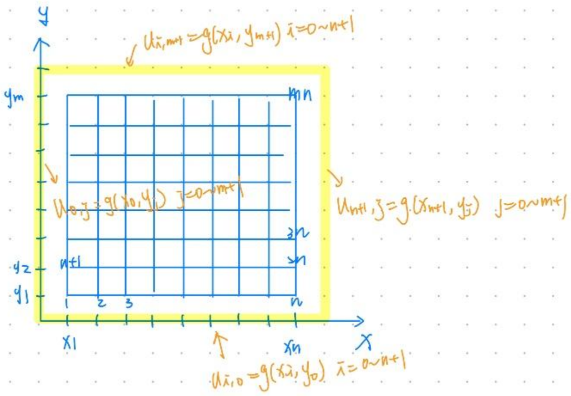
1.

先建立邊界的值

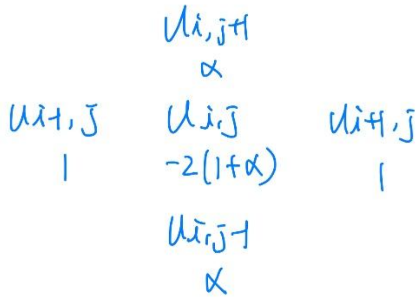



```
for (int j = 0; j <= ny; ++j) {
    double y = j * k;
    u[0][j] = cos(y);          // u(0, y) = cos(y)
    u[nx][j] = -cos(y);        // u(π, y) = -cos(y)
}

for (int i = 0; i <= nx; ++i) {
    double x = i * h;
    u[i][0] = cos(x);          // u(x, 0) = cos(x)
    u[i][ny] = 0.0;            // u(x, π/2) = 0
}
```

之後根據以下公式可知當前某點的相鄰係數關係

$$\alpha u_{i,j-1} + u_{i-1,j} - 2(1+\alpha)u_{i,j} + u_{i+1,j} + \alpha u_{i,j+1} = h^2 f_{i,j}$$



且 F = $h^2 f_{i,j}$

並依據公式做 F 邊界的修正

$$\text{Case 1} \quad j=1: \quad u_{i-1,1} - 2(1+\alpha)u_{i,1} + u_{i+1,1} + \alpha u_{i,2} = \boxed{h^2 f_{i,1} - \alpha u_{i,0}}$$

$$\text{a.} \quad i=1: \quad -2(1+\alpha)u_{1,1} + u_{2,1} + \alpha u_{1,2} = \boxed{h^2 f_{1,1} - \alpha u_{i,0} - u_{0,1}} \triangleq F_{1,1}$$

$$\text{Case 3} \quad j=m: \quad \alpha u_{i,m-1} + u_{i-1,m} - 2(1+\alpha)u_{i,m} + u_{i+1,m} = \boxed{h^2 f_{i,m} - \alpha u_{i,m+1}}$$

$$\text{a.} \quad i=1: \quad \alpha u_{1,m-1} - 2(1+\alpha)u_{1,m} + u_{2,m} = \boxed{h^2 f_{1,m} - \alpha u_{1,m+1} - u_{0,m}} = F_{1,m}$$

輸出結果:

```
PS C:\Users\yunyu\Documents\大學\三下\數值方法\Numerical_class\F74114095_numerical_hw12> ./1
   y/x          0   0.314159   0.628319   0.942478    1.25664     1.5708    1.88496    2.19911    2.51327    2.82743    3.14159
1.5708     0.0000     0.0000     0.0000     0.0000     0.0000     0.0000     0.0000     0.0000     0.0000     0.0000     0.0000
1.2566     0.3090     0.1728     0.0531    -0.0589    -0.1667    -0.2699    -0.3642    -0.4413    -0.4863    -0.4679    -0.3090
0.9425     0.5878     0.3681     0.1763    -0.0050    -0.1823    -0.3539    -0.5116    -0.6420    -0.7244    -0.7255    -0.5878
0.6283     0.8090     0.5646     0.3476     0.1326    -0.0869    -0.3056    -0.5112    -0.6862    -0.8099    -0.8589    -0.8090
0.3142     0.9511     0.7532     0.5559     0.3332     0.0858    -0.1732    -0.4243    -0.6452    -0.8145    -0.9158    -0.9511
0.0000     1.0000     0.9511     0.8090     0.5878     0.3090     0.0000    -0.3090    -0.5878    -0.8090    -0.9511    -1.0000
```

2.

