Superconductivity

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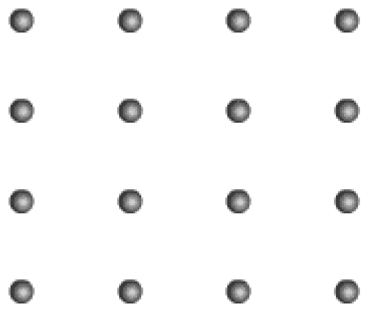
Introduction

- Cooper Pair
- Hamiltonian
  - Mean-Field Approximation
  - Bogoliubov Transformation
  - Result
- Ground State
- Gap Function
  - Experimental
  - Analytical

Cooper Pair

- A weak electron-electron bound pair mediated by phonons(quasi particles of lattice vibrations)
- Deformation of the background
- The higher density of ions leads to the attraction of another electron
- Viewed as an effective attraction between two electrons



Source: http://ffden-2.phys.uaf.edu/212_fall2003.web.dir/T.J_Barry/bcstheory.html

Hamiltonian

$$\sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} V_{\text{eff}}\left(\mathbf{k},\mathbf{k}',\mathbf{q}\right) c^{\dagger}_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}} c^{\dagger}_{\mathbf{k}'-\mathbf{q}} c_{\mathbf{k}'} \rightarrow V_{\mathbf{k}\mathbf{k}'} a^{\dagger}_{\mathbf{k}\uparrow} a^{\dagger}_{-\mathbf{k}\downarrow} a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow}$$

- Here other types of interactions are ignored, only electrons with opposite momentum and spin are allowed
- This turns out to be a great approximation
- Why opposite momentum and spin?

Hamiltonian

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} + \frac{1}{N} \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow}$$

- $\xi_{\mathbf{k}} = E_{\mathbf{k}} - \mu$, where μ is the chemical potential
- $a^\dagger_{\mathbf{k}\sigma}$ creates an electron with momentum k and spin σ
- The second term describes the destruction of a Cooper pair (two electrons with opposite momentum and spin) and the subsequent creation of another Cooper pair

Hamiltonian – Mean-Field Approximation

$$a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow} \approx \left\langle a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} \right\rangle a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow} + a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} \left\langle a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow} \right\rangle - \left\langle a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} \right\rangle \left\langle a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow} \right\rangle$$

- One of the most used method to decouple quartic term
- Unlike the usual form, the last term could not be negelected, because it corresponds to one Cooper pair in the superconducting state
- The last term turns out to be related to the gap function $\Delta_{\mathbf{k}} = -\frac{1}{N} \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \left\langle a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow} \right\rangle$

Hamiltonian – Bogoliubov Transformation

$$a_{\mathbf{k}\uparrow} = u^*_{\mathbf{k}}\gamma_{\mathbf{k}\uparrow} + v_{\mathbf{k}}\gamma^\dagger_{-\mathbf{k}\downarrow}$$
$$a^\dagger_{-\mathbf{k}\downarrow} = u_{\mathbf{k}}\gamma^\dagger_{-\mathbf{k}\downarrow} - v^*_{\mathbf{k}}\gamma_{\mathbf{k}\uparrow}$$

- The exact value of u_k and v_k are unknown, and it follows $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$ because of the fermionic anticommutation rule
- Later, u_k and v_k could be determined by diagonalizing the Hamitonian

Hamiltonian – Result

$$H_0 = \sum_{\mathbf{k}} \left(\xi_{\mathbf{k}} - \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2} + \Delta_{\mathbf{k}} \left\langle a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} \right\rangle \right)$$

$$H_1 = \sum_{\mathbf{k}} \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2} \left(\gamma^\dagger_{\mathbf{k}\uparrow} \gamma_{\mathbf{k}\uparrow} + \gamma^\dagger_{-\mathbf{k}\downarrow} \gamma_{-\mathbf{k}\downarrow} \right)$$

$$\Longrightarrow$$

$$H = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}\sigma} \gamma_{\mathbf{k}\sigma} + E_0$$

$$E_0 = \sum_{\mathbf{k}} \left(\xi_{\mathbf{k}} - E_{\mathbf{k}} + \Delta_{\mathbf{k}} \left\langle a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} \right\rangle \right)$$

- E_0 is the ground state energy
- At the Fermi level, $\xi_{\mathbf{k}} = 0$, the energy spectrum still has a gap of size $|\Delta_{\mathbf{k}}|$

