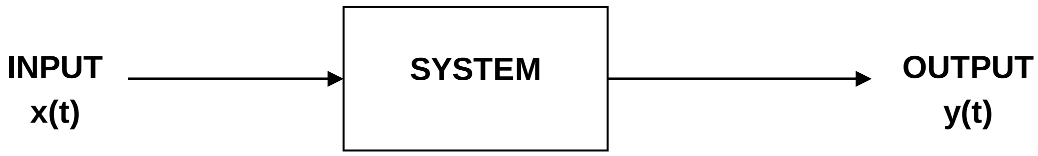
# 2. Mathematics of Digital Control Engineering

## 2.1 Continuous Systems and Transfer Function Revision

### 2.1.1 Continuous time signals

Continuous time signals are used in analogue control systems. Signals are simply represented using a single input to the system and a single output (**response**). These are labelled ***x(t)*** and ***y(t)*** respectively, showing their dependence upon the variable time.




The system of interest (a process action) cannot usually be described in a simple manner since time delays (phase shifting) and changes in amplitude (attenuation/gain) are likely to occur to any given input signal that passes through the system. The input and output relationships are therefore best described using the differential equation:

$$b_k \frac{d^k y}{dt^k} + b_{k-1} \frac{d^{k-1} y}{dt^{k-1}} + \cdots b_0 y = a_l \frac{d^l x}{dt^l} + a_{l-1} \frac{d^{l-1} x}{dt^{l-1}} + \cdots a_0 x$$

Where ***a*** and ***b*** are the coefficients and ***k*** and ***l*** represent the equation order.

### 2.1.2 The Order of Dynamic Systems

There are a number of ways this can be established but three standard 'test' input signals are the step input, the ramp input and the sinusoidal input.

The **step** input gives information on the transducers transient response.

The **ramp** input gives information on the systems steady-state response

The **sinusoidal** input gives information on the systems response to a cyclic input

The response can then be described as:

**First Order**

A system where the dynamics are represented by a first order differential equation (DE) such that

$$a\frac{d\theta_o}{dt} + b\theta_o = c\theta_i$$

This equation can be rewritten as a transfer function:

$$\frac{\theta_o}{\theta_i} = \frac{K}{1+\tau D}$$

Where **$K$** = sensitivity gain/attenuation
$\tau$ = time constant
**$D$** = differential operator ***d/dt.***

An example of a first order transducer is mercury in a glass thermometer since the heat transfer is described by a first order DE.

**Second Order**

A system where the dynamics are represented by a second order differential equation such that:

$$a\frac{d^2\theta_o}{dt}+b\frac{d\theta_o}{dt}+c\theta_o=e\theta_i$$

This equation can be rewritten as a transfer function

$$\frac{\theta_o}{\theta_i}=\frac{K\omega_n^2}{D^2+2.\xi.\omega_n D+\omega_n^2}$$

Where **$K$** = sensitivity gain/attenuation
$\xi$ = system damping
**$D$** = differential operator ***d/dt***
$\omega_n$ = natural frequency.

In the second order system there is an important trade off between the response and the degree of overshoot of the system. A value of 0.7 is often chosen for $\xi$ as a compromise.

All zero, first and second order systems apply in digital systems as well as analogue systems. In digital systems a first order system can be recognized by the highest order of ***z***.

There are several ways to solve continuous equations represented in this form with the three most common methods being discussed briefly simply for revision purposes. Further information and more examples are available in the MSc pre-course reading material and in an appropriate text such as 'Modern Control Engineering' by Ogarta.

### 2.1.3 Differential Equation Approach

Various classical procedures are commonly used to solve such differential equations as described in engineering texts. Many solutions to complex equations however are not easily derived so their use is generally restricted to the most recognisable forms (usually 1st and 2nd order).

### 2.1.4 Convolution Approach

An alternative means of finding a system's response to an input is to find the **convolution** of the input with the system's **impulse response**, as explained below.

The **impulse response** of a system is the output that results when the input is a delta function ***δ(t)*** known as a unit impulse function.

