MECH0089
Control and Robotics: Digital Control Systems

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Research interests include the design and creation of:

- bio-inspired, soft and stiffness-controllable robotics
- innovative haptic interfaces
- sensor development
- robotic art

Feedback and Consultation Hours, MS Teams:
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Digital Control Systems

Aim of the course

The main aims of this course in digital control systems are:

- To give students an appreciation of the need for modern control systems.
- To explain and discuss the various hardware and software systems that make up modern control systems.
- To provide students with mathematical tools needed in the design and undertake performance analysis of digital control systems.

Assessment

Control – 50%
Robotics – 50%

Structure of the course

Continuous Systems and Transfer Function Revision

1. Introduction to Digital Control Systems

1.1 Review and Classification of Control Systems

1.2 Analogue and Digital Control Systems

1.3 Components of a Digital Control System

1.4 Sampling Theorem

2. Mathematics of Digital Control Engineering

2.1 Continuous Systems and Transfer Function Revision

2.2 Discrete Time Systems and Linear Difference Equations

2.3 z Transform

2.4 Transfer Function

2.5 Inverse z Transform

Structure of the course

3. Discrete Time Systems

3.1 z Domain Transfer Function

3.2 Stability Criteria

3.3 Time Domain Response

3.4 Frequency Response

4. Discrete Control Systems

4.1 Equivalent Continuous Time Design

4.2 Realization and Implementation

4.3 Discrete PID Controller Design

4.4 Digital Control Applications

Tutorial Sheets will be issued for the revision class, chapters 2, 3 and 4.

Literature/reading material

[1] Digital Control Systems - B.C. Kuo
Saunders College Publishing

[2] Design of Feedback Control Systems
R.T. Stefani, B. Shahian, C.J. Savant Jr., and G.H. Hostetter
Oxford University Press

[3] Feedback Control of Dynamic Systems - G.F. Franklin, J.D. Powell, & Abbas Emami-Naeini Pearson
(Older versions are titled ‘Digital Control of Dynamic Systems’ – by Franklin & Powell)

[4] Real-Time Computer Control: An Introduction - S. Bennett
Prentice Hall International

[5] Control Systems Theory - O.I. Elgerd
McGraw-Hill

[6] The Art of Control Engineering - K. Dutton, S. Thompson, B. Barraclough
Addison-Wesley

[7] Control System Design and Simulation - J. Golten, A. Verwer
McGraw-Hill

Lecture style and expectations

- Each lecture will have a set of notes
  - There will be a few blanks for you to fill in (worked examples)
  - I will speak around the topic area – annotate your notes as you feel is appropriate.
  - Notes and overhead slides will be made available on moodle within 24 hours after each lecture.

- Tutorial Sheets
  - Should be attempted soon after the lectures, to reinforce the material and give you opportunities to ask questions.

Feel free to approach me with any feedback on teaching style, etc.

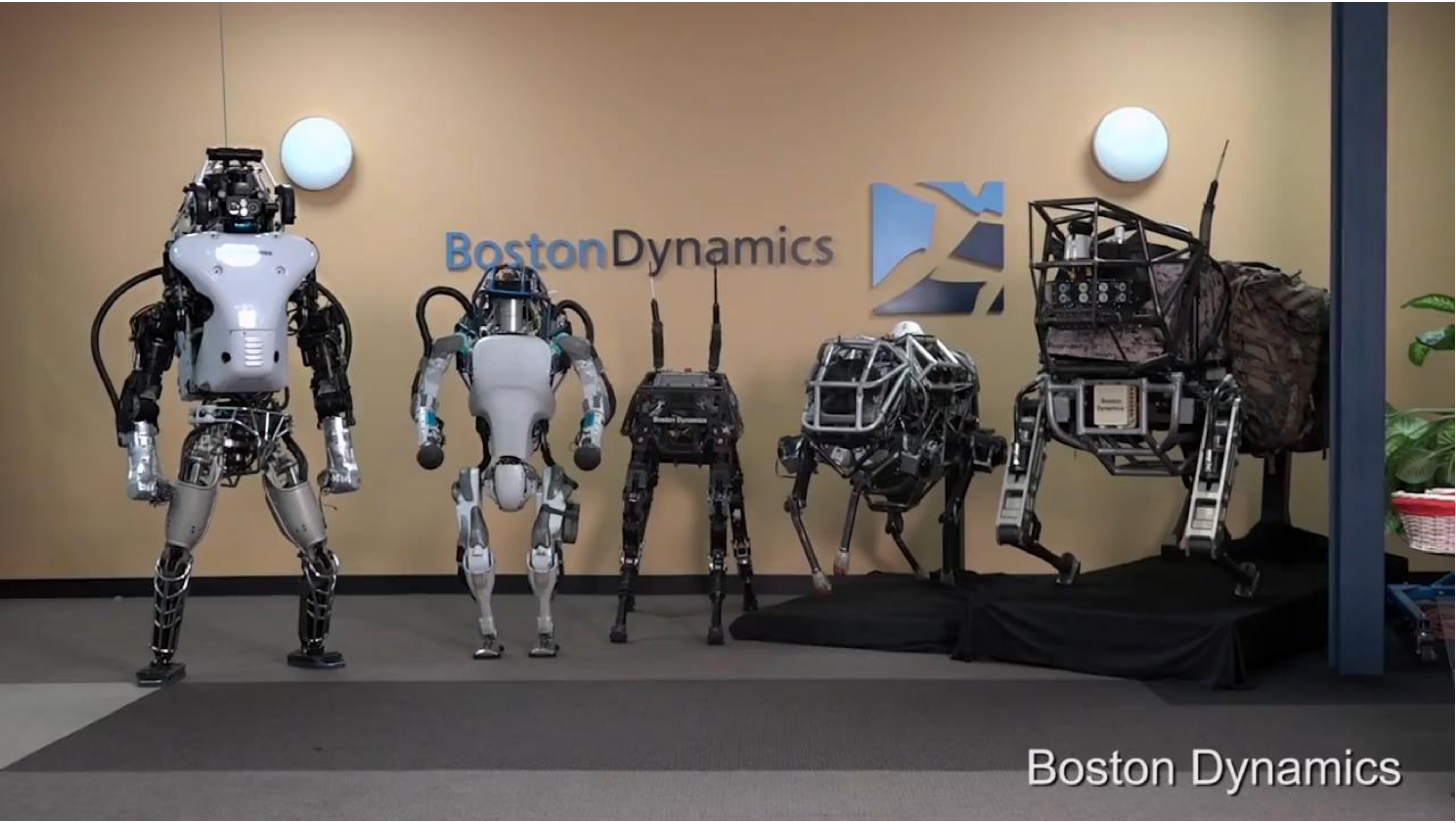
Control has a wide range of applications throughout engineering:

- Autopilot systems for aircraft and ships.
- Position control servomechanisms used in a variety of applications - radar tracking, machine tools, tv cameras etc.
- Machinery plant control.
- Robotics.

Control examples: Multi-legged robots

Control examples: Multi-legged robots

Control examples: Multi-legged robots



Control examples: Autonomous vehicles

Control examples: Drones

MECH0089
Control and Robotics:
Digital Control Systems

Lecture 1:
Continuous Systems and Transfer Function Revision (part 1)

A note about the notes

When you see something

In red like this…

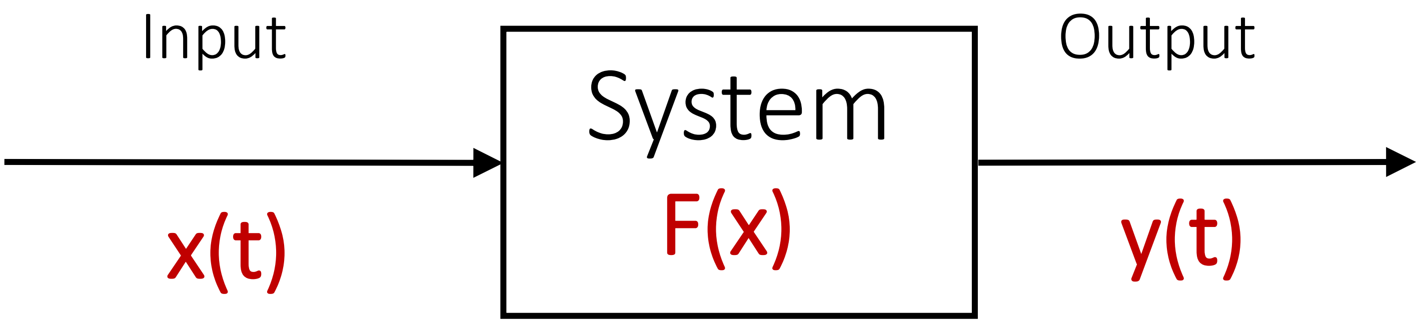
Its missing from your notes so write it down!

Continuous Systems and Transfer Function Revision

Continuous Systems and Transfer Function Revision

2.1.1 Continuous time signals

We want to derive an expression for y(t) in terms of x(t).



Now we will investigate how we model physical systems, and obtain the function block F(x) for electrical and mechanical components.

The focus of this course, and indeed much of control theory itself focuses on modelling physical systems as *linear* and *time invariant*.

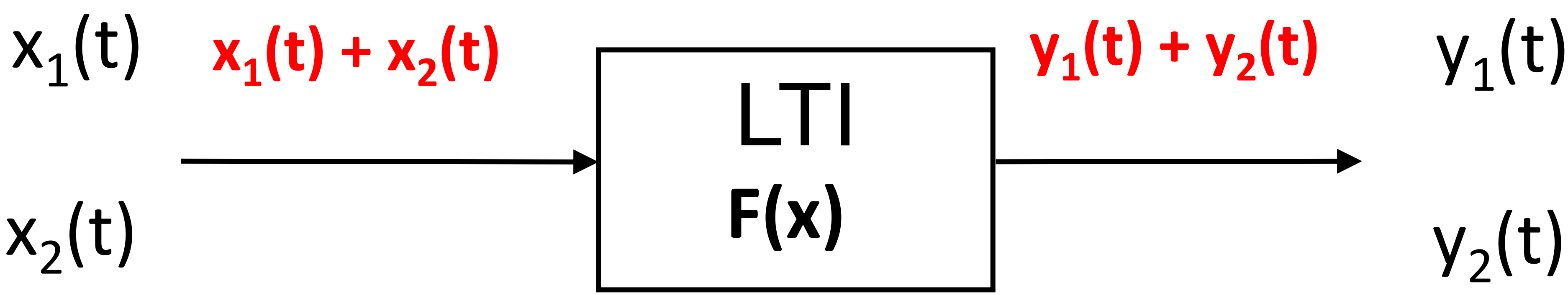
LTI systems have three key properties:

- Obey principle of superposition
- Homogeneity
- Time Invariance

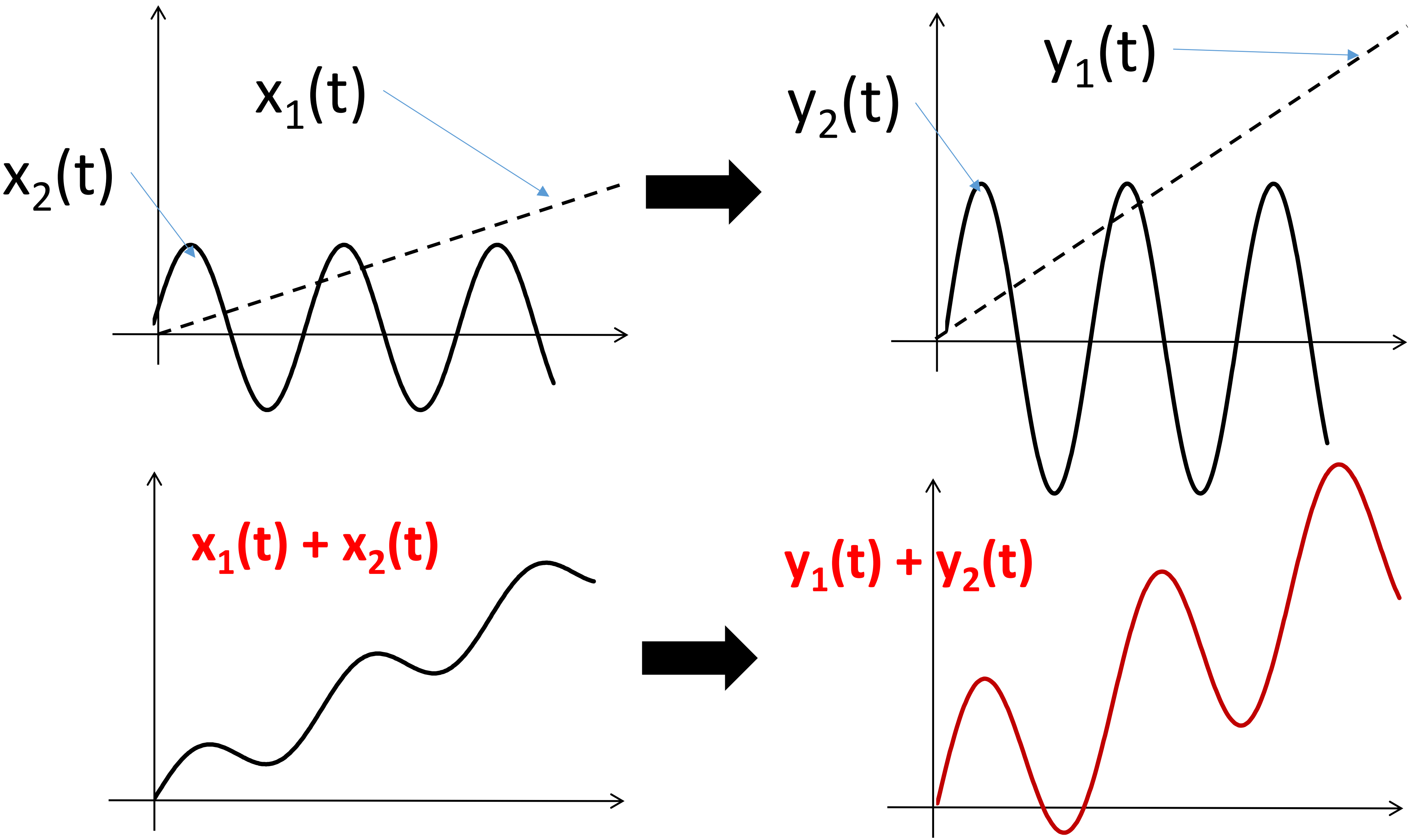
Linearity

Continuous Systems and Transfer Function Revision: Linear Time Invariant (LTI) Systems

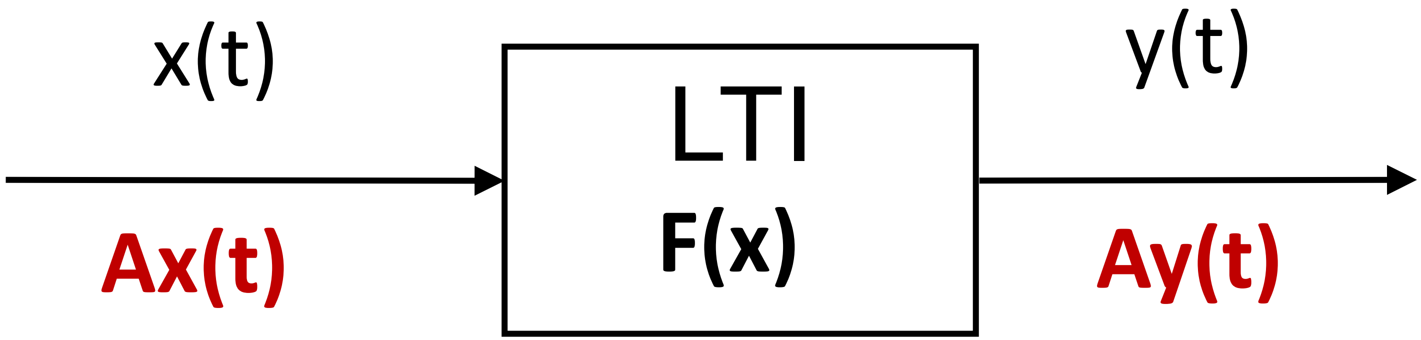
If input $x_1(t)$ produces output $y_1(t)$, and input $x_2(t)$ produces $y_2(t)$, then input $x_1(t) + x_2(t)$ produces output $y_1(t) + y_2(t)$.



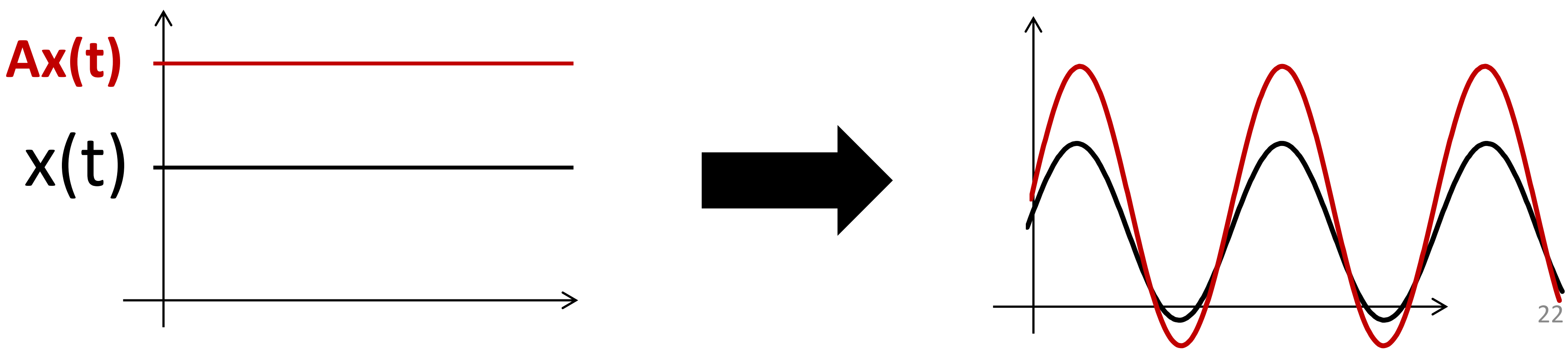
Say for a system which doubles the input F(x) =2x :



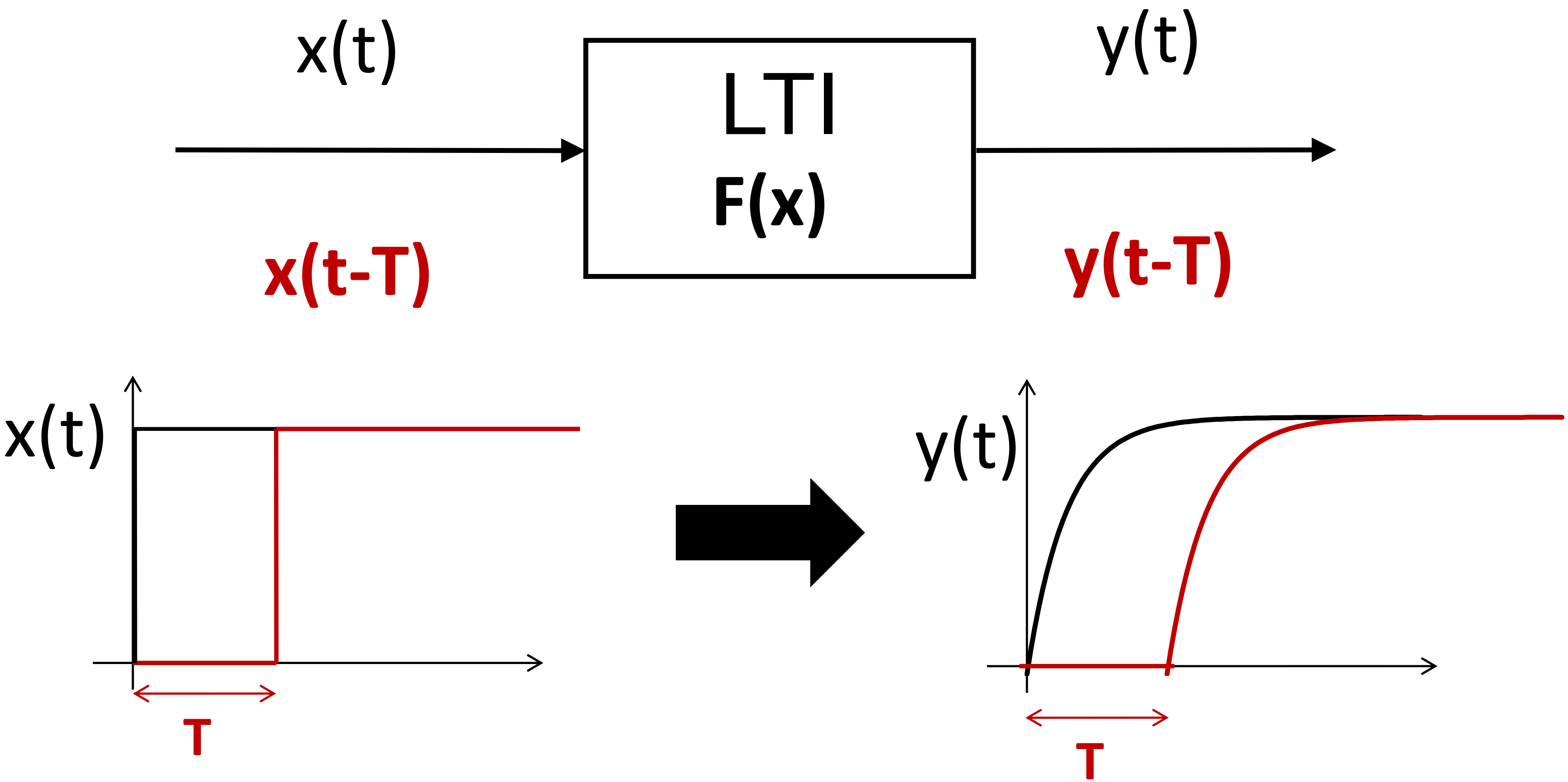
If the input to the system x(t) is scaled by a magnitude scale factor *A*, then the output y(t) is also scaled by the same factor.



For example, consider a system which generates a sine wave at a given amplitude, with a set frequency:



If input is applied at time t=0 or *T* seconds from now, the output is identical with the exception of a delay of *T* seconds.



Continuous Systems and Transfer Function Revision: LTI systems example

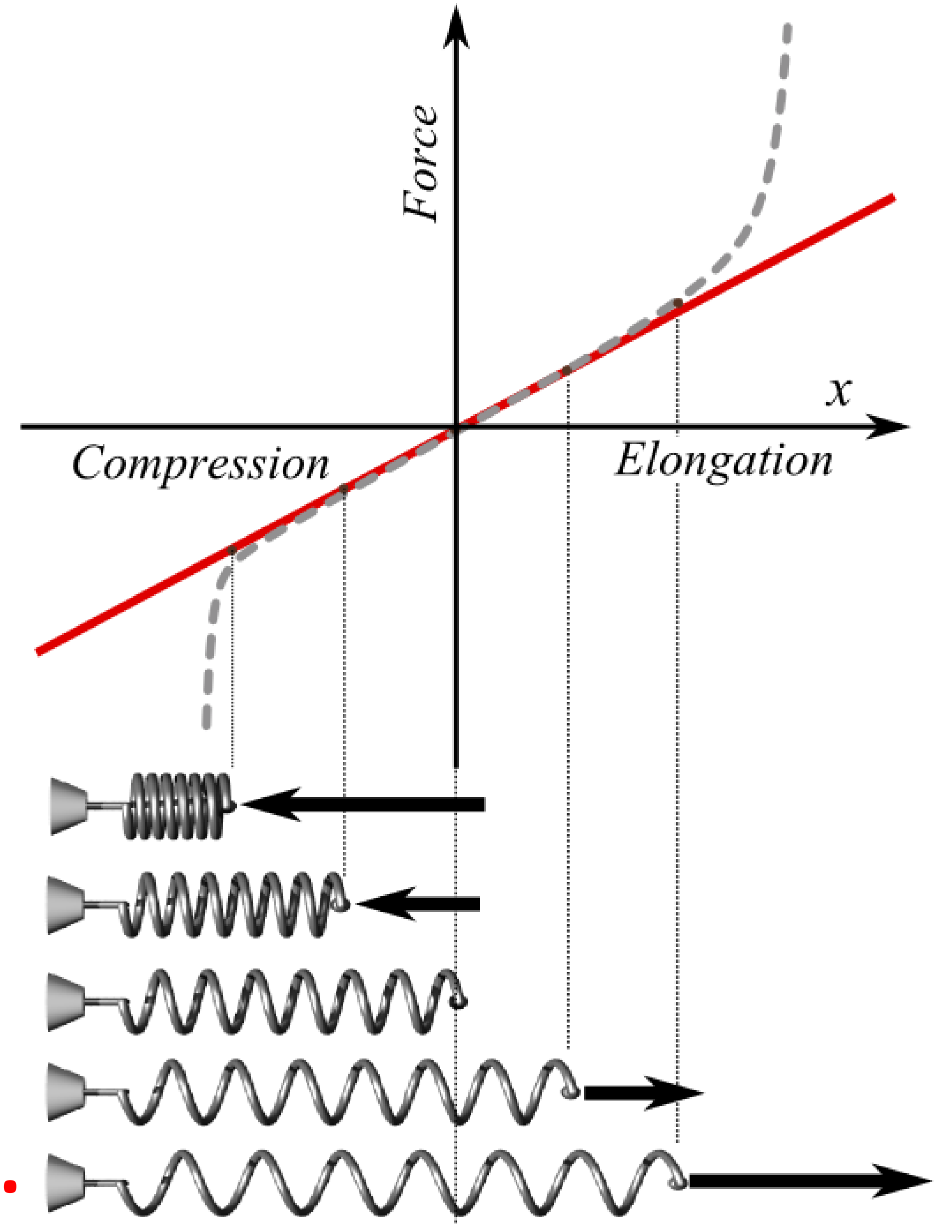
For a simple system such as a spring, across all possible compressions or extensions the response is non linear:

$$F = -kX$$

Hookes law is only a linear approximation of the true response

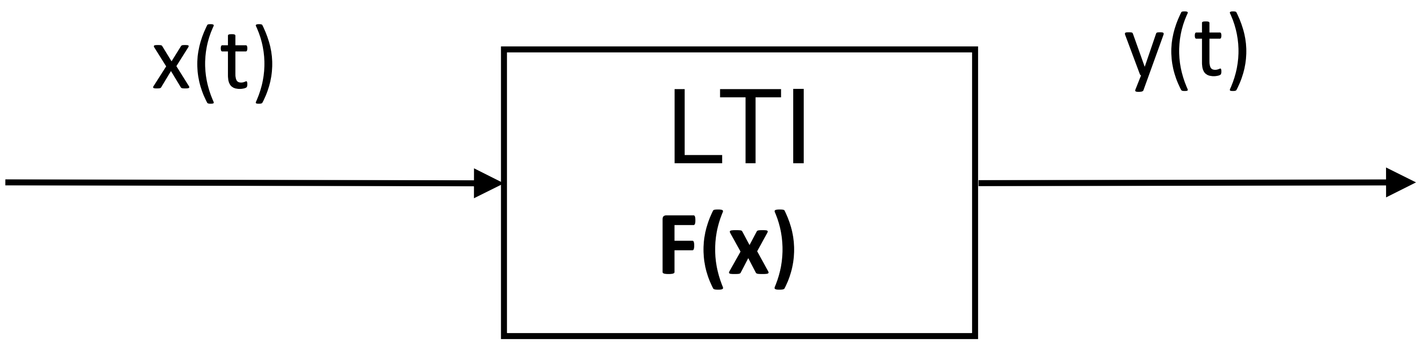
However, if we choose the operating range of the spring correctly, the response is within the linear region

And the approximation is valid.



Continuous Systems and Transfer Function Revision

As we are interested in describing something that *changes* with time, it is useful to express the function block of the system F(t) as an ordinary differential equation (ODE).



$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_2\frac{d^2 y}{dt^2} + a_1\frac{dy}{dt} + a_o = bx$$

x is input function or forcing function

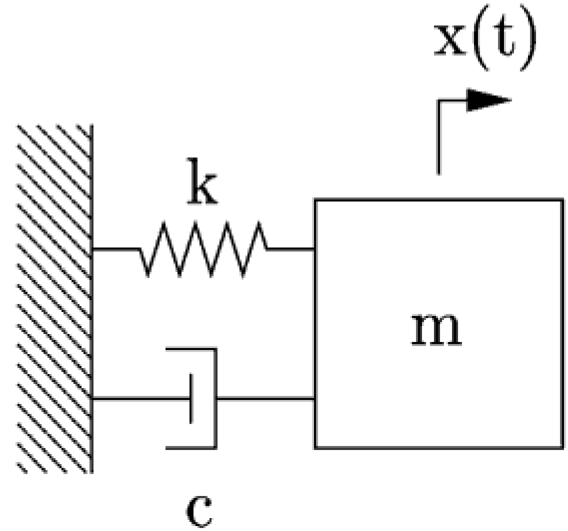
y is output

n is *order* of the ODE

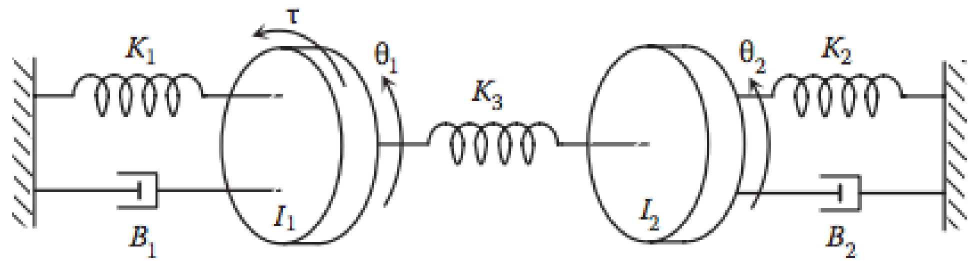
a_0… are coefficients. These *completely characterise the system*

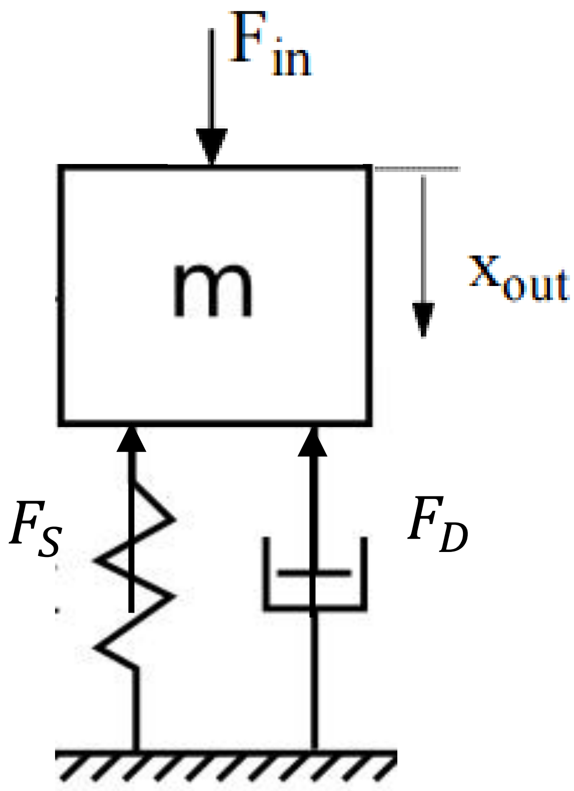
Mechanical systems consist of three basic types of elements:

- Inertia
  Examples: mass, moment of inertia
- Spring
  Examples: translational/rotational spring
- Damper
  Examples: dashpot, friction, wind drag



	Translational
Spring	$F = kx$
Dashpot	$F = c\, dx/dt$
Mass	$F = m\, d^2x/dt^2$
	Rotational
Spring	$T = k\theta$
Damper	$T = c\, d\theta/dt$
Moment of inertia	$T = J\, d^2\theta/dt^2$





Inertia: $F = ma = m\frac{d^2x}{dt^2}$

Damping: $F_D = Dv = D\frac{dx}{dt}$

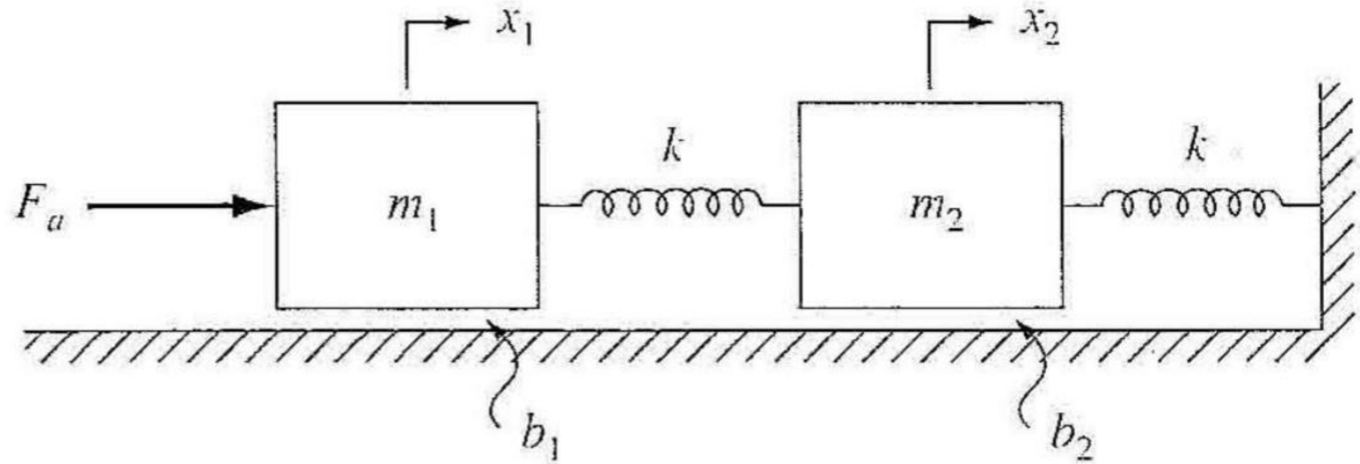
Spring: $F_S = kx$

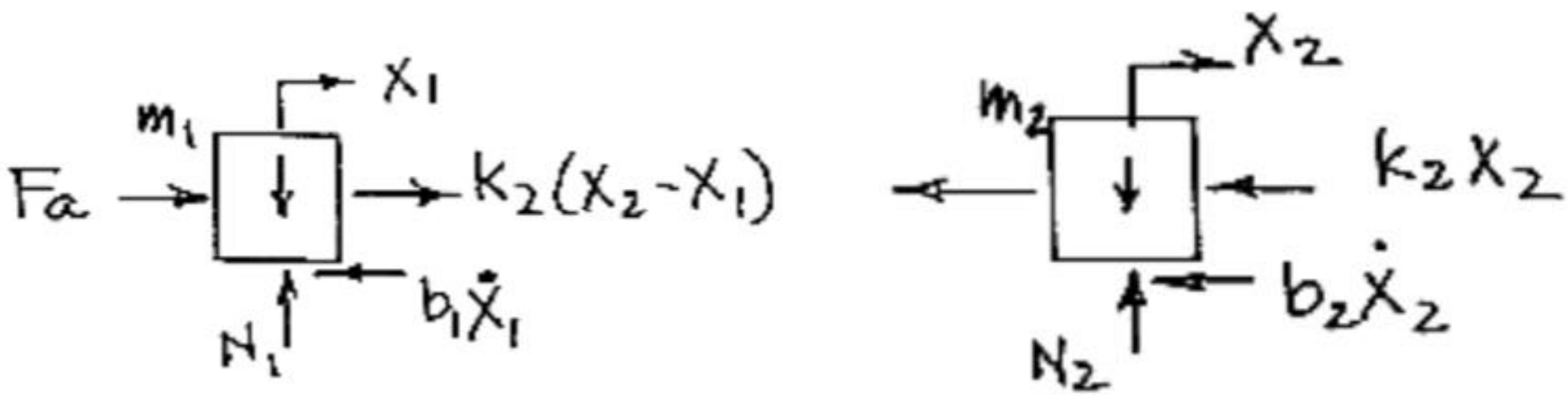
$$m\frac{d^2x_{out}}{dt^2} = -kx - D\frac{dx}{dt} + F_{in}$$

$$m\frac{d^2x_{out}}{dt^2} + D\frac{dx}{dt} + kx = F_{in}$$

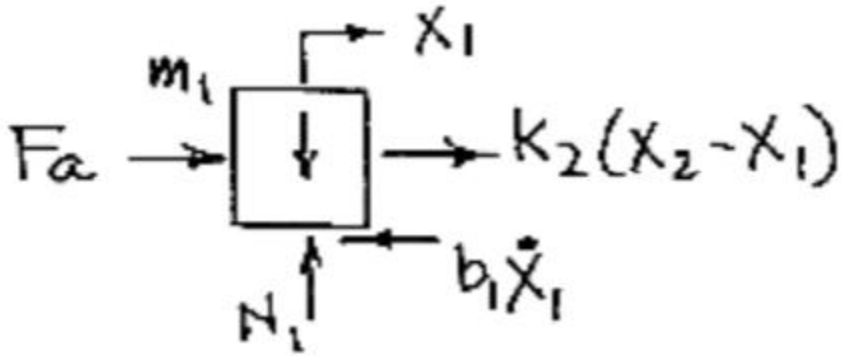
Continuous Systems and Transfer Function Revision: Mass/spring systems

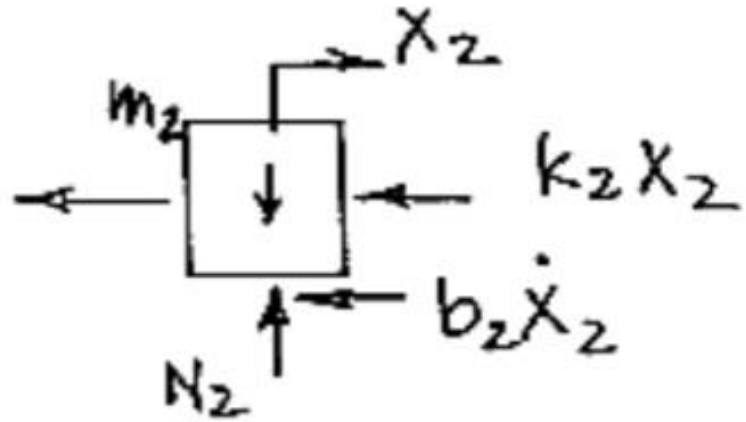
Derive the equation of motion for x_2 as a function of F_a.





