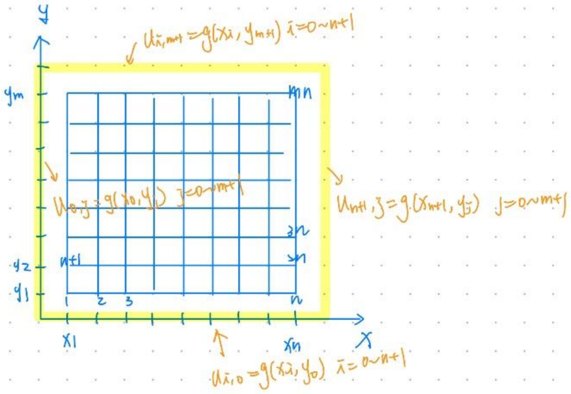
1.

先建立邊界的值




```
for (int j = 0; j <= ny; ++j) {
    double y = j * k;
    u[0][j] = cos(y);          // u(0, y) = cos(y)
    u[nx][j] = -cos(y);        // u(π, y) = -cos(y)
}

for (int i = 0; i <= nx; ++i) {
    double x = i * h;
    u[i][0] = cos(x);          // u(x, 0) = cos(x)
    u[i][ny] = 0.0;            // u(x, π/2) = 0
}
```

之後根據以下公式可知當前某點的相鄰係數關係

$$\alpha u_{i,j-1} + u_{i-1,j} - 2(1+\alpha)u_{i,j} + u_{i+1,j} + \alpha u_{i,j+1} = h^2 f_{i,j}$$

$$\begin{array}{ccc}
 & u_{i,j+1} & \\
 & \alpha & \\
u_{i-1,j} & u_{i,j} & u_{i+1,j} \\
1 & -2(1+\alpha) & 1 \\
 & u_{i,j-1} & \\
 & \alpha &
\end{array}$$

且 F = $h^2 f_{i,j}$

並依據公式做 F 邊界的修正

$$\text{Case 1} \quad j=1: \quad u_{i-1,1} - 2(1+\alpha)u_{i,1} + u_{i+1,1} + \alpha u_{i,2} = \boxed{h^2 f_{i,1} - \alpha u_{i,0}}$$

$$\text{a.} \quad i=1: \quad -2(1+\alpha)u_{1,1} + u_{2,1} + \alpha u_{1,2} = \boxed{h^2 f_{1,1} - \alpha u_{i,0} - u_{0,1}} \triangleq F_{1,1}$$

$$\text{Case 3} \quad j=m: \quad \alpha u_{i,m-1} + u_{i-1,m} - 2(1+\alpha)u_{i,m} + u_{i+1,m} = \boxed{h^2 f_{i,m} - \alpha u_{i,m+1}}$$

$$\text{a.} \quad i=1: \quad \alpha u_{1,m-1} - 2(1+\alpha)u_{1,m} + u_{2,m} = \boxed{h^2 f_{1,m} - \alpha u_{1,m+1} - u_{0,m}} = F_{1,m}$$

輸出結果:

```
PS C:\Users\yunyu\Documents\大學\三下\數值方法\Numerical_class\F74114095_numerical_hw12> ./1
```

y/x	0	0.314159	0.628319	0.942478	1.25664	1.5708	1.88496	2.19911	2.51327	2.82743	3.14159
1.5708	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.2566	0.3090	0.1728	0.0531	-0.0589	-0.1667	-0.2699	-0.3642	-0.4413	-0.4863	-0.4679	-0.3090
0.9425	0.5878	0.3681	0.1763	-0.0050	-0.1823	-0.3539	-0.5116	-0.6420	-0.7244	-0.7255	-0.5878
0.6283	0.8090	0.5646	0.3476	0.1326	-0.0869	-0.3056	-0.5112	-0.6862	-0.8099	-0.8589	-0.8090
0.3142	0.9511	0.7532	0.5559	0.3332	0.0858	-0.1732	-0.4243	-0.6452	-0.8145	-0.9158	-0.9511
0.0000	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000	-0.3090	-0.5878	-0.8090	-0.9511	-1.0000

2.

