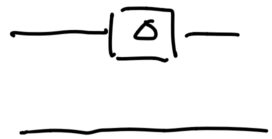
$\boxed{\text{Quantum Teleportation}}$

(a)

$$\begin{cases} \Delta \equiv (n-n_0)L/c_0 \\ P|0\rangle = |0\rangle \\ P|1\rangle = e^{i\Delta}|1\rangle \end{cases}$$



For a dual state

$$|\psi\rangle = C_0|01\rangle + C_1|10\rangle$$

After the "circuit"

$$|\psi_1\rangle = C_0 \cdot |01\rangle + C_1 \cdot e^{i\Delta} \cdot |10\rangle$$

$$= \underbrace{e^{+i\frac{\Delta}{2}}}_{\text{global phase}} \cdot \left(e^{-i\frac{\Delta}{2}} \cdot C_0|01\rangle + C_1 \cdot e^{i\frac{\Delta}{2}} C_1 |10\rangle \right)$$

By our definition of dual states.

$$|\psi\rangle = C_0 \cdot |\tilde{0}\rangle + C_1|\tilde{1}\rangle$$

$$|\psi_1\rangle = e^{-i\frac{\Delta}{2}} C_0|\tilde{0}\rangle + C_1 \cdot e^{i\frac{\Delta}{2}}|\tilde{1}\rangle$$

The transformation is $\begin{bmatrix} e^{-i\frac{\Delta}{2}} & 0 \\ 0 & e^{i\frac{\Delta}{2}} \end{bmatrix}$

in the matrix form.

The matrix is nothing but $R_Z(\Delta) = e^{-iZ\frac{\Delta}{2}}$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{pauli-Z matrix}$$

$$e^{-iZ\frac{\Delta}{2}} = \cos\frac{\Delta}{2} \cdot I - i \cdot \sin\frac{\Delta}{2} \cdot Z = \begin{bmatrix} e^{-i\frac{\Delta}{2}} & 0 \\ 0 & e^{i\frac{\Delta}{2}} \end{bmatrix}$$

$$e^{-iZ\frac{\Delta}{2}} = \sum_{n=0}^{\infty} \frac{\left(-iZ\frac{\Delta}{2}\right)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left(\frac{\Delta}{2}\right)^n \cdot Z^n$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^3 = Z$$

$$\Rightarrow e^{-iZ\frac{\Delta}{2}} = \sum_{\substack{n=0 \\ n,\text{even}}}^{\infty} \frac{(-i)^n}{n!} \left(\frac{\Delta}{2}\right)^n \cdot I + \sum_{\substack{n=0 \\ n,\text{odd}}}^{\infty} \frac{(-i)^n}{n!} \left(\frac{\Delta}{2}\right)^n \cdot Z$$

$$= I \cdot \sum_{n=0}^{\infty} \frac{(-i)^{2n}}{(2n)!} \left(\frac{\Delta}{2}\right)^{2n} + Z \sum_{n=0}^{\infty} \frac{(-i)^{2n+1}}{(2n+1)!} \left(\frac{\Delta}{2}\right)^{2n+1}$$

$$\boxed{\begin{aligned} &= I \cdot \cos\frac{\Delta}{2} + Z\,(-i) \cdot \sin\frac{\Delta}{2} \\ &= \begin{bmatrix} \cos\frac{\Delta}{2} & 0 \\ 0 & \cos\frac{\Delta}{2} \end{bmatrix} + \begin{bmatrix} -i\sin\frac{\Delta}{2} & 0 \\ 0 & i\sin\frac{\Delta}{2} \end{bmatrix} = \begin{bmatrix} e^{-i\frac{\Delta}{2}} & 0 \\ 0 & e^{+i\frac{\Delta}{2}} \end{bmatrix} \end{aligned}}$$

(b)

$$H_{bs} = i\theta(ab^{\dagger} - a^{\dagger}b)$$

a & b commute with eath other

$$B = \exp[\theta(a^{\dagger}b - ab^{\dagger})]$$

$$\Rightarrow B^{\dagger} = \exp[\theta(ab^{\dagger} - a^{\dagger}b)]$$
$$= \exp[-\theta(a^{\dagger}b - ab^{\dagger})]$$

$$BaB^{\dagger} \stackrel{?}{=} a\cos\theta + b\sin\theta$$

$$e^{\lambda G} A e^{-\lambda G} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n$$

$$C_0 = A, \quad C_1 = [G, C_0], \quad C_2 = [G, C_1]$$
$$\cdots \quad C_n = [C, C_{n-1}]$$