Continuous Systems and Transfer Function Revision: Mass/spring systems





$$F_a + k_2(x_2 - x_1) - b_1\frac{dx_1}{dt} = m_1\frac{d^2x_1}{dt^2}$$

$$-k_2(x_2 - x_1) - k_2x_2 - b_2\frac{dx_2}{dt} = m_2\frac{d^2x_2}{dt^2}$$

$$F_a = m_1\frac{d^2x_1}{dt^2} + b_1\frac{dx_1}{dt} + k_2x_1 - k_2x_2$$

$$0 = m_2\frac{d^2x_2}{dt^2} + b_2\frac{dx_2}{dt} + 2k_2x_2 - k_2x_1$$

Time vs frequency domain
Transfer function

2.1.4 Convolution Approach

Time domain

$$u(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- u is the impulse function to the system
- h is called the impulse response of the system

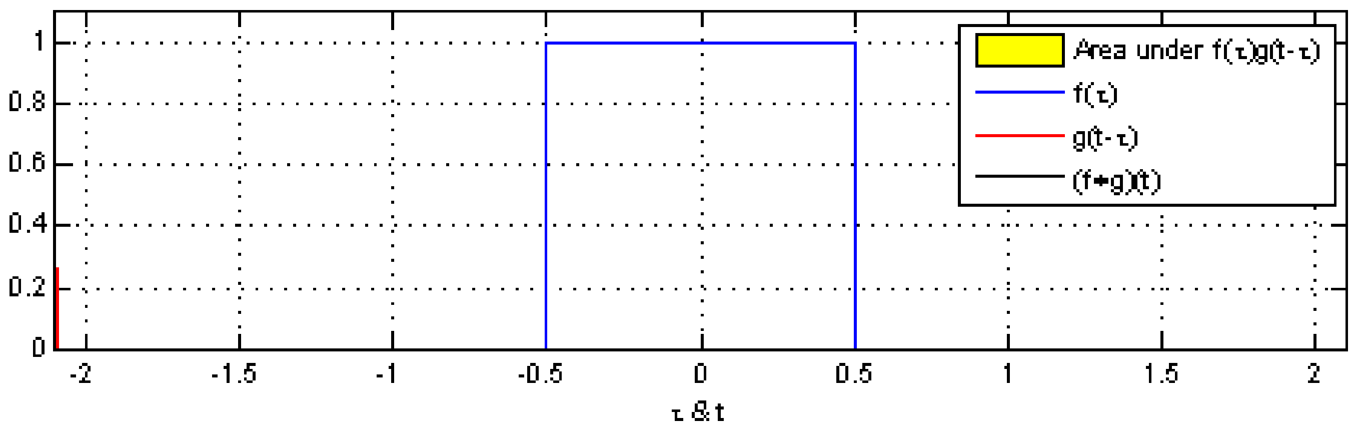
$$y(t) = \int_0^\infty h(\tau)u(t-\tau)d\tau = \int_0^\infty h(\tau - t)u(\tau)d\tau$$

with $0 \leq \tau \leq t$

Convolution

$$y(t) = h(t) * u(t)$$

Essentially, the steps for convolving two signals are to first reflect the signal *g*, then offset the reflected signal. Then calculate the area under the graph for every offset, by sliding –*g*. The convolution at each time point is equal to the area under the intersection of functions. For two pulses, the result is a triangle wave:



However, the calculations to obtain this result in the time domain are complicated, but are only multiplication in the Laplace domain.

The impulse response of a certain system is given by ***g(t)*** below. Use the convolution integral to determine the response ***y(t)*** due to a ramp input ***x(t)*** also below:

$g(t) = 0$ for $t < 0$ and by
$g(t) = e^{-2t}$ for $t \geq 0$

$x(t) = 0$ for $t < 0$ and by
$x(t) = 4t$ for $t \geq 0$

$$y(t) = x(t) * g(t) = \int_{-\infty}^{\infty} x(t-\tau)g(\tau)d\tau$$

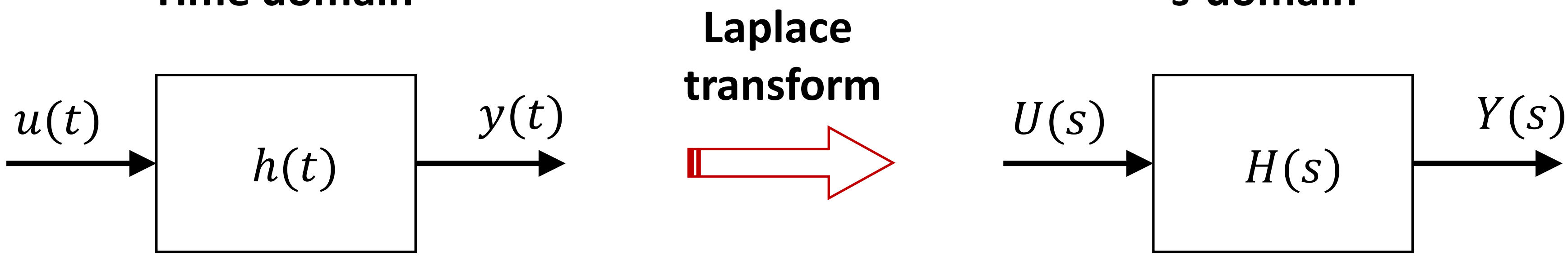
$$= \int_{-\infty}^{0} x(t-\tau)\cancel{g(\tau)}d\tau + \int_{0}^{t} x(t-\tau)g(\tau)d\tau + \int_{t}^{\infty} \cancel{x(t-\tau)}g(\tau)d\tau$$

$$= \int_{0}^{t} 4(t-\tau)e^{-2\tau}d\tau = 4t\int_{0}^{t} e^{-2\tau}d\tau - 4\int_{0}^{t} \tau e^{-2\tau}d\tau$$

$$= e^{-2t} + (2t-1) \text{ for } t \geqq 0$$

2.1.4 Convolution Approach

Time domain | **s-domain**



- u is the impulse function to the system
- h is called the impulse response of the system

- H is called the transfer function (TF) of the system

$$Y(s) = H(s) \cdot U(s)$$

Multiplication

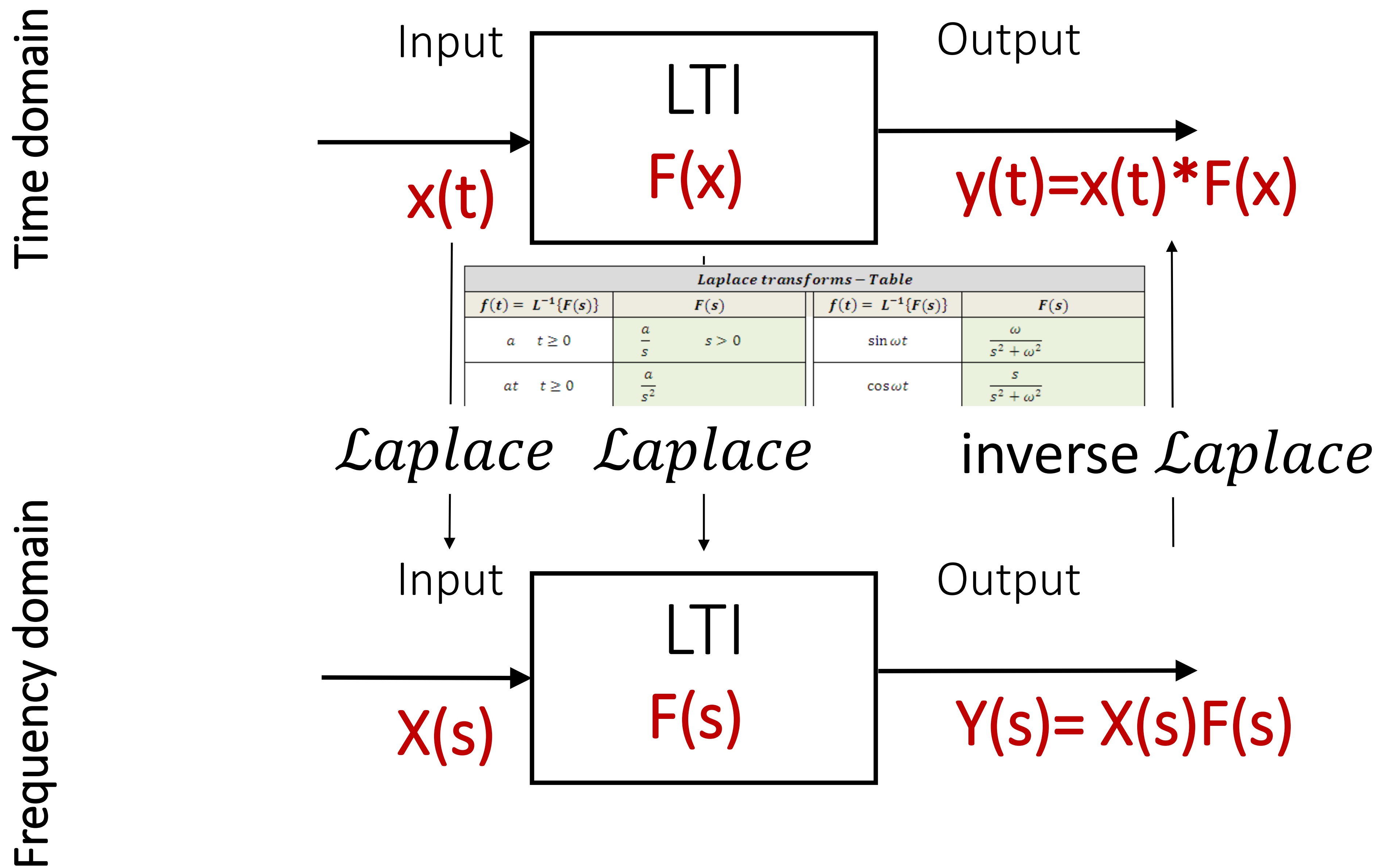
$$y(t) = \int_0^\infty h(\tau)u(t-\tau)d\tau = \int_0^\infty h(\tau - t)u(\tau)d\tau$$

with $0 \le \tau \le t$

Convolution

$$y(t) = h(t) * u(t)$$

Continuous Systems and Transfer Function Revision: Time vs frequency domain



2.1.5 Laplace Transforms

Because of our **linear assumptions** we can use Laplace transforms to simplify solving the ODEs.

The Laplace transform of a signal (function) x is the Function $X = \mathcal{L}(x)$ defined by

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

for those $s \in \mathbf{C}$ for which the integral makes sense.

- X is a complex-valued function of complex numbers.
- s is called the (complex) *frequency variable*, with units $\sec^{-1}$, t is called the *time variable* (in sec); st is unitless.
- $s = \sigma + j\omega$

Continuous Systems and Transfer Function Revision: Laplace transform table

$Laplace\ transforms - Table$			
$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \quad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
e^{-at}	$\frac{1}{s+a}$	$\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\cos(\omega t + \theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$\frac{1}{2}t^2 e^{-at}$	$\frac{1}{(s+a)^3}$	$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s+a)^n}$	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
e^{at}	$\frac{1}{s-a} \quad s > a$	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2} \quad s > \lvert\omega\rvert$
te^{at}	$\frac{1}{(s-a)^2}$	$\cosh \omega t$	$\frac{s}{s^2-\omega^2} \quad s > \lvert\omega\rvert$
$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$\frac{1}{a^2}[1 - e^{-at}(1+at)]$	$\frac{1}{s(s+a)^2}$	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}} \quad n = 1,2,3..$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}} \quad s > a$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$t^n e^{-at}$	$n!$	$1 - e^{-at}$	a

Continuous Systems and Transfer Function Revision: Laplace transforms

The Laplace variable, s, can be considered to represent the differential operator (VERY useful for control engineering):

$$\boxed{s \equiv \frac{d}{dt}}$$

$$\boxed{\frac{1}{s} \equiv \int_{0^-}^{\infty} dt}$$

Continuous Systems and Transfer Function Revision: Laplace transforms

Laplace transform of time derivative dx/dt:

$$L\left\{\frac{dx}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx}{dt} e^{-st} dt$$

Integrating by parts:

$$L\left\{\frac{dx}{dt}\right\} = s\int_{0^-}^{\infty} x(t)e^{-st} dt + \left[x(t)e^{-st}\right]_{0^-}^{\infty}$$

The initial condition $x(0^-)$ is often zero in practice

$$\boxed{L\left\{\frac{dx}{dt}\right\} = sX(s) - \cancel{x(0^-)}}$$

We can substitute this result to solve higher order derivatives:

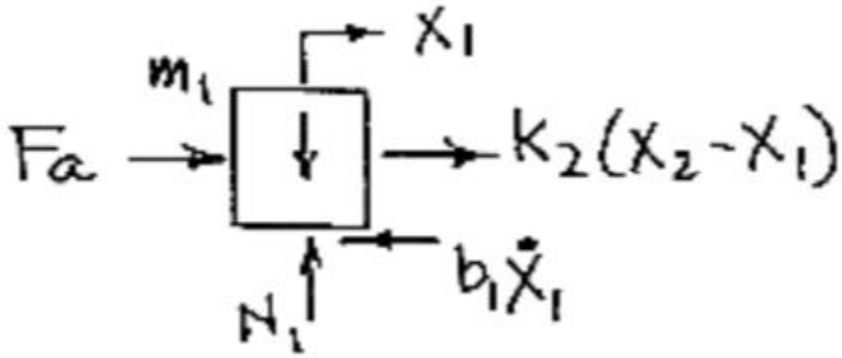
$$L\left\{\frac{d^2x}{dt^2}\right\} = sL\left\{\frac{dx}{dt}\right\} - \frac{dx}{dt}(0^-)$$

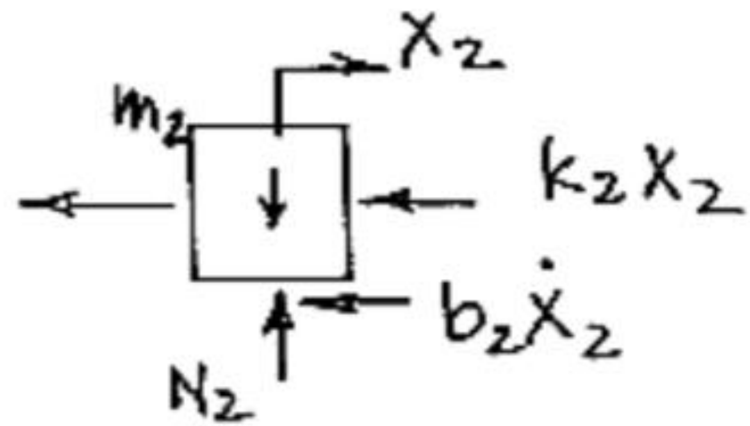
$$\boxed{L\left\{\frac{d^2x}{dt^2}\right\} = s^2X(s) - sx(0^-) - \frac{dx}{dt}(0^-)}$$

So more generally, with all initial conditions set to zero:

$$L\left\{\frac{d^nx}{dt^n}\right\} = s^nX(s)$$

Continuous Systems and Transfer Function Revision: Mass/spring systems





$$F_a + k_2(x_2 - x_1) - b_1\frac{dx_1}{dt} = m_1\frac{d^2x_1}{dt^2}$$

$$-k_2(x_2 - x_1) - k_2x_2 - b_2\frac{dx_2}{dt} = m_2\frac{d^2x_2}{dt^2}$$

$$F_a = m_1\frac{d^2x_1}{dt^2} + b_1\frac{dx_1}{dt} + k_2x_1 - k_2x_2$$

$$0 = m_2\frac{d^2x_2}{dt^2} + b_2\frac{dx_2}{dt} + 2k_2x_2 - k_2x_1$$

$$\mathcal{L}(F_a) = m_1s^2X_1(s) + b_1sX_1(s) + k_2X_1(s) - k_2X_2(s)$$

$$F_a(s) = X_1(s)\left(m_1s^2 + b_1s + k_2\right) - k_2X_2(s)$$

$$0 = m_2s^2X_2(s) + b_2sX_2(s) + 2k_2X_2(s) - k_2X_1(s)$$

$$0 = X_2(s)\left(m_2s^2 + b_2s + 2k_2\right) - k_2X_1(s)$$

Continuous Systems and Transfer Function Revision: Mass/spring systems

$$F_a = X_1(s)\left(m_1 s^2 + b_1 s + k_2\right) - k_2 X_2(s)$$

$$0 = X_2(s)\left(m_2 s^2 + b_2 s + 2k_2\right) - k_2 X_1(s)$$

$$\frac{X_2(s)\left(m_2 s^2 + b_2 s + 2k_2\right)}{k_2} = X_1(s)$$

$$F_a = \frac{X_2(s)\left(m_2 s^2 + b_2 s + 2k_2\right)\left(m_1 s^2 + b_1 s + k_2\right)}{k_2} - k_2 X_2(s)$$

$$F_a = X_2(s)\left[\frac{\left(m_2 s^2 + b_2 s + 2k_2\right)\left(m_1 s^2 + b_1 s + k_2\right)}{k_2} - k_2\right]$$

Continuous Systems and Transfer Function Revision: Modelling electrical systems

- The voltage across the capacitor.

$$v(t) = \frac{1}{C}\int i(t)dt$$

- For the current, it yields:

$$i(t) = C\frac{dv(t)}{dt}$$

- The voltage drop $v(t)$ across the inductor is:

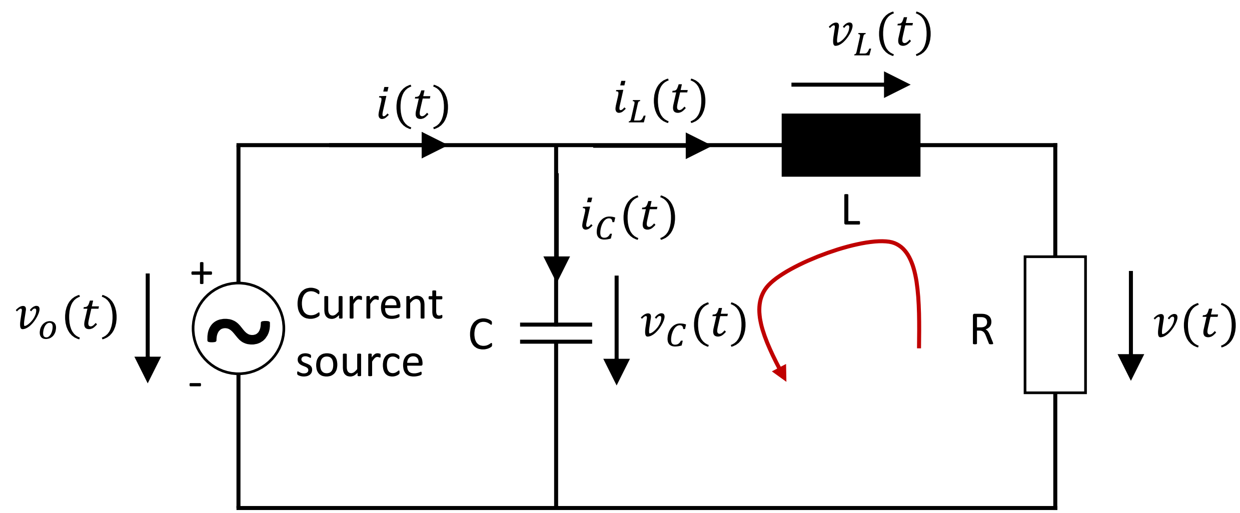
$$v(t) = \frac{d\Phi_B(t)}{dt} = \mathrm{L}\frac{di(t)}{dt}$$

- Hence, the current yields:

$$i(t) = \frac{1}{\mathrm{L}}\int v(t)dt$$

Revision: Solutions to differential equations

$$v_o(t) = v_C(t)$$



KCL and KVL:

$$i(t) = i_C(t) + i_L(t)$$

$$v_C(t) - v_L(t) - v(t) = 0$$
$$v_C(t) = v_L(t) + v(t)$$

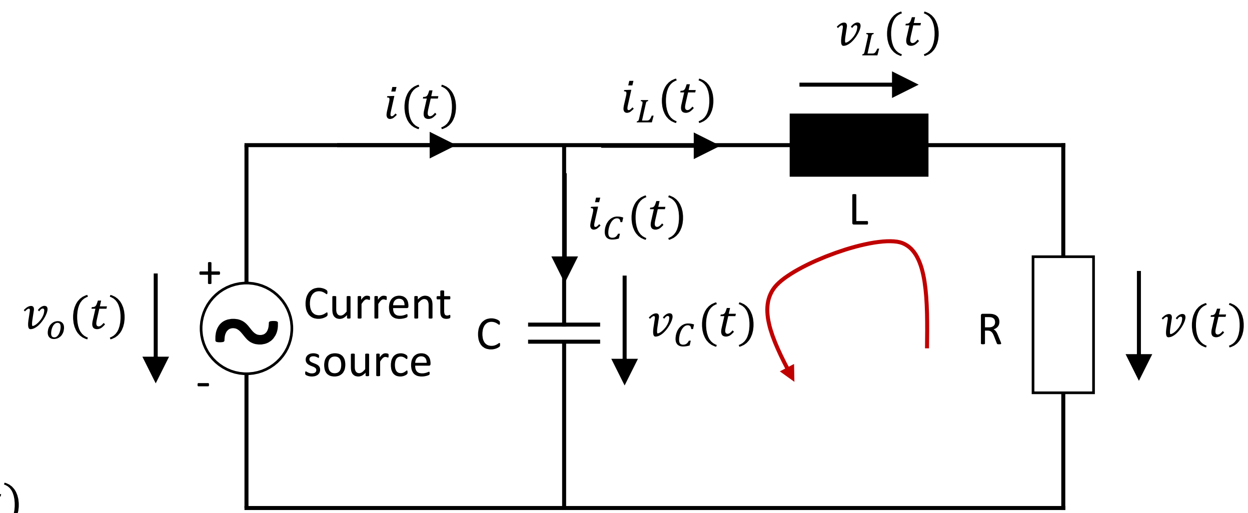
$$i(t) = C\frac{dv_c(t)}{dt} + \frac{v(t)}{R}$$

$$v_C(t) = v_L(t) + v(t)$$

$$v_C(t) = L\frac{di_L(t)}{dt} + v(t)$$

$$i(t) = C\frac{d\left(L\frac{di_L(t)}{dt} + v(t)\right)}{dt} + \frac{v(t)}{R}$$

Revision: Solutions to differential equations



$$i(t) = C\frac{d\left(L\frac{di_L(t)}{dt} + v(t)\right)}{dt} + \frac{v(t)}{R}$$

$$i(t) = C\frac{d\left(L\frac{\frac{dv(t)}{R}}{dt} + v(t)\right)}{dt} + \frac{v(t)}{R} = C\frac{d\left(\frac{L}{R}\frac{dv(t)}{dt} + v(t)\right)}{dt} + \frac{v(t)}{R}$$

$$i(t) = C\left(\frac{L}{R}\frac{d^2v(t)}{dt^2} + \frac{dv(t)}{dt}\right) + \frac{v(t)}{R} = \frac{CL}{R}\frac{d^2v(t)}{dt^2} + C\frac{dv(t)}{dt} + \frac{v(t)}{R}$$

$$\frac{R}{CL}i(t) = \frac{d^2v(t)}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{CL}v(t)$$

Revision: Solutions to differential equations - Laplace

$$\mathcal{L}\left\{\frac{R}{CL}i(t)\right\} = \mathcal{L}\left\{\frac{d^2v(t)}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{CL}v(t)\right\}$$

$$\frac{R}{CL}I(s) = s^2V(s) + s\frac{R}{L}V(s) + \frac{1}{CL}V(s)$$

Transfer function:

$$\frac{R}{CL}I(s) = V(s)\left(s^2 + s\frac{R}{L} + \frac{1}{CL}\right)$$

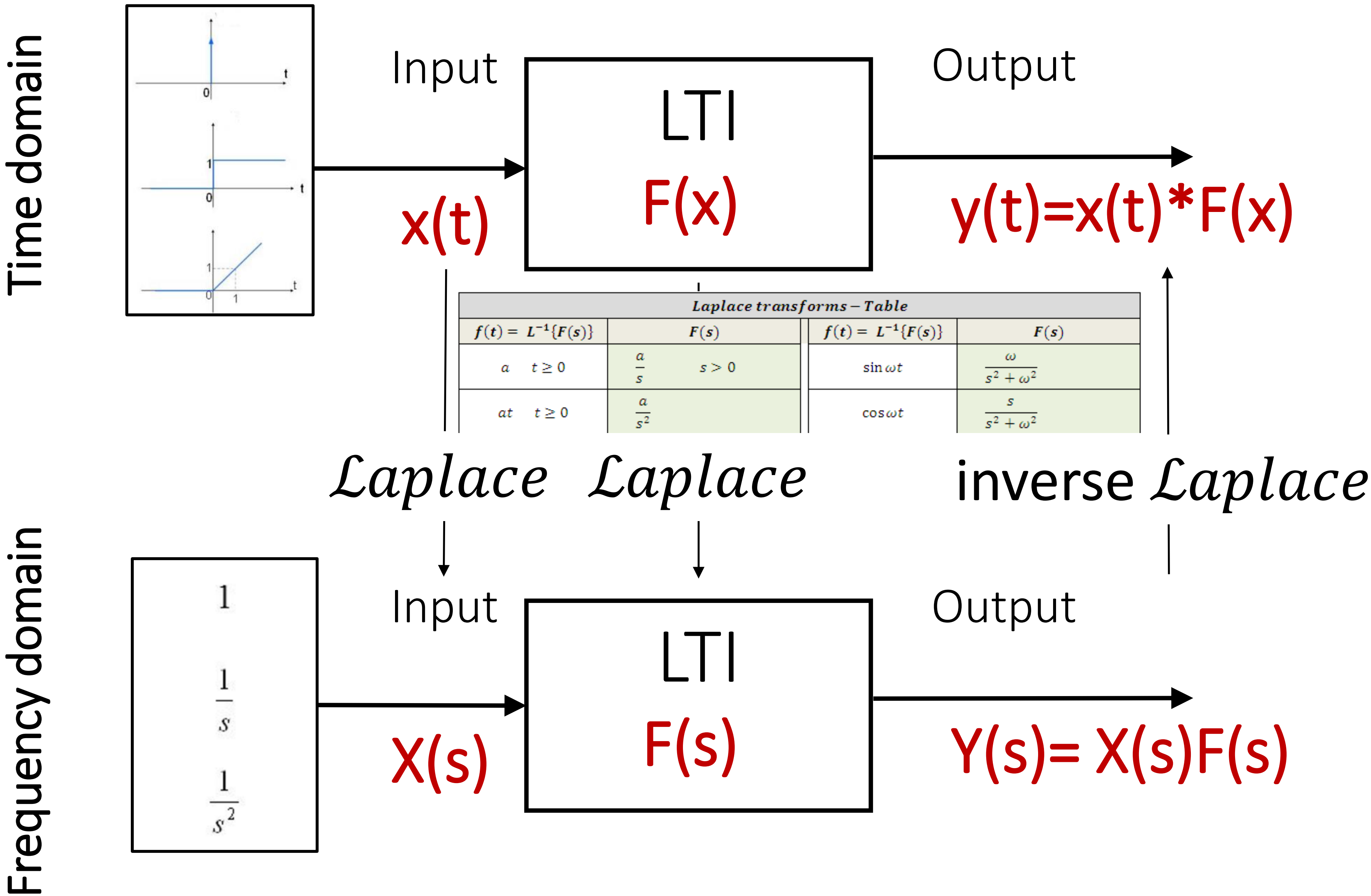
$$\frac{R}{CL\left(s^2 + s\frac{R}{L} + \frac{1}{CL}\right)} = \frac{V(s)}{I(s)} = G(s)$$

$$G(s) = \gamma\frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

MECH0089
Control and Robotics:
Digital Control Systems

Lecture 2:
Continuous Systems and Transfer Function Revision (part 2)

Continuous Systems and Transfer Function Revision: Time vs frequency domain



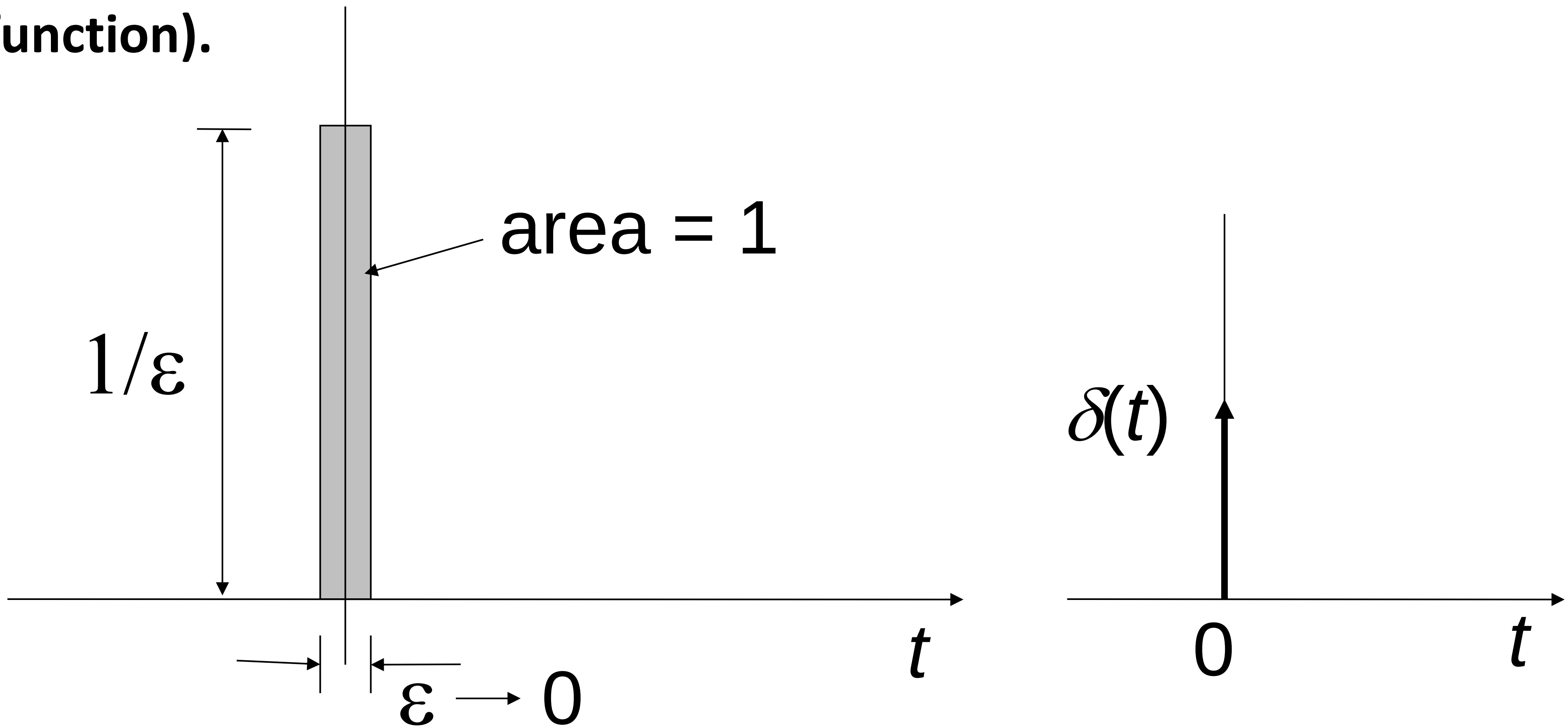
Input functions:
Impulse – Step - Ramp

Now we will consider some standard inputs and look at the response of first and second order systems:

- Impulse (In practice, we cannot create an input of $\boldsymbol{x(t) = \delta(t)}$ to characterise a system.)
- Step
- Ramp

There are many others, particularly sinusoidal inputs or other discontinuous inputs, which are important in control loops, but we will focus on the two classic examples.

A useful tool in analysing the transient response of a system is the impulse signal, a unit (amplitude=1) pulse infinitesimally small, with area =1. Formally this is known as the **Dirac delta function (impulse function).**



The Dirac delta function is a non-physical, singularity function with the following definition

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0 \\ \text{undefined} & \text{for } t = 0 \end{cases}$$

but with the requirement that

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

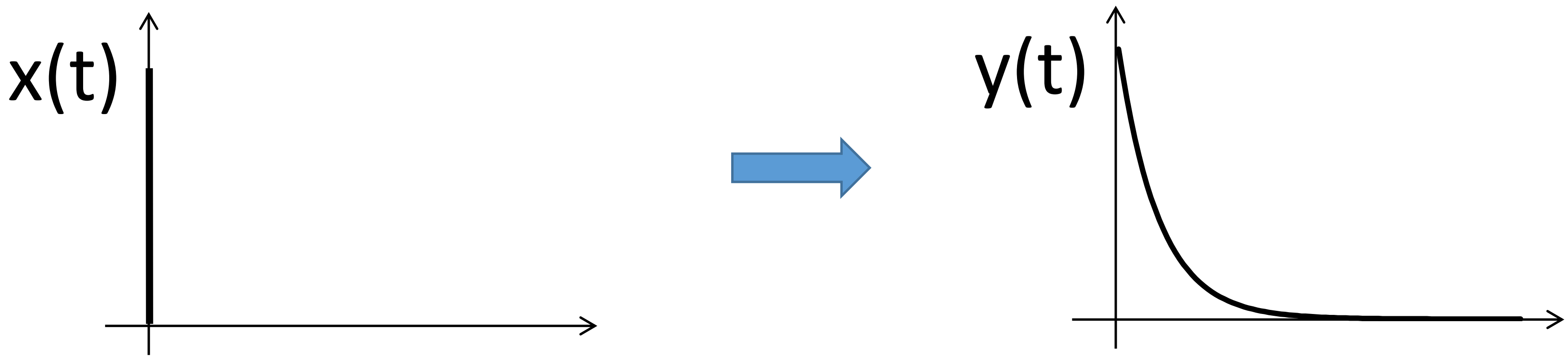
so taking the Laplace transform of this is also just 1

$$\mathcal{L}\big(\delta(t)\big) = \int_{0-}^{\infty} \delta(t)e^{-st}dt = 1$$

Thus the impulse response of the system is equal to the transfer function, and from this it can be shown that *any* arbitrary signal can be described as a summation of impulse responses.

Continuous Systems and Transfer Function Revision: Impulse response on a first order system

Take a first order response for example, the transfer function *and thus the impulse response* looks like this:



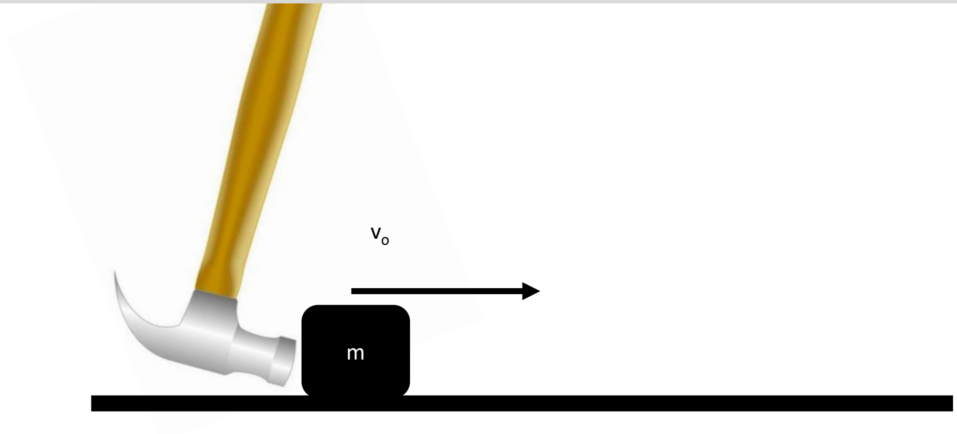
Input

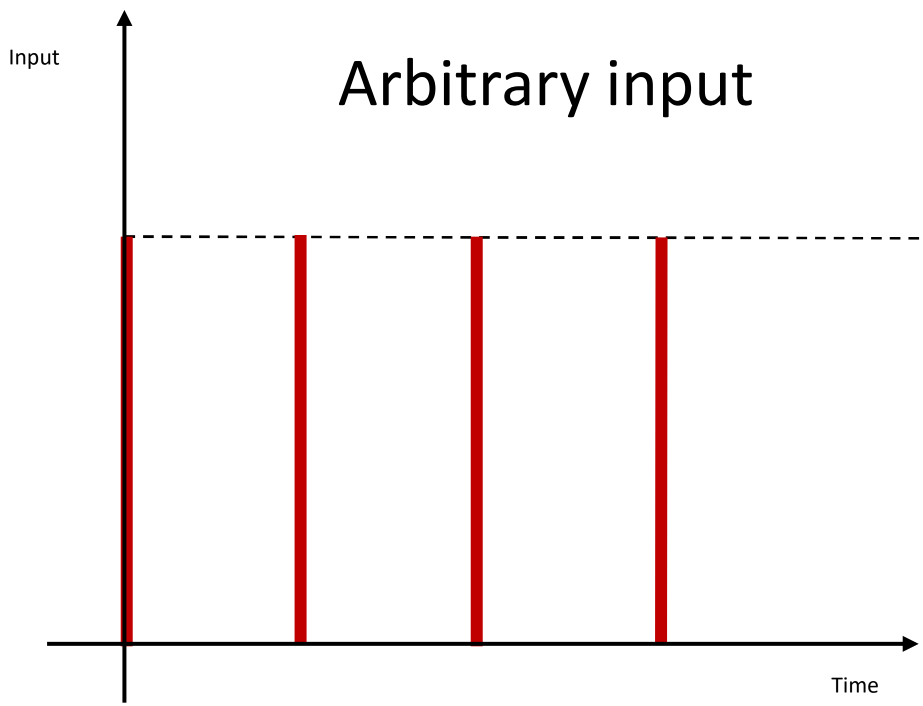
Output

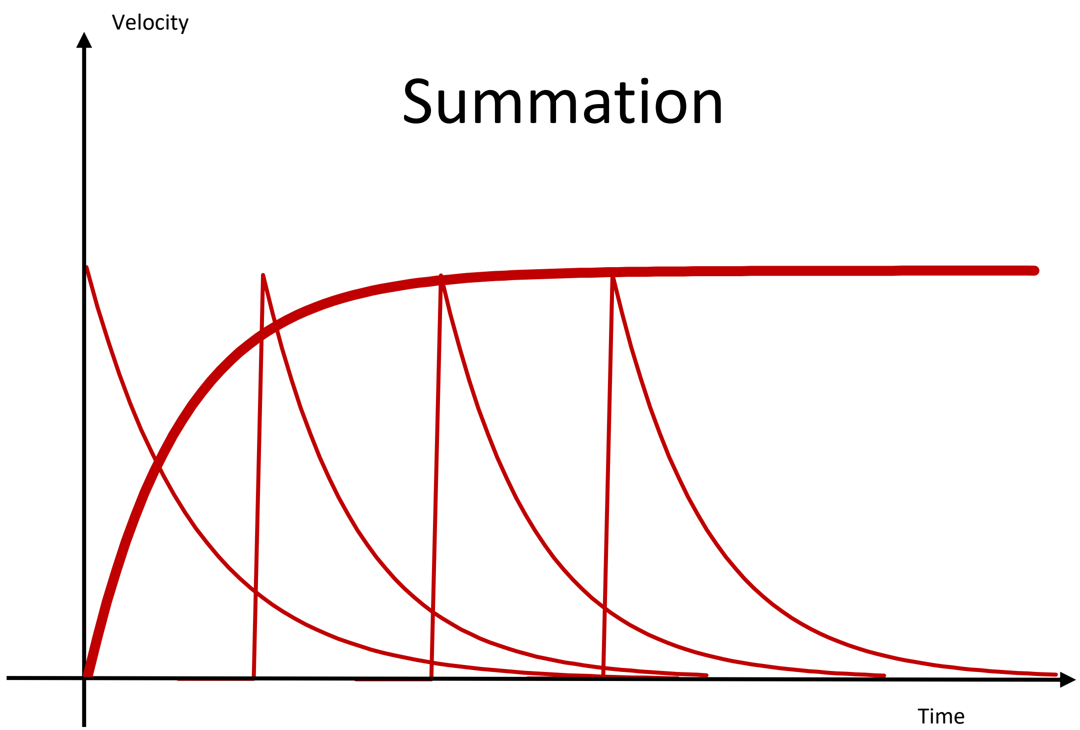
Due to our LTI assumptions:

- Scaling the input scales the output,
- Superposition of inputs equals superposition of outputs
- Time invariance