§ $\dfrac{\partial u}{\partial t}=\alpha^2\dfrac{\partial^2 u}{\partial x^2}+g(x,t)$, $0<x<l$, $t>0$,

$u(0,t)=q(t)$, $u(l,t)=r(t)$ for $t>0$, and $u(x,0)=f(x)$ for $0\le x\le l$.

Set $x_0=0$, $x_{n+1}=l$, $h=l/(n+1)$, $x_i=x_0+ih$, $k=\Delta t$, $t_j=jk$, $u(x_i,t_j)=u_{i,j}$

a. Forward-difference method

$$\begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_{j+1} = \begin{bmatrix} 1-2\lambda & \lambda & 0 & 0 \\ \lambda & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & \lambda \\ 0 & 0 & \lambda & 1-2\lambda \end{bmatrix} \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_j + k\begin{Bmatrix} g_1 \\ \bullet \\ \bullet \\ g_n \end{Bmatrix}_j + \lambda\begin{Bmatrix} q \\ 0 \\ 0 \\ r \end{Bmatrix}_j , \quad \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_0 = \begin{Bmatrix} f(x_1) \\ \bullet \\ \bullet \\ f(x_n) \end{Bmatrix}.$$

但是他必須要是 $\lambda<\frac{1}{2}$ 才是 Stability condition，而這題的 $\lambda>\frac{1}{2}$ 所以無法收斂

```
PS C:\Users\yunyu\Documents\大學\三下\數值方法\Numerical_class\F74114095_numerical_hw12> ./a
     t      r=0.50      r=0.60      r=0.70      r=0.80      r=0.90      r=1.00
  0.00        0.00       20.00       40.00       60.00       80.00      100.00
  0.50       20.00       86.67       97.14      110.00      124.44      120.00
  1.00       98.67     -346.67      154.29      160.00      -53.33      140.00
  1.50     -287.47     9208.89    -4074.29    -2290.00     3991.11      160.00
  2.00     9036.41  -224897.78   130173.33    52760.00   -94408.89      180.00
  2.50  -219475.93  5867050.07 -3798017.14  -925523.33  2264746.67      200.00
  3.00  5735364.51-157613117.51111873916.83  9830693.33-51254838.52      220.00
  3.50-154171898.804347639633.30-3364223217.4021549665.561061228317.04      240.00
  4.004255136494.01-123139189725.59103648542736.55-21592571231.10-18247809440.99      260.00
  4.50-120586107829.193584337074162.87-3271573323712.201111895746124.58154774415769.10      280.00
  5.003511985409465.36-107275644084313.391055901623133 51.69-48011073581446.536942803846827.93      300.00
  5.50-105168452838634.173299389009669934.50-3474416588464328.501914230861566125.00-558678371588144.19      320.00
  6.003236287937966754.00-104099655156134528.00116171316538748800.00-73056663199185856.0027630274496321624.00      340.00
  6.50-102157882393354480.003360194540401680896.00-3934471066422361088.002715416208582201344.00-1174367777200647936.00      360.00
  7.003298899810965668352.00-110597174395837415424.001345899458492644 06528.00-99231167287764746240.0046450020731987427328.00      380.00
  7.50-108617834509258014720.003699053180187042643968.00-4639255234960511991808.003585616937883451523072.00-17646052142434473 73824.00      400.00
  8.003633882479481487622144.00-125311803452336925310976.00160830498870235228012544.00-128572013785928809054208.0065399649629589504786432.00      420.00
  8.50-123131473964648030535680.004287562439742823665238016.00-5599275094250828796526592.004585881185380086851829760.00-2385455687726012076916736.00      440.00
  9.004213683555364034695921664.00-147814658071078712694013952.001955468326786194765722419 20.00-162961776083833999230763008.0086087046381283871337480192.00      460.00
  9.50-145286447937860293594447872.005124972247562422281037676544.00-6844792902461118663524089856.005775787803320208141167099904.00-30842030019895843332680 58112.00      480.00
 10.005037800378799706530941763584.00-178441110298547882408542208000.0023998840216381291395284153 1392.00-204329189932001513960718204928.0010994019306731503463310557 1840.00
 500.00
```

