$

$a_{ij} = \int_0^l (p\dfrac{d\phi_i}{dx}\dfrac{d\phi_j}{dx} + q\phi_i\phi_j)dx$, $b_i = \int_0^l f\phi_i dx$

Finally, $y(x) = \sum_{i=1}^{n} c_i\phi_i(x) + y_1(x)$

Ex. $\phi_i(x) = \sin(\dfrac{i\pi x}{l})$

執行結果:

```
PS C:\Users\yunyu\Documents\大學\三下\數值方法\Numerical_class\F74114095_numerical_hw11> ./aa
x          Shooting        FiniteDiff      Variation
0.000000   1.000000        1.000000        1.000000
0.100000   1.016187        1.016532        1.245777
0.200000   1.058561        1.059102        1.652246
0.300000   1.123629        1.124251        1.637690
0.400000   1.208273        1.208890        0.971795
0.500000   1.309759        1.310313        0.578444
0.600000   1.425740        1.426194        1.162088
0.700000   1.554235        1.554570        2.017215
0.800000   1.693609        1.693822        2.225878
0.900000   1.842544        1.842642        2.026533
1.000000   2.000000        2.000000        2.000000
```

Variation 的結果看起來有波動，我覺得是因為使用的基底 $\sin(i\pi x)$會波動