§ $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2} + g(x,t)$, $0 < x < l$, $t > 0$,

$u(0,t) = q(t)$, $u(l,t) = r(t)$ for $t > 0$, and $u(x,0) = f(x)$ for $0 \le x \le l$.

Set $x_0 = 0$, $x_{n+1} = l$, $h = l/(n+1)$, $x_i = x_0 + ih$, $k = \Delta t$, $t_j = jk$, $u(x_i, t_j) = u_{i,j}$

a. Forward-difference method

$$\begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_{j+1} = \begin{bmatrix} 1-2\lambda & \lambda & 0 & 0 \\ \lambda & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & \lambda \\ 0 & 0 & \lambda & 1-2\lambda \end{bmatrix} \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_j + k \begin{Bmatrix} g_1 \\ \bullet \\ \bullet \\ g_n \end{Bmatrix}_j + \lambda \begin{Bmatrix} q \\ 0 \\ 0 \\ r \end{Bmatrix}_j, \quad \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_0 = \begin{Bmatrix} f(x_1) \\ \bullet \\ \bullet \\ f(x_n) \end{Bmatrix}.$$

但是他必須要是 $\lambda < \frac{1}{2}$ 才是 Stability condition，而這題的 $\lambda > \frac{1}{2}$ 所以無法收斂

```
PS C:\Users\yunyu\Documents\大學\三下\數值方法\Numerical_class\F74114095_numerical_hw12> ./a
    t        r=0.50       r=0.60       r=0.70       r=0.80       r=0.90       r=1.00
  0.00     0.00    20.00    40.00    60.00    80.00   100.00
  0.50    20.00    86.67    97.14   110.00   124.44   120.00
  1.00    98.67  -346.67   154.29   160.00   -53.33   140.00
  1.50  -287.47  9208.89 -4074.29 -2290.00  3991.11   160.00
  2.00  9036.41-224897.78130173.33 52760.00-94408.89   180.00
  2.50-219475.935867050.07-3798017.14-925523.332264746.67   200.00
  3.005735364.51-157613117.51111873916.839830693.33-51254838.52   220.00
  3.50-154171898.804347639633.30-3364223217.40215496665.561061228317.04   240.00
  4.004255136494.01-123139189725.59103648542736.55-21592571231.10-18247809440.99   260.00
  4.50-120586107829.193584337074162.87-3271573323712.201111895746124.58154774415769.10   280.00
  5.003511985409465.36-107275644084313.39105590162313351.69-48011073581446.536942803846827.93   300.00
  5.50-105168452838634.173299389009669934.50-3474416588464328.501914230861566125.00-558678371588144.19   320.00
  6.003236287937966754.00-104099655156134528.00116171316538748800.00-73056663199185856.0027630274496321624.00   340.00
  6.50-102157882393354480.003360194540401680896.00-3934471066422361088.002715416208582201344.00-1174367777200647936.00   360.00
  7.003298899810965668352.00-110597174395837415424.00134589945849264406528.00-99231167287764746240.0046450020731987427328.00   380.00
  7.50-108617834509258014720.003699053180187042643968.00-4639255234960511991808.003585616937883451523072.00-1764605214243447373824.00   400.00
  8.003633882479481487622144.00-125311803452336925310976.00160830498870235228012544.00-128572013785928809054208.0065399649629589504786432.00   420.00
  8.50-123131473964648030535680.004287562439742823665238016.00-5599275094250828796526592.004585881185380086851829760.00-2385455687726012076916736.00   440.00
  9.004213683555364034695921664.00-147814658071078712694013952.00195546832678619476572241920.00-162961776083833999230763008.0086087046381283871337480192.00   460.00
  9.50-145286447937860293594447872.005124972247562422281037676544.00-6844792902461118663524089856.005775787803320208141167099904.00-3084203001989584333268058112.00
  480.00
 10.005037800378799706530941763584.00-178441110298547882408542208000.00239988402163812913952841531392.00-204329189932001513960718204928.0010994019306731503463310557
1840.00   500.00
```