b. Backward-difference method

$$\begin{bmatrix} 1+2\lambda & -\lambda & 0 & 0 \\ -\lambda & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & -\lambda \\ 0 & 0 & -\lambda & 1+2\lambda \end{bmatrix} \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_j - k\begin{Bmatrix} g_1 \\ \bullet \\ \bullet \\ g_n \end{Bmatrix}_j - \lambda\begin{Bmatrix} q \\ 0 \\ 0 \\ r \end{Bmatrix}_j = \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_{j-1} , \quad \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_0 = \begin{Bmatrix} f(x_1) \\ \bullet \\ \bullet \\ f(x_n) \end{Bmatrix}$$

PS C:\Users\yunyu\Documents\大學\三下\數值方法\Numerical_class\F74114095_numerical_hw12> ./b

t	r=0.50	r=0.60	r=0.70	r=0.80	r=0.90	r=1.00
0.00	0.00	20.00	40.00	60.00	80.00	100.00
0.50	20.00	33.01	50.00	70.35	93.72	120.00
1.00	45.01	56.56	71.79	90.77	113.51	140.00
1.50	83.57	90.95	102.29	117.59	136.85	160.00
2.00	141.09	140.90	144.76	152.65	164.46	180.00
2.50	225.55	212.96	204.27	199.40	198.10	200.00
3.00	348.29	316.44	287.99	262.76	240.31	220.00
3.50	525.41	464.57	406.10	349.71	294.68	240.00
4.00	779.82	676.14	573.09	470.19	366.35	260.00
4.50	1144.03	977.85	809.52	638.31	462.59	280.00
5.00	1664.26	1407.64	1144.64	874.14	593.74	300.00
5.50	2406.20	2019.44	1619.96	1206.15	774.50	320.00
6.00	3463.16	2889.84	2294.52	1674.84	1025.75	340.00
6.50	4967.73	4127.70	3252.18	2337.75	1377.17	360.00
7.00	7108.34	5887.69	4612.08	3276.61	1870.90	380.00
7.50	10152.70	8389.59	6543.56	4607.60	2566.86	400.00
8.00	14481.21	11945.68	9287.20	6495.75	3550.17	420.00
8.50	20634.41	16999.69	13184.84	9175.60	4941.78	440.00
9.00	29380.34	24182.13	18722.25	12980.38	6913.55	460.00
9.50	41810.33	34388.90	26589.62	18383.59	9709.68	480.00
10.00	59475.10	48892.99	37767.70	26058.07	13677.17	500.00

c. Crank-Nicolson method

$$\begin{bmatrix} 1+\lambda & -0.5\lambda & 0 & 0 \\ -0.5\lambda & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & -0.5\lambda \\ 0 & 0 & -0.5\lambda & 1+\lambda \end{bmatrix} \{U\}^{(j+1)} = \begin{bmatrix} 1-\lambda & 0.5\lambda & 0 & 0 \\ 0.5\lambda & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0.5\lambda \\ 0 & 0 & 0.5\lambda & 1-\lambda \end{bmatrix} \{U\}^{(j)}$$

```
PS C:\Users\yunyu\Documents\大學\三下\數值方法\Numerical_class\F74114095_numerical_hw12> ./c
```

t	r=0.50	r=0.60	r=0.70	r=0.80	r=0.90	r=1.00
0.00	0.00	20.00	40.00	60.00	80.00	100.00
0.50	20.00	63.03	83.70	103.03	121.65	120.00
1.00	75.03	118.87	125.02	130.92	135.57	140.00
1.50	163.89	202.32	189.60	181.86	179.15	160.00
1.50	163.89	202.32	189.60	181.86	179.15	160.00
1.50	163.89	202.32	189.60	181.86	179.15	160.00
2.00	300.66	331.91	286.21	248.13	212.94	180.00
2.50	512.30	524.78	424.26	341.90	275.18	200.00
3.00	832.16	814.08	632.75	480.23	344.54	220.00
3.50	1313.38	1245.29	935.60	675.07	453.22	240.00
4.00	2033.32	1886.09	1387.56	966.53	596.26	260.00
4.50	3106.08	2838.36	2052.01	1385.99	810.06	280.00
5.00	4702.01	4250.04	3037.62	2010.92	1111.86	300.00
5.50	7071.25	6343.52	4493.75	2923.75	1558.11	320.00
6.00	10586.26	9444.50	6649.43	4278.70	2206.06	340.00
6.50	15796.26	14038.33	9839.23	6272.75	3163.00	360.00
7.00	23516.08	20840.62	14559.56	9227.29	4568.27	380.00
7.50	34950.26	30912.93	21546.99	13590.05	6645.04	400.00
8.00	51883.09	45824.83	31887.43	20049.78	9708.90	420.00
8.50	76954.69	67900.90	47194.88	29602.01	14239.90	440.00
9.00	114073.71	100581.22	69849.94	43742.28	20937.26	460.00
9.50	169025.45	148958.22	103386.37	64664.39	30846.06	480.00
10.00	250373.49	220569.86	153023.81	95633.62	45504.24	500.00