**Convolution** is a mathematical operation; it is a way of combining two signals. It is defined as follows:

$$y(t) = x(t) * g(t) = \int_{-\infty}^{\infty} x(t-\tau)g(\tau)d\tau$$

The star or asterisk symbol ***** indicates convolution. The above equation can be expressed in words as "y is the convolution of x and g". It can be interpreted as 'sliding' ***x*** over ***g***, and multiplying the overlapping waveforms together. The 'sliding' necessitates a second time variable, $\tau$.

**Example:**

The impulse response of a certain system is given by ***g(t)*** below. Use the convolution integral to determine the response ***y(t)*** due to a ramp input ***x(t)*** also below:

$g(t) = 0$ for $t < 0$ and by
$g(t) = e^{-2t}$ for $t \geq 0$

$x(t) = 0$ for $t < 0$ and by
$x(t) = 4t$ for $t \geq 0$

answer: $= e^{-2\tau} + (2t - 1)$ for $t \geq 0$

### 2.1.5 Laplace Transforms

The technique of Laplace transforms provides certain powerful conceptual and analytical methods having direct application to the solution of control system differential equations. Classically, the Laplace method is regarded as an operational method for solving linear differential equations with constant coefficients. Both sides of the differential equation are transformed (into the ***S*** plane) by means of a certain function and operational pairs. Transformation results in a set of algebraic equations which may then be manipulated by standard algebraic operations. To obtain the solution the algebraic equations must be transformed back into the time domain.

The Laplace transform (transference to the s-plane) of some function x(t) is expressed mathematically as follows:

$$X(s) = L[x(t)] = \int_0^\infty x(t)e^{-st}dt$$

and its inverse

$$x(t) = L^{-1}[X(s)] = \frac{1}{2\pi j}\int_C X(s)e^{st}ds$$

where ***C*** is a contour chosen to include all singularities of ***X(S).***

It is not intended in this course to discuss Laplace transforms in any great detail since such mathematical methods are well covered in all engineering undergraduate courses whether electrical, marine, electronic, mechanical. Common Laplace pairs and operational pairs are provided below.

| $x(t)$ | $X(s) = \mathrm{L}[x(t)]$ |
|---|---|
| $\delta(\mathrm{t})$ | $1$ |
| $1\,\mathrm{or}\,\mathrm{u(t)}$ | $\frac{1}{\mathrm{s}}$ |
| $t$ | $\frac{1}{s^2}$ |
| $e^{-at}$ | $\frac{1}{s+a}$ |
| $\sin \omega t$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos \omega t$ | $\frac{s}{s^2+\omega^2}$ |
| $e^{-at}\sin \omega t$ | $\frac{\omega}{(s+a)^2+\omega^2}$ |
| $e^{-at}\cos \omega t$ | $\frac{s+a}{(s+a)^2+\omega^2}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $e^{-at}t^n$ | $\frac{n!}{(s+a)^{n+1}}$ |

**Example**

The input-output relationship of a certain system is described by a differential equation shown below. Find the response when ***x(t)*** is a step function of 10 units applied at ***t = 0*** and when the initial conditions are ***y(0) = 2, y'(0) = -10***.

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = x$$

Steps:

- Apply Laplace transforms to both sides of the equation
- Rearrange to solve for $L[y(t)]$
- Use partial fractions to break the expression into components to which we can easily apply the inverse Laplace transform.

Remember: $L[x'] = sX - x(0)$

$$L[x''] = s^2X - sx(0) - x'(0)$$

Answer: $y(t) = 13e^{-2t} - 16e^{-t} + 5$

# 2.2 Discrete Time Systems and Linear Difference Equations

### 2.2.1 Time domain representation of a discrete time signal

As covered in chapter 1, a discrete time signal can be considered as a continuous signal sampled at periodic intervals, known as the **sampling period, *T***. At this stage, it is convenient to assume that the sampling period remains constant. It is usual to represent a discrete signal with an asterix, e.g., ***$x^*(t)$***.

Discrete time signals can be expressed mathematically as:

$$x^*(t) = x(t)p_\delta(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

Hence the only values of ***x(t)*** that have significance in ***$x^*(t)$*** are those at ***t = nT***, where ***nT*** is a train of impulse functions. An alternative form for the sampled-data signal is then:

$$x^*(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

The most straightforward notation of a discrete signal is ***x(n)*** where ***n*** is an integer defined across some range $n_1 \leq n \leq n_2$.

