EECS 501: Probability and Random Processes — Prof. Sandeep Pradhan

Probability Measure, Conditional Probability and Independence

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Recap Problem

Assume $\Omega = \{1, 2, 3, 4, 5, 6\}$, and A $= \{2, 4, 6\}$, what is the smallest sigma-field containing A?

$$\mathcal{F} = \{A, A^c, \Omega, \varnothing\}$$

This is a model that lack a certain amount of information, because event A only represents the even number but cannot give further information on what specific number we are having.

Probability Measure

1. Axioms of Probability $P : \mathcal{F} \to \mathbb{R}$

   - Normalization:
   $$P(\Omega) = 1$$
   - Non-negativity:
   $$P(A) \geq 0, \text{for } A \in \mathcal{F}$$
   - Additivity:
   $$\text{If } A_1, ..., A_n, ... \in \mathcal{F}, \text{which are pairwise disjoint} \left(A_i \cap A_j = \varnothing, \text{for } i \neq j\right)$$
   $$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \lim_{N\to\infty} \sum_{i=1}^{N} P(A_i)$$

2. Derived Properties

   - Empty Set
   $$P(\varnothing) = 0$$
   Proof:
   $$\varnothing \cap \Omega = \varnothing, \varnothing \cup \Omega = \Omega$$
   $$\text{Apply Axiom 3, } P(\varnothing \cup \Omega) = P(\Omega) + P(\varnothing)$$
   $$\text{Using second condition, } P(\Omega) = P(\Omega) + P(\varnothing)$$
   $$\text{Cancel same term both sides, } P(\varnothing) = 0$$

- Law of Complement

$$P\left(A^c\right) = 1 - P\left(A\right)$$

Proof:

$$A \cap A^c = \varnothing, A \cup A^c = \Omega$$
$$P\left(A \cup A^c\right) = P\left(\Omega\right) = P\left(A\right) + P\left(A^c\right)$$
$$P\left(A^c\right) = 1 - P\left(A\right)$$

- Inclusion and Exclusion Principle

$$\text{Suppose } A \cap B \neq \varnothing, P\left(A \cup B\right) = P\left(A\right) + P\left(B\right) - P\left(A \cap B\right)$$

Proof:

$$P\left(A \cup B\right) = P\left(A \cap B^c\right) + P\left(A^c \cap B\right) + P\left(A \cap B\right) \text{(Pairwise Disjoint)}$$
$$P\left(A^c \cap B\right) + P\left(A \cap B\right) = P\left(B\right), P\left(A \cap B^c\right) + P\left(A \cap B\right) = P\left(A\right) \text{(Axiom 3)}$$
$$P\left(A^c \cap B\right) + P\left(A \cap B\right) + P\left(A \cap B^c\right) + P\left(A \cap B\right) = P\left(B\right) + P\left(A\right)$$
$$P\left(A \cup B\right) + P\left(A \cap B\right) = P\left(B\right) + P\left(A\right)$$
$$P\left(A \cup B\right) = P\left(A\right) + P\left(B\right) - P\left(A \cap B\right)$$

- Continuity of P

$$\text{If } A_1 ... A_n ... \in \mathcal{F},$$
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{N \to \infty} P\left(\bigcup_{i=1}^{N} A_i\right), P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{N \to \infty} P\left(\bigcap_{i=1}^{N} A_i\right)$$

Conditional Probability

1. Definition $P : \mathcal{F} \to \mathbb{R}$

$$P_B(A) = \frac{P\left(A \cap B\right)}{P(B)} = \underbrace{P\left(A|B\right)}_{\text{Standard Notation}}$$

2. Axiom Verification

- Normalization

$$P_B\left(\Omega\right) = \frac{P\left(\Omega \cap B\right)}{P(B)} = \frac{P\left(B\right)}{P(B)} = 1$$

- Non-negativity

Assume E,B are in σ - field, then E $\cap$ B is in the field as well,

$$P_B\left(E\right) = \frac{P\left(E \cap B\right)}{P(B)} \geq 0$$

- Additivity
  Let $A_1 ... A_n ...$ pairwise disjoint

$$P_B\left(\bigcup_{i=1}^{\infty} A_i\right) = \frac{P\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap B\right)}{P(B)} = \frac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)}{P(B)}$$
$$= \frac{\sum_{i=1}^{\infty} P\left(A_i \cap B\right)}{P(B)} = \sum_{i=1}^{\infty} P_B\left(A_i\right)$$

Independence

1. Definition: Two events B & C are said to be independent

$$P\left(C|B\right) = P\left(C\right) \iff P\left(C \cap B\right) = P\left(B\right) \cdot P\left(C\right)$$