-05

Th 5~6 4122

Fri 12~1 4240

Historic Perspective

CM

* Diffq and initial conditions

*. Lack of information of I.C.

* Classical probability [observation does not affect the outcome of a random experiment]

* Microscopic Quantity → * Quantum Prob.

Church-Turing Thesis:

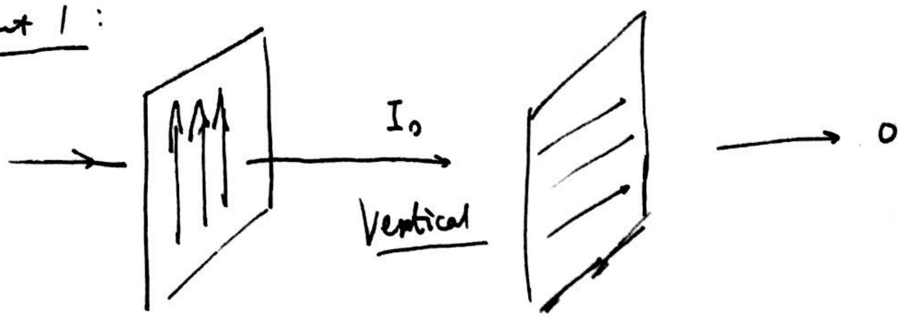
A classical (probabilistic) turing machine can efficiently (polynomial) simulate any realistic model of computation

Problem: factoring a composite number

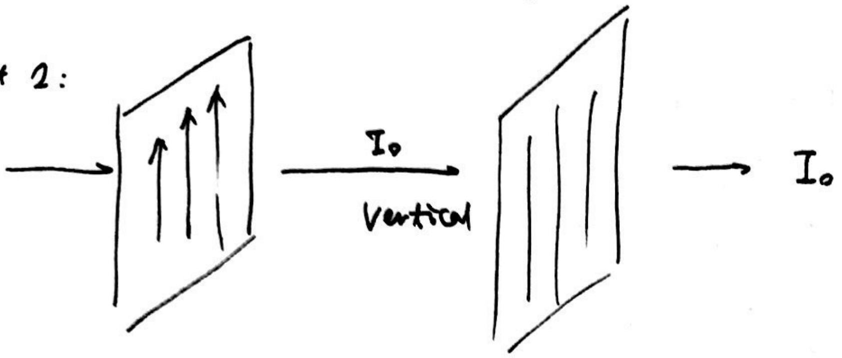
→ scale exponentially in the number of digits

1994 Shor provided a polynomial-time. quantum algorithm

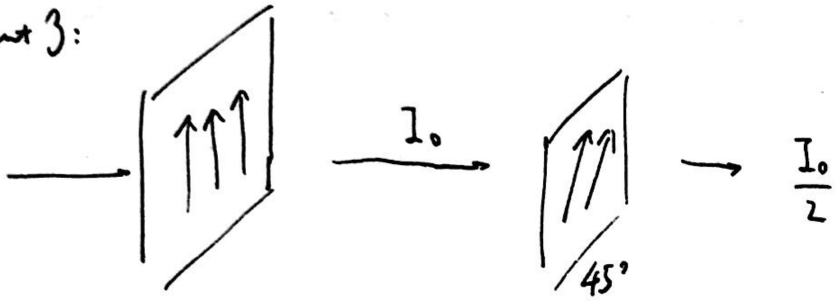
Experiment 1:



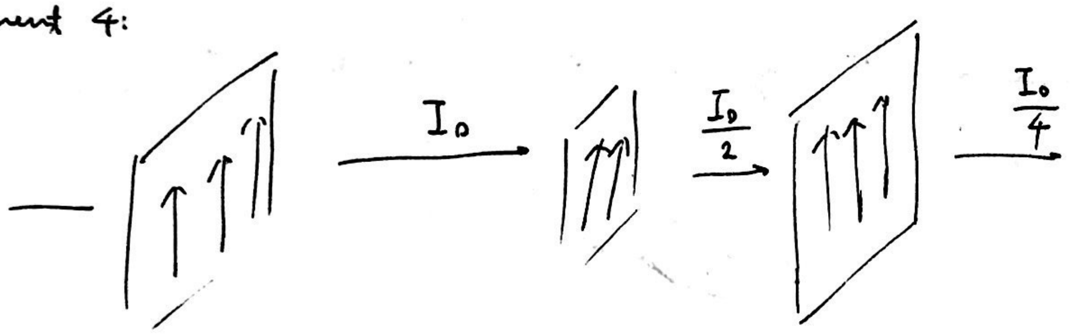
Experiment 2:



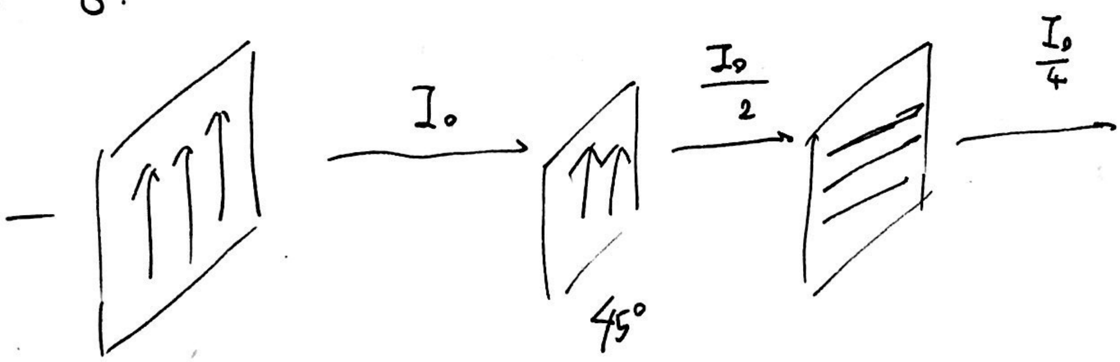
Experiment 3:



Experiment 4:



.... 5:



Stern
-gerlach

4 axioms of Quantum Mechanics

Axiom 1 Associated with any isolated physical system, ~~there~~ is a Hilbert space (complex-valued) known as ~~the~~ state space. The system is completely ~~complex~~ described by its state vector which has unit norm.

Hilbert Space: Complete inner product space

Operation: $V \times V \to \mathbb{C}$ vector space (V)

(a) $(\underline{x}, \underline{y}) = (\underline{y}, \underline{x})^*$

(b) $(\alpha \underline{x}_1 + \beta \underline{x}_2, \underline{y})$

$= \alpha(\underline{x}_1, \underline{y}) + \beta(\underline{x}_2, \underline{y})$

(c) $(\underline{x}, \underline{x}) \geq 0$ equality iff $\underline{x} = 0$

Notation

$\langle x|$ Bra

$|x\rangle$ Ket

Norm $\|x\| = \sqrt{(\underline{x}, \underline{x})} = \sqrt{\langle x|x\rangle}$

Axiom 2 A closed/isolated physical system evolves according to a linear transformation, which is unitary.

At t_1 $|\psi\rangle_{t_1}$

t_2 $|\psi\rangle_{t_2}$

$$|\psi\rangle_{t_2} = U|\psi\rangle_{t_1}$$

$${}_{t_2}\langle\psi|\psi\rangle_{t_2} = {}_{t_1}\langle\psi|U^\dagger U|\psi\rangle_{t_1}$$

$$= {}_{t_1}\langle\psi|\psi\rangle_{t_1}$$

$\mathbb{I}$

$\Rightarrow U^\dagger U = \mathbb{I}$

Unitarity

$UU^\dagger U^\dagger U = \mathbb{I}$ $\quad U^\dagger U U^\dagger U$

$U^\dagger (UU^\dagger) U \quad U^\dagger \mathbb{I} U$

Adjoint : A is an operator (linear transformation)

$$A : H \to H$$

$A^{\dagger}$ is that unique operator, satisfies

$$(\underline{y}, A\underline{x}) = (A^{\dagger}\underline{y}, \underline{x})$$ for all vector $\underline{x}$ & $\underline{y}$

$$\langle y | A | x \rangle = \langle y | Ax \rangle = \langle A^{\dagger} y | x \rangle$$

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Recall: Axioms

① Any isolated system is a unit vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$

② State (vector) evolves according to unitary transformation.