$x(n) = x_1(nT)$ for ***n*** an integer
$x(n) = 0$ otherwise

### 2.2.2 Linear Difference Equations

The information available to a digital controller about a process is a set of time series of the process inputs and outputs. The digital controller itself may be considered as a discrete time process and its input and output time series examined. As with a continuous time system the system inputs and outputs are related by the process transfer function, except with a discrete time process the transfer function is not a function of the differential operator, ***D***, but of a time shift operator, which involves the previous data values in the transfer function.

Discrete time processes may be modeled and analyzed using linear difference equations. These are simply constant coefficient linear equations relating present and historic values of discrete time signals. In order to use these it is convenient to change the notation from that used when looking at data sampling. For data sampling the discrete time signal was denoted ***f*(kT)***, where ***k*** denoted the ***k-th*** sample of the signal from ***t=0*** with sampling interval ***T***.

Linear difference equations involve the discrete time signal at any arbitrary sampling instant and the sampling interval, ***T*** is fixed and so is dropped from the notation. The ***k*** remaining refers to the current sample. Only discrete time signals are involved in the equation so the star is also dropped.

A signal ***u*** is thus denoted as:

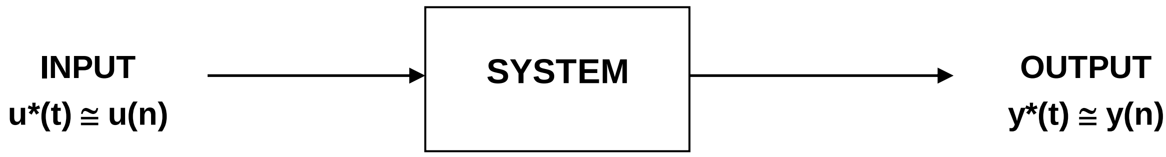
| | |
|---|---|
| ***u(k)*** | for the current sample |
| ***u(k-n)*** | for the n-th previous sample |
| ***u(k+n)*** | for the n-th future sample |

A linear difference equation relating signals ***u*** and ***y*** may thus, for example be written

$$a_0\, u(k) + a_1\, u(k-1) + \dots\, a_m\, u(k-m) = b_0\, y(k) + b_1\, y(k-1) + \dots\, b_n\, y(k-n)$$

where $\boldsymbol{a_0 \,..\, a_m, b_0 \,..\, b_n}$ are the equation coefficients and the difference equation is of order ***m*** or ***n***, whichever is greater.

Hence in a digital control system the input and output signals can be described by a **linear difference equation** as follows




Difference equations occur in algebra and numerical methods as approximations to the derivative of a function. In particular backward approximations to the first and second derivative of a function ***u*** would be written

$$D(u) \cong \frac{u(k) - u(k-1)}{T}$$
$$D^2(u) \cong \frac{u(k) - 2u(k-1) + u(k-2)}{T^2}$$

Thus the second order difference equation

$$a_0\, u(k) + a_1\, u(k-1) + a_2\, u(k-2) = 0$$

can be considered a numerical approximation to a second order differential equation in ***u*** and to contain the same information about the signal ***u*** (providing the sampling is fast enough for the approximation to hold).

This relationship is of particular interest when considering control systems, as the derivatives of a signal contain information about the states or energy stores in the system producing the signal, as demonstrated by the relationship between the coefficients in a state-space model and a transfer function model of a system.

Difference equations are thus of use practically in describing the dynamic behaviour of discrete time systems or in forming discrete time equivalent model of a continuous time process.

Difference equations may be used to model the behaviour of dynamic systems by rearranging an equation to put the current (or next future) system output on the left hand side. This then gives the system output as a linear function of a certain number of previous system inputs and outputs, depending on the order of the system. For multi-input multi-output systems the outputs are in general a function of all the previous system inputs and outputs. The order of the model is a function of the underlying physical processes of the system.

For a system second order with respect to the system output and zero order with respect to the system input the equation would be

$$y(k) = c_1\, y(k-1) + c_2\, y(k-2) + d_0\, u(k)$$

Note that this could equivalently be expressed

$$y(k+1) = c_1\, y(k) + c_2\, y(k-1) + d_0\, u(k+1)$$

This form of system model is called an 'Auto-Regressive' system model or ARMA model, because the system output depends on its own past values (auto-regression) as well as a moving average of the input values. These past outputs and inputs are weighted by the coefficients *c* and *d*, respectively. ARMA models are often used in System Identification.