Hamiltonian – Bogoliubov Transformation Revisit

$$\begin{aligned} a_{\mathbf{k}\uparrow} &= u_{\mathbf{k}}^{*}\gamma_{\mathbf{k}\uparrow} + v_{\mathbf{k}}\gamma^{\dagger}_{-\mathbf{k}\downarrow} \\ a^{\dagger}_{-\mathbf{k}\downarrow} &= u_{\mathbf{k}}\gamma^{\dagger}_{-\mathbf{k}\downarrow} - v_{\mathbf{k}}^{*}\gamma_{\mathbf{k}\uparrow} \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \gamma_{\mathbf{k}\uparrow} &= u_{\mathbf{k}}a_{\mathbf{k}\uparrow} - v_{\mathbf{k}}a^{\dagger}_{-\mathbf{k}\downarrow} \\ \gamma^{\dagger}_{-\mathbf{k}\downarrow} &= v_{\mathbf{k}}^{*}a_{\mathbf{k}\uparrow} + u_{\mathbf{k}}^{*}a^{\dagger}_{-\mathbf{k}\downarrow} \end{aligned}$$

$$|u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}}\right)$$

$$|v_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}}\right)$$

- As $\Delta_{\mathbf{k}} \to 0$, $|u_{\mathbf{k}}|^2 \to 1$ for $\xi_{\mathbf{k}} > 0$ and $|u_{\mathbf{k}}|^2 \to 0$ for $\xi_{\mathbf{k}} < 0$
- As $\Delta_{\mathbf{k}} \to 0$, $|v_{\mathbf{k}}|^2 \to 0$ for $\xi_{\mathbf{k}} > 0$ and $|v_{\mathbf{k}}|^2 \to 1$ for $\xi_{\mathbf{k}} < 0$.
- At the normal state, creating a Bogoliubon excitation corresponds to creating an electron for energies above the Fermi level and creating a hole (destroying an electron) of opposite momentum and spin for energies below the Fermi level.
- At the superconducting state, a Bogoliubon becomes a superposition of both an electron and a hole state.

Ground State

$$|\Psi_{\mathrm{BCS}}\rangle = \mathcal{N}\prod_{\mathbf{q}} \mathrm{e}^{\alpha_{\mathbf{q}} a^\dagger_{\mathbf{q}} a^\dagger_{-\mathbf{q}\downarrow}} |0\rangle$$

$$|\Psi_{\mathrm{BCS}}\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow}\right) |0\rangle$$

- The transformed Bogoliubov operators could act on the ground state and generate excited states by $\gamma^\dagger_{\mathbf{k}_1\sigma 1}\gamma^\dagger_{\mathbf{k}_2\sigma 2}\gamma^\dagger_{\mathbf{k}_3\sigma 3}\cdots\gamma^\dagger_{\mathbf{k}_n\sigma n}\,|\Psi_{\mathrm{BCS}}\rangle$
- This is analogous to the raising and lowering operators acting on the ground state of the harmonic oscillators

The Gap Function (Experimental)

$$\Delta_{\mathbf{k}} = -\frac{1}{N} \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \langle a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow} \rangle$$

- From the experimental evidence, the existence of a critical temperature suggests there is a small energy gap separating the charge carriers from the state of normal conduction
- The critical temperature for superconductivity must be a measure of the band gap
- The critical temperature was found to depend upon isotopic mass(Isotope Effect).

Gap Function (Analytical)

$$\Delta_{\mathbf{k}} = -\frac{1}{N}\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\left\langle a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow}\right\rangle$$

$$\Downarrow$$

$$1 = \frac{V_0}{N}\sum_{k<k_D}\frac{1}{2E_{\mathbf{k}}}\tanh\left(\frac{E_{\mathbf{k}}}{2k_B T}\right)$$

$$\Downarrow$$

$$\frac{1}{V_0\rho_F} = \ln\left(\frac{2\hbar\omega_D}{\Delta_0}\right); \Delta_0 = 2\hbar\omega_D \mathrm{e}^{-\frac{1}{V_0\rho_F}}$$

- Switch to the Bogoliubov basis and assume the Bogoliubon follows the Fermi-Dirac distribution
- Assume a constant attractive potential $V_{\mathbf{k},\mathbf{k}'}$ around Fermi energy, and $\mathbf{k}$ independent
- Change sum into integral and approximate the tanh to 1 since we are considering low-temerature effect
- An arbitrarily small attractive interaction V_0 gives rise to a finite gap at zero temperature

Gap Function (Analytical)

$$1 = \frac{V_0}{N} \sum_{k<k_D} \frac{1}{2E_{\mathbf{k}}} \tanh\left(\frac{E_{\mathbf{k}}}{2k_B T}\right)$$

$$\Downarrow$$

$$\frac{1}{V_0 \rho_F} = \int_0^{\hbar\omega_D} \frac{d\varepsilon}{\varepsilon} \tanh\left(\frac{\varepsilon}{2k_B T_c}\right)$$

$$\Downarrow$$

$$T_c = \frac{2\mathrm{e}^{\gamma_E}}{\pi} \frac{\hbar\omega_D}{k_B} \mathrm{e}^{-\frac{1}{V_0 \rho_F}}$$

$$\Downarrow$$

$$\frac{\Delta_0}{k_B T_c} \approx 1.76$$

- Change sum into integral and change T to T_c as critical temperature also set $\Delta_{\mathbf{k}} \to 0$
- The universal ratio between the zero-temperature gap and the critical temperature was one of the major success of BCS theory for almost all the known superconductor at that time
- T_c depends linearly on the Debye trequency ω_D, which in turn varies as the inverse square root of the ionic mass M, i.e. $T_c \propto \omega_D \propto M^{-1/2}$, in agreement with the experimental observations(Isotope Effect).