$|u_i\rangle \otimes |v_j\rangle$
$|u_i\rangle \otimes |v_k\rangle$ } Inner product $\quad \langle u_i|u_i\rangle\langle v_k|v_j\rangle = 0$ if $k \neq j$

Orthonormal collection of vector

$$|\alpha\rangle \otimes |\psi\rangle = \left(\sum_i c_i |v_i\rangle\right) \otimes \left(\sum_j d_j |u_j\rangle\right) = \sum_{i,j} c_i d_j \left(\underbrace{|v_i\rangle \otimes |u_j\rangle}_{\text{orthonormal basis}}\right)$$

Seperable $\quad |\alpha\rangle \otimes |\psi\rangle$

A vector is said to be seperable if it can written as $\underbrace{|\alpha\rangle \otimes |\psi\rangle}_{\text{From two Hilbert space}}$, for some $|\alpha\rangle \in \mathcal{H}_1$ & $|\psi\rangle \in \mathcal{H}_2$

$\mathcal{H} = \mathbb{C}^2$ (Quantum bit). $\quad |0\rangle$ & $|1\rangle$ basis

$\mathcal{H} \otimes \mathcal{H} \quad$ $|0\rangle \otimes |1\rangle$
$|0\rangle \otimes |0\rangle$
$|1\rangle \otimes |1\rangle$
$|1\rangle \otimes |0\rangle$

$$|0\rangle \otimes |0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$|0\rangle \otimes |1\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$|\psi\rangle = \frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \cdots |01\rangle + \frac{1}{2\sqrt{2}}|10\rangle + \frac{1}{2\sqrt{2}}|11\rangle \quad \text{(Seperable)}$$

$$= \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

Entangled

A vector which is not seperable is called entangled

$$\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle = \left(\alpha|0\rangle + \beta|1\rangle\right)\left(\gamma|0\rangle + \delta|1\rangle\right)$$

$$= \underline{\alpha\gamma}|00\rangle + \alpha\cdot\delta|01\rangle + \beta\gamma|10\rangle + \underline{\beta}\delta|11\rangle$$

$\alpha, \gamma, \beta, \delta$. cannot be zero

$\Rightarrow$ $\alpha\cdot\delta \neq 0$ & $\beta\cdot\gamma \neq 0$ must hold

$\Rightarrow$ contradicts the assumption.

$$\left.\begin{aligned} &\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle \\ &\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle \end{aligned}\right\} \text{entangled.}$$

Global Phase

$\{|\psi_1\rangle, |\psi_2\rangle\}$ basis for $\mathbb{C}^2$

$$|00\rangle = (\alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle) \otimes (\alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle)$$
$$= \alpha_1\alpha_1|\psi_1\psi_1\rangle + \alpha_1\alpha_2|\psi_1\psi_2\rangle$$
$$\alpha_2\alpha_1|\psi_2\psi_1\rangle + \alpha_2\alpha_2|\psi_2\psi_2\rangle$$

$$\left.\begin{aligned} |0\rangle &= \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle \\ |1\rangle &= \beta_1|\psi_1\rangle + \beta_2|\psi_2\rangle \end{aligned}\right\} \Rightarrow \frac{1}{\sqrt{2}}\underbrace{(\alpha_2\beta_1 - \alpha_1\beta_2)}\left[|\psi_2\psi_1\rangle - |\psi_1\psi_2\rangle\right]$$

$$\downarrow$$

$$\text{abs}(\quad) = 1 = e^{j\phi}$$

global phase $\underbrace{\underline{|\psi\rangle}, \underline{e^{j\theta}|\psi\rangle}}$

No measurement cannot distinguish these two.

$$|01\rangle = \cancel{\alpha_1\beta_1|\psi_1\psi_1\rangle} + \alpha_1\beta_2|\psi_1\psi_2\rangle$$
$$+ \alpha_2\beta_1|\psi_2\psi_1\rangle + \cancel{\alpha_2\beta_2|\psi_2\psi_2\rangle}$$

$$|10\rangle = \cancel{\beta_1\alpha_1|\psi_1\psi_1\rangle} + \beta_1\alpha_2|\psi_1\psi_2\rangle + \beta_2\alpha_1|\psi_2\psi_1\rangle + \cancel{\beta_2\alpha_2|\psi_2\psi_2\rangle}$$

$\{M_{\alpha_1} \cdots M_{\alpha_K}\}$

$\|\psi\rangle$	$e^{j\theta}\|\psi\rangle$
$p[X=\alpha_i] = \langle\psi\|M_{\alpha_i}^{\dagger} M_{\alpha_i}\|\psi\rangle$	$p[X=\alpha_i] = \langle\psi\| M_{\alpha_i}^{\dagger} M_{\alpha_i}\|\psi\rangle (e^{j\theta})^* e^{j\theta}$ $= \langle\psi\| M_{\alpha_i}^{\dagger} M_{\alpha_i}\|\psi\rangle$

$$\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle = |\psi\rangle$$

Proj. measurement $\{\underbrace{|00\rangle}_{0}, \underbrace{|01\rangle}_{1}, \underbrace{|10\rangle}_{2}, \underbrace{|11\rangle}_{3}\}$

$p[X=0] = 0$

$p[X=1] = 1/2$

$p[X=2] = 1/2$.

$p[X=3] = 0$

—— End of Lecture 3 ——