$$|\psi\rangle_{t_2} = U \cdot |\psi\rangle_{t_1}$$

Adjoint $(\underline{y}, A\underline{x}) = (A^{+}\underline{y}, \underline{x})$ for all $\underline{x}, \underline{y} \in \mathcal{H}$

* $(A^{+})^{+} = A$
* $\langle y|A|x\rangle$
* ${}_{t_2}\langle\psi|\psi\rangle_{t_2} = (\underline{\psi}_{t_2}, \underline{\psi}_{t_2}) = (U\underline{\psi}_{t_1}, U\underline{\psi}_{t_1})$

can apply back and forth by adjoint

$$= (\underline{\psi}_{t_1}, U^{+}U\underline{\psi}_{t_1})$$

$$= {}_{t_1}\langle\psi|\underbrace{U^{+}U}_{\mathbb{1}}|\psi\rangle_{t_1}$$

unitary

③ Measurement axiom

POVM

(Projective)

$$\langle V_i | V_j \rangle = \delta_{ij}$$

A measurement is an orthonormal basis $\{|V_1\rangle, |V_2\rangle \dots |V_d\rangle\}$ $d = \dim(\mathcal{H})$

* Classical output : X

* Quantum output $P[X=i] = |\langle V_i | \psi \rangle|^2$. The state collapse to $|V_i\rangle$

$$|\psi\rangle = \sum_{i=1}^{d} \alpha_i |V_i\rangle$$

$\boxed{\text{if the event } \{X=i\} \text{ is observed}}$

$$1 = \langle \psi | \psi \rangle = \sum_{i,j} \alpha_i \alpha_j^* \langle V_j | V_i \rangle = \sum_{i,j} |\alpha_i|^2 \delta_{ij} = \sum_i |\alpha_i|^2$$

$\boxed{\text{Inner Product}}$

$$\langle \underline{y}, \alpha_1 x_1 + \alpha_2 x_2 \rangle = \alpha_1 \langle \underline{y} | \underline{x_1} \rangle + \alpha_2 \langle \underline{y} | \underline{x_2} \rangle$$

$$\langle \alpha_1 \underline{y_1} + \alpha_2 \underline{y_2}, \underline{x} \rangle = \alpha_1^* \langle \underline{y_1}, \underline{x} \rangle + \alpha_2^* \langle y_2, \underline{x} \rangle$$

$$\langle V_j | \psi \rangle = \sum_{i=1}^{d} \alpha_i \langle V_j | V_i \rangle = \sum_{i=1}^{d} \alpha_i \delta_{ij} = \alpha_j$$

Axiom 3 : A classical measurement device is modeled
(generalized) $\{M_{\alpha_1}, M_{\alpha_2} \dots, M_{\alpha_k}\}$ a collection of operators
$\hookrightarrow$ for some finite k, α_i is real
that satisfies $\sum_{i=1}^{k} M_{\alpha_i}^{+} M_{\alpha_i} = \mathbb{I}$

$\rightarrow$ produces a classical & a quantum output

$$P[X = \alpha_m] = \langle \psi | M_{\alpha_m}^{+} M_{\alpha_m} | \psi \rangle \Rightarrow \sum_{m=1}^{k} \langle \psi | M_{\alpha_m}^{+} M_{\alpha_m} | \psi \rangle$$

$$= (M_{\alpha_m} \psi , M_{\alpha_m} \psi \rangle \qquad = \langle \psi | \underbrace{\sum_{m=1}^{k} M_{\alpha_m}^{+} M_{\alpha_m}}_{\mathbb{1}} | \psi \rangle$$

If $\{ x = \alpha_m \}$ is observed then $|\psi\rangle$ collapse to $\dfrac{M_{\alpha_m} |\psi\rangle}{\sqrt{P[X = \alpha_m]}}$

normalize

$\dfrac{M_{\alpha_m} |\psi\rangle}{C_m}$, C_m is such that the state has unit norm

$$L = \sqrt{(M_{\alpha_m} \psi , M_{\alpha_m} \psi)} = \sqrt{P[x = \alpha_m]}$$

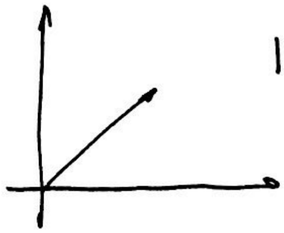
$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \qquad M_0^{+} M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} ; \quad M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad M_1^{+} M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|\psi\rangle = \sum_{m=1}^{k} \alpha_m \frac{M_{\alpha_m} |\psi\rangle}{\sqrt{P[x = \alpha_m]}} \qquad \langle \psi | \psi \rangle = \sum_{\substack{n=1 \\ m=1}}^{k} \langle \psi | M_{\alpha_n}^{+} M_{\alpha_m} | \psi \rangle$$