b. Backward-difference method

$$\begin{bmatrix} 1+2\lambda & -\lambda & 0 & 0 \\ -\lambda & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & -\lambda \\ 0 & 0 & -\lambda & 1+2\lambda \end{bmatrix} \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_j - k \begin{Bmatrix} g_1 \\ \bullet \\ \bullet \\ g_n \end{Bmatrix}_j - \lambda \begin{Bmatrix} q \\ 0 \\ 0 \\ r \end{Bmatrix}_j = \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_{j-1}, \quad \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_0 = \begin{Bmatrix} f(x_1) \\ \bullet \\ \bullet \\ f(x_n) \end{Bmatrix}$$

PS C:\Users\yunyu\Documents\大學\三下\數值方法\Numerical_class\F74114095_numerical_hw12> ./b

t	r=0.50	r=0.60	r=0.70	r=0.80	r=0.90	r=1.00
0.00	0.00	20.00	40.00	60.00	80.00	100.00
0.50	20.00	46.24	68.43	87.76	104.86	120.00
1.00	58.24	80.99	99.15	114.45	127.87	140.00
1.50	115.94	127.40	136.68	144.93	152.65	160.00
2.00	196.96	189.18	184.12	181.29	180.11	180.00
2.50	307.35	270.74	244.43	225.28	211.02	200.00
3.00	455.15	377.59	321.21	278.99	246.33	220.00
3.50	650.69	516.72	418.99	345.13	287.24	240.00
4.00	907.13	697.04	543.59	427.12	335.32	260.00
4.50	1241.32	929.93	702.41	529.34	392.53	280.00
5.00	1674.72	1229.91	904.90	657.38	461.39	300.00
5.50	2234.74	1615.50	1163.13	818.37	545.14	320.00
6.00	2956.34	2110.37	1492.48	1021.40	647.88	340.00
6.50	3884.17	2744.69	1912.60	1278.10	774.85	360.00
7.00	5075.19	3556.97	2448.58	1603.29	932.75	380.00
7.50	6602.09	4596.39	3132.40	2015.89	1130.11	400.00
8.00	8557.64	5925.66	4004.90	2540.03	1377.86	420.00
8.50	11060.25	7624.87	5118.20	3206.53	1689.88	440.00
9.00	14261.02	9796.18	6538.82	4054.71	2083.95	460.00
9.50	18352.79	12570.00	8351.63	5134.75	2582.73	480.00
10.00	23581.68	16112.76	10664.97	6510.70	3215.13	500.00

c. Crank-Nicolson method

$$\begin{bmatrix} 1+\lambda & -0.5\lambda & 0 & 0 \\ -0.5\lambda & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & -0.5\lambda \\ 0 & 0 & -0.5\lambda & 1+\lambda \end{bmatrix} \{U\}^{(j+1)} = \begin{bmatrix} 1-\lambda & 0.5\lambda & 0 & 0 \\ 0.5\lambda & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0.5\lambda \\ 0 & 0 & 0.5\lambda & 1-\lambda \end{bmatrix} \{U\}^{(j)}$$