b, c 結果差異這麼大我覺得是因為 Crank-Nicolson method 比較容易受 λ 的值影響(阿這題的 λ 又蠻大的)，有試過把 Δt 調小一點，兩個出來的結果差異就沒有這麼大了

3.

和第一題概念類似，只是公式不太一樣

$$\alpha u_{i,j-1}+(r_i^2-\frac{h}{2}r_i)u_{i-1,j}-2(\alpha+r_i^2)u_{i,j}+(r_i^2+\frac{h}{2}r_i)u_{i+1,j}+\alpha u_{i,j+1}=r_i^2h^2f_{i,j}$$

Case 1. $j=1$: $(r_i^2-\frac{h}{2}r_i)u_{i-1,1}-2(\alpha+r_i^2)u_{i,1}+(r_i^2+\frac{h}{2}r_i)u_{i+1,1}+\alpha u_{i,2}=\boxed{r_i^2h^2f_{i,1}-\alpha u_{i,0}}$

a. $i=1$: $-2(\alpha+r_1^2)u_{1,1}+(r_1^2+\frac{h}{2}r_1)u_{2,1}+\alpha u_{1,2}=\boxed{r_1^2h^2f_{1,1}-\alpha u_{1,0}-(r_1^2-\frac{h}{2}r_1)u_{0,1}}\triangleq F_{1,1}$

Case 3 $j=m$:

$$\alpha u_{i,m-1}+(r_i^2-\frac{h}{2}r_i)u_{i-1,m}-2(\alpha+r_i^2)u_{i,m}+(r_i^2+\frac{h}{2}r_i)u_{i+1,m}=\boxed{r_i^2h^2f_{i,m}-\alpha u_{i,m+1}}$$

a. $i=1$:

$$\alpha u_{1,m-1}-2(\alpha+r_1^2)u_{1,m}+(r_1^2+\frac{h}{2}r_i)u_{2,m}=\boxed{r_1^2h^2f_{1,m}-\alpha u_{1,m+1}-(r_1^2-\frac{h}{2}r_1)u_{0,m}}\triangleq F_{1,m}$$

但是這題比較有問題的是他的邊界條件在四個交點的地方會有衝突，我是依照他邊界條件給的順序依序做的，所以那四個點會是 50 或 100 而非 0

```
PS C:\Users\yunyu\Documents\大學\三下\數值方法\Numerical_class\F74114095_numerical_hw12> ./3
   t/r        0.5        0.6        0.7        0.8        0.9          1
1.0472     50.0000     0.0000     0.0000     0.0000     0.0000   100.0000
0.8727     50.0000    32.2567    32.9444    43.1044    63.5365   100.0000
0.6981     50.0000    45.8602    50.5938    62.4550    79.7182   100.0000
0.5236     50.0000    49.6786    55.8351    67.7253    83.3054   100.0000
0.3491     50.0000    45.8602    50.5938    62.4550    79.7182   100.0000
0.1745     50.0000    32.2567    32.9444    43.1044    63.5365   100.0000
0.0000     50.0000     0.0000     0.0000     0.0000     0.0000   100.0000
```

4.