Continuous Systems and Transfer Function Revision: Impulse response of a system

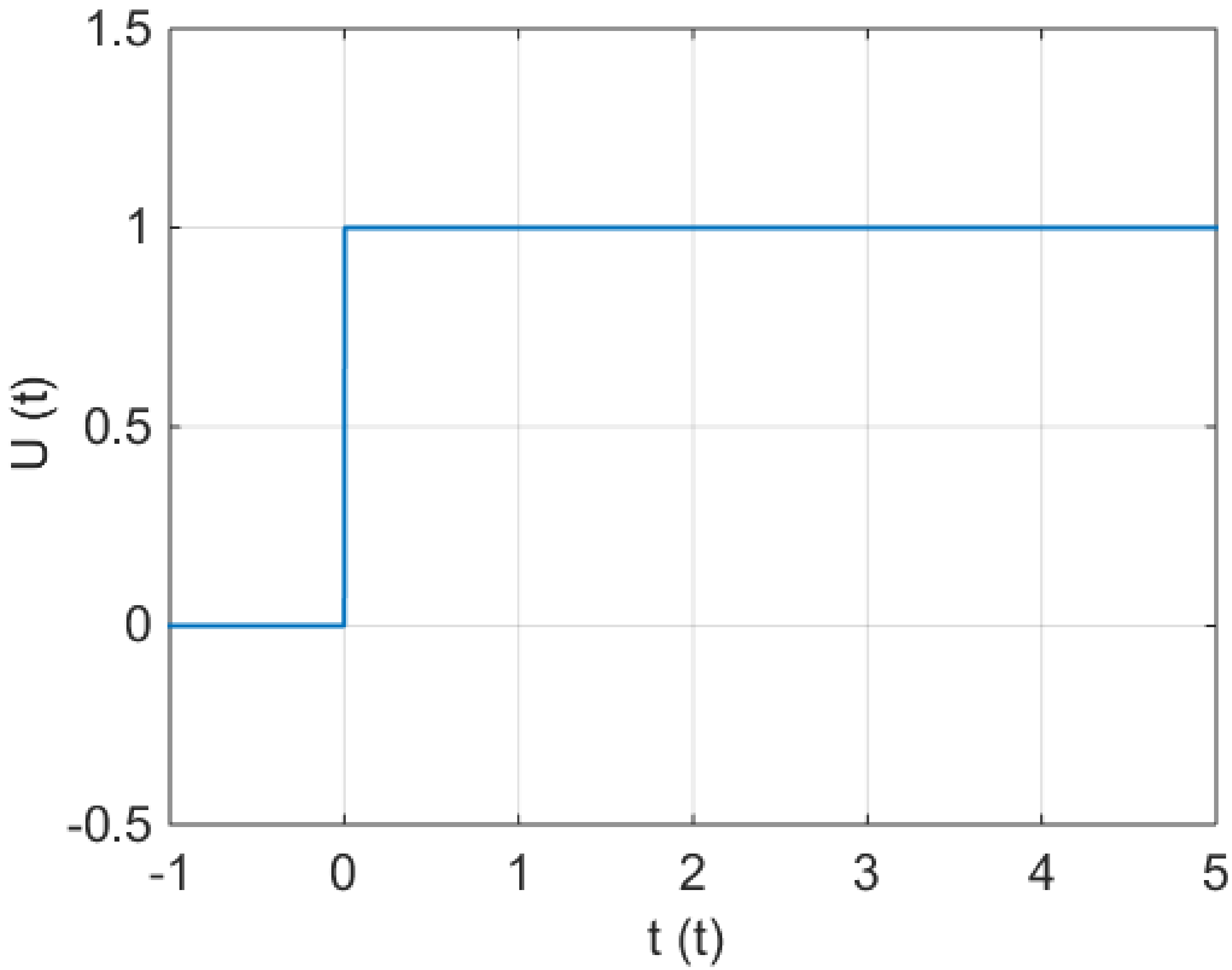






Continuous Systems and Transfer Function Revision: Step Input

A step input is a discontinuous function, which is zero for all negative values of *t* and 1 for all positive values



Continuous Systems and Transfer Function Revision: Step Input – Laplace transform

$$x(t) = U(t)$$

$$L\{U(t)\} = \int_{0^-}^{\infty} e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_{0^-}^{\infty}$$

$$\boxed{\frac{1}{s}}$$

Or for a gain of *A*

$$x(t) = AU(t) \quad L\{AU(t)\} = \frac{A}{s}$$

Continuous Systems and Transfer Function Revision: Step Input – Applications

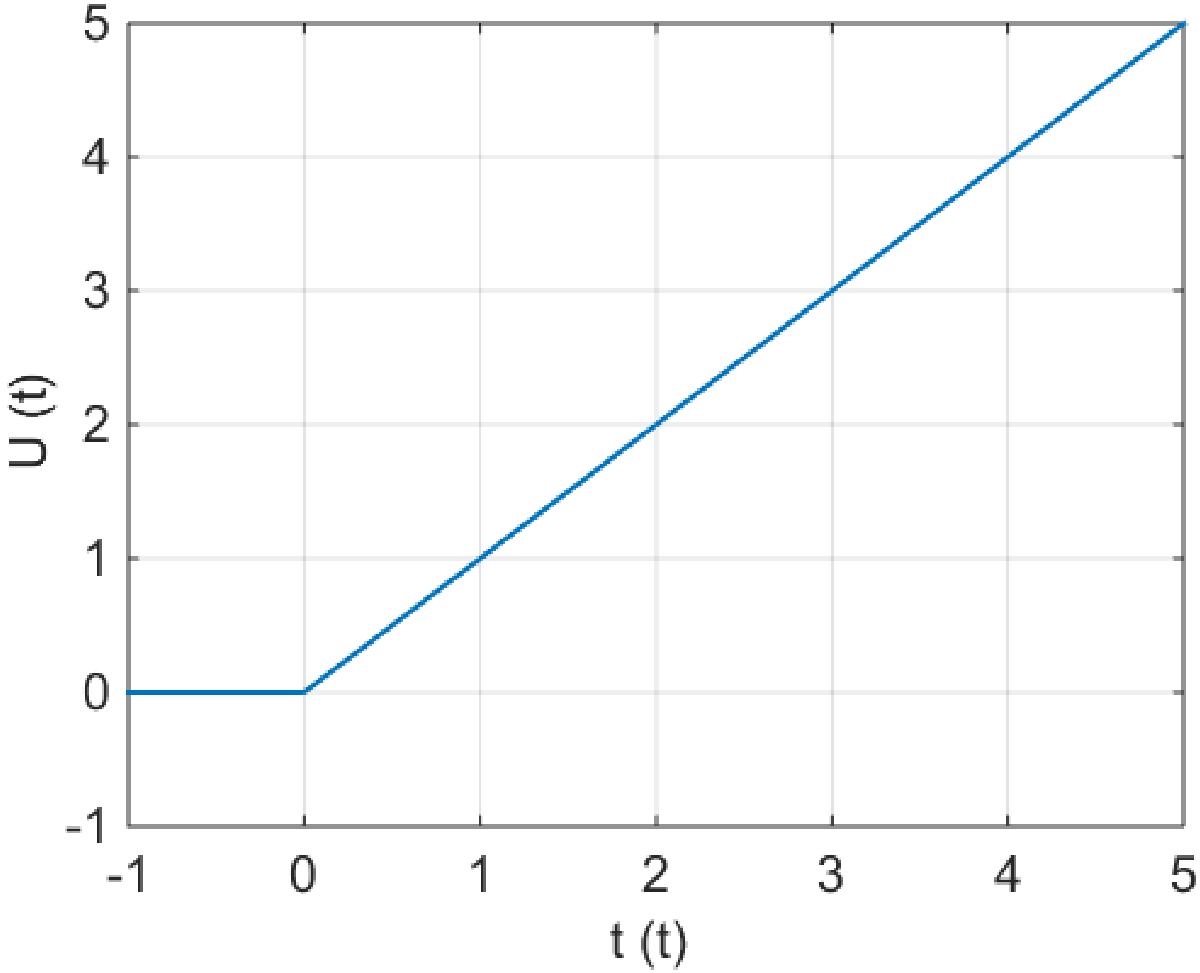
The step response is extremely useful in control theory for describing the behaviour of the system. In part because it incorporates the "transient" behaviour – from the sudden change from zero to one, as well as the "steady state" behaviour as the system settles down to a single value.

It also replicates many real world control applications such as

- Position control – Move to a X=10mm position and stay
- Speed control – go to 33.333 RPM
- Temperature – Heat element on 3D printer to 230°C

Continuous Systems and Transfer Function Revision: Ramp Input

A Ramp input has a value of *t* for all *t* value above zero and zero elsewhere, often is scaled by a gain *A*



$$x(t) = at$$

$$L\{at\} = \int_{0^-}^{\infty} ate^{-st}dt = -a\left[\frac{t}{s}e^{-st}\right]_0^{\infty} + a\int_0^{\infty}\frac{1}{s}e^{-st}dt$$

$$= a\left[-\frac{1}{s^2}e^{-st}\right]_0^{\infty} \quad \boxed{\frac{a}{s^2}}$$

Ramp inputs are useful in understanding the steady state behaviour of a system i.e. when *t* goes to infinity

Practical examples of control applications using ramp inputs are

- Servo motors – Shaft *position* rather than speed
- Ovens for PCB manufacturing etc. – strict linear *profile* of temperature required as opposed to “get to the this temperature quickly”
- CNC milling machine, move in X direction and constant rate

Continuous Systems and Transfer Function Revision: Summary of input functions

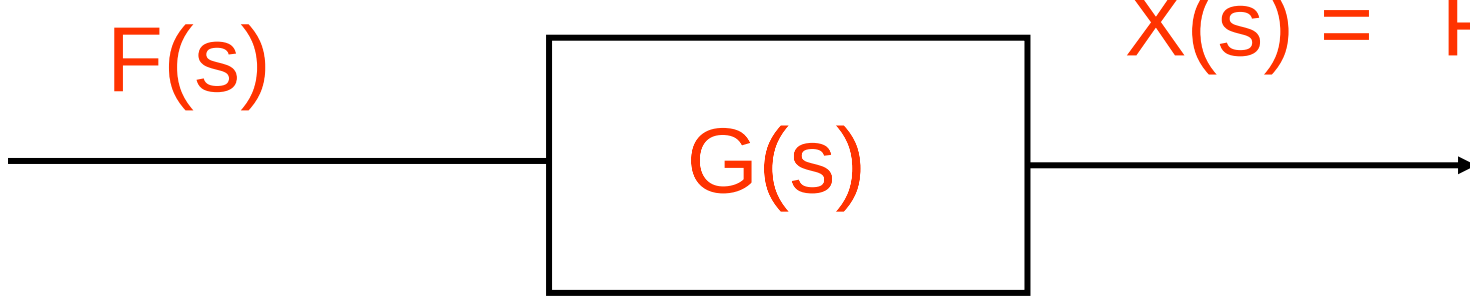
Impulse $$F(s) = A$$

Step $$F(s) = \frac{A}{s}$$

Ramp $$F(s) = \frac{A}{s^2}$$

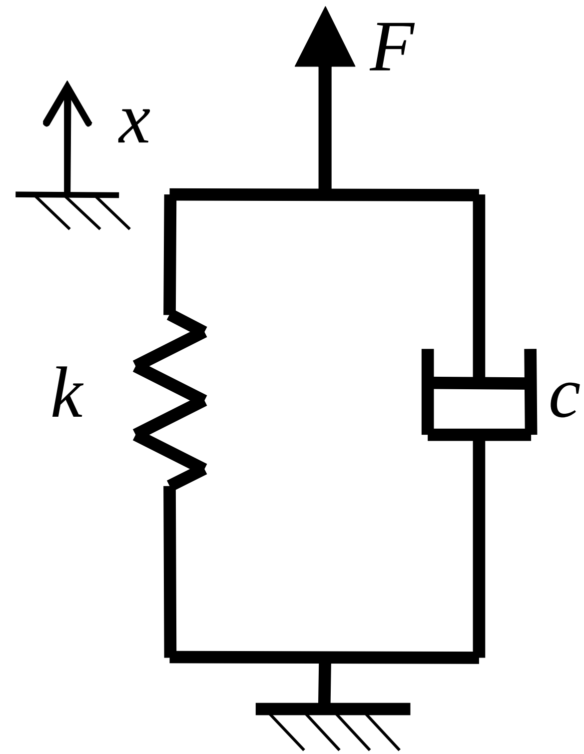
For *unit* response
A=1

We can apply these inputs to the LTI system by multiplying the transfer function by the input, both in terms of *s*



$$X(s) = F(s)G(s)$$

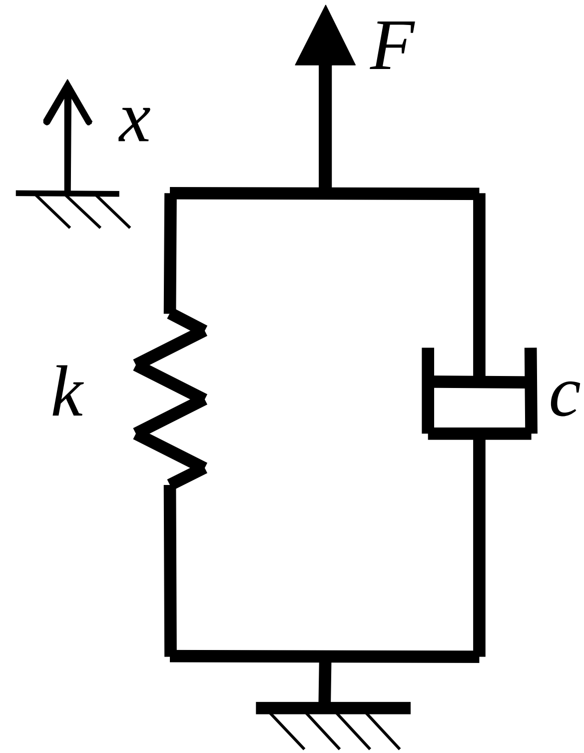
First Order System Step Response



In a shock absorber in a car, damping added in parallel to spring suspension in vehicles to damp oscillations and absorb impulses

Transfer function desired:

$$G(s) = \frac{X(s)}{F(s)}$$



Balancing forces as function of time:

$$f(t) = f_{spring}(t) + f_{damper}(t) = kx(t) + c\frac{dx}{dt}$$

Rewriting as function of s

$$F(s) = kX(s) + csX(s) = X(s)(k + cs)$$

Transfer function is thus:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{(k+cs)} = \boxed{\frac{\alpha}{(1+Ts)}}$$

$$\alpha = {}^{1}\!/_{k}$$

$$T = {}^{c}\!/_{k}$$

Continuous Systems and Transfer Function Revision: First Order Systems

All first order systems i.e. those with only $\frac{dx}{dt}$
Take the following "standard" forms

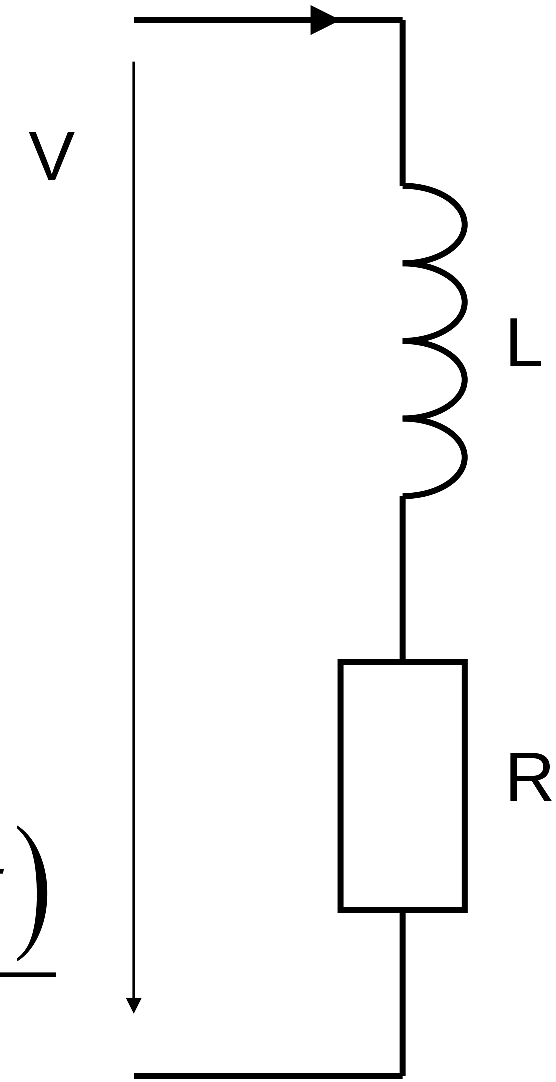
$$\frac{X(s)}{Y(s)} = \frac{\alpha}{(1+Ts)} = \frac{\gamma}{1+\tau s}$$

α, γ Gain

T, τ Time Constant

This function is commonly known as an exponential time delay, or lag.
This is an incredibly common function, they turn up everywhere!

Electromagnets are used in motors both rotary and linear, as well as in power transfer, and also magnetic levitation in trains



First obtain the transfer function $\dfrac{I(s)}{V(s)}$

Balancing voltages gives:

$$v(t) = v_R(t) + v_I(t)$$

where

$$v_R(t) = i(t)R \qquad v_I(t) = L\frac{di(t)}{dt}$$

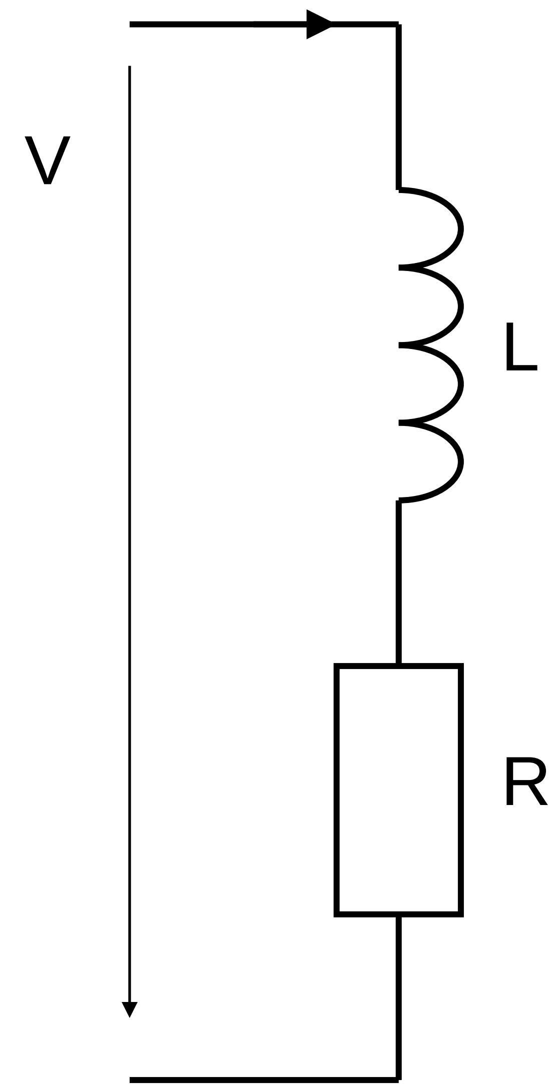
Substitution gives:

$$v(t) = Ri(t) + L\frac{di(t)}{dt}$$

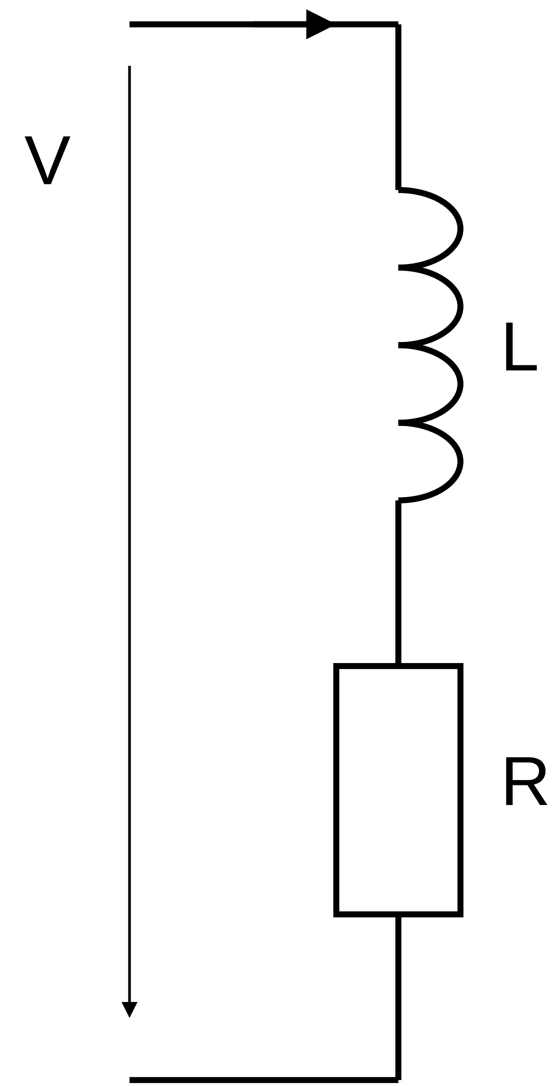
In the Laplace domain:

$$V(s) = RI(s) + LsI(s)$$

$$\frac{I(s)}{V(s)} = \frac{1}{(Ls+R)}$$



For a **step** input of 1V (unit step) $V(s) = \frac{1}{s}$



$$I(s) = V(s)\frac{1}{(Ls+R)}$$

$$= \frac{1}{s(Ls+R)}$$

Now we can look to the Laplace transform tables to find out I(t)

The first task is to split the expression up using partial fractions, into expressions that *are* given in the tables:

$$I(s) = \frac{1}{s(Ls+R)} = \frac{1/L}{s\left(s+R/L\right)}$$

$$= \frac{k_1}{s} + \frac{k_2}{\left(s+R/L\right)}$$

Partial Fraction Expansion

where $$k_1\left(s+R/L\right)+k_2 s = 1/L$$

Looking at $s = 0$ $\qquad k_1 \, {}^{R}\!/\!_{L} = {}^{1}\!/\!_{L} \longrightarrow k_1 = {}^{1}\!/\!_{R}$

Looking at $s = -\frac{R}{L}$ $\qquad -k_2 \frac{R}{L} = \frac{1}{L} \longrightarrow k_2 = -{}^{1}\!/\!_{R}$

Which yields: $$I(s) = \frac{1}{R}\left[\frac{1}{s} - \frac{1}{\left(s + {}^{R}\!/\!_{L}\right)}\right]$$

So looking at the Laplace tables, we can now use entries for $\frac{1}{s}$ and $\frac{1}{(s+a)}$

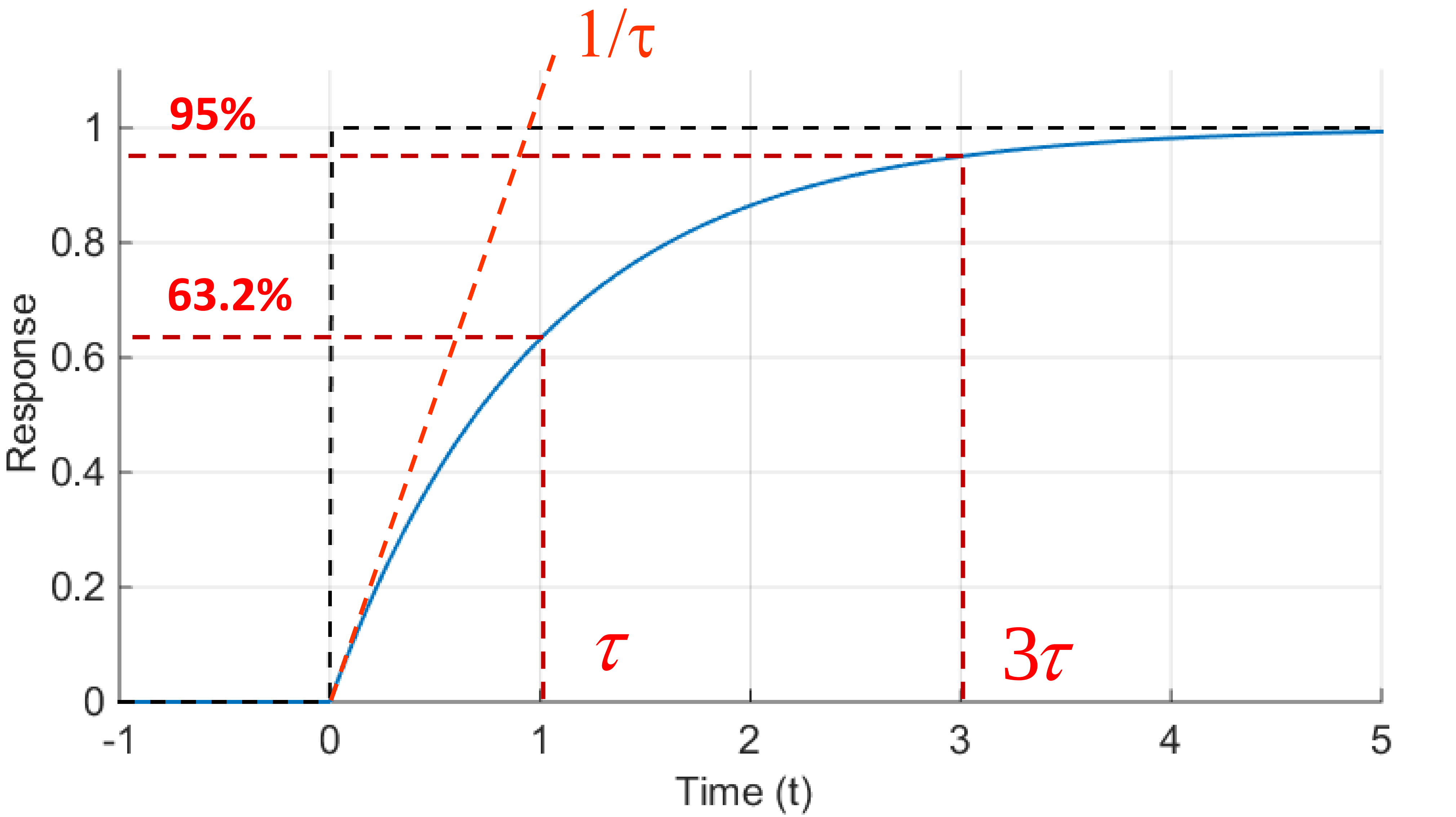
$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\boxed{\frac{a}{s}} \quad s > 0$
$at \quad t \geq 0$	$\frac{a}{s^2}$
e^{-at}	$\boxed{\frac{1}{s+a}}$

Only consider $t > 0$, So $u(t) = 1$

$$I(t) = \frac{1}{R}\left[1 - e^{-\frac{R}{L}t}\right]$$

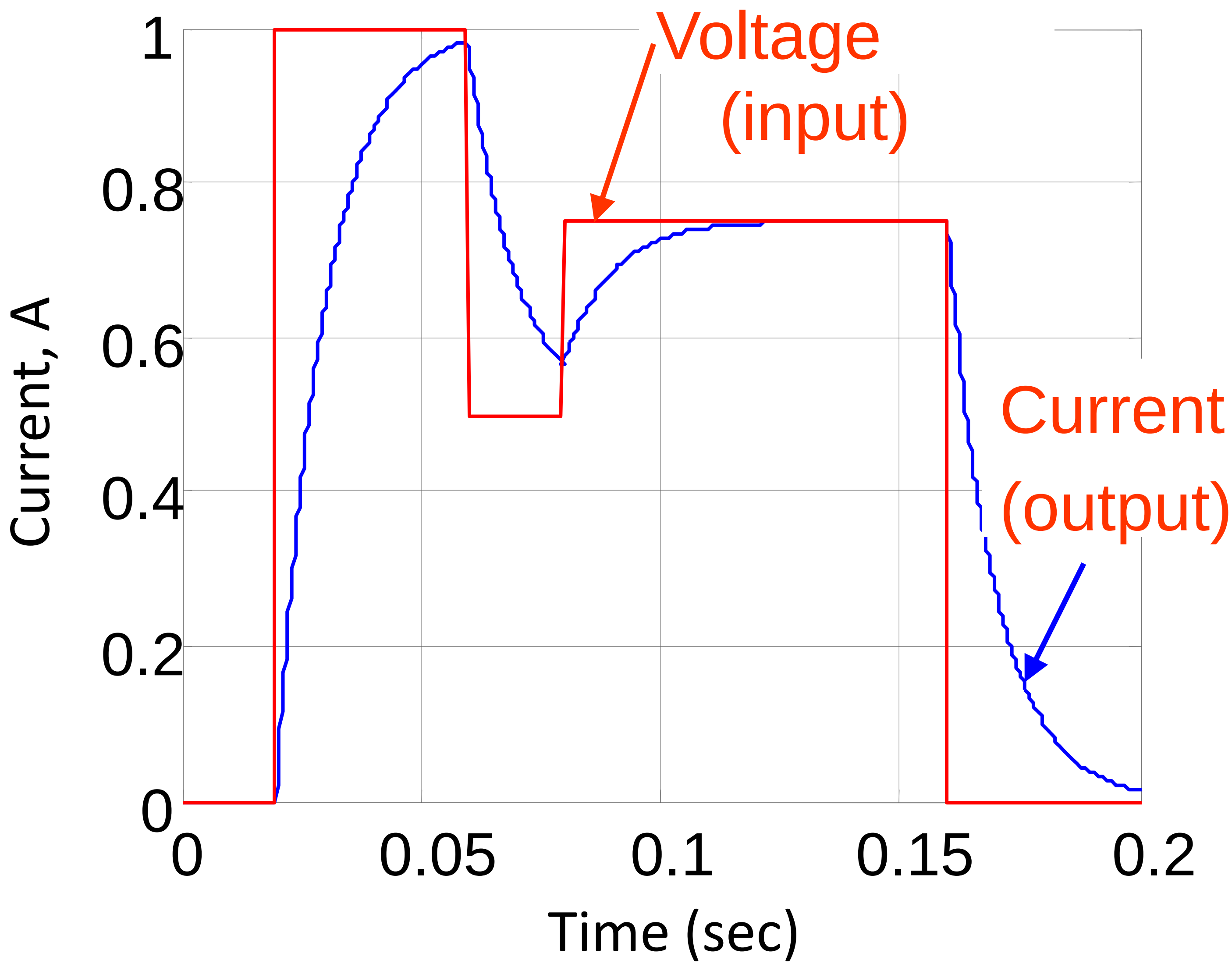
Which has the familiar form of a first order exponential rise, with a steady state gain of 1/R. Time constant τ = L/R

So (for now) assume L and R = 1, so gain and time constant =1



These "step responses" can be repeated for different values of V and for different starting values of *I(t)*

The response will always be behind the input for all time constants > 0, and thus these systems are referred to as *first order lags*



Second Order Systems

Continuous Systems and Transfer Function Revision: Mass Spring Damper

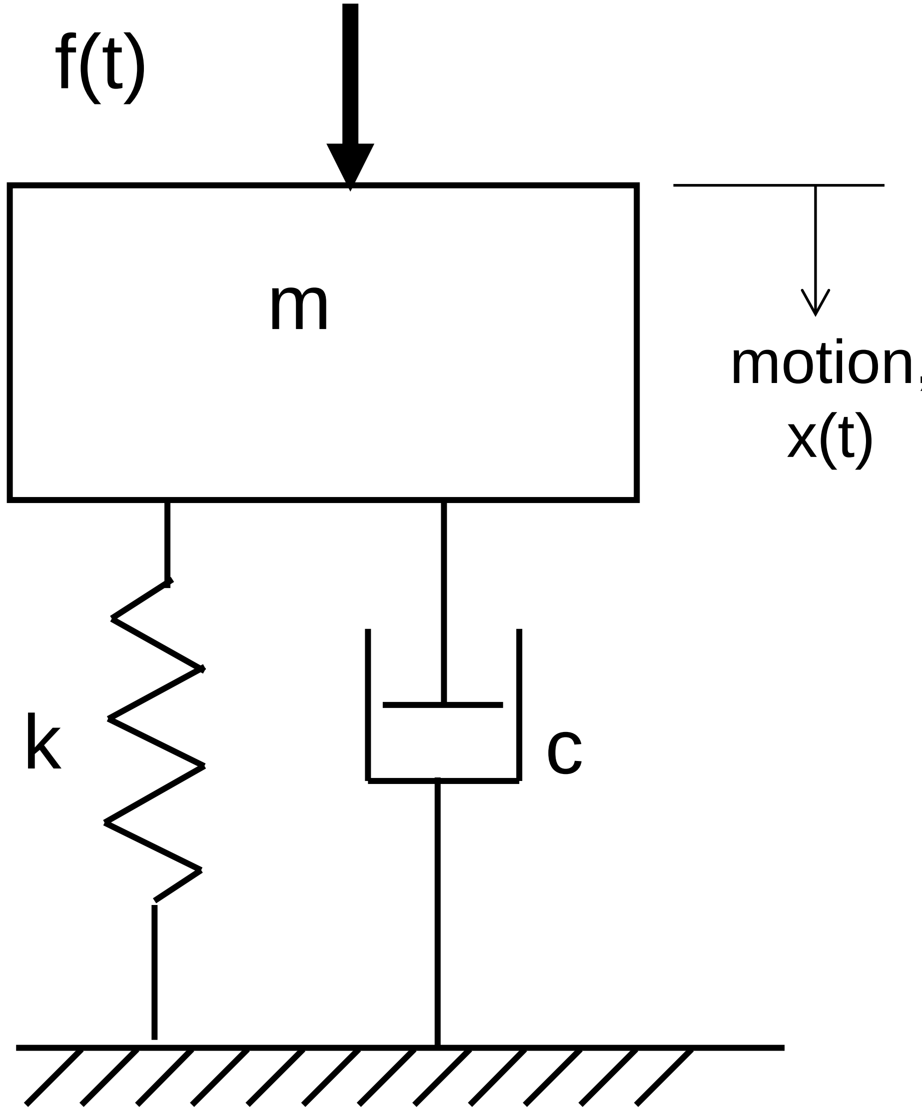
Consider a simple mechanical system
Mass/Spring/Damper (MSD)

As before, we wish to relate the input force to the output displacement
i.e. Transfer function desired:

$$G(s) = \frac{X(s)}{F(s)}$$

Using Newton's Second Law
Balancing forces as function of time:

$$f(t) = f_S(t) + f_D(t) + f_I(t)$$



Continuous Systems and Transfer Function Revision: Mass Spring Damper

$$f_S(t) = kx(t) \qquad f_D(t) = c\frac{dx(t)}{dt} \qquad f_I(t) = m\frac{d^2x(t)}{dt^2}$$

Combining yields the following time domain equations

$$f(t) = kx(t) + c\frac{dx(t)}{dt} + m\frac{d^2x(t)}{dt^2}$$

Initial conditions zero

$$x(0) = \frac{dx(0)}{dt} = 0$$

Rewriting as function of s

$$F(s) = kX(s) + csX(s) + ms^2X(s)$$

$$F(s) = X(s)\left(k + cs + ms^2\right)$$

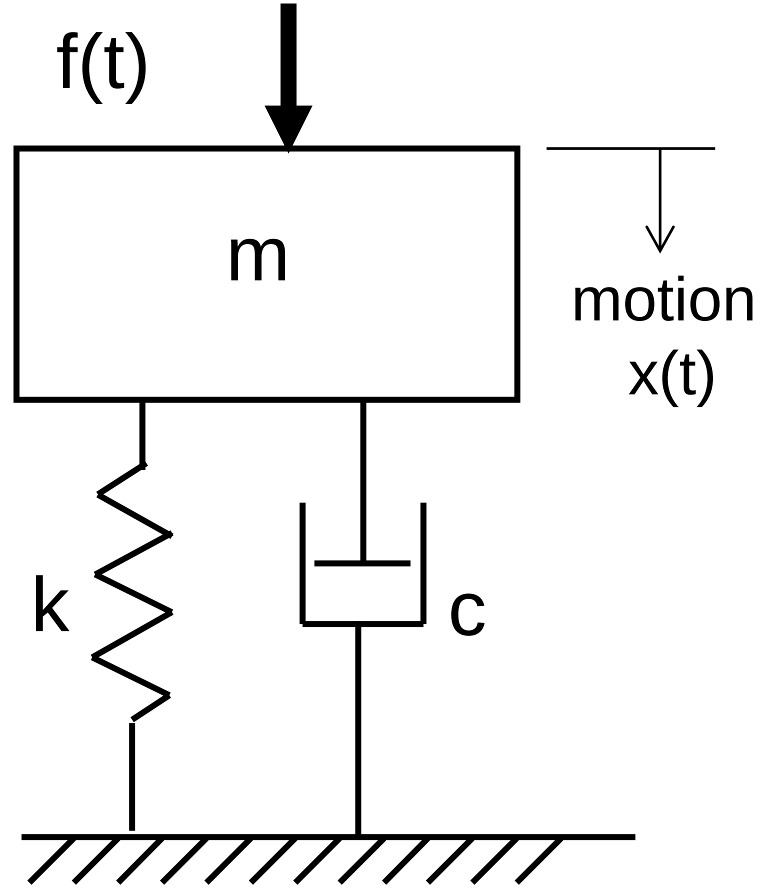
$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} = G(s)$$

Difficult looking 2^{nd} order ODE, converted to quadratic equation

Laplace transforms – Table			
$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \quad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
e^{-at}	$\frac{1}{s+a}$	$\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\cos(\omega t + \theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$\frac{1}{2}t^2e^{-at}$	$\frac{1}{(s+a)^3}$	$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s+a)^n}$	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
e^{at}	$\frac{1}{s-a} \quad s > a$	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2} \quad s > \lvert\omega\rvert$
te^{at}	$\frac{1}{(s-a)^2}$	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2} \quad s > \lvert\omega\rvert$
$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
1	1		$s + a$

Continuous Systems and Transfer Function Revision: Mass Spring Damper

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$



However, we want to write it in a "standard form", not least because that's what it will look like in the Laplace transfer tables! This means the coefficient of the highest order of s on the denominator is 1.

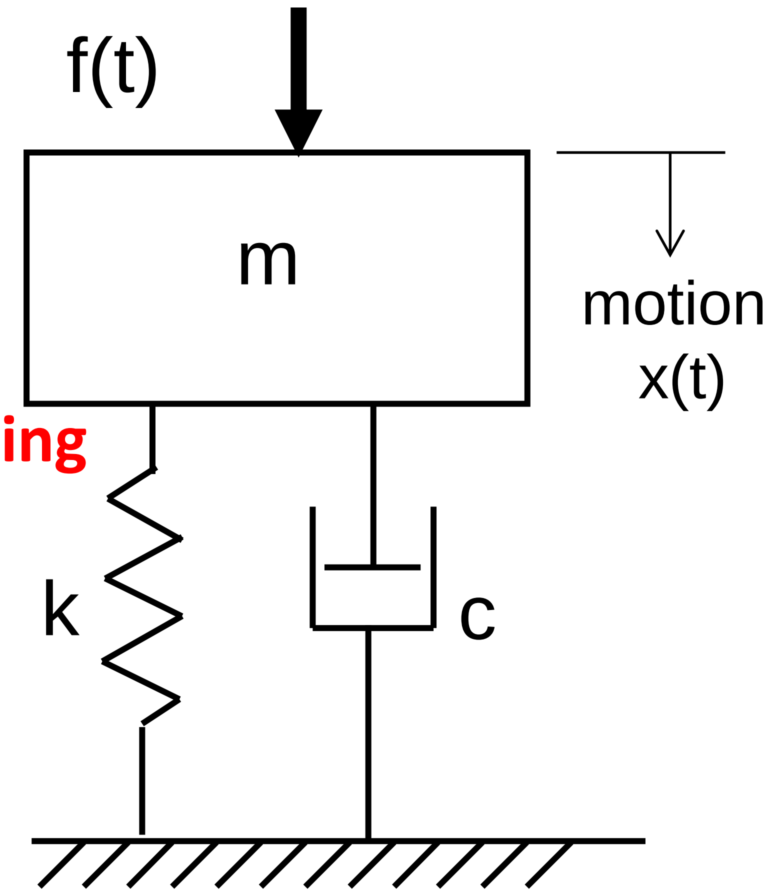
$$\boxed{\frac{\omega_n^{\;2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\;2}}}$$

Continuous Systems and Transfer Function Revision: Mass Spring Damper

$$G(s) = \frac{k/m}{s^2 + c/m\, s + k/m} \boxed{\frac{1}{k}}$$

$$= \frac{\omega_n^{\,2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}} \frac{1}{k}$$

Just a scaling factor
Or gain



Where,

$$\omega_n = \sqrt{\frac{k}{m}}$$ Natural frequency

$$\zeta = \frac{c}{2\sqrt{km}}$$ Damping ratio

These have physical meanings

Continuous Systems and Transfer Function Revision: Second Order systems

The standard form for second order systems is shown below:

$$G(s) = \gamma \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

γ **Gain**

ω_n **Natural Frequency**

ζ **Damping Ratio**

This function is known as a damped oscillator, in that it produces harmonic sinusoidal oscillations which decay over time. This type of system appears *everywhere* in physics, as well as in engineering. Even to the extent that some higher order systems are simplified to become second order, just because it is so well understood.