Difference equations offer an intuitive description of a system’s dependence on individual factors, but they do not allow us to derive a transfer function form for more general analysis. In order to express the system as a discrete time transfer function we need an operator notation to denote the time shifting of data. This is provided by the ***z*** transform.

## 2.3 z Transform

The Z transform, like the Laplace transform, is an operational function that provides a very useful tool to solve discrete linear equations. In fact, it is simply the Laplace transform of a discrete time signal, with the substitution $z = e^{sT}$ used for convenience and ease of interpretation.

**Notation:**

The ***z*** transform of a discrete signal ***x(n)*** is denoted by ***X(z)***. The symbolic forms are expressed as:

$$X(z) = Z[x(n)] \qquad \text{(notation for z transform)}$$

and

$$x(n) = Z^{-1}[X(z)] \qquad \text{(notation for inverse z transform)}$$

**Definition:**

Assuming that the signal ***x*(nT)*** is defined only in the positive time region then its Laplace transform may be described as

$$X^*(s) = \sum_{n=0}^{\infty} x(nT)e^{-nTs} .$$

Expressing ***z*** in terms of the Laplace operator ***s*** as

$$z = e^{sT} \text{ or } s = \frac{1}{T}\ln z ,$$

we achieve

$$X(z) = [X^*(s)]_{z=e^{sT}} = \sum_{n=0}^{\infty} x(n)z^{-n} .$$

Note that the ***z*** transform of a signal is identical for all signals with the same values at all sampling instants and that ***Z{f(t)} = Z{f*(t)}*** (the z transform of the sampled signal is the same as the z transform of the original continuous signal).

**Examples**

1. Derive the ***z*** transform of the discrete unit pulse function $\delta$ defined as:

***$\delta$(n) = 1 for n = 0***
***$\delta$(n) = 0 for n ≠ 0***

2. Given that $Z[r^n] = \frac{z}{z-r}$, derive the ***z*** transform of

***e(n) = $r^n$ cos(n$\theta$) for n $\geq$ 0***

answer: $\frac{z^2 - rz\cos(\theta)}{z^2 - 2rz\cos(\theta) + r^2}$

Transform tables are a convenient way of finding the ***z*** transform of a particular function of time or its Laplace transform in a more convenient fraction form. A summary of ***z*** transform function and operational pairs is provided below.

| $x(n)$ | $X(z)$ | $X(s)$ |
|---|---|---|
| $\delta(\mathrm{n})$ | $1$ | $1$ |
| $1\,\mathrm{or}\,\mathrm{u(n)}$ | $\dfrac{z}{z-1}$ | $\dfrac{1}{\mathrm{s}}$ |
| $nT$ | $\dfrac{Tz}{(z-1)^2}$ | $\dfrac{1}{s^2}$ |
| $e^{-anT}$ | $\dfrac{z}{z-e^{-aT}}$ | $\dfrac{1}{s+a}$ |
| $a^n$ | $\dfrac{z}{z-a}$ | $\dfrac{1}{s-\ln(a)/T}$ |
| $\sin naT$ | $\dfrac{z\sin aT}{z^2-2z\cos aT+1}$ | $\dfrac{a}{s^2+a^2}$ |
| $\cos naT$ | $\dfrac{z^2-z\cos aT}{z^2-2z\cos aT+1}$ | $\dfrac{s}{s^2+a^2}$ |

The ***z*** transform and ***z*** operator are the analytical and algebraic tools used to manipulate discrete time data and system models. The use of ***z*** as an operator is roughly analogous to that of the ***D*** operator in continuous time systems.

The effect of the ***z*** operator on a signal is to shift it forward in time by one sampling interval so that ***z u*** is equivalent in meaning to ***u(k+1)*** in the previous notation. Similarly ***u*** is equivalent to ***u(k)*** and $\boldsymbol{z^{-1} u}$ equivalent to ***u(k-1)***.