In our case

$$A = a \qquad C_1 = [(a^{\dagger}b - ab^{\dagger}), a]_{+}$$
$$= b[a^{\dagger}, a]_{+} - b^{\dagger}\underbrace{\cancel{[a, a]_{+}}}_{0}$$
$$= -b$$

$$C_2 = [(a^{\dagger}b - ab^{\dagger}), -b]_{+}$$
$$= -\left(a^{\dagger}[b, b]_{+} - a[b^{\dagger}, b]_{+}\right)$$
$$= -a$$

$$C_3 = [\ (a^{+}b - ab^{+})\ ,\ -\ a]_{+}$$

$$= (-1)\cdot(-b) = b$$

$\Rightarrow B\cdot a\cdot B^{+}$

$n =$	0	1	2	3	4
$C_n =$	$a,$	$-b,$	$-a,$	b	$a \cdots$

$$= \sum_{n=0}^{\infty} \frac{\theta^n}{n!} C_n.$$

$$= \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!}\cdot a + \sum_{n=0}^{\infty} \frac{i\cdot(i\theta)^{2n+1}}{(2n+1)!}\cdot b$$

$$= \sum_{n=0}^{\infty} \underbrace{(i^2)^n}_{=(-1)^n}\cdot \frac{\theta^{2n}}{(2n)!}\, a + \sum_{n=0}^{\infty} (i\cdot i)\underbrace{(i^2)^n}_{=(-1)^n}\cdot \frac{\theta^{2n+1}}{(2n+1)!}\cdot b$$

(using (29))

$$= a\cdot\cos\theta - b\sin\theta$$

For BbB^{+} $\quad C_0 = A = b \Rightarrow C_1 = a$
$\Rightarrow C_2 = -b \Rightarrow C_3 = -a \cdots\cdots$

$$\Rightarrow BbB^{+} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} C_n$$

$$= \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!}\cdot b - \sum_{n=0}^{\infty} \frac{i(i\theta)^{2n+1}}{(2n+1)!}\cdot a$$

$$= b\cdot\cos\theta + a\sin\theta$$

$$B|00\rangle = |00\rangle$$

$$B|01\rangle = B\cdot a^{+}|00\rangle = B\cdot a^{+}\underbrace{B^{+}B}_{\mathbb{I}}|00\rangle$$

$$= \left(a^{+}\cos\theta - b^{+}\sin\theta\right)\cdot B|00\rangle$$

$$= a^{+}\cos\theta|00\rangle - b^{+}\sin\theta|00\rangle$$

$$= \cos\theta|01\rangle - \sin\theta|10\rangle$$

$$\Rightarrow B\cdot|\tilde{0}\rangle = \cos\theta\cdot|\hat{0}\rangle - \sin\theta|\tilde{1}\rangle$$

$$B|10\rangle = B\cdot b^{+}|00\rangle = Bb^{+}B^{+}B|00\rangle$$

$$= \left(b^{+}\cos\theta + a^{+}\sin\theta\right)B\cdot|00\rangle$$

$$= \cos\theta|10\rangle + \sin\theta|01\rangle$$

$$\Rightarrow B\cdot|\hat{1}\rangle = \sin\theta|\hat{0}\rangle + \cos\theta|\tilde{1}\rangle$$

$\Rightarrow$ B in matrix form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = e^{-i\theta Y} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$e^{-i\theta Y} \stackrel{?}{=} \cos\theta \cdot I - i\sin\theta \cdot Y \stackrel{?}{=} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$e^{-i\theta Y} = \sum_{n=0}^{\infty} \frac{(-i\theta Y)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-i\theta)^n}{n!} \cdot Y^n$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Y^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\Rightarrow e^{-i\theta Y} = \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \frac{(-i\theta)^n}{n!} \cdot I + \sum_{\substack{n=0 \\ \text{odd}}}^{\infty} \frac{(-i\theta)^n}{n!} \cdot Y$$

$$\boxed{\begin{aligned} &= \cos\theta \cdot I - i \cdot \sin\theta \cdot Y \\ &= \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix} - i \begin{bmatrix} 0 & -i\sin\theta \\ i\sin\theta & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \end{aligned}}$$

(C)

$$|01\rangle = |\tilde{0}\rangle$$

$$B \cdot |\hat{0}\rangle = \cos\theta \cdot |\hat{0}\rangle - \sin\theta |\hat{1}\rangle$$

$$\Rightarrow B \cdot |01\rangle = \cos\theta \cdot |01\rangle - \sin\theta |10\rangle$$

The upper π shift will make the state

$$\cos\theta \cdot |01\rangle - \sin\theta \cdot e^{-i\pi} \cdot |10\rangle$$

$$= \cos\theta \cdot |01\rangle + \sin\theta |10\rangle$$

$$= \boxed{\frac{\sqrt{2}}{2} |01\rangle + \frac{\sqrt{2}}{2} |10\rangle}$$ (for this special beam splitter)

$$|10\rangle = |\tilde{1}\rangle$$

$$B \cdot |\hat{1}\rangle = \sin\theta \cdot |\hat{0}\rangle + \cos\theta |\hat{1}\rangle$$

$$\Rightarrow B \cdot |10\rangle = \sin\theta |01\rangle + \cos\theta |10\rangle$$

The upper π shift will make the state

$$\frac{\sqrt{2}}{2} |01\rangle + \frac{\sqrt{2}}{2} \cdot e^{-i\pi} \cdot |10\rangle = \boxed{\frac{\sqrt{2}}{2}\left(|01\rangle - |10\rangle\right)}$$