EECS 501: Probability and Random Processes Prof. Sandeep Pradhan

Set operations, Event Space

2020/01/08 Yunjie Wang

Set Theory

1. Notation
   - $A = \underbrace{\{1, 2, ..., 100\}}_{\text{exhausted listing}} = \underbrace{\{x|1 \leq x \leq 100, x \text{ is an integer}\}}_{\text{listing by property}}$
     No repetition allowed
   - $\varnothing$: empty set(No element)
2. Set Operations
   - Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$
   - Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$
   - Complement: $A \cap B^c = \{x : x \in A \text{ and } x \notin B\}$
   - Set Difference: $B \setminus A := B \cap A^c$, i.e., $B \setminus A$ is the set of $x \in B$ that do not belong to A
   - Disjoint Sets: $A \cap B = \varnothing$ (mutually exclusive)
3. Set Identities
   - Commutative Laws
     $$A \cup B = B \cup A \text{ and } A \cap B = B \cap A$$
   - Associative Laws
     $$A \cup (B \cup C) = (A \cup B) \cup C$$
     $$A \cap (B \cap C) = (A \cap B) \cap C$$
   - Distributive Laws
     $$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
     $$B \cap \left(\bigcup_{i=1}^{\infty} A_i \right) = \bigcup_{i=1}^{\infty} (B \cap A_i) \text{ (Generalized)}$$
     $$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
     $$B \cup \left(\bigcap_{i=1}^{\infty} A_i \right) = \bigcap_{i=1}^{\infty} (B \cup A_i) \text{ (Generalized)}$$

- De Morgans

$$(A \cap B)^c = A^c \cup B^c$$
$$\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c$$
$$(A \cup B)^c = A^c \cap B^c$$
$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

4. Limit Notation/ Infinite collections of subsets of Ω (Important)

- $\sum_{i=1}^{\infty} a_i = \lim\limits_{N\to\infty} \sum_{i=1}^{N} a_i$
- Countable Union and Intersection

$$\bigcup_{i=1}^{N} A_i = \{x : \exists \text{i is an integer such that x belongs to at least } A_i\}$$
$$\bigcap_{i=1}^{N} A_i = \{x : \text{x belongs to } A_i \text{ for all i}\}$$

- Some useful properties: Let Ω denote the real numbers, $\Omega = \mathcal{R} := (-\infty, \infty)$

$$\bigcap_{n=1}^{\infty} (-\infty, 1/n) = (-\infty, 0]$$
$$\bigcup_{n=1}^{\infty} (-\infty, -1/n] = (-\infty, 0)$$
$$\bigcap_{n=1}^{\infty} [0, 1/n) = \{0\}$$
$$\bigcup_{n=1}^{\infty} (-\infty, n] = (-\infty, \infty)$$
$$\bigcap_{n=1}^{\infty} (-\infty, n] = \varnothing$$

5. Functions

- A function consists of:

  A set X of admissible inputs (domain)

  A rule or mapping f that associates to each $x \in X$ a value $f(x)$ that belongs to a set Y (co-domain)
- $f : X \to Y$, or we say "f maps X into Y"
- The set of all possible values of $f(x)$ is called range, symbolically $f(x) : x \in X$. And since $f(x) \in Y$ for each x, it is clear that the range is a subset of Y. However, the reange may or may not be equal to Y.

  A function is said to be **onto** if its range is equal to its co-domain

  A function is said to be **one-to-one** if the condition $f(x_1) = f(x_2)$ implies $x_1 = x_2$

A function is said to be **invertible** if for every $y \in Y$ there is a unique $x \in X$ with $f(x) = y$. In other words, a function is invertible if and only if it is both one-to-one and onto. Or we can say for every $y \in Y$, the equation $f(x) = y$ has a unique solution

- If $f : X \to Y$ and if $B \subset Y$, then the inverse image of B is

$$f^{-1}(B) := \{x \in X : f(x) \in B\},$$

which we emphasize is a subset of X

This concept applies to any function whether or not it is invertible

6. Countable and Uncountable Sets
    - The cardinality of A: the number of points in a set A, denoted by $|A|$
    - If A and B are two disjoint sets, then

$$|A \cup B| = |A| + |B|$$

    - The cardinality of a set may be finite or infinite, and three cases are needed to be considered
        (a) A and B both have finite cardinality
        (b) One has finite cadinality and one has infinite cardinality
        (c) Both have infinite cardinality
    - A nonempty set A is said to be **coutable** if the elements of A can be enumerated or listed in a sequence. It can be written in the form:

$$A = \bigcup_{i=1}^{\infty} \{a_k\}$$

And the empty set is also said to be countable

Probability

1. Random experiment: An experiment whose outcome cannot be predicted in advance
2. Sample Space(Ω): Set of all possible observable outcomes
3. Event: Subset of a sample space
    - If ω is a point in Ω, we shall write $\omega \in \Omega$

- Let A and B be two collections of points in Ω,

  If every point A also belongs to B, we say that A is subset of B, and we denote this by writing $A \subset B$

  If $A \subset B$ and $B \subset A$, then we write $A = B$; i.e', two sets are equal if they contain exactly the same points

  If $A \subset B$ but $A \neq B$, we say that A is a proper subset of B

- $A \subset B$, equivalently, $x \in A \implies x \in B$
- In words, all the element in A is in B

4. Event Space(σ-algebra)

- A collection of events $\mathcal{F}$ is called a σ-algebra
  - $\mathcal{F}$ is not empty(at least one set)
  - It is closed under complementation: If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
  - It is closed under countable union: If countable collections of sets, $A_1, A_2, ..., A_N$, belongs to $\mathcal{F}$, then $\underset{i=1}{\cup} A_i \in \mathcal{F}$
- Example

$$\begin{aligned}
&\Omega = \{1, 2, 3, 4, 5, 6\}, A = \{2, 4, 6\}, \Omega^c = \varnothing \\
&\Rightarrow \mathcal{F}_1 = \{\Omega, \varnothing\} \text{ (simpliest event space)}, \mathcal{F}_2 = \{\Omega, \varnothing, A, A^c\} \\
&\Rightarrow \text{Finest set(Power set, contains all subsets of } \Omega\text{): } \mathcal{F} = \{A : A \subseteq \Omega\}
\end{aligned}$$

- More Properties
  - $\Omega \in \mathcal{F}$
  - $\varnothing \in \mathcal{F}$
  - $\overset{N}{\underset{i=1}{\cap}} A_i \in \mathcal{F}$

    Proof:

$$\begin{aligned}
&A_1, A_2, ..., A_N \text{ belong to } \mathcal{F} \\
\xrightarrow{\text{Axiom b}} &A_1^c, A_2^c, ..., A_N^c \text{ belong to } \mathcal{F} \\
\xrightarrow{\text{Axiom c}} &\overset{N}{\underset{i=1}{\cup}} A_N^c \in \mathcal{F} \\
\xrightarrow{\text{De Morgan}} &\left(\overset{N}{\underset{i=1}{\cap}} A_i\right)^c \in \mathcal{F} \\
\xrightarrow{\text{Axiom b}} &\left(\overset{N}{\underset{i=1}{\cap}} A_i\right) \in \mathcal{F}, \ (A^c)^c = A
\end{aligned}$$