$\alpha_i = \sqrt{P[X = \alpha_m]}$

Example



$$|\psi\rangle = \left[\frac{1}{\sqrt{2}} ; \frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

computational basis

Quantum State $|0\rangle$

$\{ |0\rangle, |1\rangle \}$

$$p[X=0] : |\langle 0|\psi\rangle|^2 = \left([1 \quad 0] \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right)^2 = \frac{1}{2}$$

$$p[X=1] : \langle 1|\psi\rangle = \frac{1}{2}$$

Quantum state $|\phi\rangle$

$\boxed{\text{Decoherence}}$ *

$\{ |+\rangle, |-\rangle \}$

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$p[x=+] = \langle +|\psi\rangle = 1$$

$$p[x=-] = \langle -|\psi\rangle = 0$$

Axiom 4 A composite physical system consisting of subsystems $|\psi_1\rangle$ in $\mathcal{H}_1$ & $|\psi_2\rangle$ in $\mathcal{H}_2$ is a state $|\psi_1\rangle \otimes |\psi_2\rangle$ in Hilbert space $\mathcal{H}$ which is given by $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

Tensor product:

$\mathcal{H}_1$ $\quad {}_1\langle \cdot | \cdot \rangle_1$ & $\mathcal{H}_2$ $\quad {}_2\langle \cdot | \cdot \rangle_2$

Take a vector $|\alpha\rangle \in \mathcal{H}_1$

$|\psi\rangle \in \mathcal{H}_2$

& create $|\alpha\rangle \otimes |\psi\rangle$

$$\mathcal{H} = \underbrace{\text{span}}_{\text{all possible linear combination}} \{ |\alpha\rangle \otimes |\psi\rangle : |\alpha\rangle \in \mathcal{H}_1 , |\psi\rangle \in \mathcal{H}_2 \}$$

$\mathcal{H}$ has to satisfy

① $(\underset{\text{number}}{z} |\alpha\rangle) \otimes |\psi\rangle = |\alpha\rangle \otimes (z|\psi\rangle) = z(|\alpha\rangle \otimes |\psi\rangle)$

② $(|\alpha_1\rangle + |\alpha_2\rangle) \otimes |\psi\rangle = |\alpha_1\rangle \otimes |\psi\rangle + |\alpha_2\rangle \otimes |\psi\rangle$

③ Same with $|\psi\rangle$

④ $\{\theta_1\} = \sum_{i=1}^{n} c_i |\alpha_i\rangle \otimes |\psi_i\rangle$; $\{\theta_2\} = \sum_{j=1}^{n} d_j |\beta_j\rangle \otimes |\phi_j\rangle$

$$\langle \theta_2 | \theta_1 \rangle = \sum_{j,i} d_j^* c_j \ \langle \beta_i | \alpha_i \rangle \langle \phi_j | \psi_i \rangle$$

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Recall

3. Measurement (projective)

$\{ |v_1\rangle \dots |v_d\rangle \}$ $\longrightarrow X$ (classical random)

$\longrightarrow$ Quantum state

$P[X=i] = |\langle v_i | \psi \rangle|^2$, quantum state $\} \rightarrow |v_i\rangle$

collapse

general measurement

$\{ M_{\alpha_1} \dots M_{\alpha_K} \}$ $\quad \sum_{i=1}^{K} M_{\alpha_i}^{+} M_{\alpha_i} = \mathbb{I}$

$P[X=\alpha_i] : \langle \psi | M_{\alpha_i}^{+} M_{\alpha} | \psi \rangle$

Q. state $\quad \dfrac{M_{\alpha_i} |\psi\rangle}{\sqrt{P[X=\alpha_i]}}$

A composite system $|\psi_1\rangle, |\psi_2\rangle$ is a state $|\psi_1\rangle \otimes |\psi_2\rangle$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$