PS C:\Users\yunyu\Documents\大學\三下\數值方法\Numerical_class\F74114095_numerical_hw12> ./c

t	r=0.50	r=0.60	r=0.70	r=0.80	r=0.90	r=1.00
0.00	0.00	20.00	40.00	60.00	80.00	100.00
0.50	20.00	87.63	148.79	231.70	348.95	120.00
1.00	99.63	182.52	211.23	230.67	215.57	140.00
1.50	242.30	297.77	287.81	302.43	365.94	160.00
2.00	443.15	491.32	442.78	400.65	329.26	180.00
2.50	757.21	776.08	634.56	524.98	469.53	200.00
3.00	1230.41	1208.57	960.44	745.54	520.62	220.00
3.50	1946.82	1863.17	1420.33	1036.24	720.30	240.00
4.00	3031.26	2848.98	2138.62	1513.16	914.94	260.00
4.50	4667.74	4339.95	3201.97	2191.75	1290.01	280.00
5.00	7140.60	6587.03	4820.21	3249.28	1781.37	300.00
5.50	10871.39	9980.05	7249.79	4809.55	2578.35	320.00
6.00	16502.89	15096.83	10921.57	7190.64	3726.89	340.00
6.50	24998.57	22817.69	16454.52	10753.61	5498.11	360.00
7.00	37816.83	34463.21	24802.72	16149.60	8130.28	380.00
7.50	57153.31	52031.19	37393.06	24265.62	12127.60	400.00
8.00	86323.18	78530.44	56383.70	36524.39	18127.42	420.00
8.50	130324.34	118502.96	85028.92	54996.49	27196.15	440.00
9.00	196697.57	178797.07	128234.83	82872.16	40852.54	460.00
9.50	296815.62	269744.64	193406.88	124903.95	61464.34	480.00
10.00	447834.01	406928.53	291708.22	188313.35	92537.04	500.00

3.

和第一題概念類似，只是公式不太一樣

$$\alpha u_{i,j-1} + (r_i^2 - \frac{h}{2} r_i) u_{i-1,j} - 2(\alpha + r_i^2) u_{i,j} + (r_i^2 + \frac{h}{2} r_i) u_{i+1,j} + \alpha u_{i,j+1} = r_i^2 h^2 f_{i,j}$$

$$\text{Case 1.} \quad j=1: \quad (r_i^2 - \frac{h}{2} r_i) u_{i-1,1} - 2(\alpha + r_i^2) u_{i,1} + (r_i^2 + \frac{h}{2} r_i) u_{i+1,1} + \alpha u_{i,2} = \boxed{r_i^2 h^2 f_{i,1} - \alpha u_{i,0}}$$

$$\text{a.} \quad i=1: \quad -2(\alpha + r_1^2) u_{1,1} + (r_1^2 + \frac{h}{2} r_1) u_{2,1} + \alpha u_{1,2} = \boxed{r_1^2 h^2 f_{1,1} - \alpha u_{1,0} - (r_1^2 - \frac{h}{2} r_1) u_{0,1}} \triangleq F_{1,1}$$

$$\text{Case 3} \quad j=m:$$

$$\alpha u_{i,m-1} + (r_i^2 - \frac{h}{2} r_i) u_{i-1,m} - 2(\alpha + r_i^2) u_{i,m} + (r_i^2 + \frac{h}{2} r_i) u_{i+1,m} = \boxed{r_i^2 h^2 f_{i,m} - \alpha u_{i,m+1}}$$

$$\text{a.} \quad i=1:$$

$$\alpha u_{1,m-1} - 2(\alpha + r_1^2) u_{1,m} + (r_1^2 + \frac{h}{2} r_1) u_{2,m} = \boxed{r_1^2 h^2 f_{1,m} - \alpha u_{1,m+1} - (r_1^2 - \frac{h}{2} r_1) u_{0,m}} \triangleq F_{1,m}$$

但是這題比較有問題的是他的邊界條件在四個交點的地方會有衝突，我是依照他邊界條件給的順序依序做的，所以那四個點會是 50 或 100 而非 0

PS C:\Users\yunyu\Documents\大學\三下\數值方法\Numerical_class\F74114095_numerical_hw12> ./3

t/r	0.5	0.6	0.7	0.8	0.9	1
1.0472	50.0000	0.0000	0.0000	0.0000	0.0000	100.0000
0.8727	50.0000	32.2567	32.9444	43.1044	63.5365	100.0000
0.6981	50.0000	45.8602	50.5938	62.4550	79.7182	100.0000
0.5236	50.0000	49.6786	55.8351	67.7253	83.3054	100.0000
0.3491	50.0000	45.8602	50.5938	62.4550	79.7182	100.0000
0.1745	50.0000	32.2567	32.9444	43.1044	63.5365	100.0000
0.0000	50.0000	0.0000	0.0000	0.0000	0.0000	100.0000

4.