How do I go back to the time domain?

As before, we use the inverse Laplace transform to get the time domain response. To reiterate, the benefit of standard forms is that the transforms are given in the tables: *(e.g. from Dorf and Bishop)*

$f(t)$	$F(s)$
$\boxed{\dfrac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t, \zeta < 1}$	$\boxed{\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$

γ not shown as it is unaffected by Laplace transform

$$\frac{x(t)}{x(0)} = \gamma \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2 t}\right)$$

Continuous Systems and Transfer Function Revision: Inverse Laplace transformation

$$\frac{x(t)}{x(0)} = \gamma \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2} t\right)$$

This looks complicated, but consider one term at a time

$$\gamma \frac{\omega_n}{\sqrt{1-\zeta^2}}$$

γ ω_n ζ All constants, so whole term is just a number

$$e^{-\zeta\omega_n t}$$

Is an exponential function, which depending on whether $-\zeta\omega_n$ is positive or negative, increases or decays

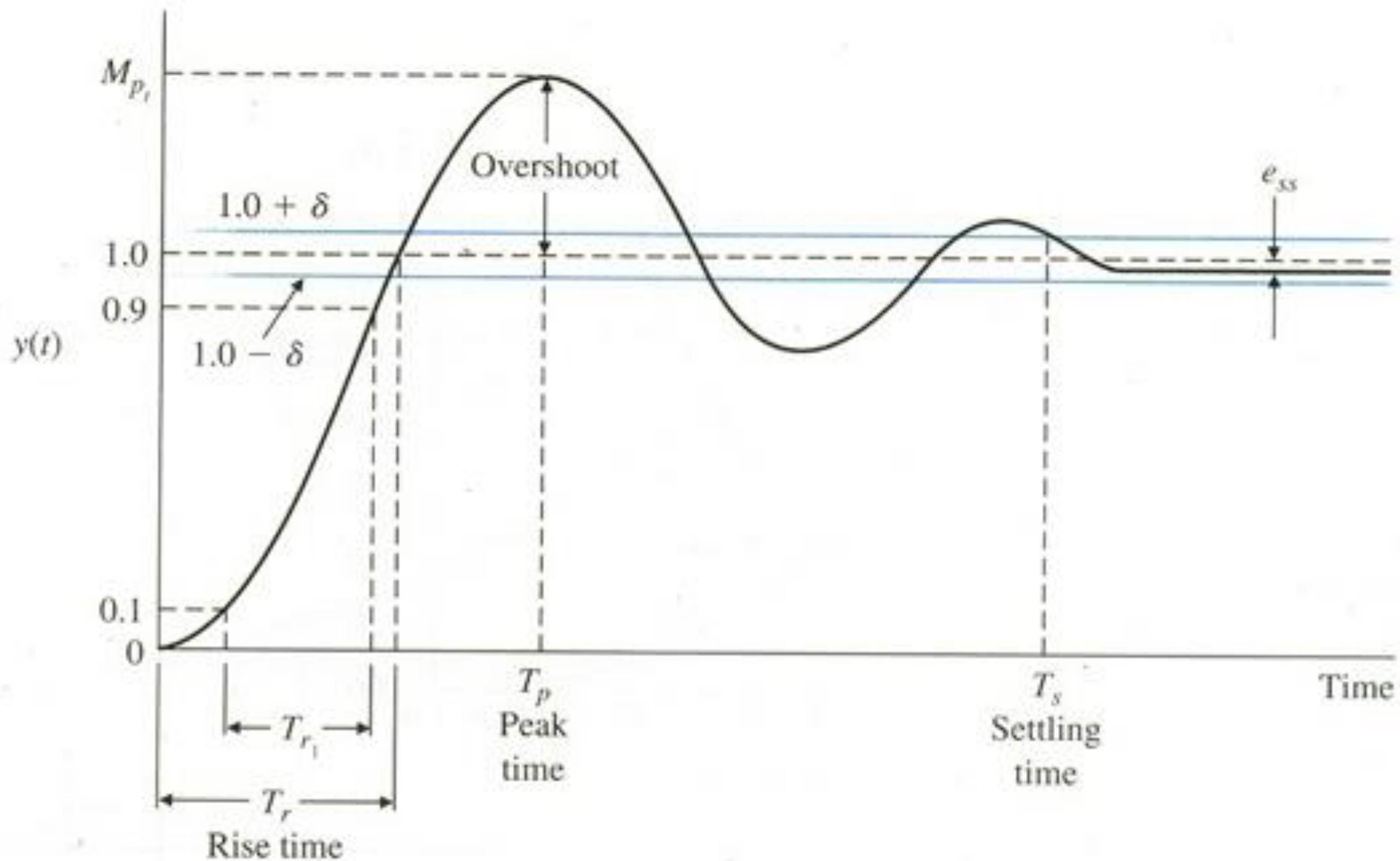
$$\sin\left(\omega_n \sqrt{1-\zeta^2} t\right)$$

ω_n ζ Are just constants, so assuming ζ Less than 1, this is just a sine wave

$$\frac{x(t)}{x(0)} = \gamma \underbrace{\frac{\omega_n}{\sqrt{1-\zeta^2}}}_{\text{Gain}} \underbrace{e^{-\zeta\omega_n t}}_{\text{+ Exponential}} \underbrace{\sin\left(\omega_n \sqrt{1-\zeta^2} t\right)}_{\text{+Sine Wave}}$$

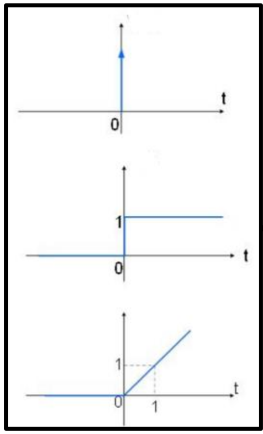
This is assuming $\boxed{\zeta < 1}$

Continuous Systems and Transfer Function Revision: Second Order Systems – Unit Step Response



Continuous Systems and Transfer Function Revision: Time vs frequency domain

Time domain



Input → LTI $F(x)$ → Output

$x(t)$ → $y(t) = x(t) * F(x)$

Laplace transforms – Table			
$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \quad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$

$\mathcal{L}aplace$ $\quad$ $\mathcal{L}aplace$ $\quad$ inverse $\mathcal{L}aplace$

Frequency domain

$$1$$

$$\frac{1}{s}$$

$$\frac{1}{s^2}$$

Input → LTI $F(s)$ → Output

$X(s)$ → $Y(s) = X(s)F(s)$

1st order

$$\frac{X(s)}{Y(s)} = \frac{\alpha}{(1 + Ts)} = \frac{\gamma}{1 + \tau s}$$

2nd order

$$G(s) = \gamma \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Continuous Systems and Transfer Function Revision: Laplace transforms example

The input-output relationship of a certain system is described by a differential equation shown below. Find the response when ***x(t)*** is a step function of 10 units applied at ***t = 0*** and when the initial conditions are ***y(0) = 2, y'(0) = -10***.

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = x$$

Apply Laplace transforms to both sides of the equation and then solve using partial fraction expansion.

Steps:

- Apply Laplace transforms to both sides of the equation
- Rearrange to solve for $L[y(t)]$
- Use partial fractions to break the expression into components to which we can easily apply the inverse Laplace transform.

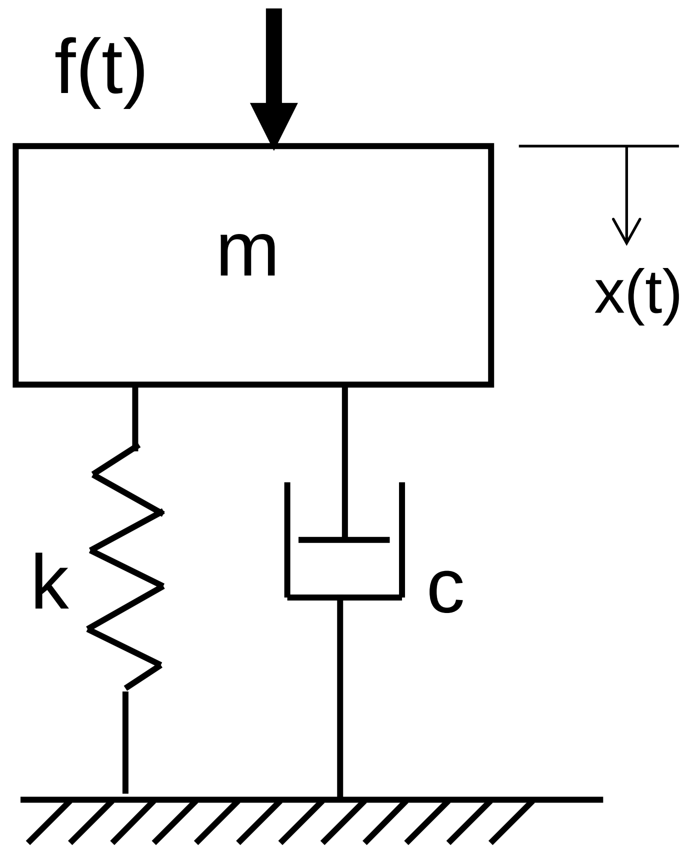
To account for initial conditions: $L[x'] = sX - x(0)$ $\quad$ $L[x''] = s^2X - sx(0) - x'(0)$

Effect of damping ratio on the transfer function of a 2nd order systems

Continuous Systems and Transfer Function Revision: 2nd Order Transfer Function

The standard form for second order systems is shown below:

$$G(s) = \gamma \frac{\omega_n^{2}}{s^2 + 2\zeta\omega_n s + \omega_n^{2}}$$



γ **Gain**

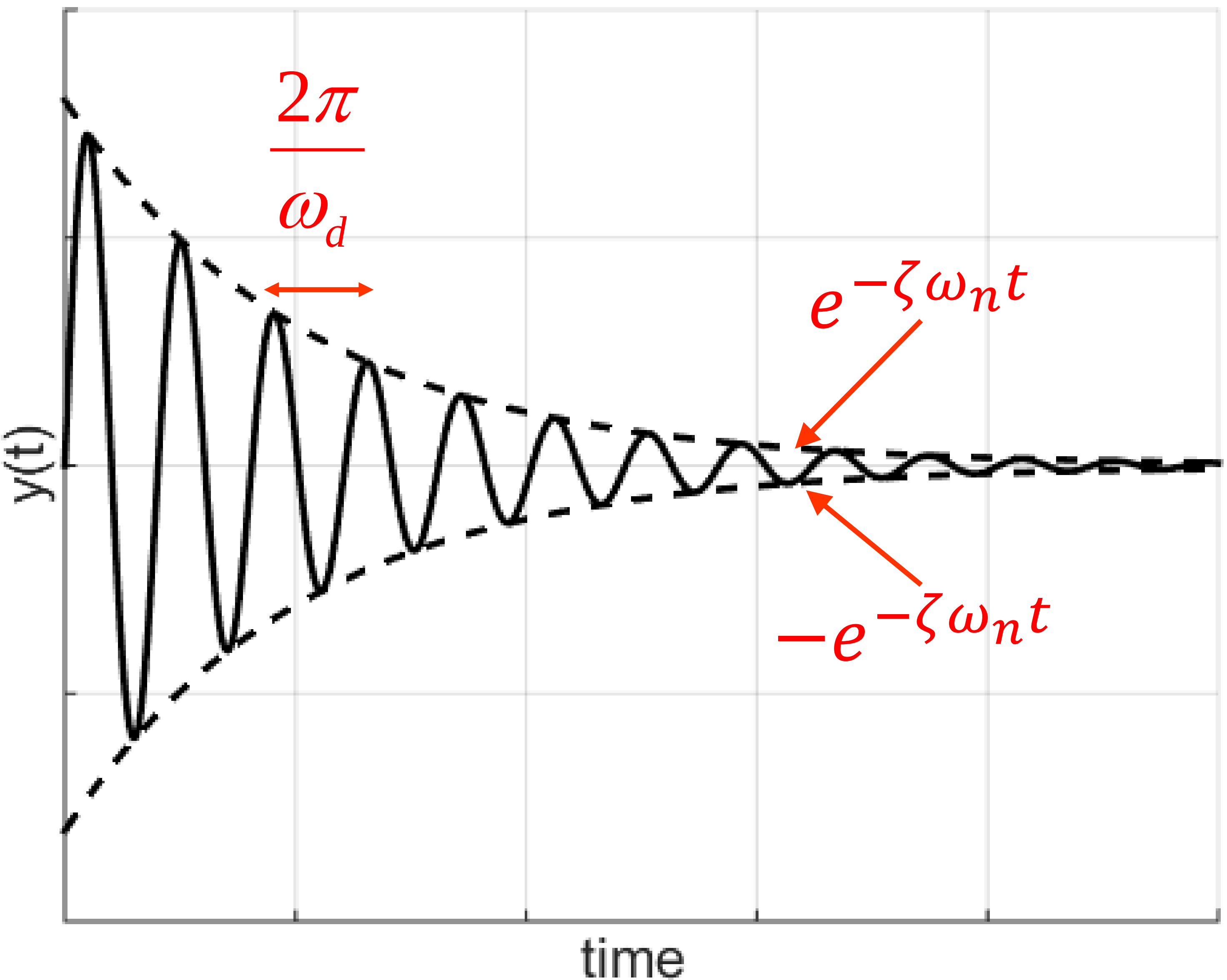
ω_n **Natural Frequency**

ζ **Damping Ratio**

Luckily, this equation is a "standard form" and the time domain result is given directly in Laplace Tables:

$$\gamma \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2} t\right)$$

Continuous Systems and Transfer Function Revision: 2nd Order Transfer Function



But this is only for $\zeta < 1$

Given the damping ratio for a mass spring damper

$$\zeta = \frac{c}{2\sqrt{km}} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

Its clear that through choice of *m, k,* and *c* you could create a system with

$$\zeta \geq 1$$

$$G(s) = \gamma \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

The denominator determines the behaviour of the transfer function, as the numerator is a constant. This is known as the *characteristic equation*. It makes life much simpler if we first try to factor the denominator into first order terms like (s+a)(s+b)

So we require the roots of the equation $s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}$

Which can be determined from the quadratic equation:

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Thus it is clear that the roots of the characteristic equations *and thus the behaviour of the transfer function* is determined by the damping ratio

Overdamped	$\zeta > 1$	$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
Critically Damped	$\zeta = 1$	$s = -\omega_n$
Underdamped	$0 < \zeta < 1$	$s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$
Undamped	$\zeta = 0$	$s = \pm j\omega_n$

Normally, only the underdamped case is given directly in the Laplace tables, so we need a bit of manipulation to get the others

Overdamped $\zeta > 1$

The characteristic equation has two real and negative roots:

$$s = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} = -\alpha,$$

$$s = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} = -\beta,$$

Note:

$$\alpha\beta = \omega_n^2$$

$$\frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}} = \frac{\alpha\beta}{(s+\alpha)(s+\beta)}$$

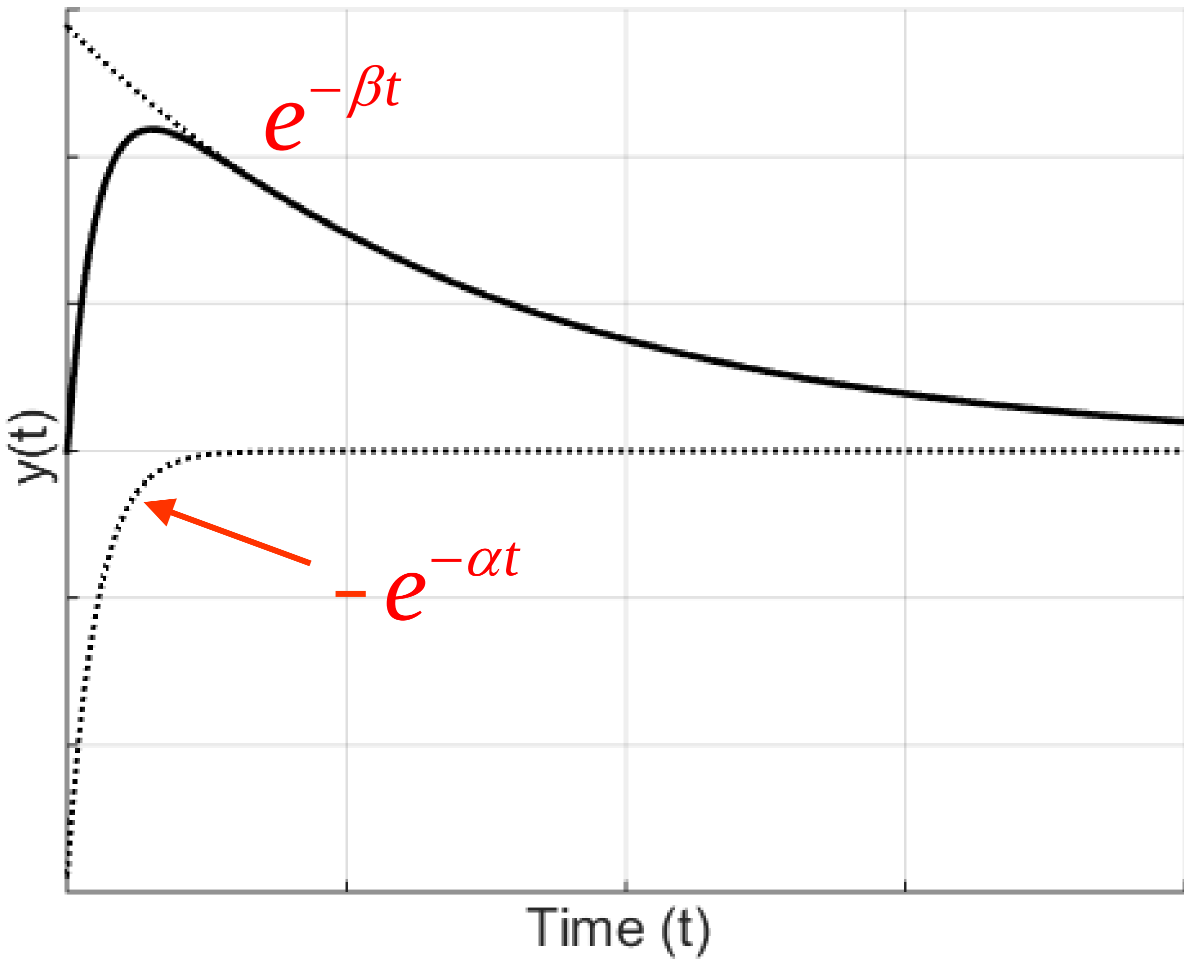
Overdamped $\zeta > 1$

Using partial fractions we can split this into two separate first order systems:

$$\left(\frac{\alpha\beta}{\beta-\alpha}\right)\frac{1}{S+\alpha}+\left(\frac{\alpha\beta}{\alpha-\beta}\right)\frac{1}{S+\beta}$$

which gives the following in the time domain:

$$x(t)=\alpha\beta\left(\boxed{\frac{e^{-\alpha t}}{\beta-\alpha}}+\boxed{\frac{e^{-\beta t}}{\alpha-\beta}}\right)$$



β Term dominates

There is *no* sinusoidal term anymore, the response is only a combination of exponential decays, a more complex but similar response to a first order system

Critically damped $\zeta = 1$

Special case with a single repeated root $s = -\omega_n$

$$\frac{\omega_n^{\,2}}{s^2 + 2\omega_n s + \omega_n^{\,2}} = \frac{\omega_n^{\,2}}{\left(s + \omega_n\right)^2}$$

To get the time response, we first take the numerator outside:

$$L^{-1}\left[\frac{\omega_n^{\,2}}{\left(s + \omega_n\right)^2}\right] = \omega_n^2 L^{-1}\left[\frac{1}{\left(s + \omega_n\right)^2}\right]$$

Critically damped $\zeta = 1$

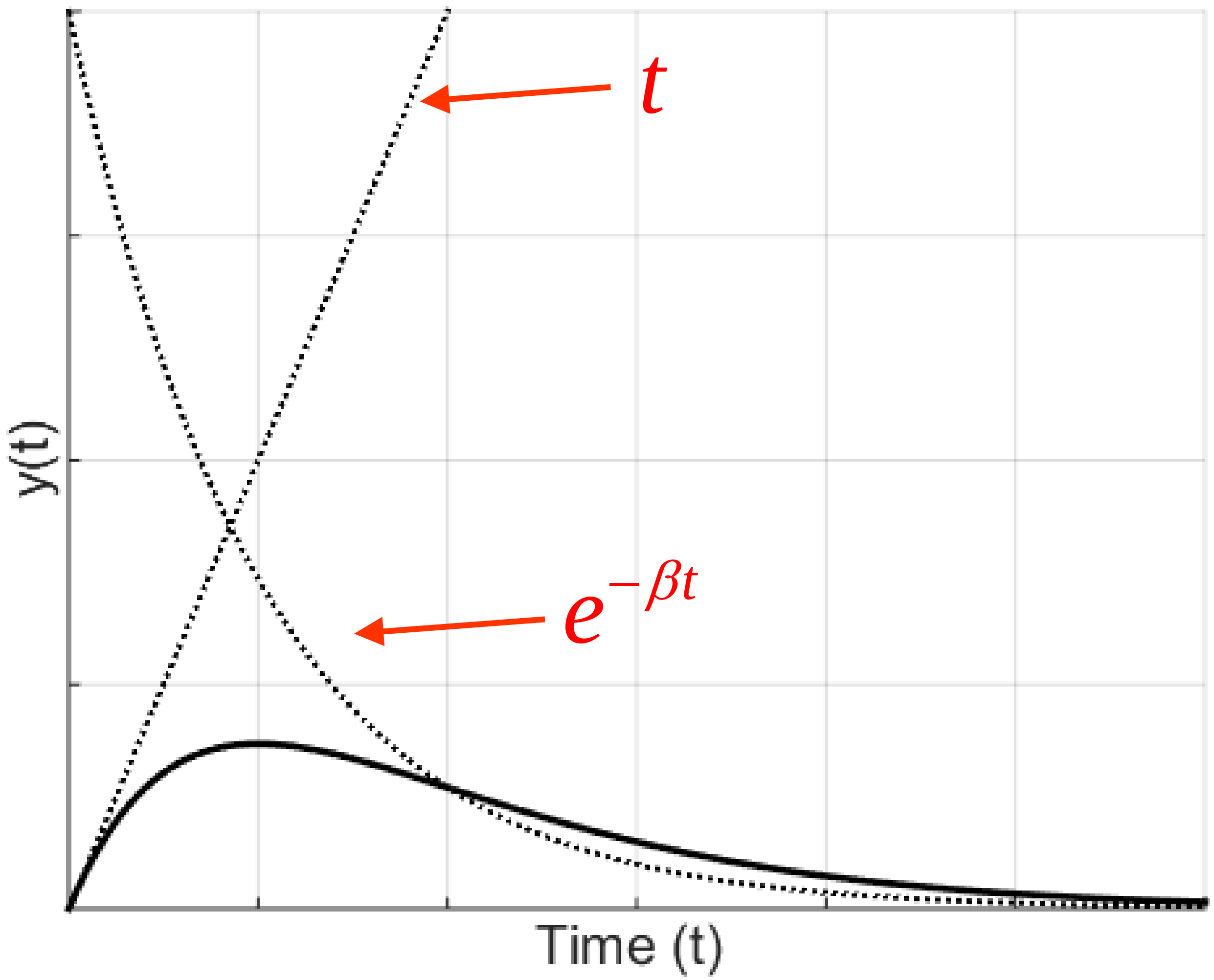
We can then use the inverse Laplace rule

if $L^{-1}\left[F(s)\right] = f(t)$ then $L^{-1}\left[F(s-a)\right] = e^{at} f(t)$

$$\omega_n^2 L^{-1}\left[\frac{1}{\left(s+\omega_n\right)^2}\right] = \omega_n^2 e^{-\omega_n t} L^{-1}\left[\frac{1}{s^2}\right]$$

$\frac{1}{s^2}$ is the standard form of a ramp, or more generally from the Laplace tables: $L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$

Critically damped $\zeta = 1$



Result is an exponential decay, multiplied by a ramp $= \omega_n^2 e^{-\omega_n t} t$

Undamped $\zeta = 0$

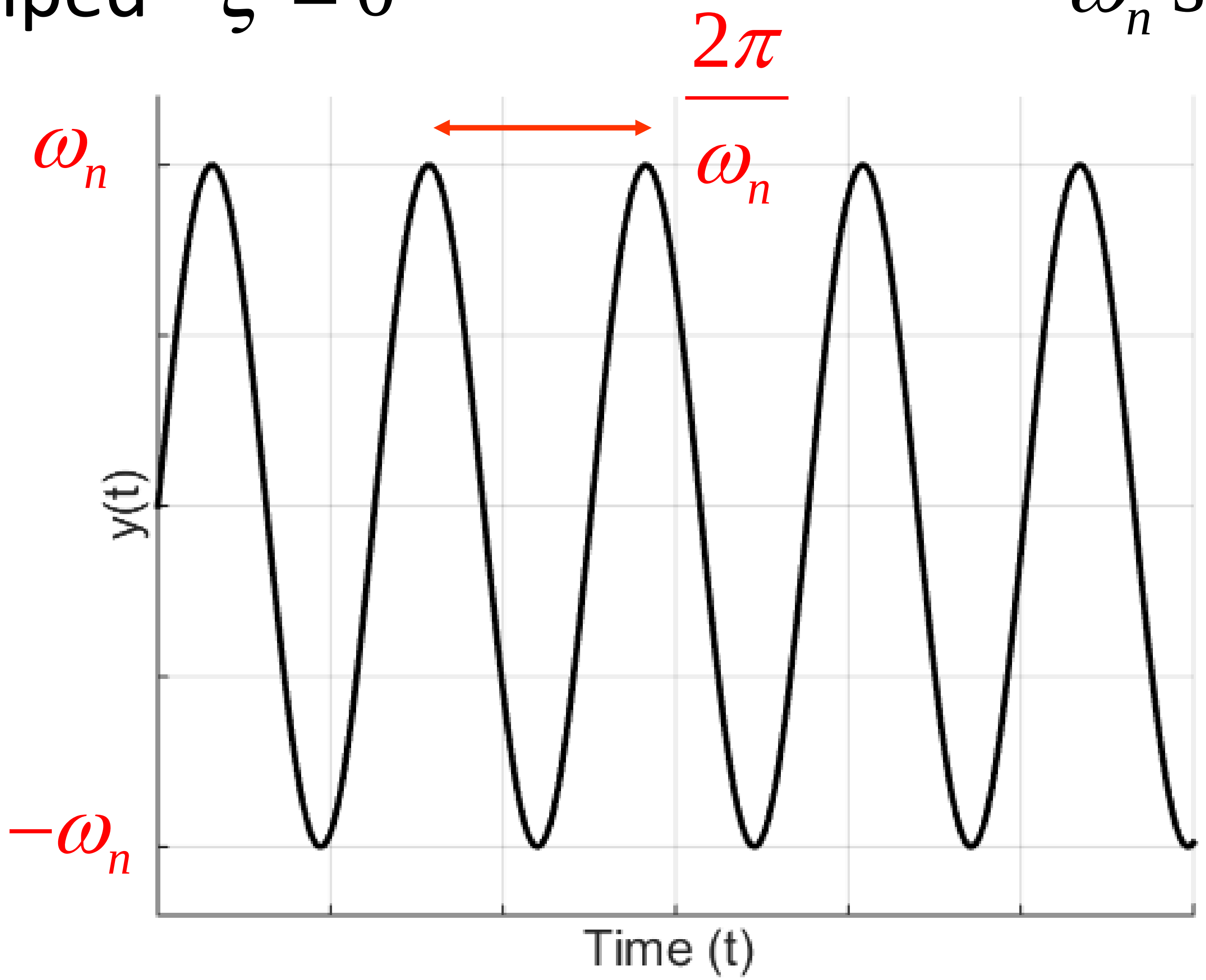
Another special case with two imaginary roots $s = \pm j\omega_n$

$$\frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}} = \frac{\omega_n^{\ 2}}{s^2 + \omega_n^{\ 2}}$$

Which, using the standard form for an oscillator is

$$\omega_n L^{-1}\left[\frac{\omega_n}{s^2 + \omega_n^{\ 2}}\right] = \omega_n \sin(\omega_n t)$$

Undamped $\zeta = 0$ $\omega_n \sin(\omega_n t)$



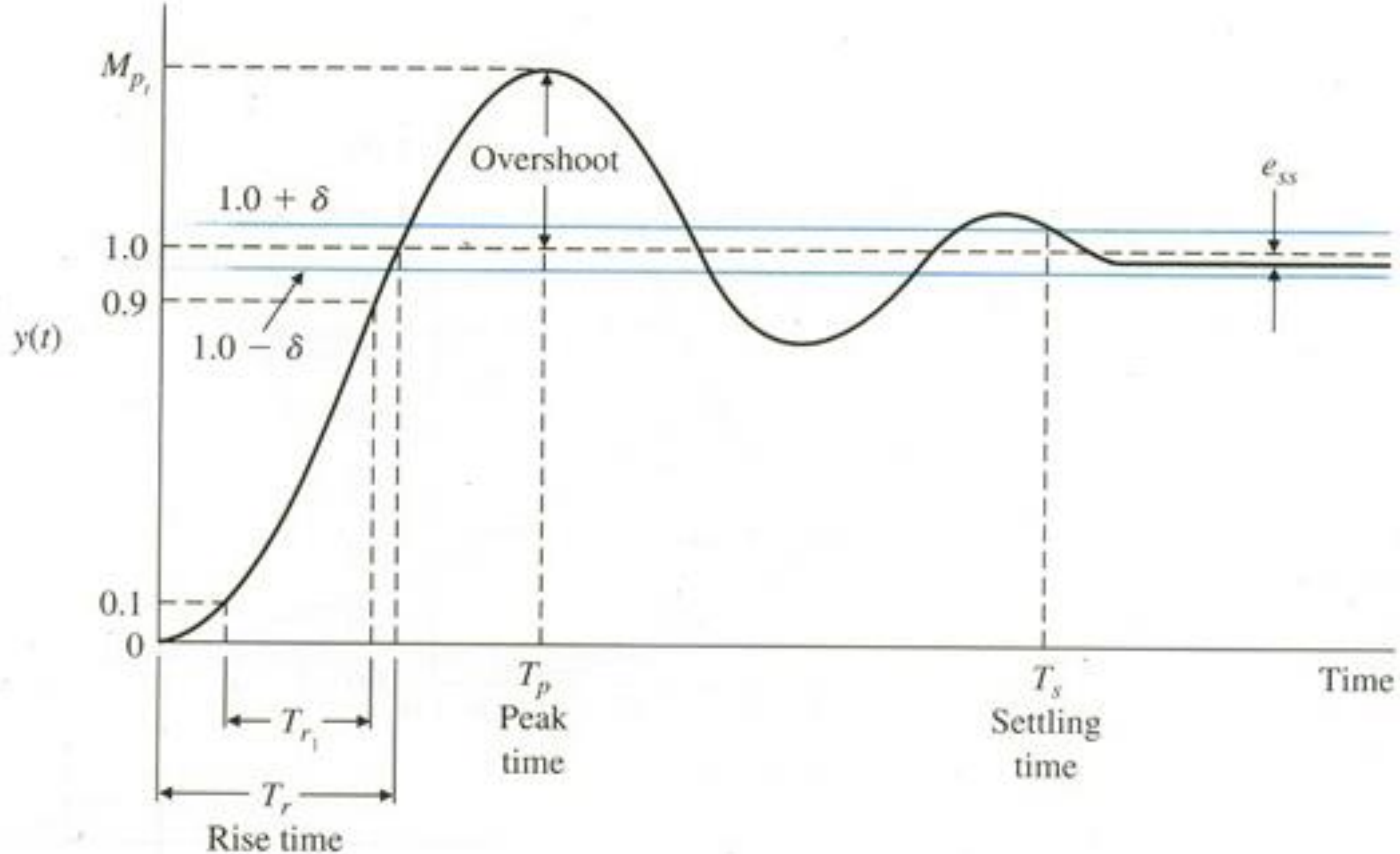
So the response is now just a sinusoid, with no exponential components

Unlike the first order system, the response of the second order system can vary considerably depending upon the parameters:

Overdamped	$\zeta > 1$	Addition of two exp. decays
Critically Damped	$\zeta = 1$	Ramp multiplied by exp. decay
Underdamped	$0 < \zeta < 1$	Exp. Decaying sinusoid
Undamped	$\zeta = 0$	Sinusoid

Applying a step input to a 2nd order systems - Damping

Continuous Systems and Transfer Function Revision: Second Order Systems – Unit Step Response



$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Thus it is clear that the roots of the characteristic equations *and thus the behaviour of the transfer function* is determined by the damping ratio

Overdamped	$\zeta > 1$	$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
Critically Damped	$\zeta = 1$	$s = -\omega_n$
Underdamped	$0 < \zeta < 1$	$s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$
Undamped	$\zeta = 0$	$s = \pm j\omega_n$

Normally, only the underdamped case is given directly in the Laplace tables, so we need a bit of manipulation to get the others

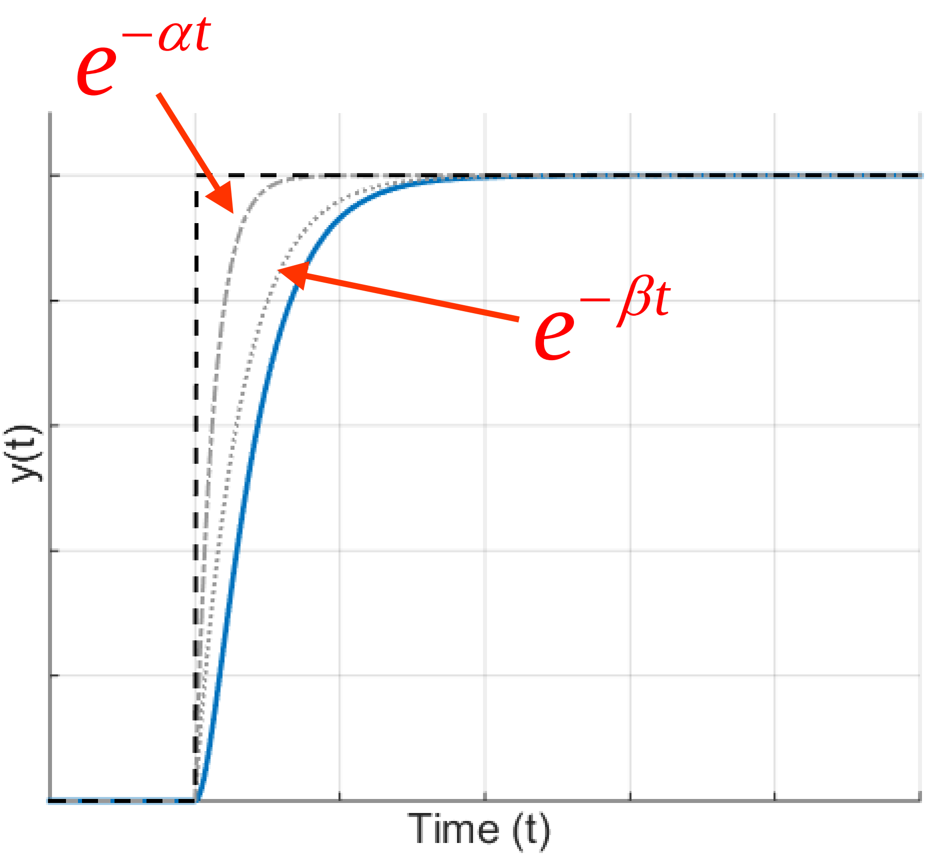
Previously we saw for a step of *A* the output can be calculated from

$$Y(s) = \frac{A}{s} G(s) \qquad Y(s) = \frac{1}{s} \cdot \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

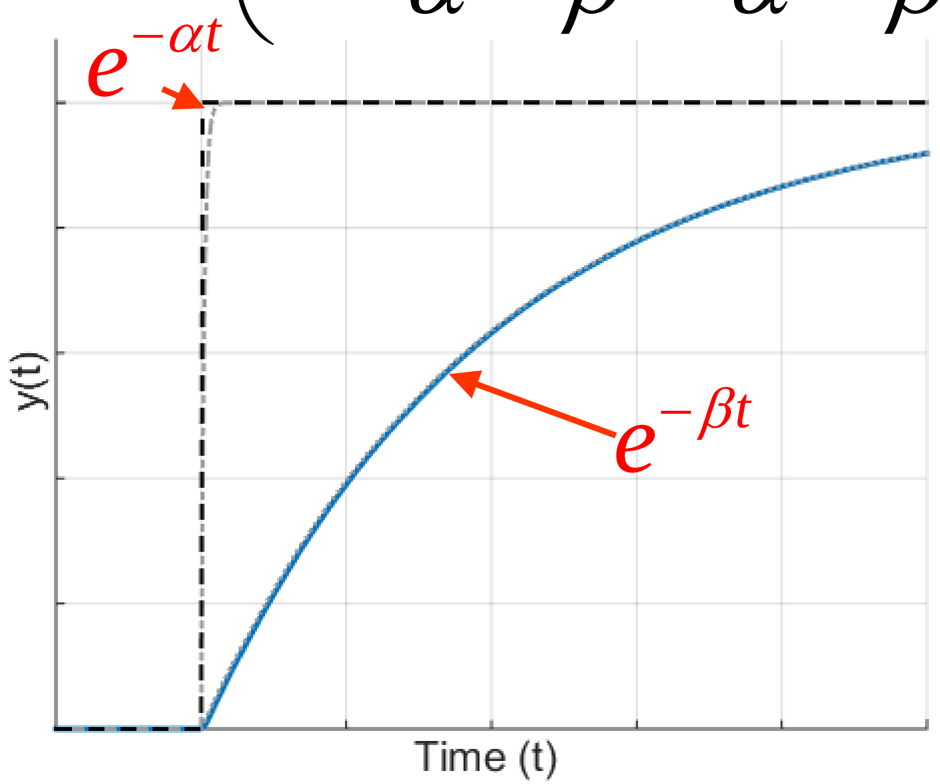
All the following can be found through partial fraction decomposition, and the standard Laplace tables.

