Mathematical Concepts 3

Exercise 1: Optimality in 2 dimensions

Let $f : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto -\cos(x_1^2 + x_2^2 + x_1 x_2)$

(a) Create a contour plot of f in the range $[-2, 2] \times [-2, 2]$ with Python.

(b) Compute ∇f

(c) Compute $\nabla^2 f$

Now, we define the restriction of f to $S_r = \{(x_1, x_2) \in \mathbb{R}^2 | \; x_1^2 + x_2^2 + x_1 x_2 < r\}$ with $r \in \mathbb{R}, r > 0$, i.e., $f_{|S_r} : S_r \to \mathbb{R}, (x_1, x_2) \mapsto f(x_1, x_2)$.

(d) Show that $f_{|S_{\overline{r}}}$ with $\overline{r} = \pi/4$ is convex.

(e) Find the local minimum $\mathbf{x}^*$ of $f_{|S_{\overline{r}}}$

(f) Is $\mathbf{x}^*$ a global minimum of f?

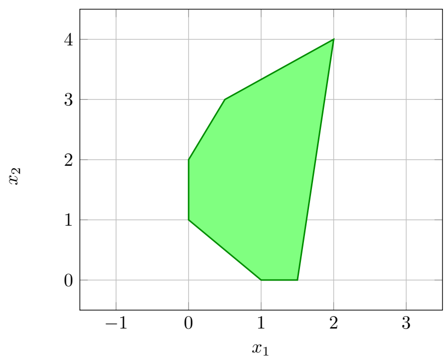
Exercise 2: Least-Squares
We have 5 datapoints given $\mathbf{x}_k = (0.7, 1.0), (0.8, 0.2), (1.5, 1.4), (1.6, 1.5), (2.0, 1.8)$ in $\mathbb{R}^2$.

(a) Formulate a least-squares linear regression optimization-problem to find the regression-direction $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$

(b) Derive the analytical solution symbolically without explicitly solving it

(c) Write a program in Python that solves the least squares problem and plot the datapoints and regression-line

Exercise 3: Constrained optimization problems
A linear optimization-problem is supposed to find that $\mathbf{x} \in \mathbb{R}^2$, which minimizes $\mathbf{x}^T \mathbf{c}$ with $\mathbf{c} = (2.0, 1.0)^T$ and the solution-space is restricted to the green area in the figure below.



(a) Formulate this optimization-problem as a Linear Program with linear constraints.

(b) Show, whether this problem is convex.

(c) Solve this problem graphically and mark the optimal point $\mathbf{x}$ in the figure.