我覺得這題比較符合講義這段的內容，但是邊界 p(0, t)≠ p(1,t) ≠ 0

§ $\frac{\partial^2 u}{\partial t^2}=\alpha^2\frac{\partial^2 u}{\partial x^2}$, $0<x<l$, $t>0$, $u(0,t)=u(l,t)=0$, $u(x,0)=f(x)$, $\frac{\partial u}{\partial t}=g(x)$

$$\frac{1}{k^2}(u_{i,j+1}-2u_{i,j}+u_{i,j-1})=\alpha^2\frac{1}{h^2}(u_{i+1,j}-2u_{i,j}+u_{i-1,j}),$$

$$u_{i,j+1}=\lambda^2u_{i-1,j}+2(1-\lambda^2)u_{i,j}+\lambda^2u_{i+1,j}-u_{i,j-1},\quad \lambda^2=\alpha^2\frac{k^2}{h^2},$$

$$u_{1,j+1}=2(1-\lambda^2)u_{1,j}+\lambda^2u_{2,j}-u_{1,j-1},$$

$$u_{i,j+1}=\lambda^2u_{i-1,j}+2(1-\lambda^2)u_{i,j}+\lambda^2u_{i+1,j}-u_{i,j-1},$$

$$u_{n,j+1}=\lambda^2u_{n-1,j}+2(1-\lambda^2)u_{n,j}-u_{n,j-1}.$$

$$\begin{Bmatrix}u_1\\ \bullet\\ \bullet\\ u_n\end{Bmatrix}_{j+1}=\begin{bmatrix}2(1-\lambda^2)&\lambda^2&0&0\\ \lambda^2&2(1-\lambda^2)&\lambda^2&0\\ 0&\lambda^2&2(1-\lambda^2)&\lambda^2\\ 0&0&\lambda^2&2(1-\lambda^2)\end{bmatrix}\begin{Bmatrix}u_1\\ \bullet\\ \bullet\\ u_n\end{Bmatrix}_j-\begin{Bmatrix}u_1\\ \bullet\\ \bullet\\ u_n\end{Bmatrix}_{j-1}$$

教授上課沒有特別說這種問題怎麼處理，所以我只能仿造 forward-difference 的方式做邊界修正不知道這樣對不對 ouo

$$\begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_{j+1} = \begin{bmatrix} 1-2\lambda & \lambda & 0 & 0 \\ \lambda & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & \lambda \\ 0 & 0 & \lambda & 1-2\lambda \end{bmatrix} \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_{j} + k \begin{Bmatrix} g_1 \\ \bullet \\ \bullet \\ g_n \end{Bmatrix}_{j} + \boxed{\lambda \begin{Bmatrix} q \\ 0 \\ 0 \\ r \end{Bmatrix}_{j}}$$

然後它的 t 只說≥0, 沒說到多少所以我先假設跟 x 一樣

PS C:\Users\yunyu\Documents\大學\三下\數值方法\Numerical class\F74114095 numerical hw12> ./4

t\x	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
0.00	1.00	0.81	0.31	-0.31	-0.81	-1.00	-0.81	-0.31	0.31	0.81	2.00
0.10	1.00	1.02	0.85	0.35	-0.29	-0.81	-1.02	-0.85	-0.35	0.29	2.00
0.20	1.00	1.04	1.06	0.87	0.35	-0.31	-0.85	-1.06	-0.87	0.84	2.00
0.30	1.00	1.04	1.06	1.06	0.85	0.31	-0.35	-0.87	0.13	0.84	2.00
0.40	1.00	1.02	1.04	1.04	1.02	0.81	0.29	0.84	0.84	1.29	2.00
0.50	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	2.00	2.00
0.60	1.00	0.98	0.96	0.96	0.98	2.19	2.71	3.16	3.16	2.71	2.00
0.70	1.00	0.96	0.94	0.94	2.15	2.69	3.35	3.87	3.87	3.16	2.00
0.80	1.00	0.96	0.94	2.13	2.65	3.31	3.85	4.06	3.87	3.16	2.00
0.90	1.00	0.98	2.15	2.65	3.29	3.81	4.02	3.85	3.35	2.71	2.00
1.00	1.00	2.19	2.69	3.31	3.81	4.00	3.81	3.31	2.69	2.19	2.00