Overdamped $\zeta > 1$

$$y(t) = \left(1 + \frac{\beta e^{-\alpha t}}{\alpha - \beta} - \frac{\alpha e^{-\beta t}}{\alpha - \beta}\right)$$



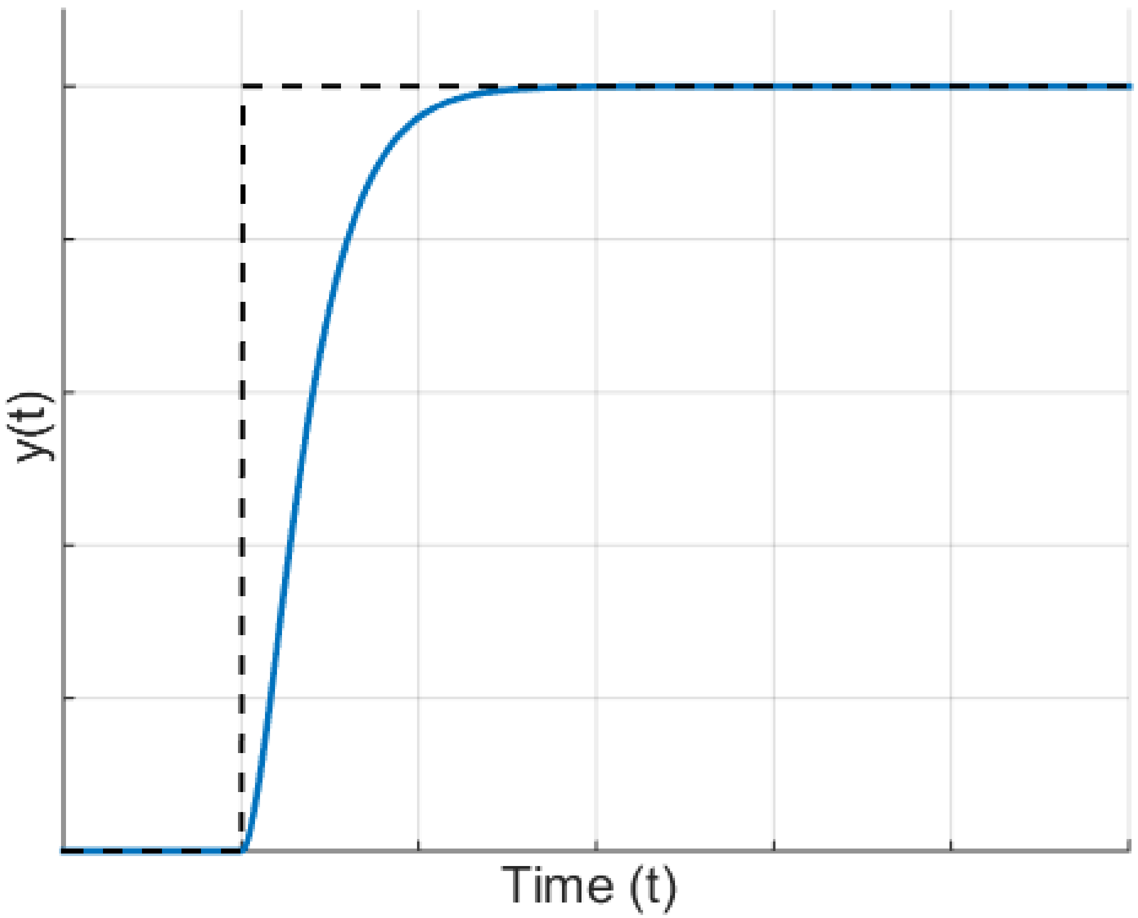
Low ζ



High ζ

The slowest rise from the β exponential dominates, as ζ increases the α term becomes negligible, and system becomes first order

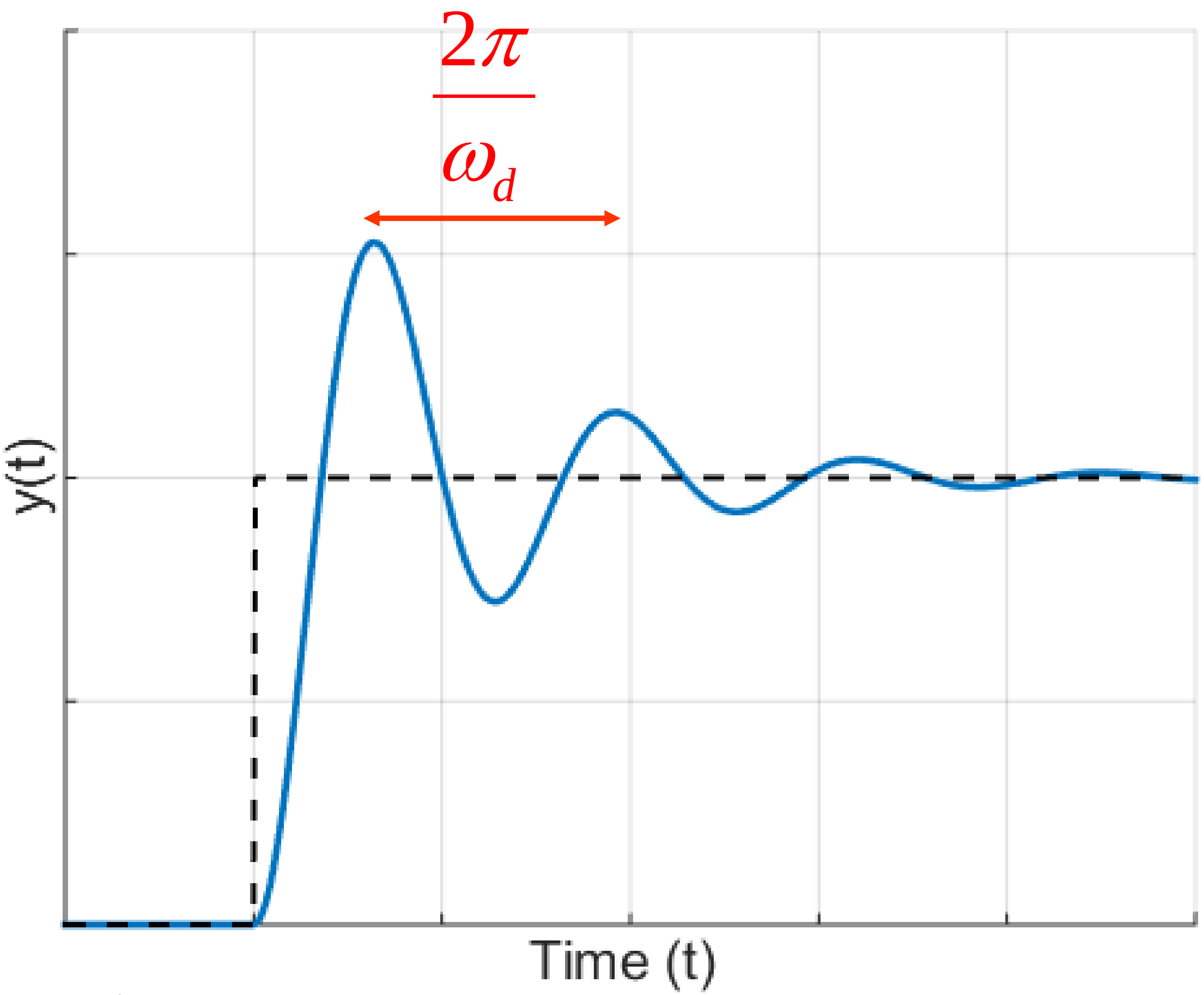
Critically Damped $\zeta = 1$ $y(t) = \left(1 - e^{-\omega_n t} - \omega_n e^{-\omega_n t} t\right)$



Fastest possible rise without oscillating

Underdamped

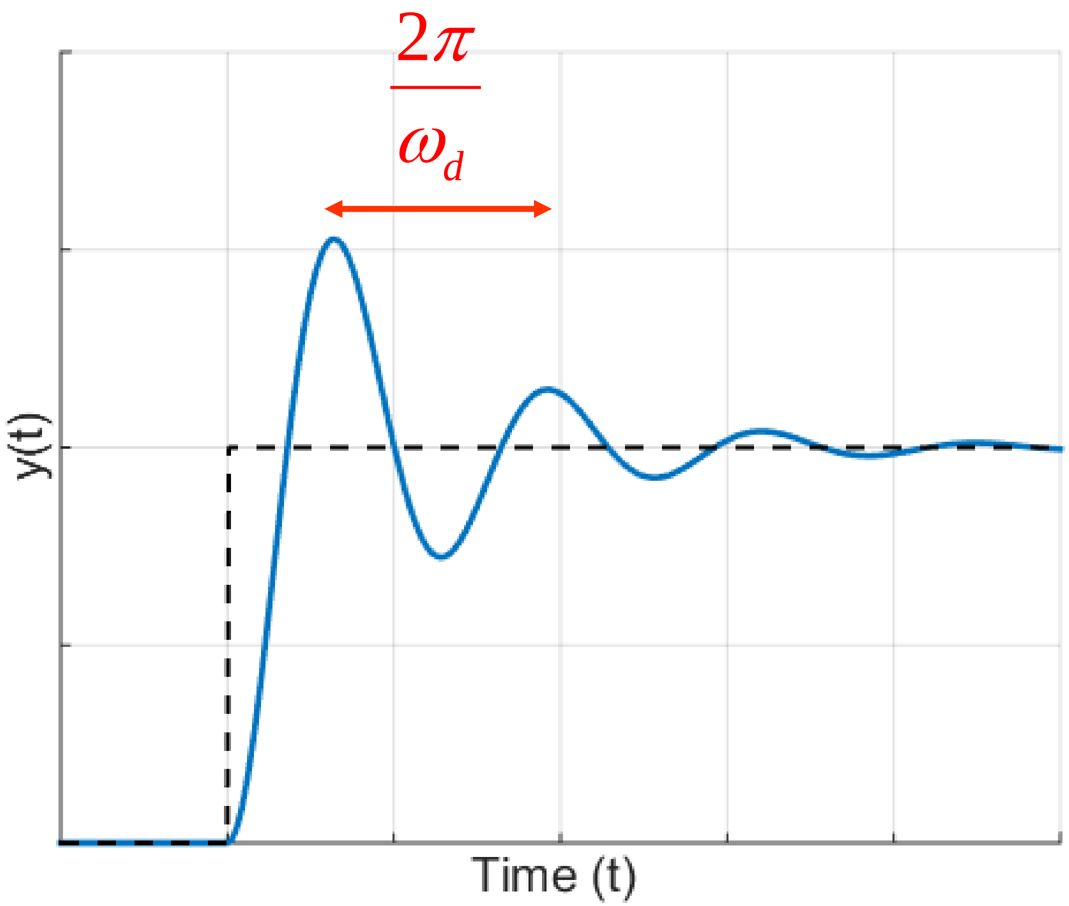
$$0 < \zeta < 1$$



$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \cos^{-1}(\zeta)\right)$$

Underdamped

$0 < \zeta < 1$



Or is sometimes written in terms of the *damped natural frequency*

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_d t + \cos^{-1}(\zeta)\right)$$

Undamped $\zeta = 0$

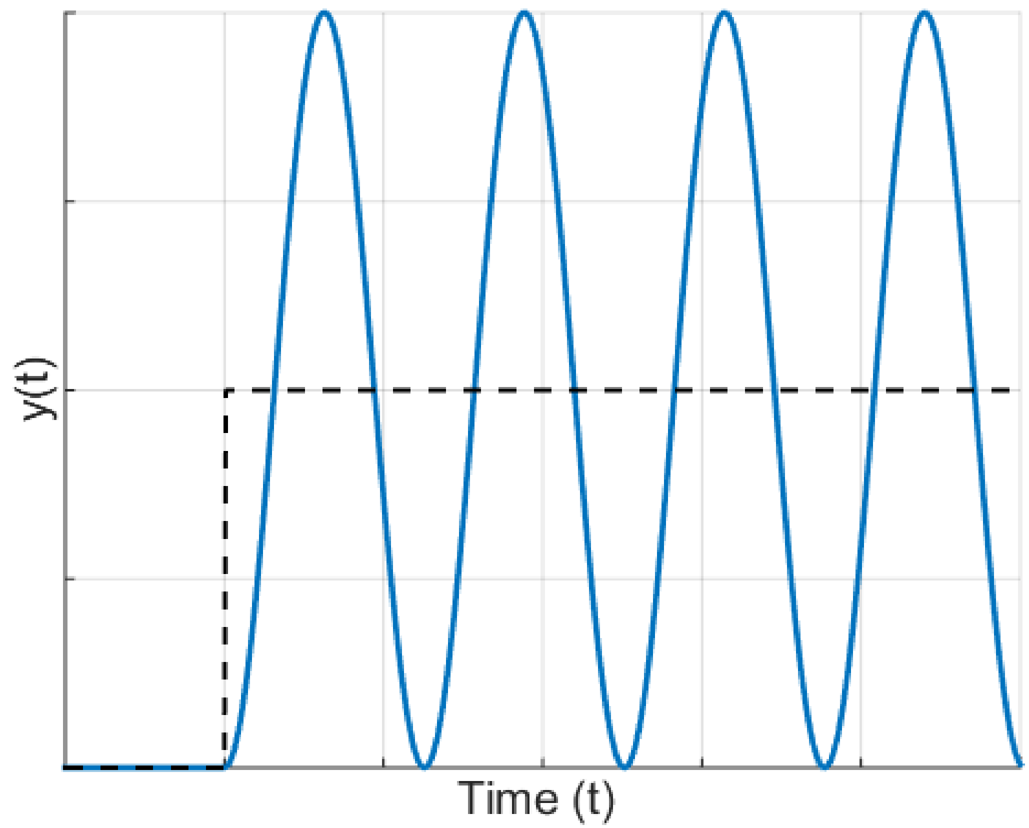
$$y(t) = \left(1 - \cos\left(\omega_n t\right)\right)$$

As damping ratio reaches zero, the oscillations no longer decay.

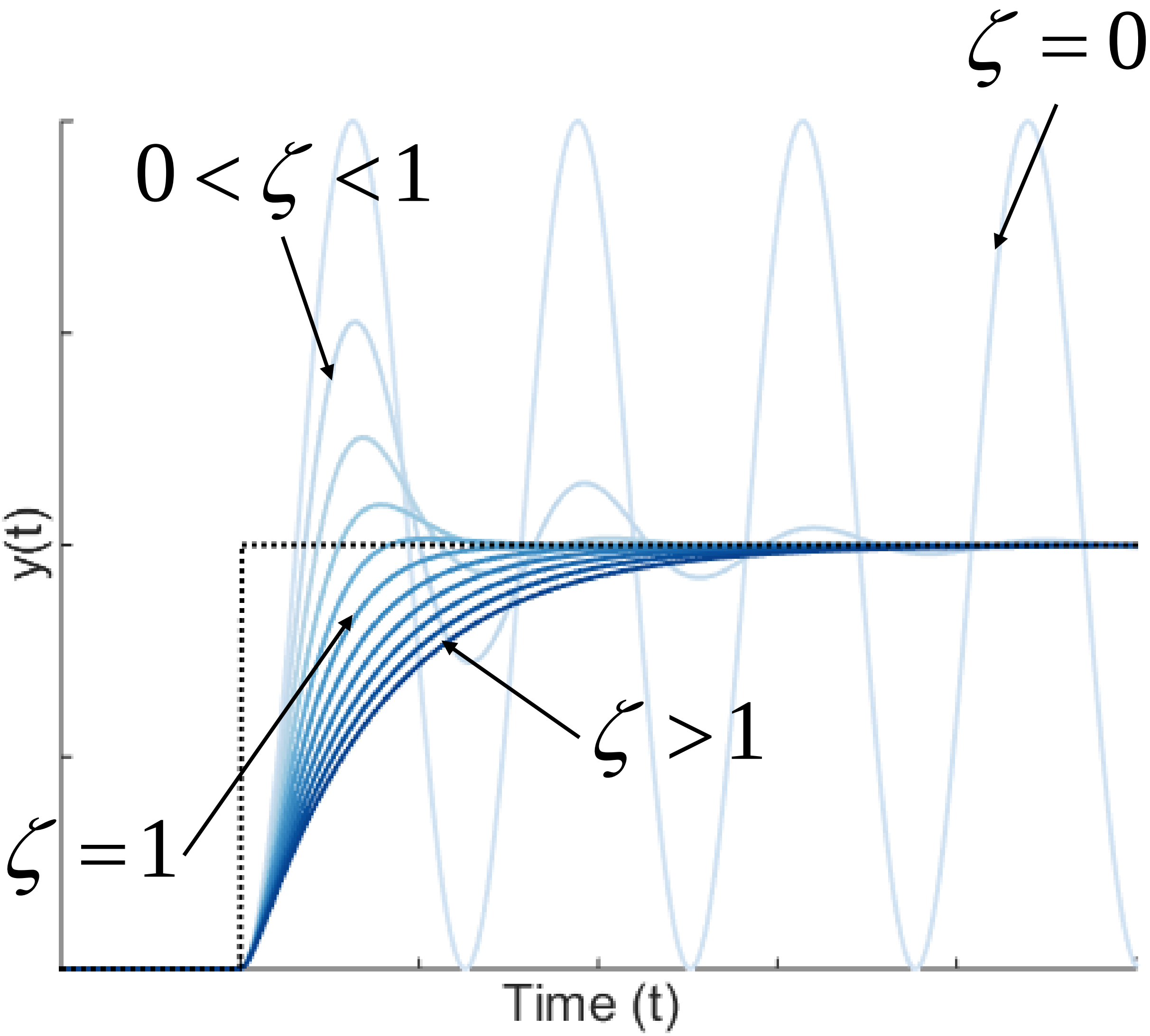
$$\omega_d = \omega_n \sqrt{1-0} = \omega_n$$

$$\cos^{-1}(-1) = \pi$$

So oscillations at natural frequency with pi phase shift



Summary: Step input response of 2nd order systems



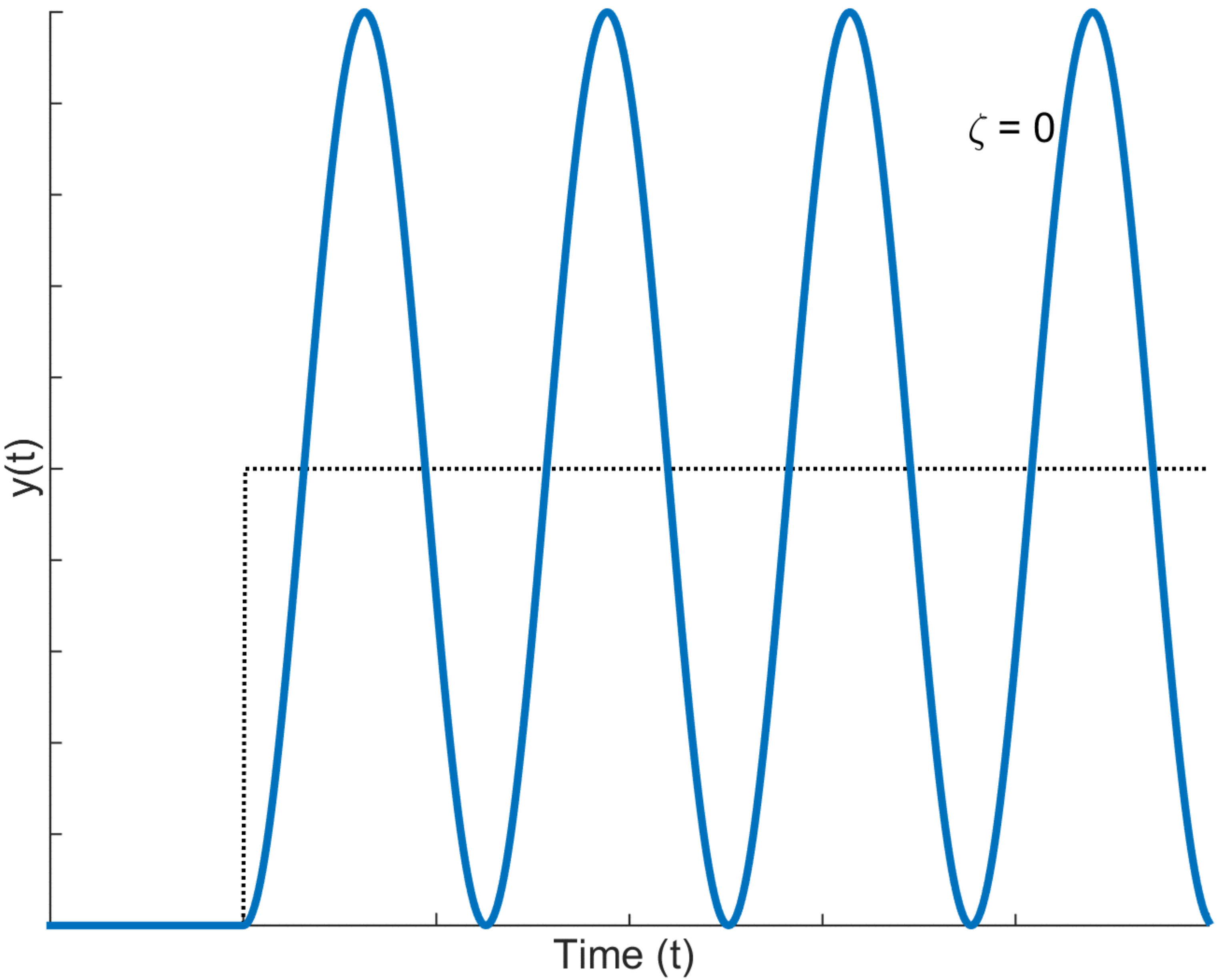
ζ=0 : undamped – Sustained oscillations

ζ<1 : under-damped – Exp Decaying Oscillatory response

ζ=1 : critically damped – Fastest response with no overshoot

ζ>1 : over-damped – cannot overshoot

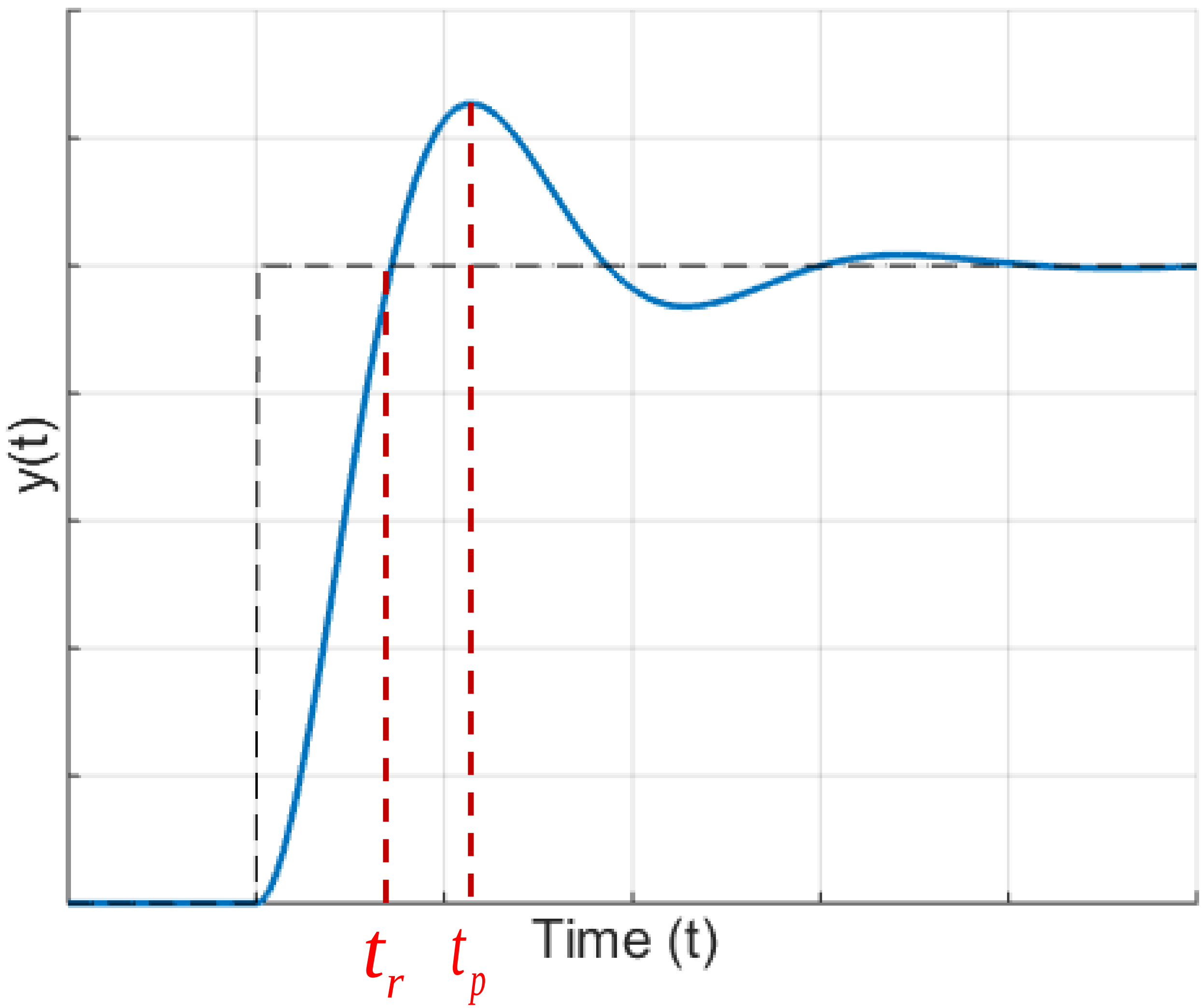
Continuous Systems and Transfer Function Revision: Summary: Step input responses of a 2nd order systems



Underdamped step input responses

Rise Time t_r
Time to reach steady state value (for first time)

Peak Time t_p
Time to initial overshoot



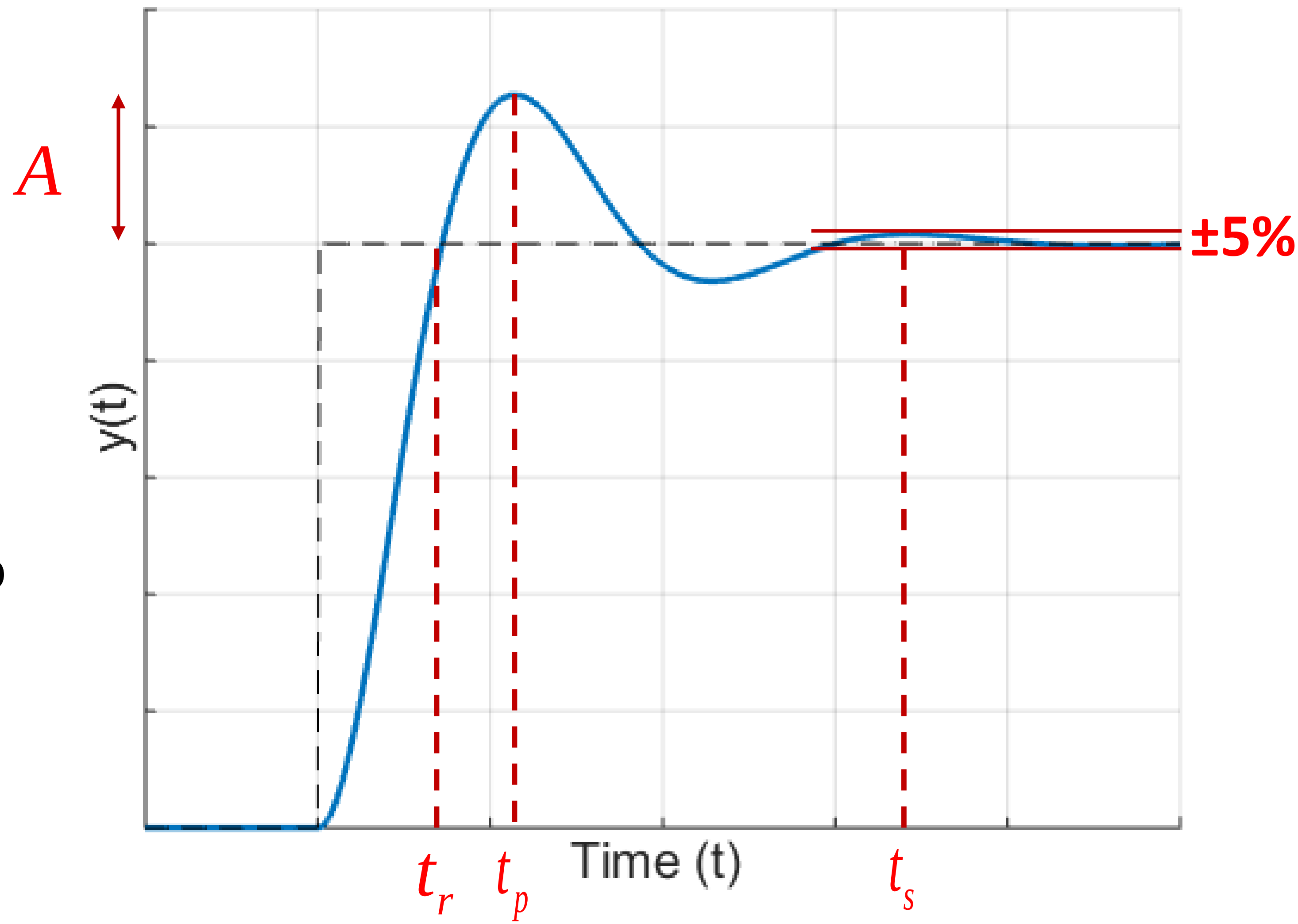
Peak overshoot

Value of overshoot above steady state

$$y(t_p) - y_{ss} = A$$

Settling Time t_s

Time for response to reach and maintain a certain ratio of the steady state value. Commonly ±5% or ± 2%



These criteria can be read from the graph, but having equations for them allows for system parameters ζ and ω_n to be estimated. It is also useful for quantifying the effect of ζ on the response.

Rise Time

This can be found by setting the response *y* to 1 and then rearranging:

$$t_r = \frac{1}{\omega_d}\left(\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

So a higher damping ratio gives a slower response, but lower damping ratios give a more oscillatory response. Also, that for *overdamped* systems where ζ > 1, the rise time is infinite.

Peak Time

Peak time occurs at half a period of oscillation, or can be found from finding first minima by setting derivative to zero

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

Inserting the above expression into the underdamped response :

$$1 + A = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\pi/\sqrt{1-\zeta^2}} \sin\left(\pi + \cos^{-1}(\zeta)\right)$$

$$A = \frac{\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}} e^{-\zeta\pi/\sqrt{1-\zeta^2}} = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

Often this is given as a percentage:

$$A = 100 e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

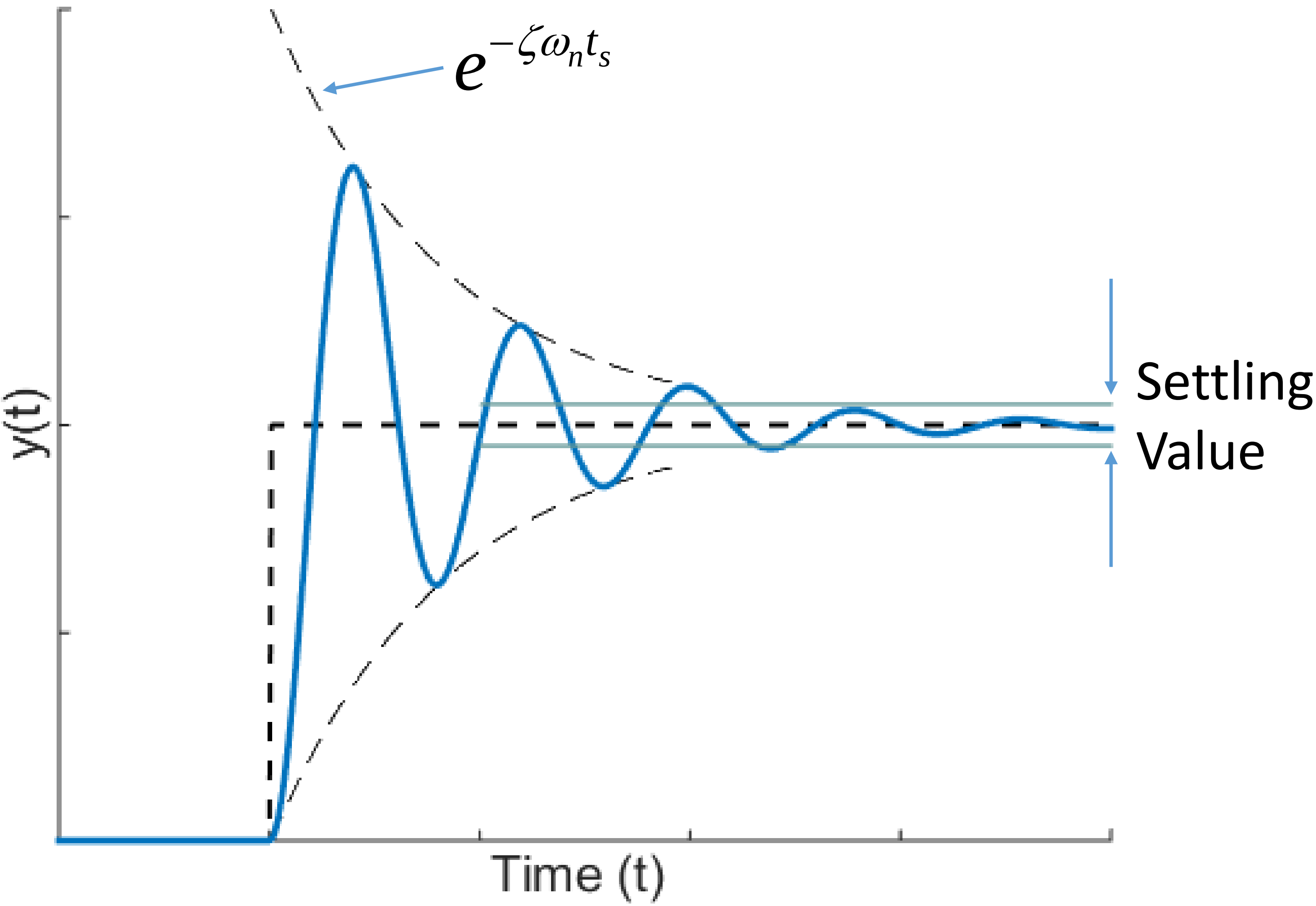
This can be rearranged to find the required damping ratio for a required overshoot

$$\zeta = \sqrt{\frac{\left(\ln\left(A/100\right)\right)^2}{\pi^2 + \left(\ln\left(A/100\right)\right)^2}}$$

$$A/100 \to 0$$

$$\zeta \to 1$$

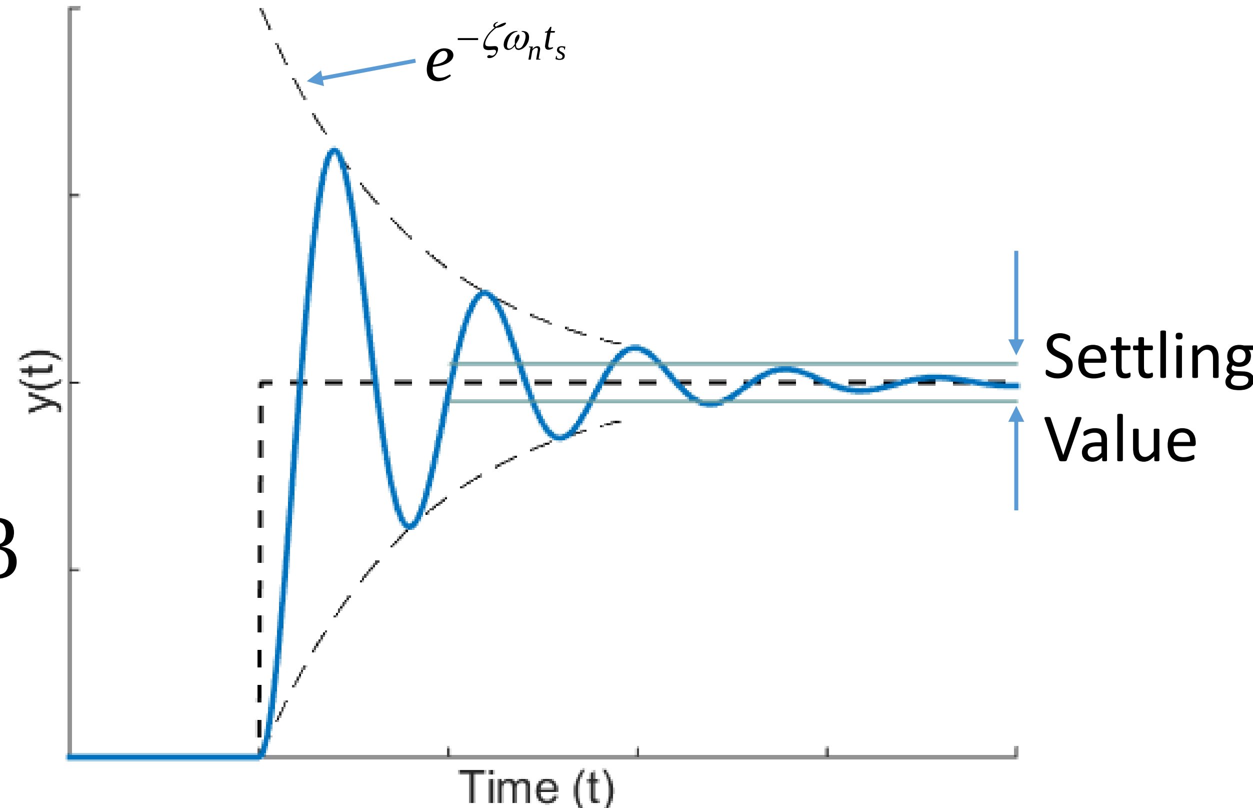
The envelope of the oscillation is described by the decaying exponential term, so to find the settling time, we can just consider this term alone



So for example, the common target is 5%:

$$e^{-\zeta\omega_n t_s} = 0.05$$

$$-\zeta\omega_n t_s = \ln(0.05) = -3$$



$$t_s(5\%) = \frac{3}{\zeta\omega_n}$$

Or similarly $t_s(2\%) = \dfrac{4}{\zeta\omega_n}$

So settling time increases with a *decreasing* damping ratio, for underdamped systems only

Understanding Poles and Zeros

Continuous Systems and Transfer Function Revision: System Poles and Zeros

- As defined, the transfer function is a rational function in the complex variable $s = \sigma + j\omega$, that is

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

- It is often convenient to factor the polynomials in the numerator and denominator, and to write the transfer function in terms of those factors:

$$G(s) = \frac{N(s)}{D(s)} = K \frac{(s - z_1)(s - z_2) \ldots (s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2) \ldots (s - p_{n-1})(s - p_n)}$$

- where the numerator and denominator polynomials, $N(s)$ and $D(s)$, have real coefficients defined by the system's differential equation and $K = \frac{b_m}{a_n}$.