Taking the difference equation

$$y(k) = c_1\, y(k-1) + c_2\, y(k-2) + d_0\, u(k),$$

and applying the z transform, gives

$$Y(z) = c_1\, z^{-1}\, Y(z) + c_2\, z^{-2}\, Y(z) + d_0\, U(z)$$

or rearranging

$$Y(z)\,(1 - c_1 z^{-1} - c_2 z^{-2}) = d_0\, U(z)$$

so, in ***z*** transfer function form

$$G(z) = \frac{Y(z)}{U(z)} = \frac{d_0}{1-c_1 z^{-1} - c_2 z^{-2}}$$

Listed below are some important properties of the ***z*** transform, similar to the properties of the Laplace transform. We will revisit the relationship between Laplace and ***z*** transform (or ***s*** and ***z*** plane) when we study the stability of systems. For further details, it is recommended to study some of the literature, e.g., Chapter 3 in ‚Digital Control Systems' by B.C. Kuo.

# z Transform Properties

- **Linearity (Addition/Subtraction; Multiplication by a constant):**
  $Z[cf(k)] = cF(z)$ constant c
  $Z[f(k) \pm g(k)] = F(z) \pm G(z)$

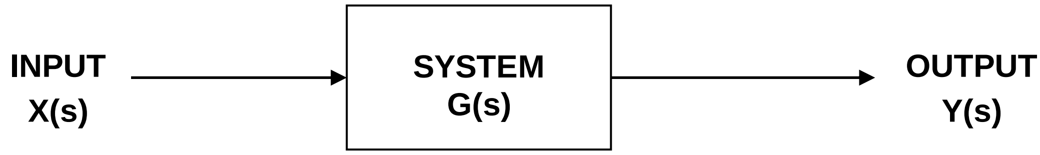
- **Multiplication by k:**
  $Z[kf(k)] = -z\dfrac{dF(z)}{dz}$
  $Z[c^k f(k)] = F(\frac{z}{c})$ constant c

- **Right Shift (Time Delay):**
  $Z[f(k-1)] = f(-1) + z^{-1}F(z)$
  $Z[f(k-2)] = f(-2) + z^{-1}f(-1) + z^{-2}F(z)$
  $Z[f(k-n)] = f(-n) + z^{-1}f(1-n) + z^{-2}f(2-n) + \ldots + z^{-n+1}f(-1) + z^{-n}F(z)$

- **Left Shift (Time Advance):**
  $Z[f(k+1)] = zF(z) - zf(0)$
  $Z[f(k+2)] = z^2F(z) - z^2 f(0) - zf(1)$
  $Z[f(k+n)] = z^n F(z) - z^n f(0) - z^{n-1} f(1) - \ldots - z^2 f(n-2) - zf(n-1)$

- **Initial and Final Value Theorem:**
  $f(0) = \lim_{z\to\infty} F(z)$
  $\lim_{k\to\infty} f(k) = \lim_{z\to 1}\left[\dfrac{z-1}{z}F(z)\right]$ if $\lim_{k\to\infty} f(k)$ exists and finite

# 2.4 Transfer Function




The transfer function (in time, ***s*** plane or ***z*** plane) is derived from the input to output relationship in any system.

**Transfer function in the *s* plane**

A useful advantage of the Laplace transform (and z transform) over time domain representations is that the transfer function can be read from the block diagram as a simple product, e.g. for the example above…

*Y(s) =G(s).X(s)*

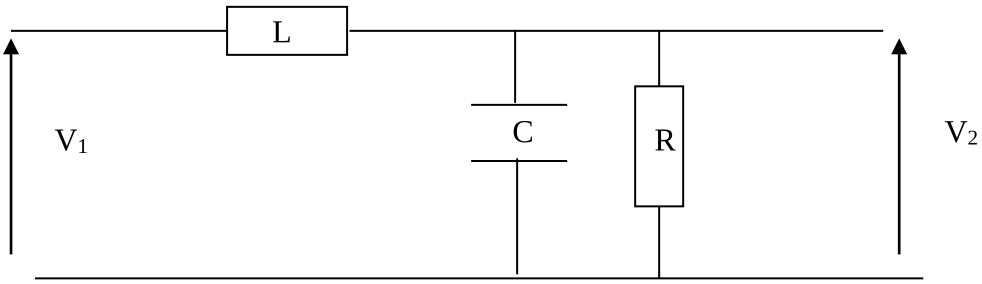
Hence the transfer function becomes

$$G(s) = \frac{Y(s)}{X(s)} = \frac{a_l s^l + a_{l-1} s^{l-1} + a_{l-2} s^{l-2} + a_{l-3} s^{l-3} \ldots + a_0}{b_k s^k + b_{k-1} s^{k-1} + b_{k-2} s^{k-2} + b_{k-3} s^{k-3} \ldots + b_0} = \frac{N(s)}{D(s)}$$