我覺得這題比較符合講義這段的內容，但是邊界 $p(0,t) \neq p(1,t) \neq 0$

$$\S \quad \frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0, \quad u(0,t) = u(l,t) = 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t} = g(x)$$

$$\frac{1}{k^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = \alpha^2 \frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}),$$

$$u_{i,j+1} = \lambda^2 u_{i-1,j} + 2(1-\lambda^2)u_{i,j} + \lambda^2 u_{i+1,j} - u_{i,j-1}, \quad \lambda^2 = \alpha^2 \frac{k^2}{h^2},$$

$$u_{1,j+1} = 2(1-\lambda^2)u_{1,j} + \lambda^2 u_{2,j} - u_{1,j-1},$$

$$u_{i,j+1} = \lambda^2 u_{i-1,j} + 2(1-\lambda^2)u_{i,j} + \lambda^2 u_{i+1,j} - u_{i,j-1},$$

$$u_{n,j+1} = \lambda^2 u_{n-1,j} + 2(1-\lambda^2)u_{n,j} - u_{n,j-1}.$$

$$\begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_{j+1} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & 0 \\ 0 & \lambda^2 & 2(1-\lambda^2) & \lambda^2 \\ 0 & 0 & \lambda^2 & 2(1-\lambda^2) \end{bmatrix} \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_j - \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_{j-1}$$

教授上課沒有特別說這種問題怎麼處理，所以我只能仿造 forward-difference 的方式做邊界修正不知道這樣對不對 ouo

$$\begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_{j+1} = \begin{bmatrix} 1-2\lambda & \lambda & 0 & 0 \\ \lambda & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & \lambda \\ 0 & 0 & \lambda & 1-2\lambda \end{bmatrix} \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_j + k \begin{Bmatrix} g_1 \\ \bullet \\ \bullet \\ g_n \end{Bmatrix}_j + \boxed{\lambda \begin{Bmatrix} q \\ 0 \\ 0 \\ r \end{Bmatrix}_j}$$

然後它的 t 只說≥0, 沒說到多少所以我先假設跟 x 一樣

```
PS C:\Users\yunyu\Documents\大學\三下\數值方法\Numerical class\F74114095 numerical hw12> ./4
```

t\x	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
0.00	1.00	0.81	0.31	-0.31	-0.81	-1.00	-0.81	-0.31	0.31	0.81	2.00
0.10	1.00	1.02	0.85	0.35	-0.29	-0.81	-1.02	-0.85	-0.35	0.29	2.00
0.20	1.00	1.04	1.06	0.87	0.35	-0.31	-0.85	-1.06	-0.87	0.84	2.00
0.30	1.00	1.04	1.06	1.06	0.85	0.31	-0.35	-0.87	0.13	0.84	2.00
0.40	1.00	1.02	1.04	1.04	1.02	0.81	0.29	0.84	0.84	1.29	2.00
0.50	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	2.00	2.00
0.60	1.00	0.98	0.96	0.96	0.98	2.19	2.71	3.16	3.16	2.71	2.00
0.70	1.00	0.96	0.94	0.94	2.15	2.69	3.35	3.87	3.87	3.16	2.00
0.80	1.00	0.96	0.94	2.13	2.65	3.31	3.85	4.06	3.87	3.16	2.00
0.90	1.00	0.98	2.15	2.65	3.29	3.81	4.02	3.85	3.35	2.71	2.00
1.00	1.00	2.19	2.69	3.31	3.81	4.00	3.81	3.31	2.69	2.19	2.00