Continuous Systems and Transfer Function Revision: System Poles and Zeros

- Poles and zeros are found by

$$N(s) = 0 \text{ and } D(s) = 0$$

- All of the coefficients of polynomials $N(s)$ and $D(s)$ are real, therefore the poles and zeros must be either purely real, or appear in complex conjugate pairs.

$$\boxed{\text{The poles and zeros are properties of the transfer function, and therefore of the differential equation describing the input-output system dynamics. Together with the gain constant, they completely characterize the differential equation, and provide a complete description of the system.}}$$

System Poles and Zeros - Example

- A linear system is described by the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 3\frac{du}{dt} + 12u$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y\right\} = s^2Y(s) + 2sY(s) + 5Y(s)$$

$$\mathcal{L}\left\{3\frac{du}{dt} + 12u\right\} = 3sU(s) + 12U(s)$$

$$N(s) = 3s + 12 = 0 \Rightarrow z_1 = -4$$

$$ax^2 + bx + c{=}0$$

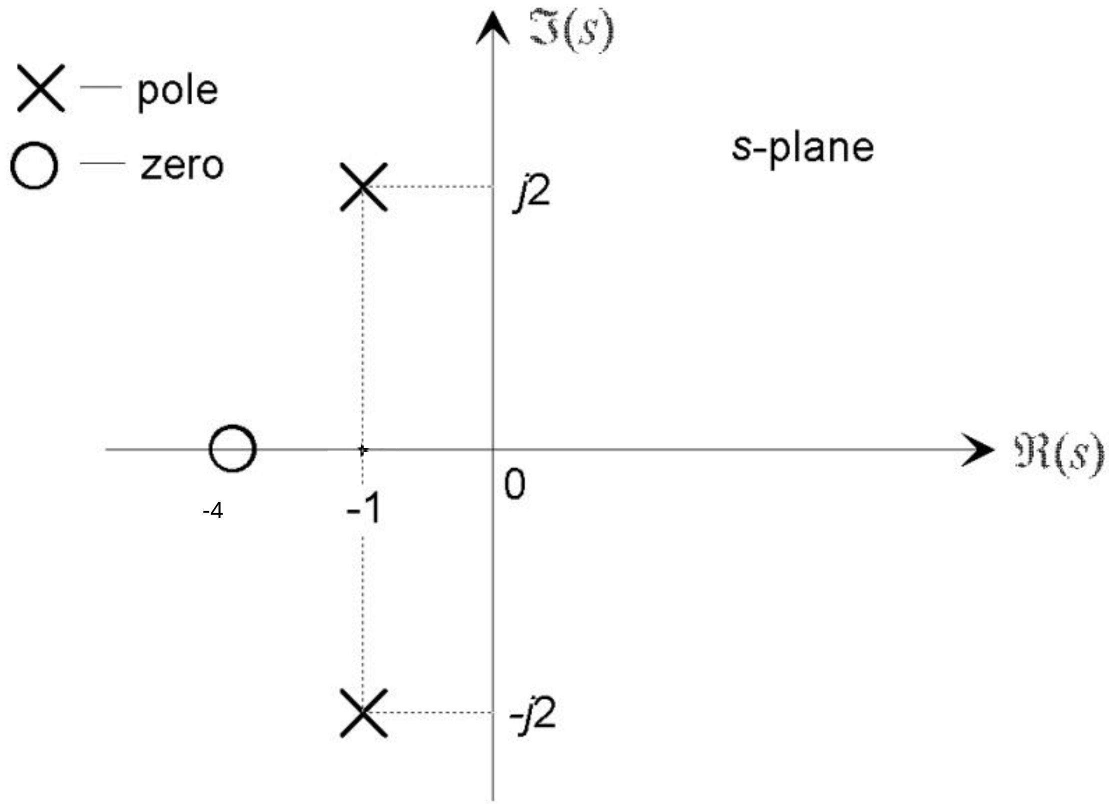
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D(s) = s^2 + 2s + 5 \Rightarrow p_{1/2} = \frac{-2 \pm \sqrt{2^2 - 4 \times 5}}{2} = -1 \pm j2$$

$$G(s) = \frac{N(s)}{D(s)} = 3\frac{(s - (-4))}{(s - (-1 + j2\,))(s - (-1 - j2\,))}$$

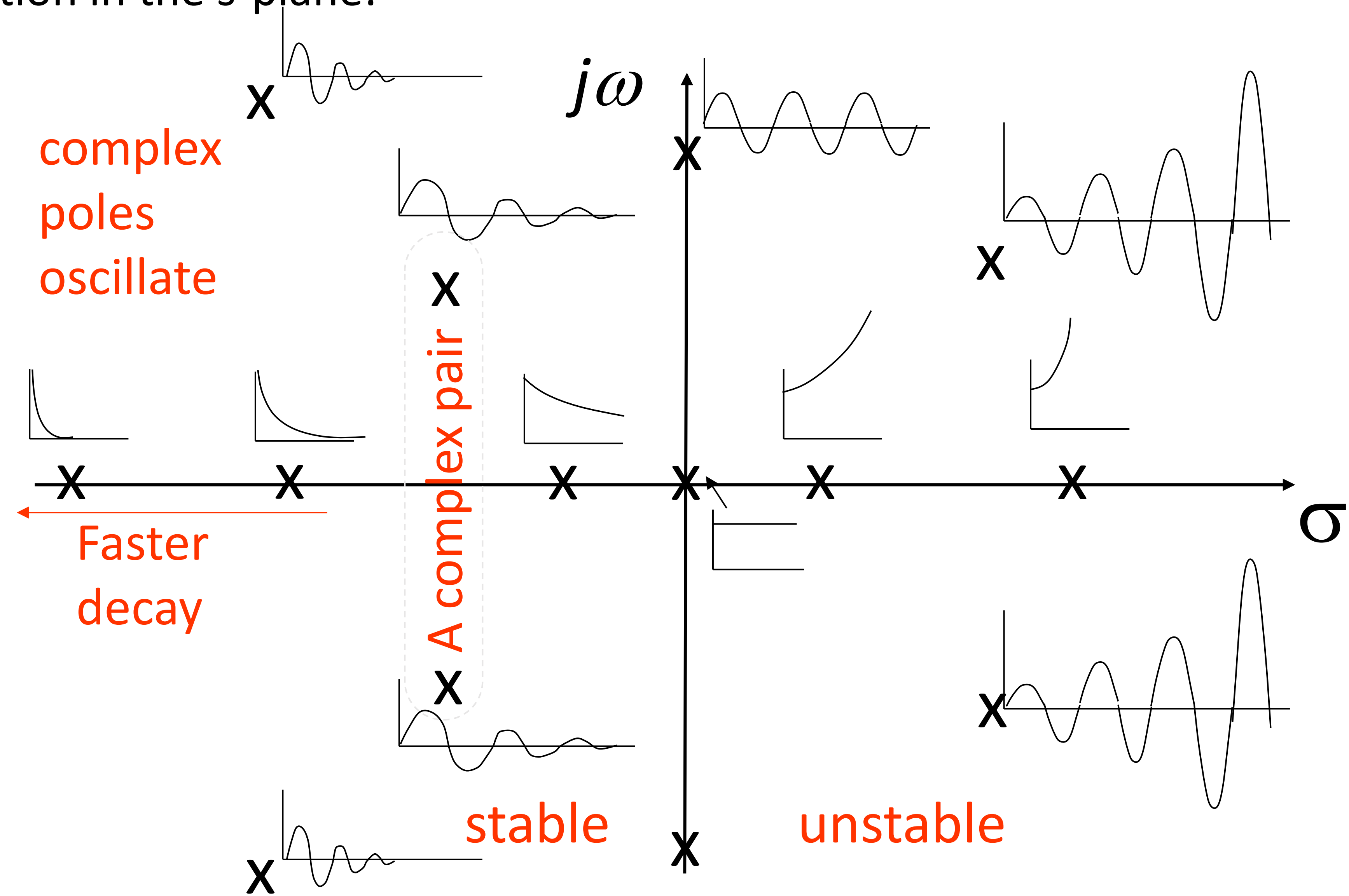
System Poles and Zeros - Example

$$G(s) = \frac{N(s)}{D(s)} = 3\,\frac{(s-(-4))}{(s-(-1+j2\,))(s-(-1-j2\,))}$$



Summary: Transfer functions in the s-plane - Zeros

The impulse response of each pole in the transfer function depends upon its location in the s-plane:



Effects of the damping ratio on poles

As we have seen, the roots of the transfer function of a second order system depends upon the damping coefficient:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

We can plot the poles of the characteristic equation as with any other transfer function:

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

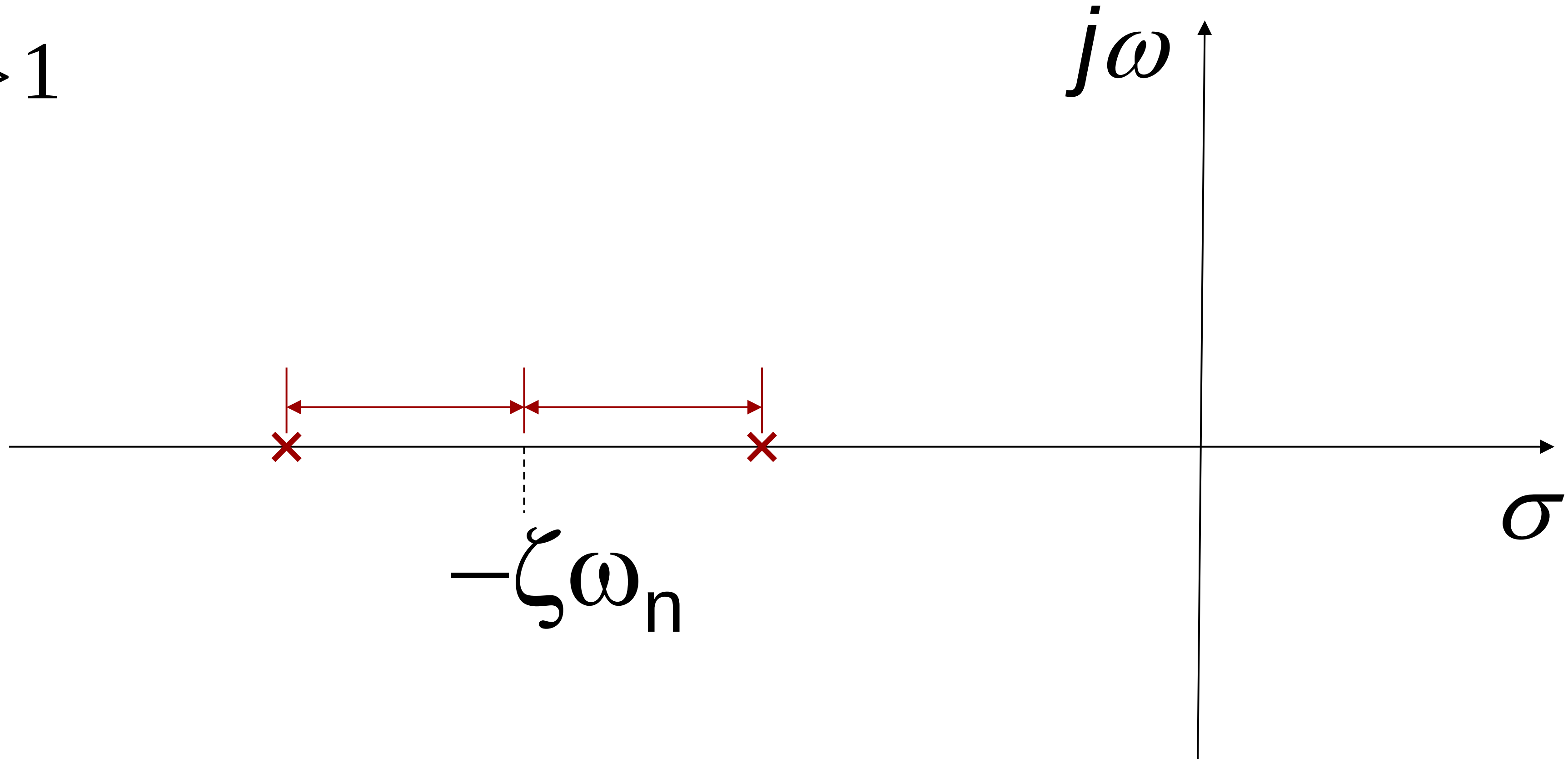
Both roots are real, and thus are on real axis

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

No oscillations in step response

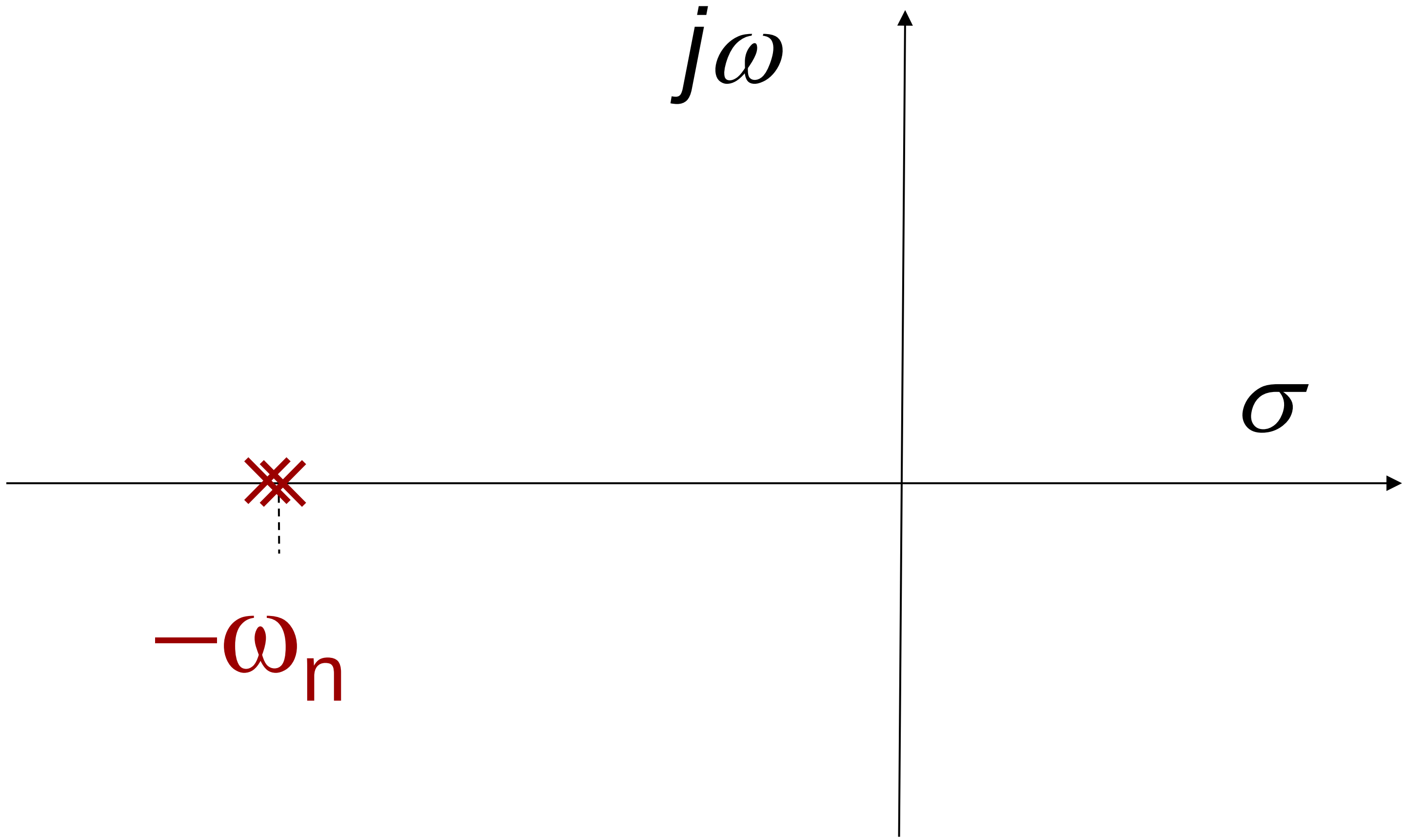
$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$\zeta > 1$



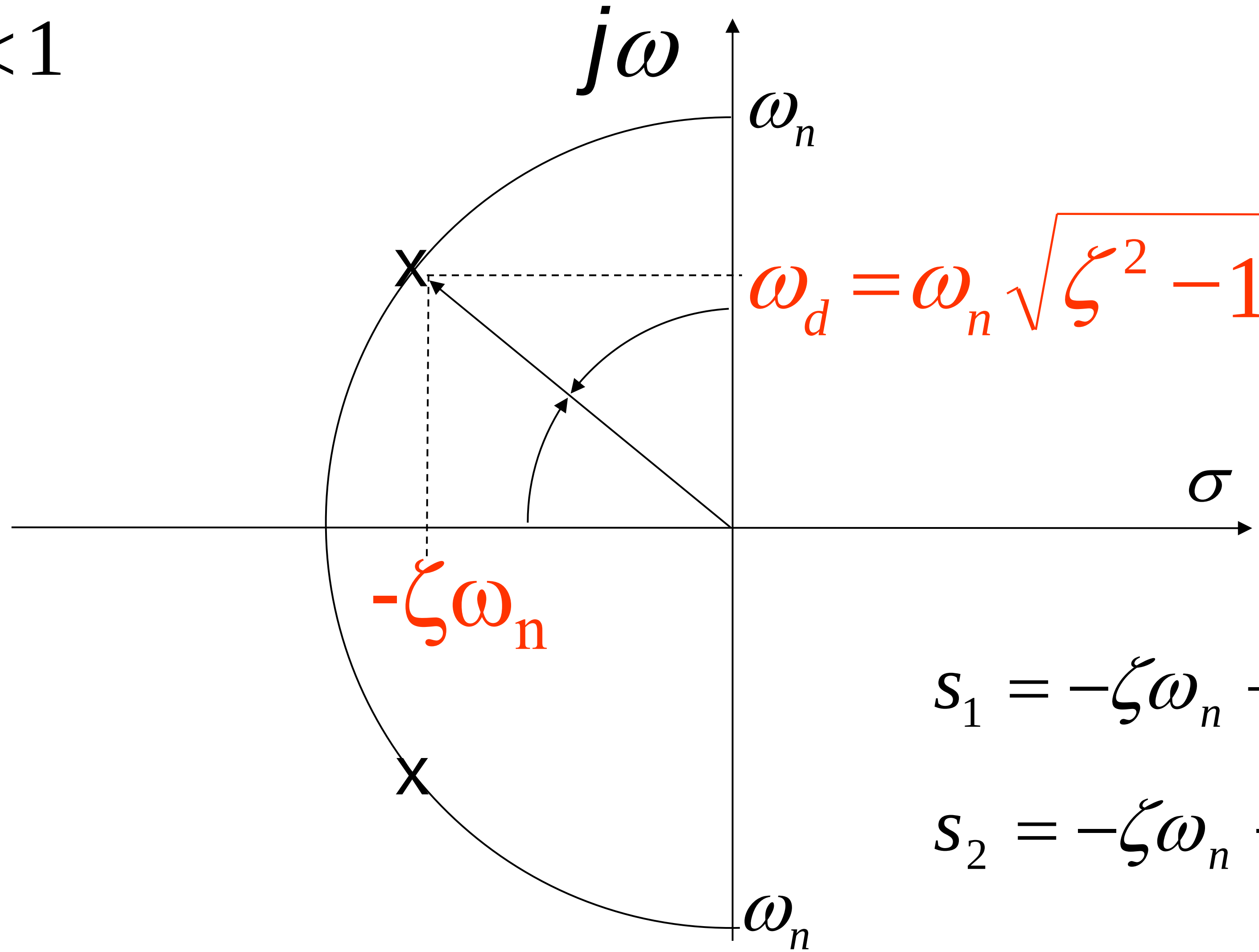
$\zeta = 1$

$s = -\omega_n$



The roots of the equations coincide, and the system is a product of two equal first order lags.

$\zeta < 1$



$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

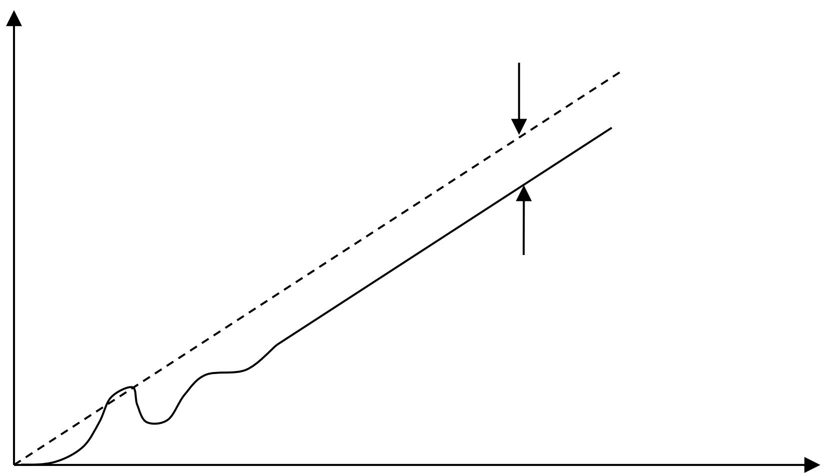
$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

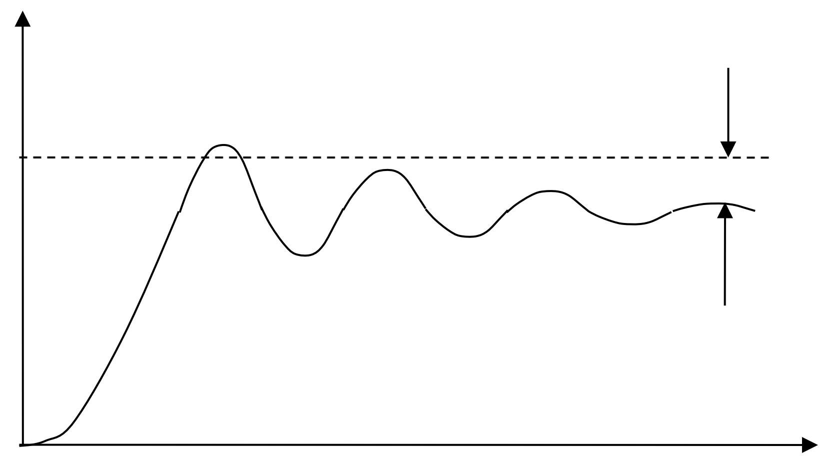
Both roots complex, and form conjugate pair. System is a product of exponential function and oscillatory component.

Previously we have looked at performance criteria for the transient response of a system: *settling time, peak time, overshoot etc.*

In addition to this it is important to know how accurately the control system tracks the demand once it has settled down, i.e. it is in the *steady state.*

For example, in the case of a ramp or step input:





Steady state doesn't imply system is not in motion! – e.g. can be moving at constant velocity, or oscillating.

The final value theorem gives the final value reached:

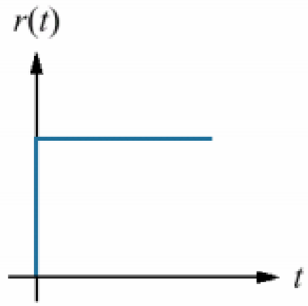
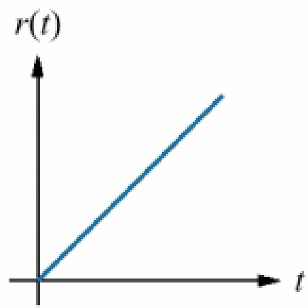
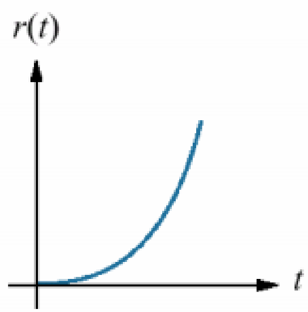
EXTRA S TERM

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

So, the final value is found by setting s to zero in the Laplace representation of the output, and multiplying by s.

There are two checks performed in Control theory which confirm valid results for the Final Value Theorem:
1) Y(s) should have no poles in the right half of the complex plane.
2) Y(s) should have no poles on the imaginary axis, except at most one pole at s=0.

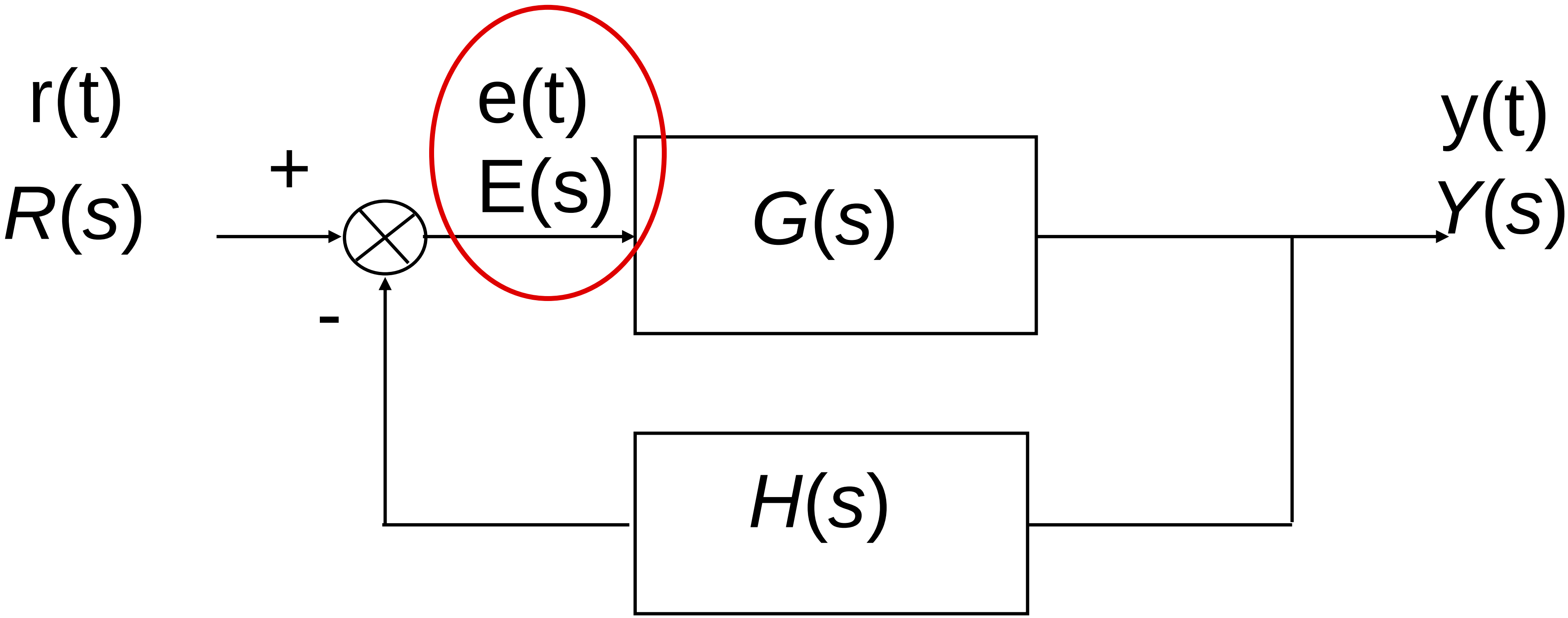
Continuous Systems and Transfer Function Revision: Final Value Theorem (FVT)

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

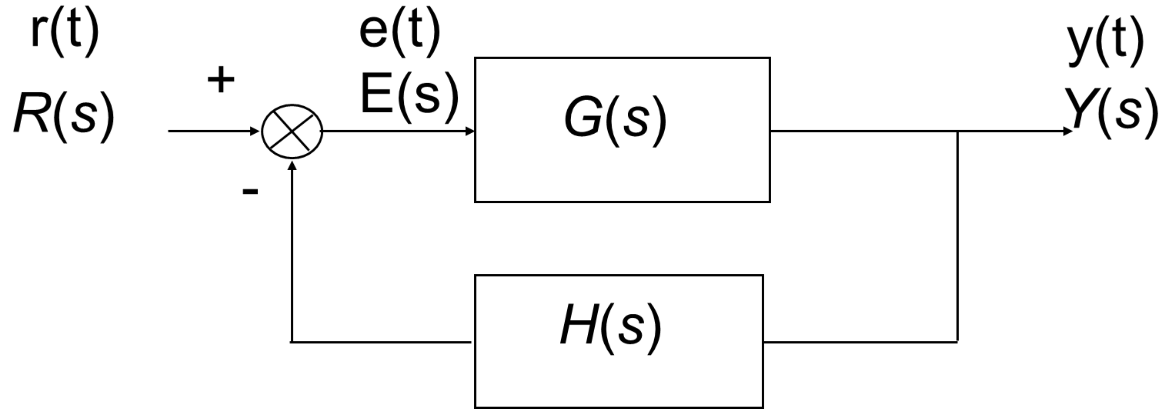
Continuous Systems and Transfer Function Revision: Steady State Error

We are normally more interested in the final value of the *error* rather than the output, as the goal of the controller is to drive the error as close to zero.

Further for some inputs such as a ramp or sinusoidal input, the final output isn't really meaningful.



Continuous Systems and Transfer Function Revision: Steady State Error



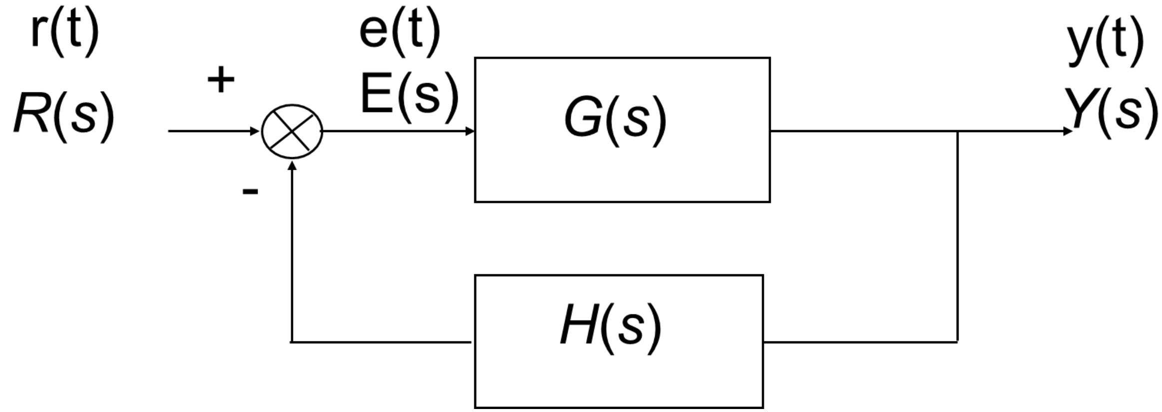
So, we first need to write the error signal with respect to the input:

$$E(s) = R(s) - H(s)Y(s) \qquad Y(s) = G(s)E(s)$$

So: $E = R - HGE \quad \rightarrow E + HGE = R$

$$E = \frac{R}{1+GH}$$

Continuous Systems and Transfer Function Revision: Steady State Error



Therefore the final value of the error signal e(t) would be:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

with

$$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

so

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$$

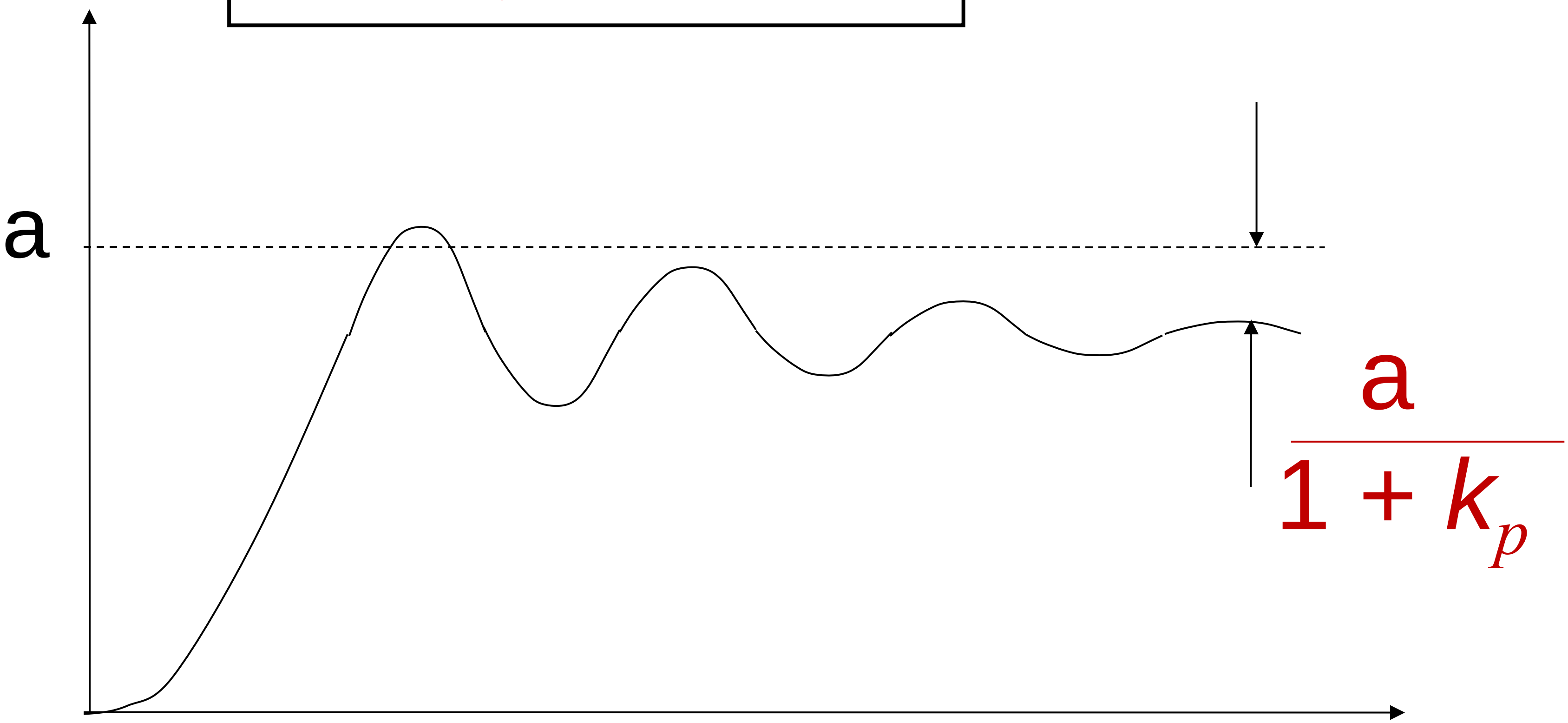
$$R(s) = \frac{A}{s} \qquad E(s) = \frac{R(s)}{1+G(s)H(s)}$$

Thus the final error is:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{s\,{}^{a}\!/_{s}}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{a}{1 + G(s)H(s)} = \frac{a}{1 + k_p}$$

where $\boxed{k_p = \lim_{s \to 0} G(s)H(s)}$ Is the **position error constant**



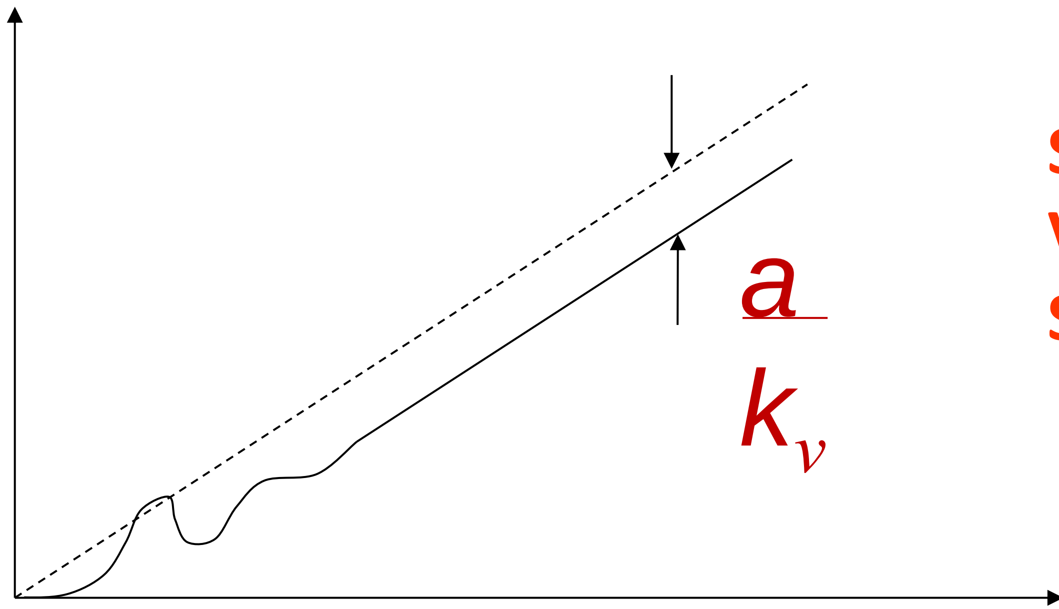
$$R(s) = \frac{A}{s^2} \qquad E(s) = \frac{R(s)}{1+G(s)H(s)}$$

Thus the final error is:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{s\,{}^{a}\!/_{s^2}}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{a}{s + sG(s)H(s)} = \frac{a}{k_v}$$

where $\boxed{k_v = \lim_{s \to 0} sG(s)H(s)}$ Is the **velocity error constant**



Steady State Velocity Lag, S.S.V.L.

Once again, lets return to the servo example to see how this system would perform given these inputs.

System types

Continuous Systems and Transfer Function Revision: System Type

Let's look more generally at the transfer function of our system and introduce the concept of system **type.** By factoring out any s terms from the denominator, we can write the transfer function in the following form:

$$G(s)H(s)=\frac{(s-z_1)(s-z_2)(s-z_3)\ldots}{s^p(s-\sigma_1)(s-\sigma_2)(s-\alpha_k+j\omega_k)(s-\alpha_k-j\omega_k)\ldots}$$

So, if there are *p poles at the origin*, the system is said to be a 'type *p*' system.

System Type IS NOT THE ORDER OF THE SYSTEM!!!

For example, a servo motor can be expressed as:

$$\frac{\Theta_o(s)}{\Theta_i(s)}=\frac{k}{s(Is+f)}$$

Type 1 System

Continuous Systems and Transfer Function Revision: System Type

Or an electro magnet system

$$\frac{I(s)}{V(s)} = \frac{1}{(Ls+R)}$$

Type 0 System

Or a mass spring damper system

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

Type 0 System

Essentially, for a zero error we want k_p and k_v to be infinite, or at least be a constant for a finite error, depending on our requirements.

$$SSE = \frac{a}{1+k_p} \qquad SSVL = \frac{a}{k_v}$$

$$k_p = \lim_{s \to 0} G(s)H(s) \qquad k_v = \lim_{s \to 0} sG(s)H(s)$$

The **position error constant** k_p for a *type p* system is given by:

p > 0 $k_p = \lim_{s \to 0} G(s) = \infty$ no steady-state position error

p = 0 k_p is finite finite position error

The **velocity error constant** for a *type p* system is given by:

p > 1 $k_v = \lim_{s \to 0} sG(s) = \infty$ no velocity error

p = 1 k_v is finite steady state velocity lag

p = 0 $k_v = 0$ infinite lag (completely fails to track)

No. Integrators in denominator = system TYPE	Input type		
	Step $r(t) = a$ $R(s) = a/s$	Ramp $r(t) = at$ $R(s) = a/s^2$	Acceleration $r(t) = at^2/2$ $R(s) = a/s^3$
0	$e_{ss} = a/(1+k_p)$	$e_{ss} = \infty$	$e_{ss} = \infty$
1	$e_{ss} = 0$	$e_{ss} = a/k_v$	$e_{ss} = \infty$
2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = a/k_a$

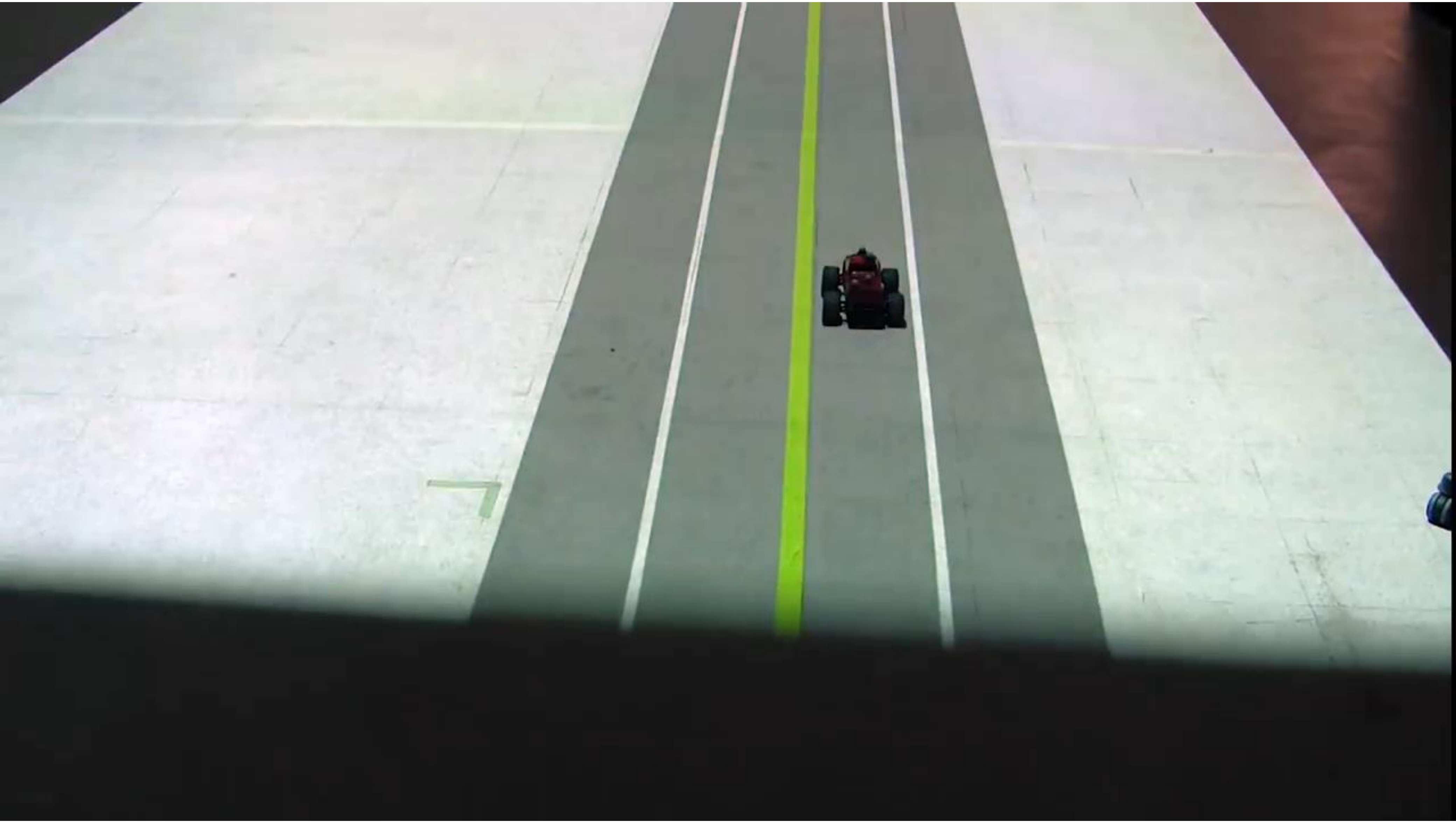
∞ **means system never settles**

The more poles there are at the origin of the open-loop system, the better the steady-state tracking performance.