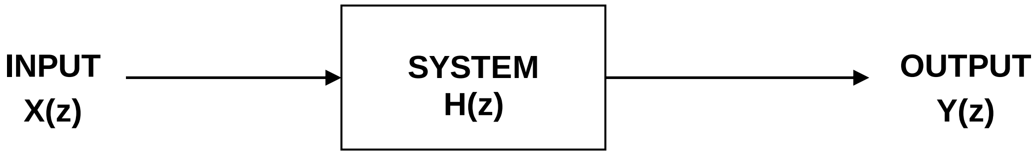
Where ***N(s)*** is the numerator polynomial and ***D(s)*** is the denominator polynomial.

**Example**

Determine the transfer function of the circuit below using Laplace transforms



## The transfer function in the *z* plane




Similarly in the ***z*** plane, transfer functions are described in a similar way describing the input-output relationship of a system.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_0 + a_l z^{-l} + a_2 z^{-2} + a_3 z^{-3} \ldots + a_k z^{-k}}{1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} \ldots + b_k z^{-k}} = \frac{N(z)}{D(z)}$$

It is accepted practise to express a transfer function in negative powers of ***z***.

## Example

Find the transfer function of an initially relaxed system

y(n) – y(n-1) +0.5y(n-2) = x(n) + x(n-1)

## 2.5 Inverse *z* transform

Having expressed a **discrete-time linear time-invariant signal** in ***z*** domain and having undertaken various manipulations it is necessary to convert back to the time-domain using inverse ***z*** transforms. The inverse **z** transform can be written as

$$x(n) = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz$$

Solving this contour integral requires advanced mathematical techniques beyond the scope of this course, but in practice we can usually calculate the inverse z transform using one of two alternative approaches: the partial fraction expansion method or the power series expansion method.

**Partial fraction Expansion**

Whilst in the ***z*** domain, express the resulting equation in partial fraction form and then transform back to the time domain using the standard functional pairs. This method is rather similar to the partial fraction expansion method used in the inverse Laplace transform with one important modification. Looking at the table of Laplace and ***z*** transform pairs in chapter 2.3, we see that most of the common ***z*** transform functions have at least a zero at ***z = 0***. E.g., the ***z*** transform of the step function has a ***z*** in the numerator not found in the matching Laplace transform, e.g.,

$$x(n) = u(n) = 1 \quad X(z) = \frac{z}{z-1} \quad X(s) = \frac{1}{s}.$$

In order to carry over this ***z***, we modify the partial fraction expansion, and expand $\frac{Y(z)}{z}$, as shown in the example below. This allows the use of the standard ***z*** transform pairs tabled above.

**Example**
Using the partial fractions method, obtain the inverse Z-transform of:

$$Y(z) = \frac{1}{(1-z^{-1})(1-0.5z^{-1})}$$

**Power Series Expansion Method**

If the function can be written as a power series in z…

e.g. $Y[z] = a+bz^{-1}+cz^{-2}+dz^{-3}$

… then any value in the sequence can be determined from the appropriate coefficient in the series, by remembering that $z^{-1}$ can be interpreted as a delay of one sample

e.g. $y[n] = a\delta[n]+b\delta[n-1]+c\delta[n-2]+d\delta[n-3]$, which is the same as the finite series {y[0]=a, y[1]=b, y[2]=c, y[3]=d}. This sequence is the system's impulse response (the output resulting from an impulse delivered at n=0.

The method is equally applicable to series of infinite length. It is arguably at its most powerful and useful in cases where a simple rule can be identified to describe that series, e.g. $y[n] = 2^n$.

As you have seen, applying the inverse transform to the power series is not difficult. The prior step of achieving the power series form is sometimes more difficult. It may require long-division of polynomials.

**Example**
Using the power series method, inverse Z-transform the following function.

$$Y(z) = \frac{z}{z-a}$$