Tensor produt

$H_1 \quad {}_1\langle\cdot|\cdot\rangle_1 \quad \boxed{\text{dimension } d_1}$ $\qquad H_2 \quad {}_2\langle\cdot|\cdot\rangle_2 \quad \boxed{\text{dimension } d_2}$

$$H = \mathcal{H}_1 \otimes \mathcal{H}_2 = \text{span}\left\{ |\alpha\rangle \otimes |\psi\rangle \quad \begin{matrix} |\alpha\rangle \in \mathcal{H}_1 \\ |\psi\rangle \in \mathcal{H}_2 \end{matrix} \right\}$$

① $(z|\alpha\rangle) \otimes |\psi\rangle = |\alpha\rangle \otimes (z|\psi\rangle) = z(|\alpha\rangle \otimes |\psi\rangle)$

② $(|\alpha_1\rangle + |\alpha_2\rangle) \otimes |\psi\rangle = |\alpha_1\rangle \otimes |\psi\rangle + |\alpha_2\rangle \otimes |\psi\rangle$

③ same for $|\psi\rangle$

④ $\left.\begin{matrix} |\theta_1\rangle = \sum_{i=1}^{n} C_i |\alpha_i\rangle \otimes |\psi_i\rangle \\ |\theta_2\rangle = \sum_{j=1}^{m} d_j |\beta_j\rangle \otimes |\phi_j\rangle \end{matrix}\right\}$ Two linear combination.

$$\langle\theta_2|\theta_1\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} d_j^* C_i \left[{}_1\langle\beta_j|\alpha_i\rangle_1 \; {}_2\langle\phi_j|\psi_i\rangle_2 \right]$$

Act on H_1, H_2

⑤ $(A \otimes B)(|\alpha\rangle \otimes |\psi\rangle) = (A|\alpha\rangle) \otimes (B|\psi\rangle)$

⑥ $(A \otimes B)\left(\sum_i C_i |\alpha_i\rangle \otimes |\psi_i\rangle\right) = \sum_i C_i \left(A|\alpha_i\rangle \otimes B|\psi_i\rangle\right)$

⑦ $\left(\sum_i C_i (A_i \otimes B_i)\right)(|\alpha\rangle \otimes |\psi\rangle) = \sum_i C_i (A_i|\alpha\rangle \otimes B_i|\psi\rangle)$

(a) $|\text{all-zero}\rangle_1 \otimes |\psi\rangle_2 = |\alpha\rangle_1 \otimes |\text{all-zero}\rangle_2$

$= |\text{all-zero}\rangle_1 \otimes |\text{all-zero}\rangle_2$

b $\mathcal{H}_1 = \mathbb{C}^2$ $\mathcal{H}_2 = \mathbb{C}^2$

$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ $\begin{bmatrix} V_3 \\ V_4 \end{bmatrix}$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \otimes \begin{bmatrix} V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} V_1V_3 \\ V_1V_4 \\ V_2V_3 \\ V_2V_4 \end{bmatrix} = |\theta_1\rangle$$

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \otimes \begin{bmatrix} U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} U_1U_3 \\ U_1U_4 \\ U_2U_3 \\ U_2U_4 \end{bmatrix} = |\theta_2\rangle$$

Inner Product?

?

$$\langle\theta_2|\theta_1\rangle = U_1^* U_3^* V_1V_3 + U_1^* U_4^* V_1V_4 + U_2^* U_3^* V_2V_3 + U_2^* U_4^* V_2V_4$$

operator tensor product

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \xrightarrow{\text{Kronecker Delta}} \begin{bmatrix} a_{11}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{12} \cdots \\ a_{21} \cdots & a_{22} \cdots \end{bmatrix}$$

Dim $|V_1\rangle \cdots |V_{d_1}\rangle$ orthonormal basis for $\mathcal{H}_1$

$|U_1\rangle \cdots |U_{d_2}\rangle$ — — — — $\mathcal{H}_2$.

Claim $\{|V_i\rangle \otimes |U_j\rangle\}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

$1 \le i \le d_1$

$1 \le j \le d_2$.

— End of Lecture 2 —