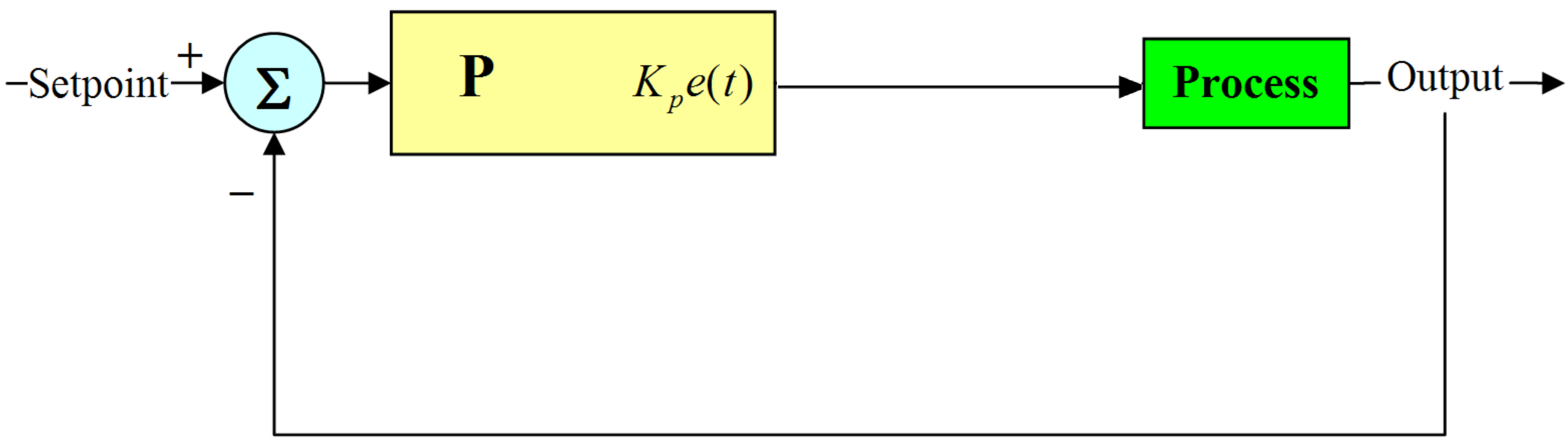
However, pure integrations have a highly destabilizing effect on the control system!

PID Control

Continuous Systems and Transfer Function Revision: Line following

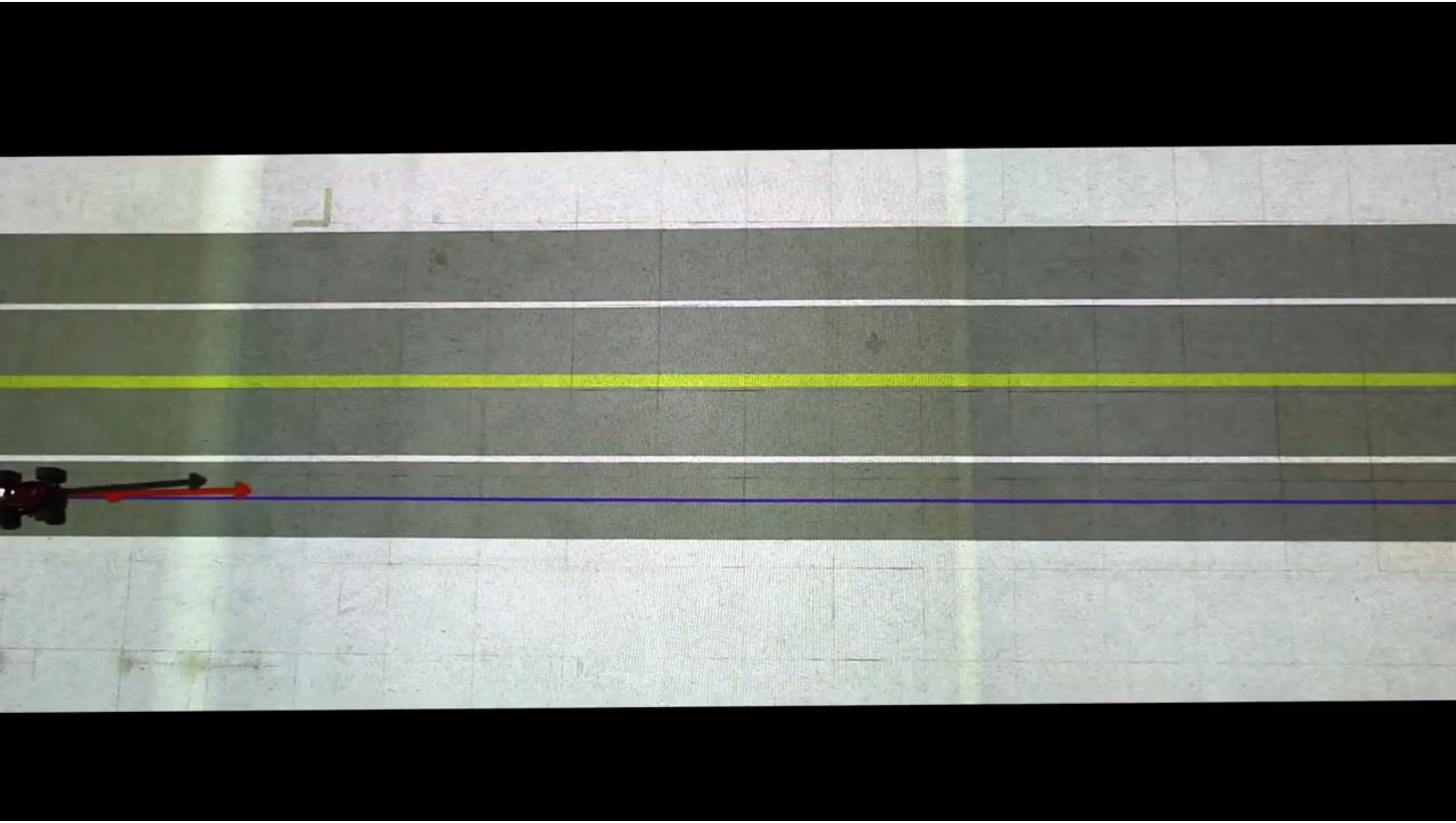


Continuous Systems and Transfer Function Revision: Proportional Control

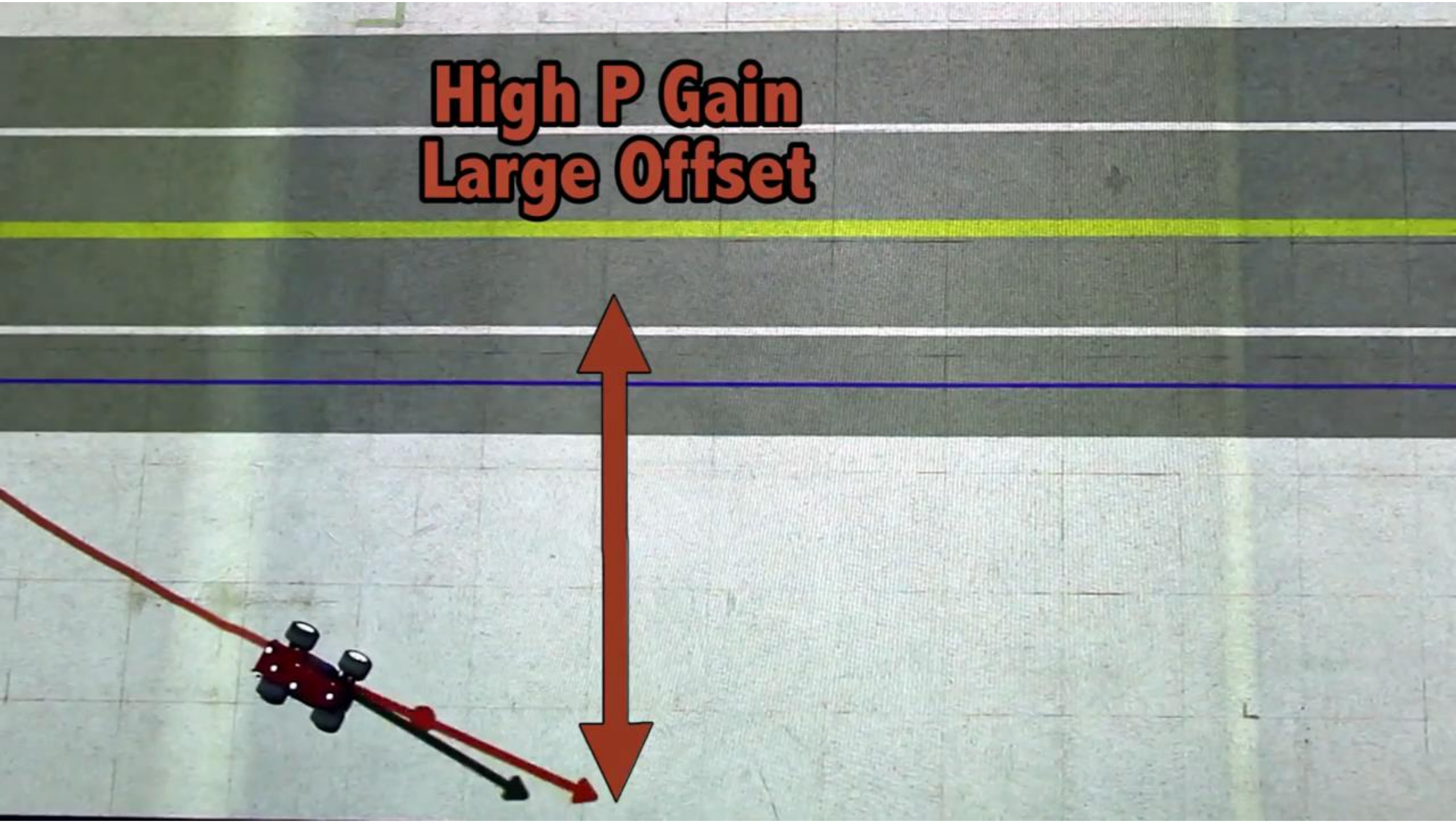


Continuous Systems and Transfer Function Revision: Proportional Error

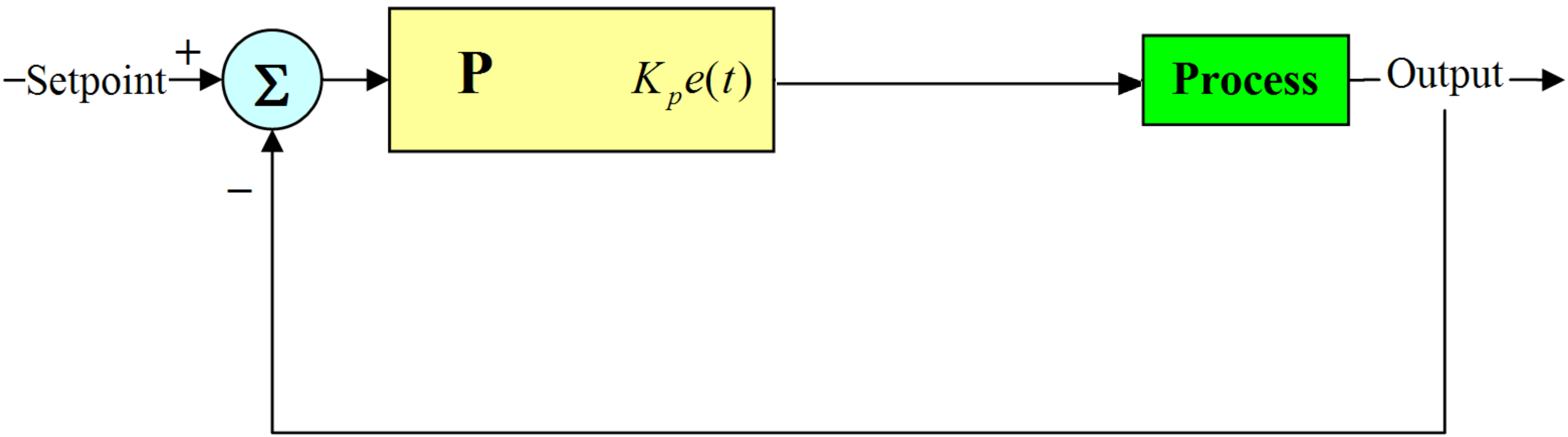
Continuous Systems and Transfer Function Revision: Trade-off: Proportional Error



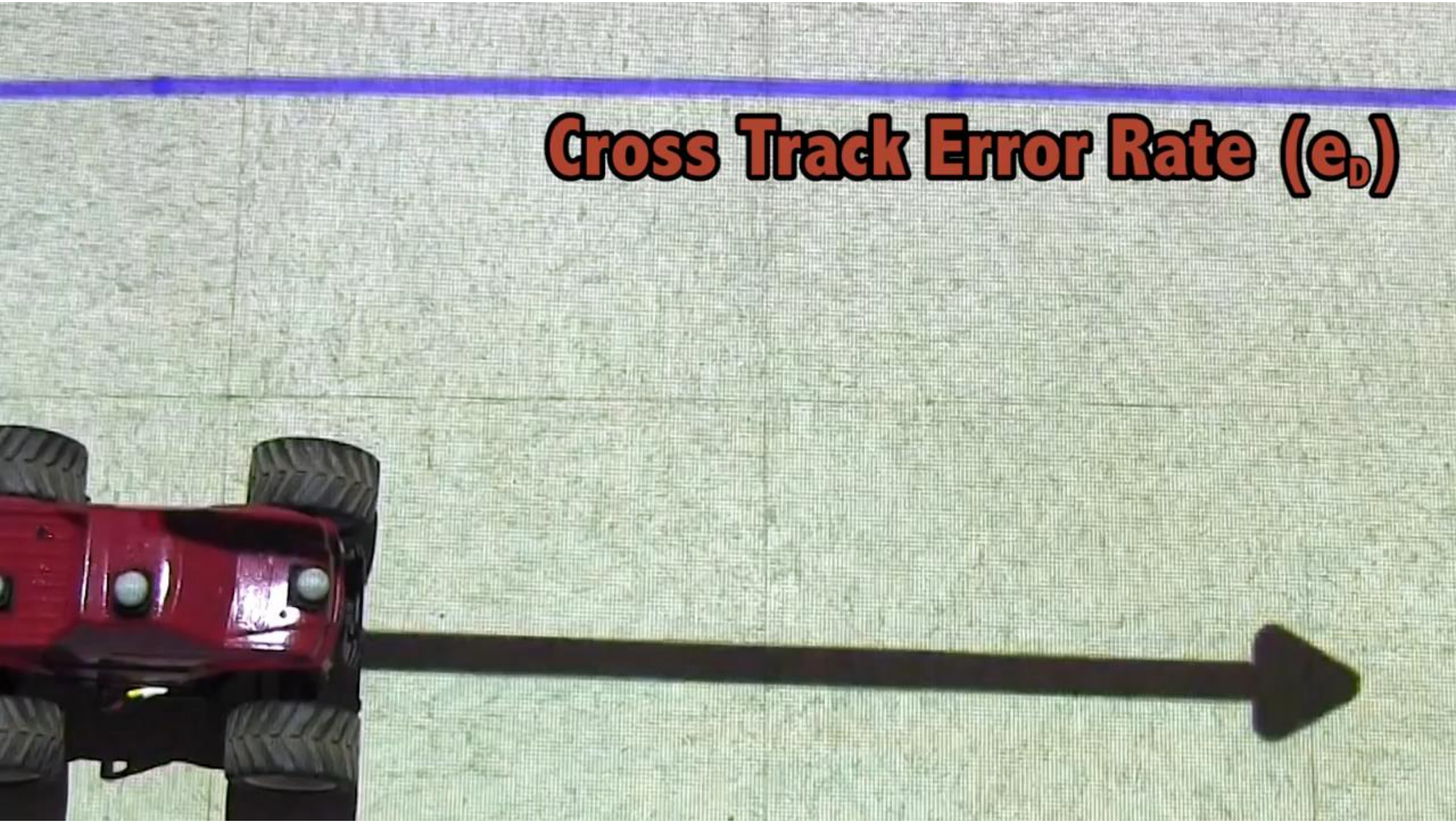
Continuous Systems and Transfer Function Revision: Trade-off: Proportional Error



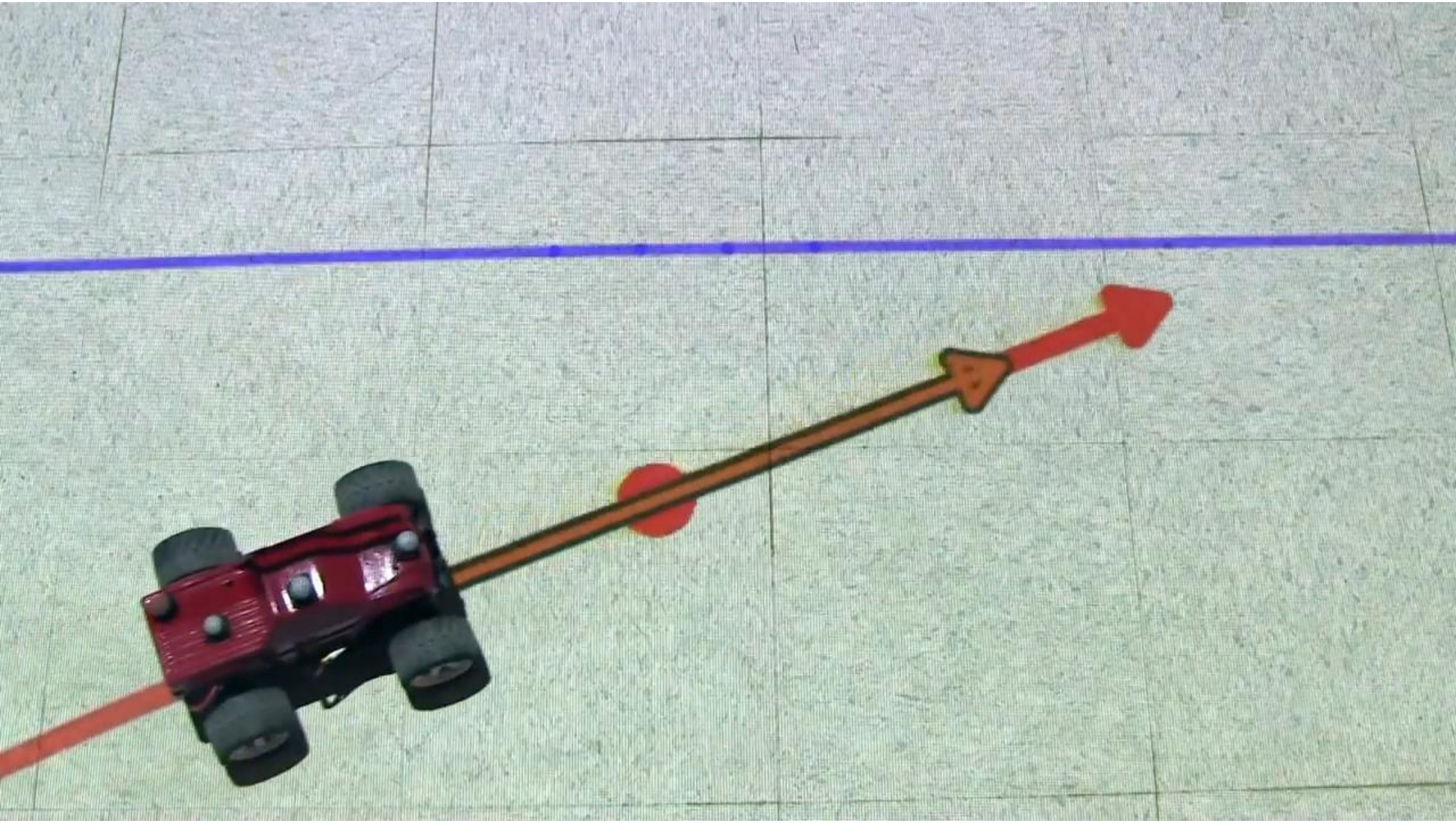
Continuous Systems and Transfer Function Revision: PID Control



Continuous Systems and Transfer Function Revision: Derivative Error

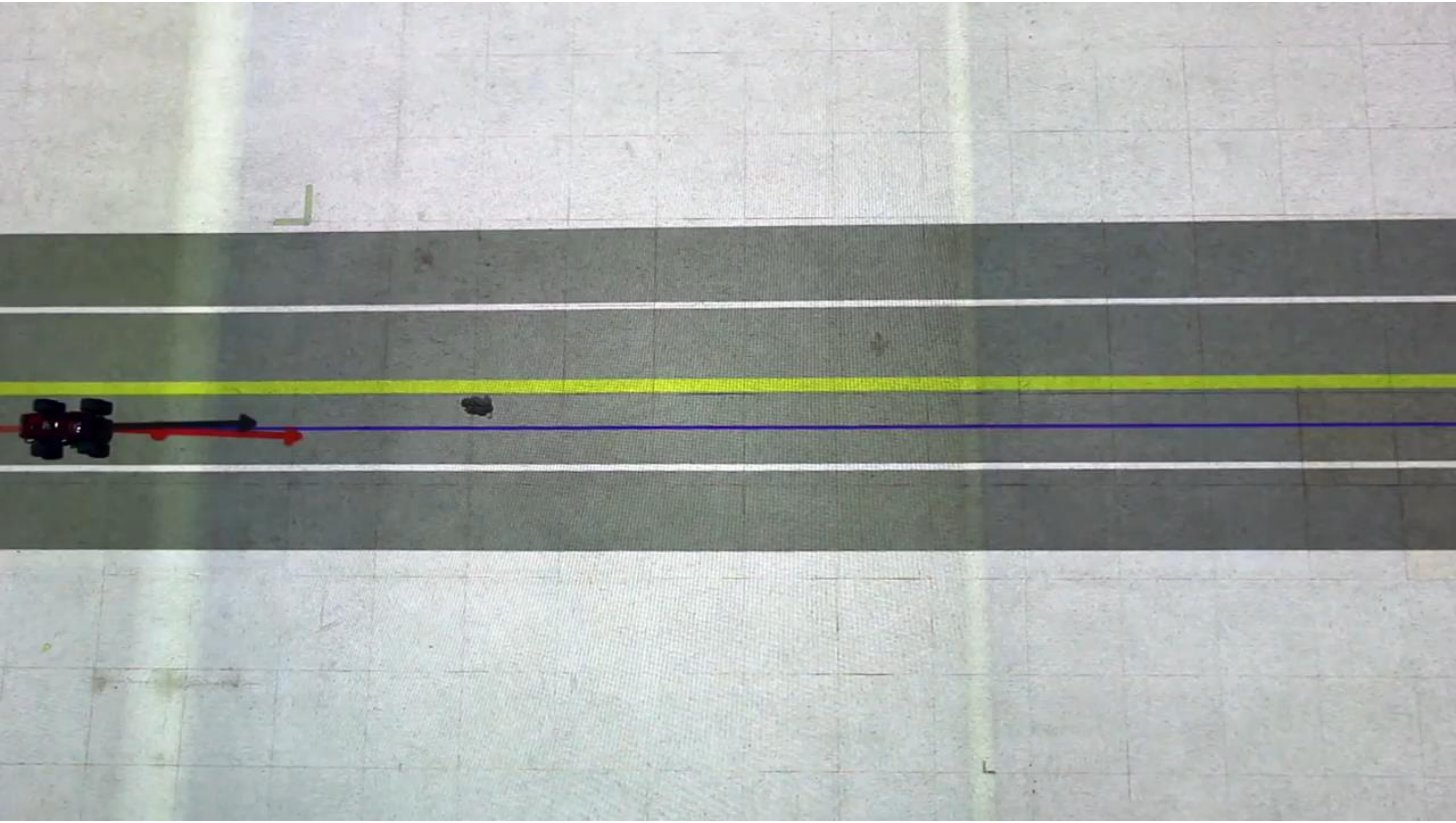


Continuous Systems and Transfer Function Revision: Derivative Error

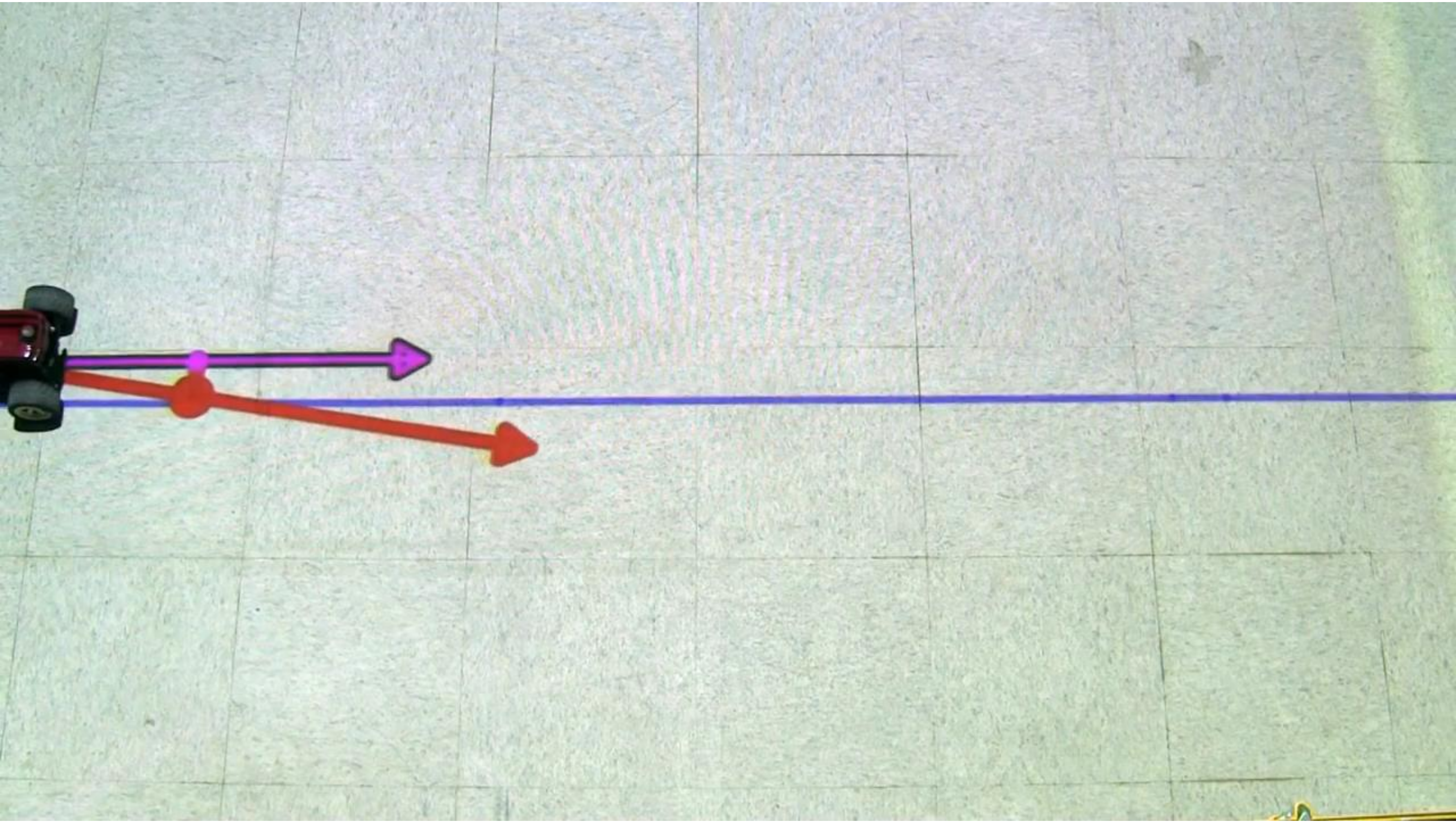


Continuous Systems and Transfer Function Revision: Integral Error

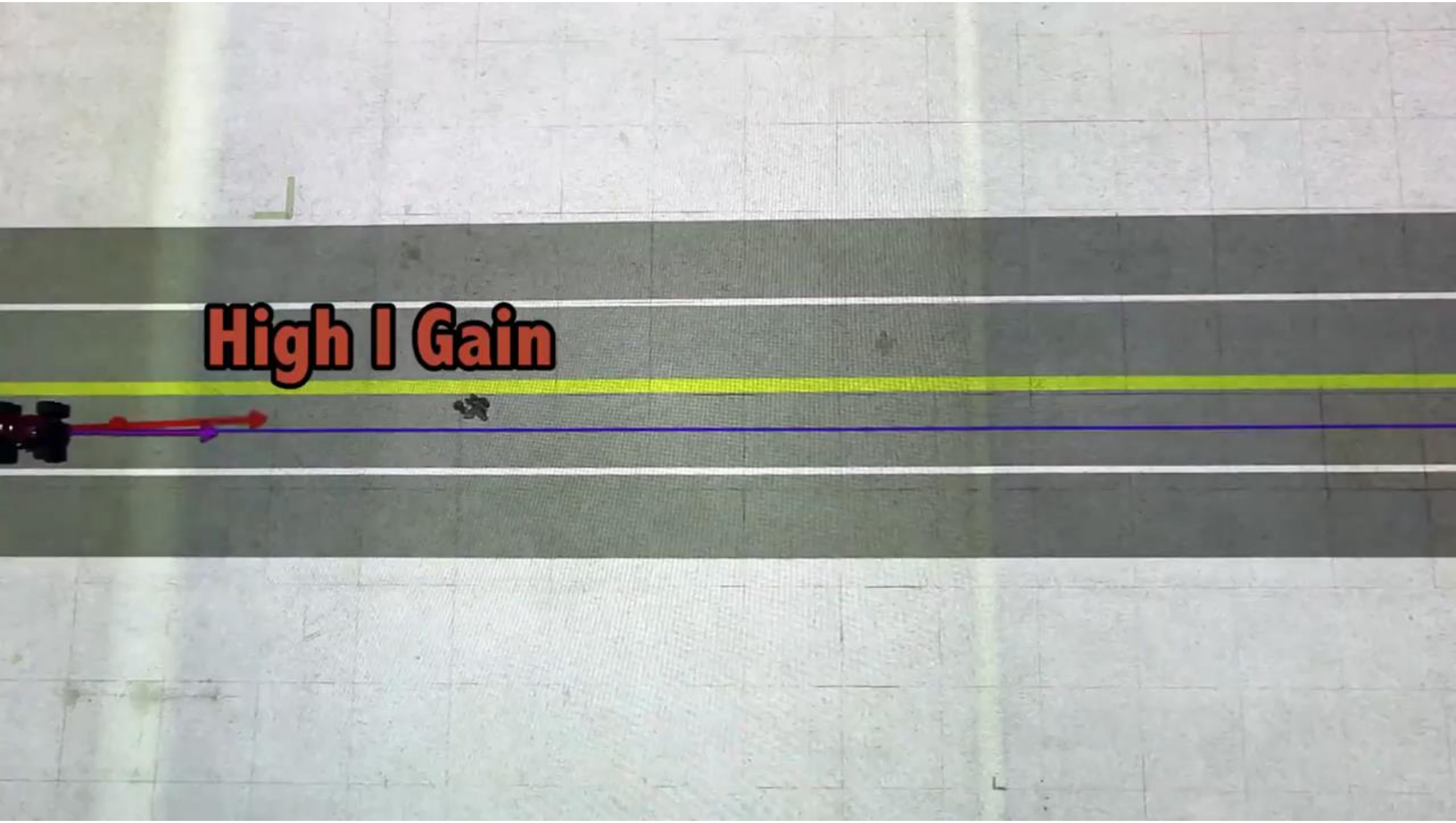
Continuous Systems and Transfer Function Revision: Integral Error



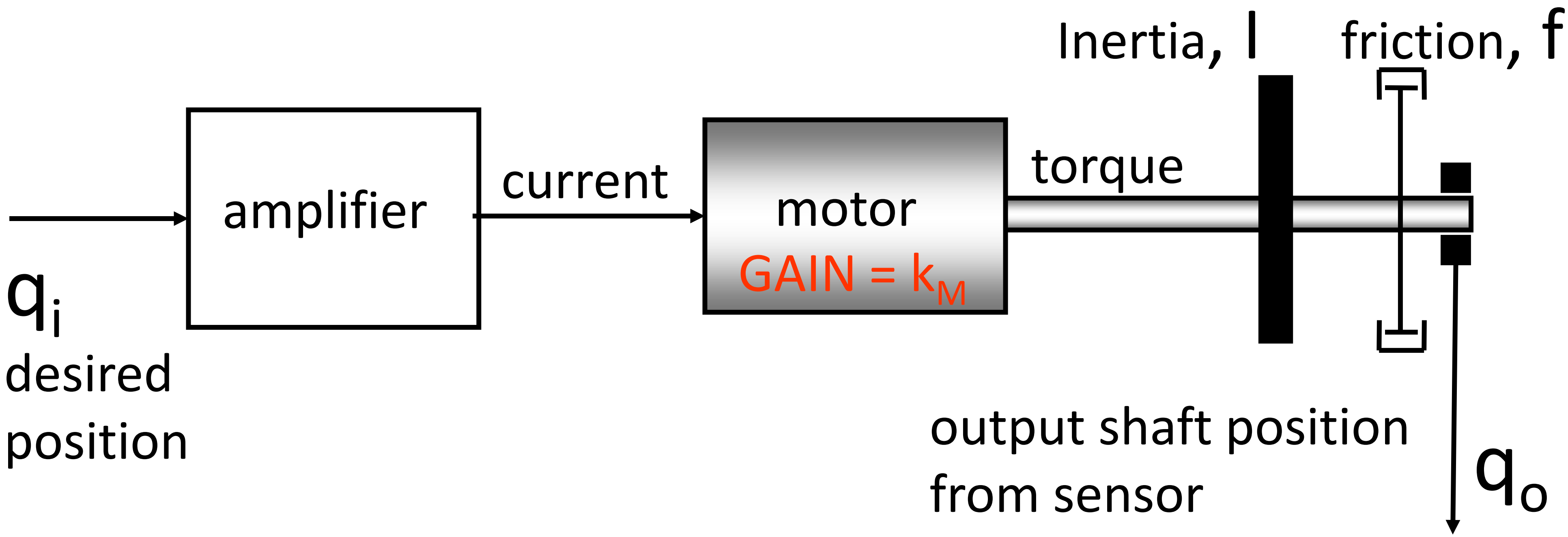
2.1 Continuous Systems and Transfer Function Revision: Integral Error



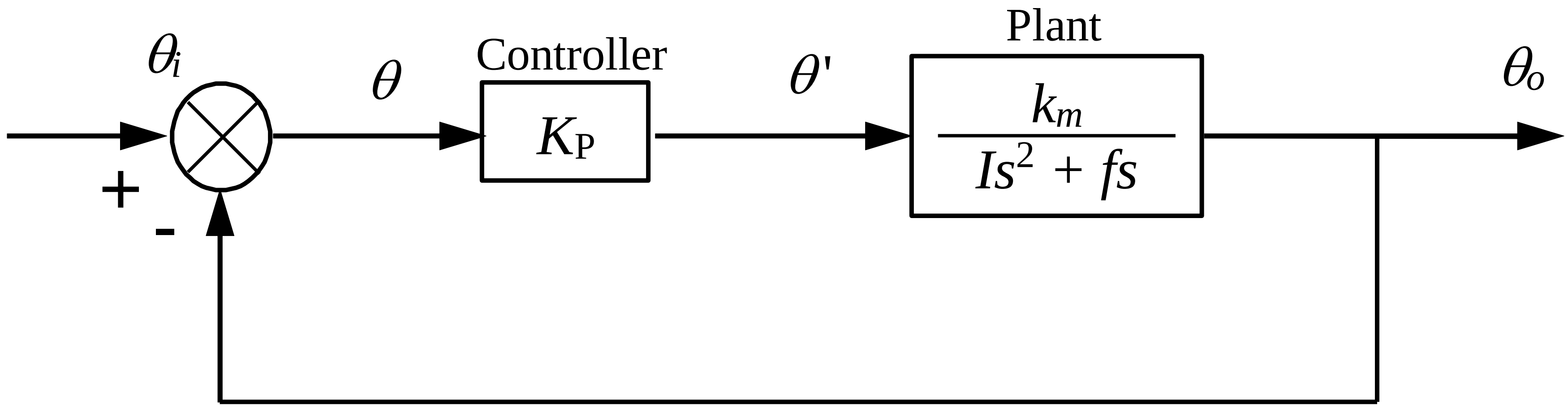
Continuous Systems and Transfer Function Revision: Integral Error



These can be modelled as shown below, with a DC motor rotating an inertial load, with a friction or damping resistance.

