EECS 541, Winter 2021

Homework 9: Introduction to BCS Theory

Yunjie Wang

1 BCS Theory: Hamiltonian and Ground State

The effective Hamiltonian of Cooper pairs is given,

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} + \frac{1}{N} \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow} \tag{1}$$

where $\xi_{\mathbf{k}} = E_{\mathbf{k}} - \mu$. And the Bogoliubov transformation is also given.

$$\begin{aligned} a_{\mathbf{k}\uparrow} &= u^*_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow} + v_{\mathbf{k}} \gamma^\dagger_{-\mathbf{k}\downarrow} \\ a^\dagger_{-\mathbf{k}\downarrow} &= u_{\mathbf{k}} \gamma^\dagger_{-\mathbf{k}\downarrow} - v^*_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow} \end{aligned} \tag{2}$$

(a) Using mean-field approximation to show that the quartic term to be

$$a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow} \approx \left\langle a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} \right\rangle a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow} + a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} \left\langle a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow} \right\rangle - \left\langle a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} \right\rangle \left\langle a_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\uparrow} \right\rangle$$

(b) Using the fermionic anticommutation rule to show that $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$

(c) Rewrite the effective Hamiltonian using the Bogoliubov transformation, and by demanding the coefficients of $\gamma^\dagger_{\mathbf{k}\uparrow}\gamma^\dagger_{-\mathbf{k}\downarrow}$ and $\gamma_{\mathbf{k}\uparrow}\gamma_{-\mathbf{k}\downarrow}$ to be zero(Diagonalization), determine the exact value of $v_{\mathbf{k}}$ and $u_{\mathbf{k}}$

$$\begin{aligned} H_0 &= \sum_{\mathbf{k}} \left[2\xi_{\mathbf{k}} |v_{\mathbf{k}}|^2 - \Delta_{\mathbf{k}} u_{\mathbf{k}} v^*_{\mathbf{k}} - \Delta^*_{\mathbf{k}} u^*_{\mathbf{k}} v_{\mathbf{k}} + \Delta_{\mathbf{k}} \left\langle a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} \right\rangle \right] \\ H_1 &= \sum_{\mathbf{k}} \left[\xi_{\mathbf{k}} \left(|u_{\mathbf{k}}|^2 - |v_{\mathbf{k}}|^2 \right) + \Delta_{\mathbf{k}} u_{\mathbf{k}} v^*_{\mathbf{k}} + \Delta^*_{\mathbf{k}} u^*_{\mathbf{k}} v_{\mathbf{k}} \right] \left(\gamma^\dagger_{\mathbf{k}\uparrow}\gamma_{\mathbf{k}\uparrow} + \gamma^\dagger_{-\mathbf{k}\downarrow}\gamma_{-\mathbf{k}\downarrow} \right) \end{aligned} \tag{3}$$

(d) Substitute the $v_{\mathbf{k}}$ and $u_{\mathbf{k}}$ into the Hamiltonian and show the final Hamiltonian to be

$$\begin{aligned} H_0 &= \sum_{\mathbf{k}} \left(\xi_{\mathbf{k}} - \sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2} + \Delta_{\mathbf{k}} \left\langle a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} \right\rangle \right) \\ H_1 &= \sum_{\mathbf{k}} \sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2} \left(\gamma^\dagger_{\mathbf{k}\uparrow}\gamma_{\mathbf{k}\uparrow} + \gamma^\dagger_{-\mathbf{k}\downarrow}\gamma_{-\mathbf{k}\downarrow} \right) \end{aligned} \tag{4}$$

(e) Assume $|\Psi_{\mathrm{BCS}}\rangle = \mathcal{N} \prod_{\mathbf{q}} \mathrm{e}^{\alpha_{\mathbf{q}} a^\dagger_{\mathbf{q}} a^\dagger_{-\mathbf{q}\downarrow}} |0\rangle$, show that the BCS ground state is

$$|\Psi_{\mathrm{BCS}}\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} \right) |0\rangle \tag{5}$$

Hint: Start from show $a_{\mathbf{k}\uparrow}\theta^n_{\mathbf{k}}|0\rangle = n\theta^{n-1}_{\mathbf{k}} \alpha_{\mathbf{k}} a^\dagger_{-\mathbf{k}\downarrow}|0\rangle$ using the method of induction, $\theta_{\mathbf{k}} = \alpha_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}$.