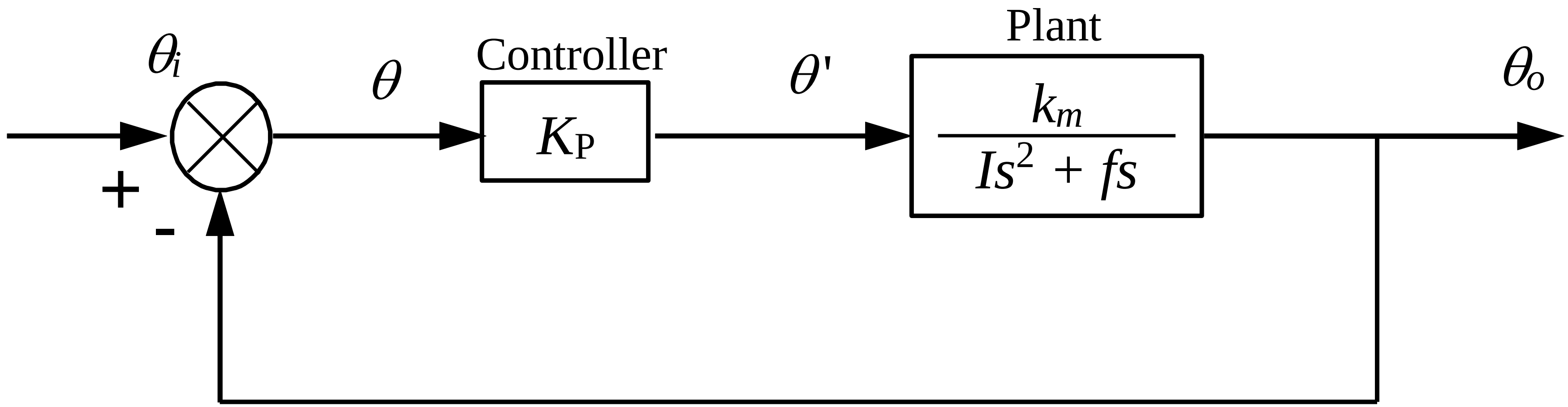


Continuous Systems and Transfer Function Revision: Proportional Error



$$F(s) = \frac{G(s)}{1+G(s)} = \frac{K_P k_m}{Is^2 + fs + K_P k_m}$$

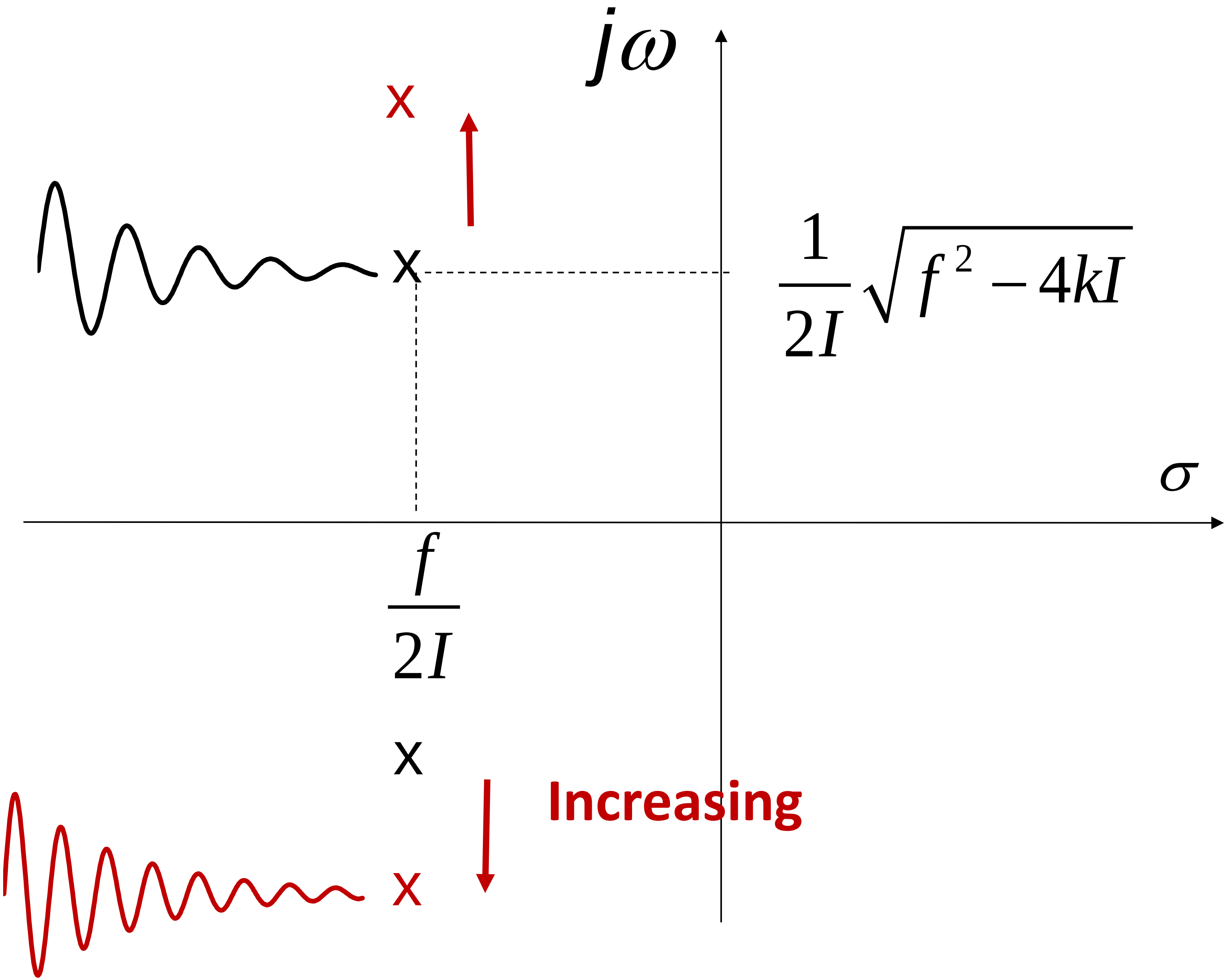
Continuous Systems and Transfer Function Revision: Proportional Error



$$F(s) = \frac{G(s)}{1+G(s)} = \frac{K_P k_m}{Is^2 + fs + K_P k_m}$$

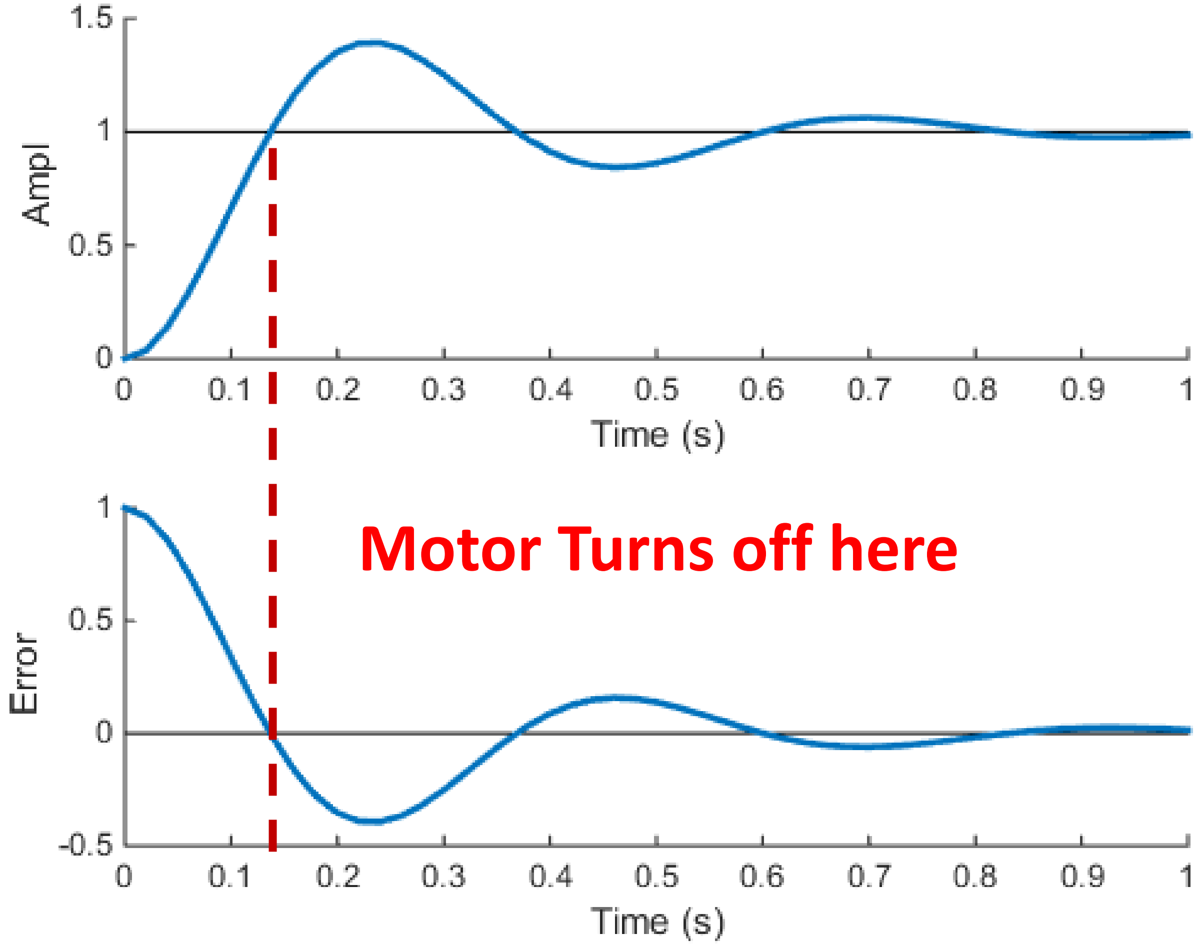
Low values of K_p give stable but slow responses, and high SSVL.
High values reduce SSVL but response overshoots considerably.

Continuous Systems and Transfer Function Revision: Servomechanism in the s-plane



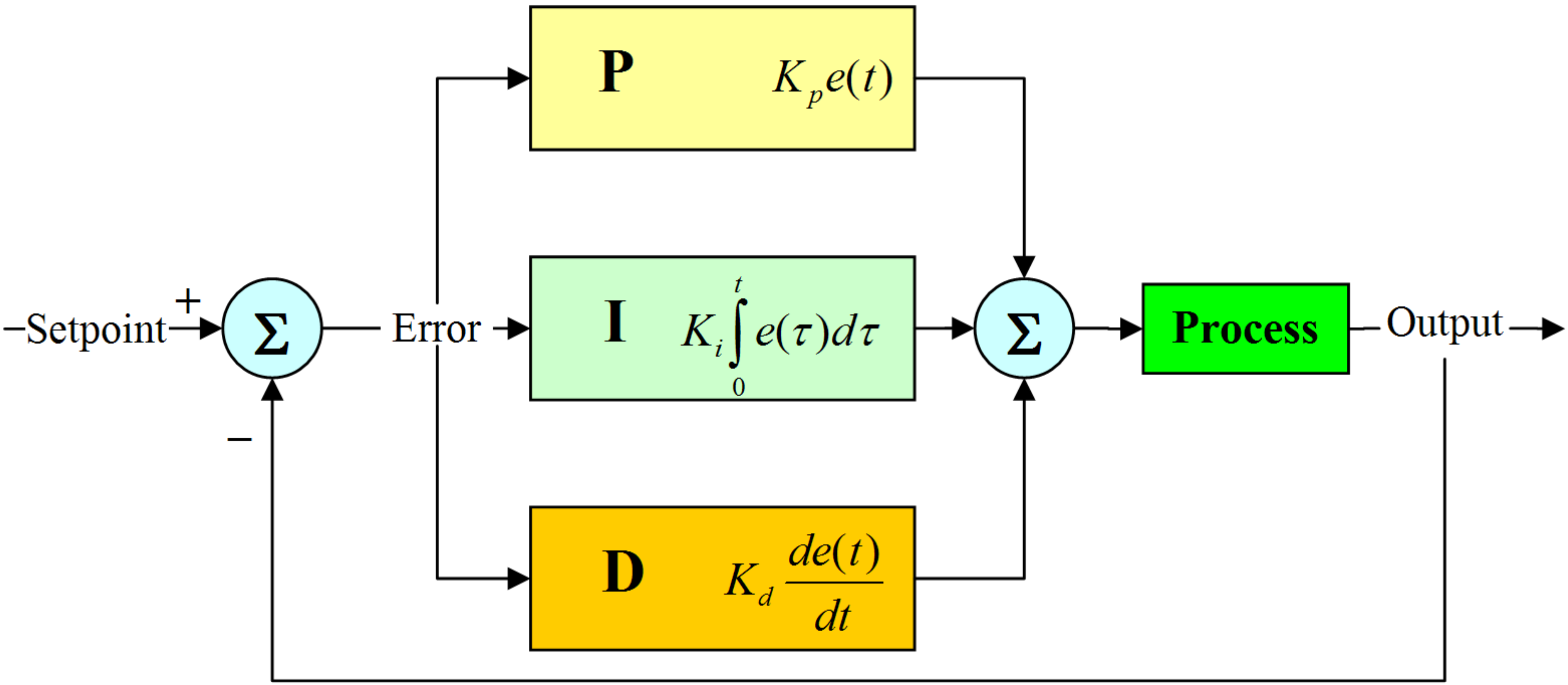
Continuous Systems and Transfer Function Revision: Proportional Error

With proportional control the error *and thus the control signal* does not reach zero until the motor is at the desired position.



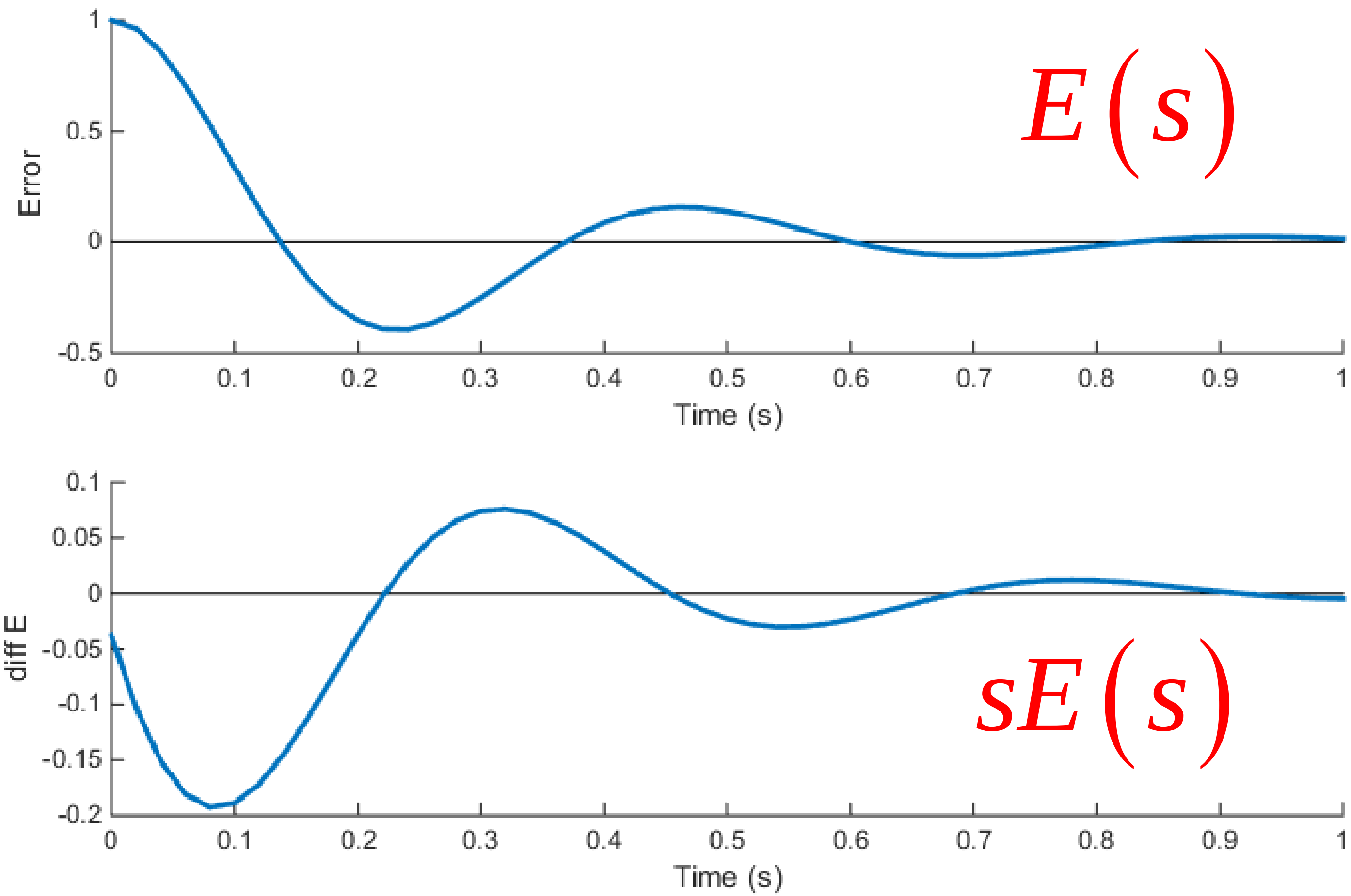
However, due to the inertia for the system, the motor continues to move even without a control signal, resulting in an overshoot.

Continuous Systems and Transfer Function Revision: PID Control



Continuous Systems and Transfer Function Revision: Derivative Error

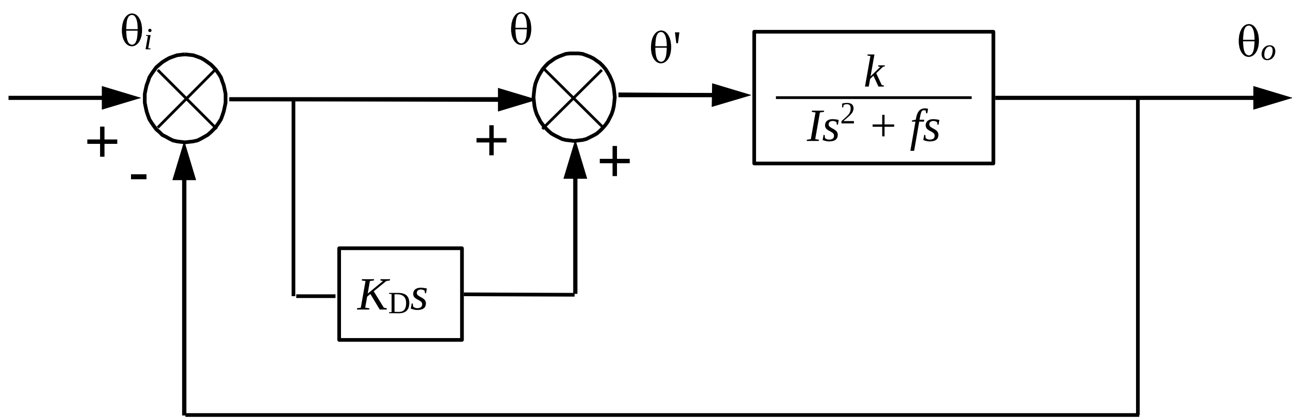
We can replicate this by considering the **derivative** of the error.



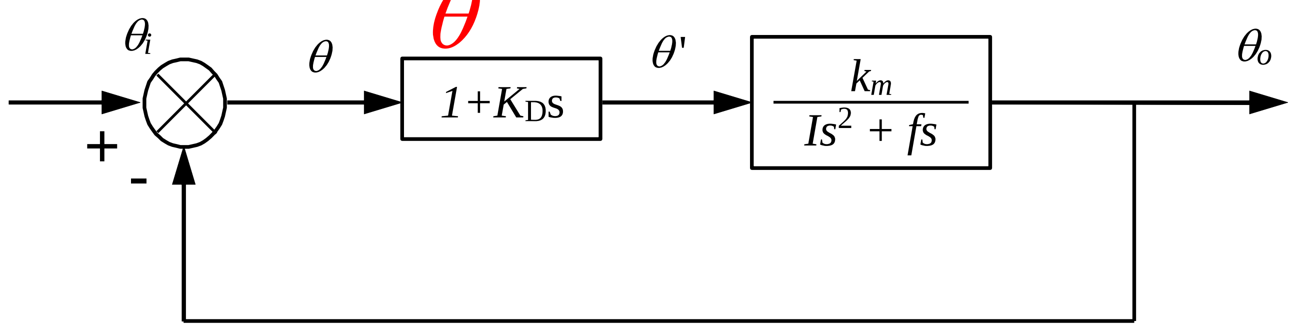
This is negative as the servo approaches the target, so offers a way of restraining the servos forward motion.

Continuous Systems and Transfer Function Revision: Derivative Error

Let’s consider a derivative controller with unity proportional gain



$$\theta' = \theta + K_D s\theta \qquad \frac{\theta'}{\theta} = 1 + K_D s \qquad \theta = \theta_i - \theta_o$$



The open loop transfer function now becomes:

$$G' = \frac{(1 + K_D s)k_m}{Is^2 + fs}$$

Type 1

Continuous Systems and Transfer Function Revision: Derivative Error

The open loop transfer function now becomes:

$$G' = \frac{\left(1 + K_D s\right) k_m}{Is^2 + fs}$$

Type 1

Giving a *closed loop transfer function* of

$$F\left(s\right) = \frac{G'\left(s\right)}{1 + G'\left(s\right)} = \frac{\left(1 + K_D s\right) k_m}{Is^2 + fs + \left(1 + K_D s\right) k_m}$$

$$\boxed{F\left(s\right) = \frac{k_m K_D s + k_m}{Is^2 + \left(f + k_m K_D\right) s + k_m}}$$

Continuous Systems and Transfer Function Revision: Derivative Error

$$\boxed{F(s) = \frac{k_m K_D s + k_m}{Is^2 + (f + k_m K_D)s + k_m}}$$

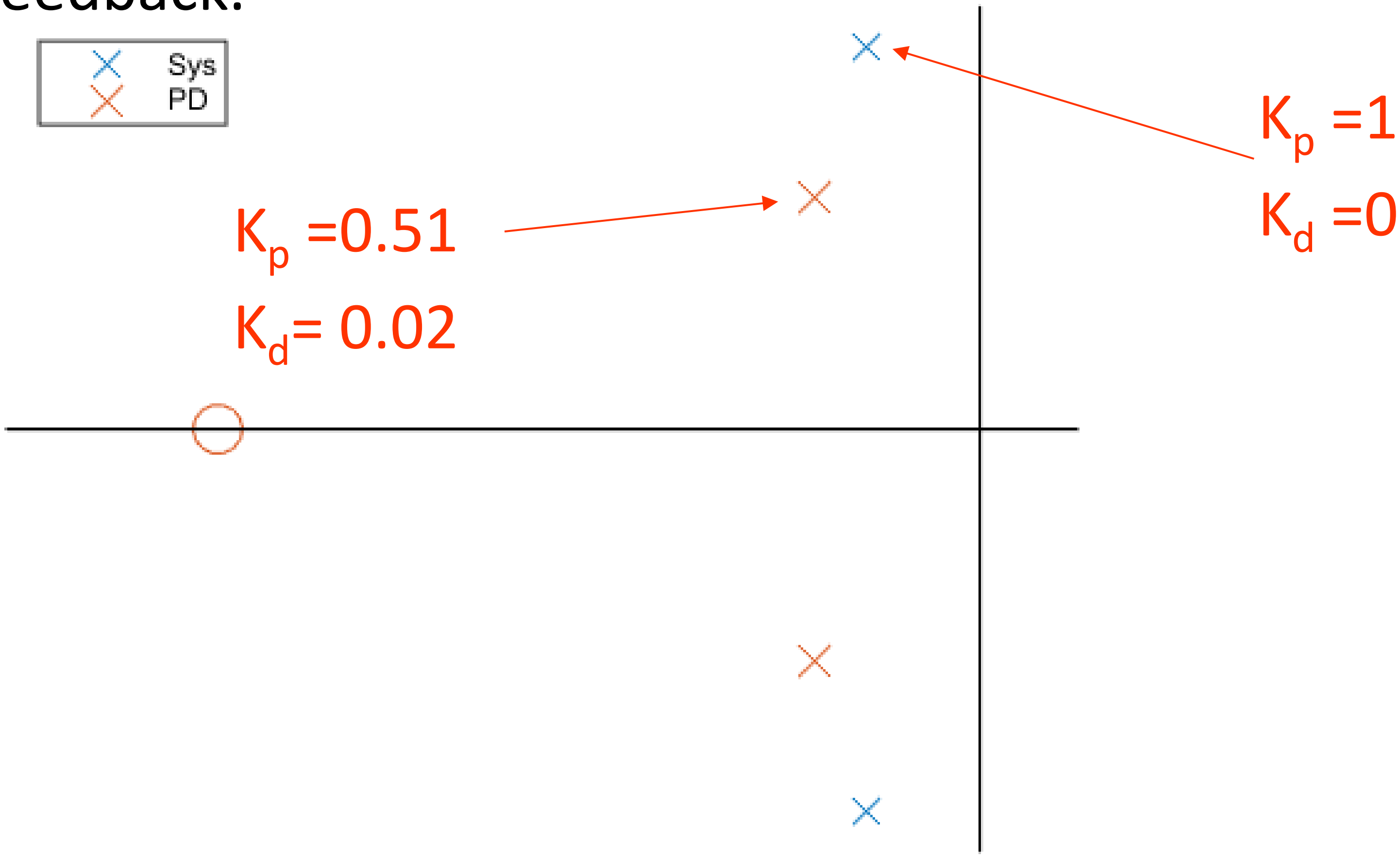
System is still second order, but now there is a zero in the numerator. The effect of this is subtle compared to the change in the poles.

More importantly, lets consider the SSVL of this system:

$$k_v = \lim_{s \to 0} sG'(s) = \frac{s(1 + K_D s)k_m}{s(Is + f)} = \frac{k_m}{f}$$

So unlike velocity feedback, the SSVL is unchanged by derivative error. So we can improve transient response without compromising steady state error.

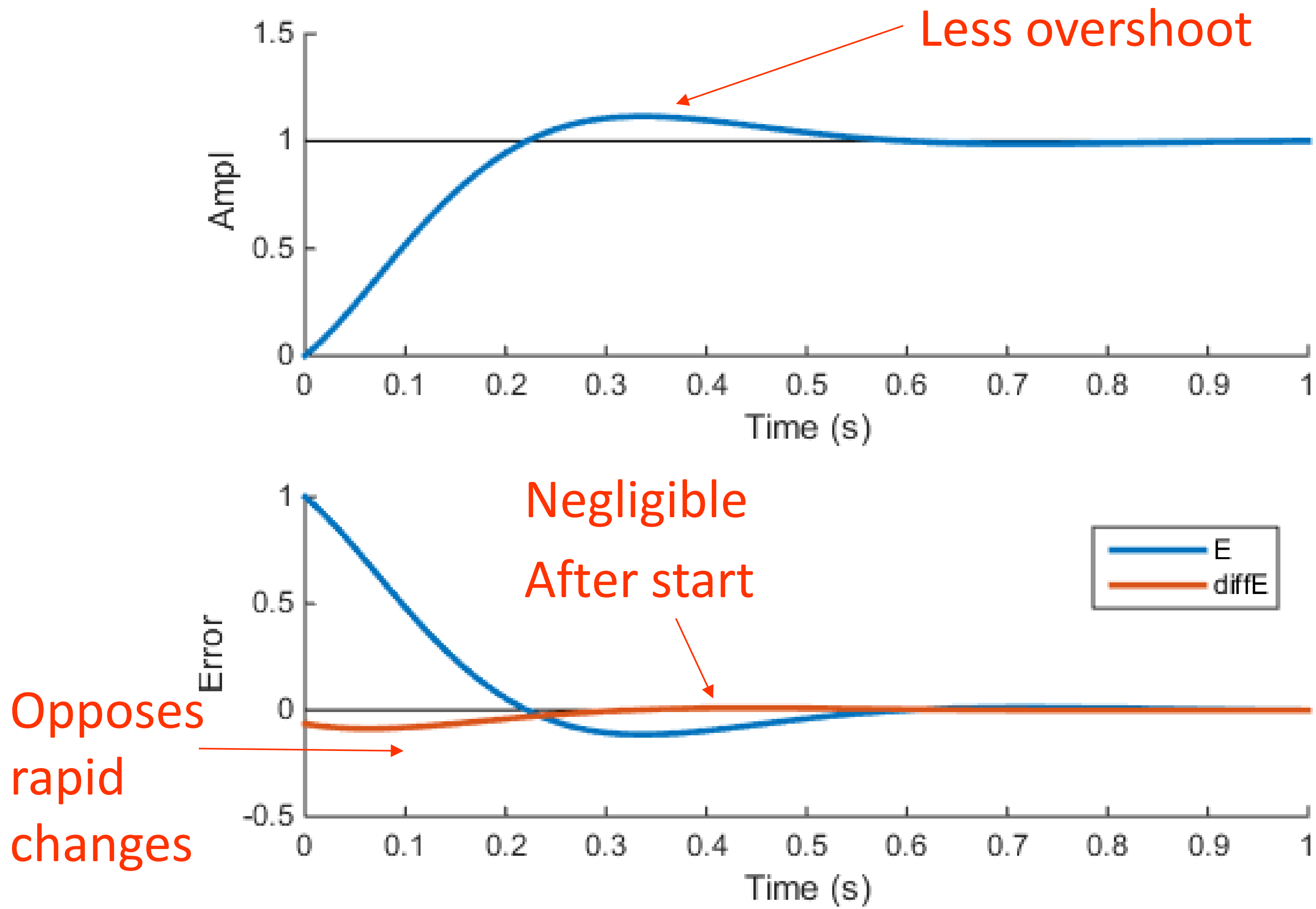
The effect on the step response is similar to that of the minor loop velocity feedback.



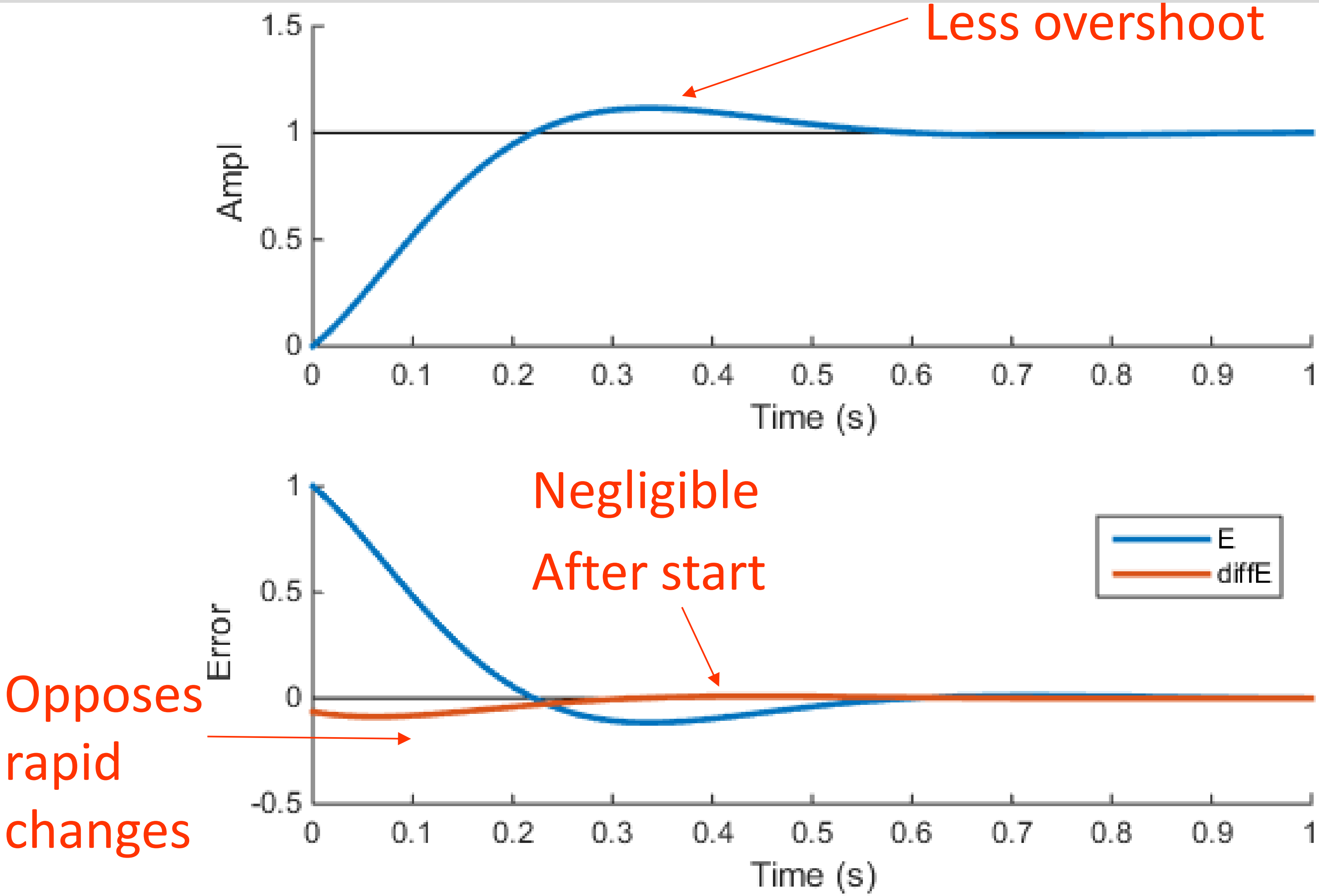
The poles of the system become less oscillatory and decay quicker. This combined with a reduced proportional gain, gives an improved transient response.

Continuous Systems and Transfer Function Revision: Derivative Error

This is clear when looking at the step response, and the relative contributions of the two error terms.

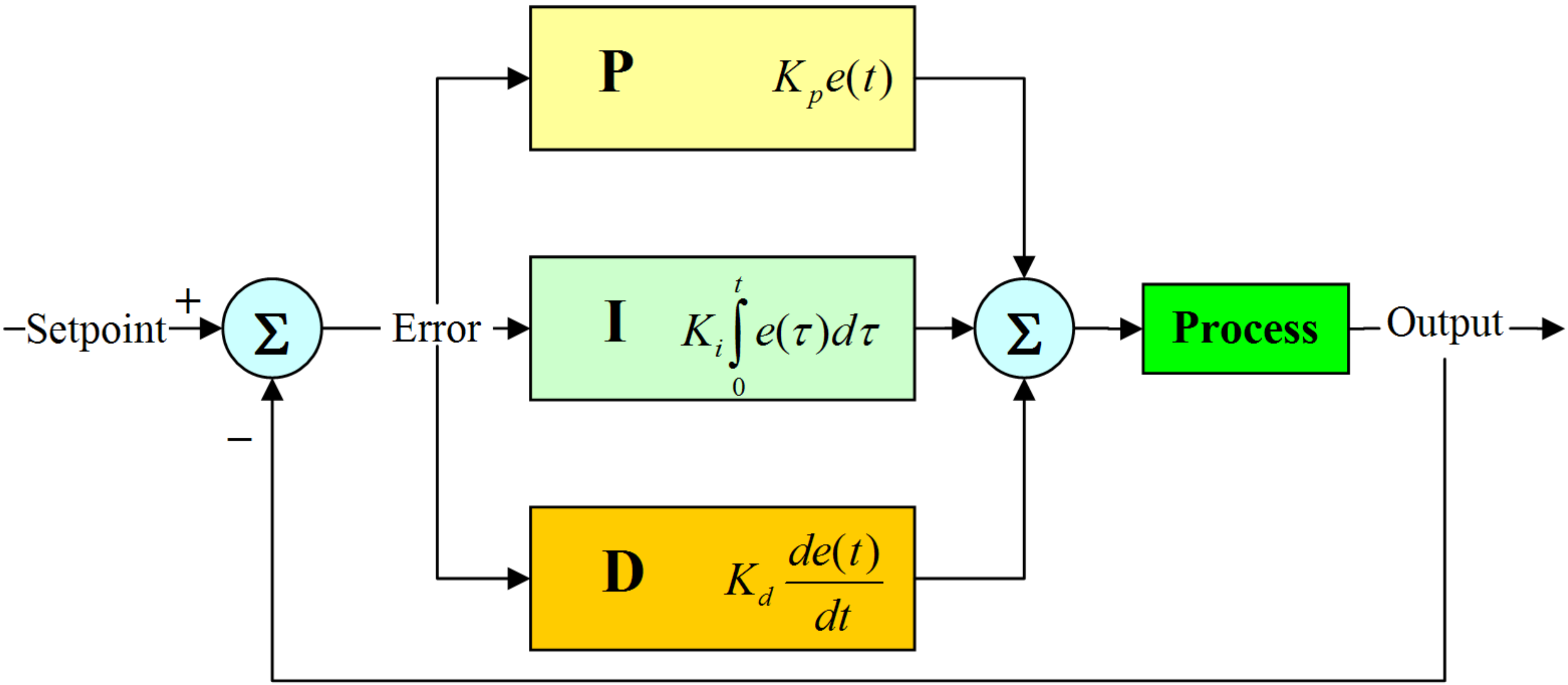


Continuous Systems and Transfer Function Revision: Derivative Error



At the start, the derivative term is significant *and in the opposite direction to* the proportional error term, but becomes negligible as the system settles.

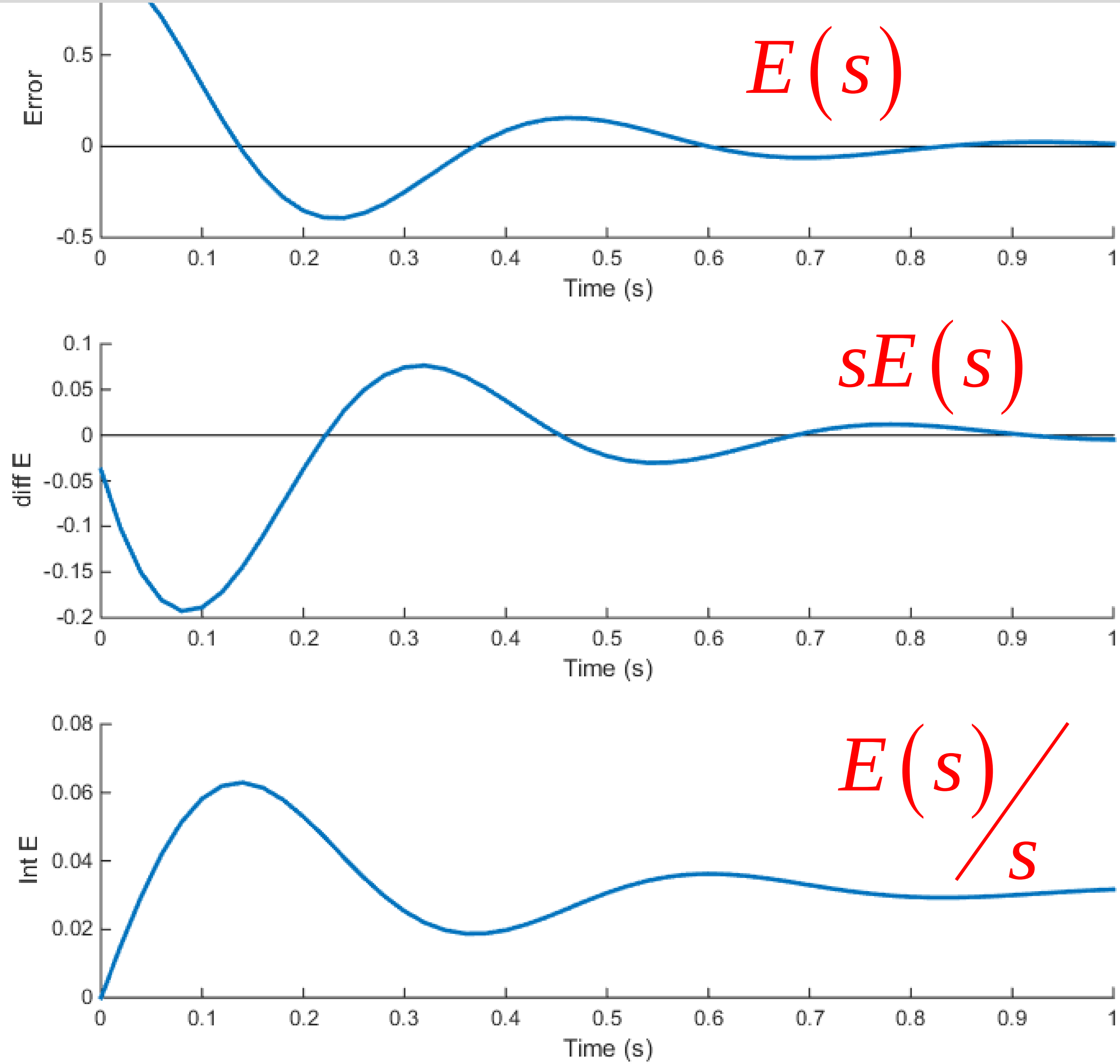
Continuous Systems and Transfer Function Revision: PID Control



Continuous Systems and Transfer Function Revision: Integral Error

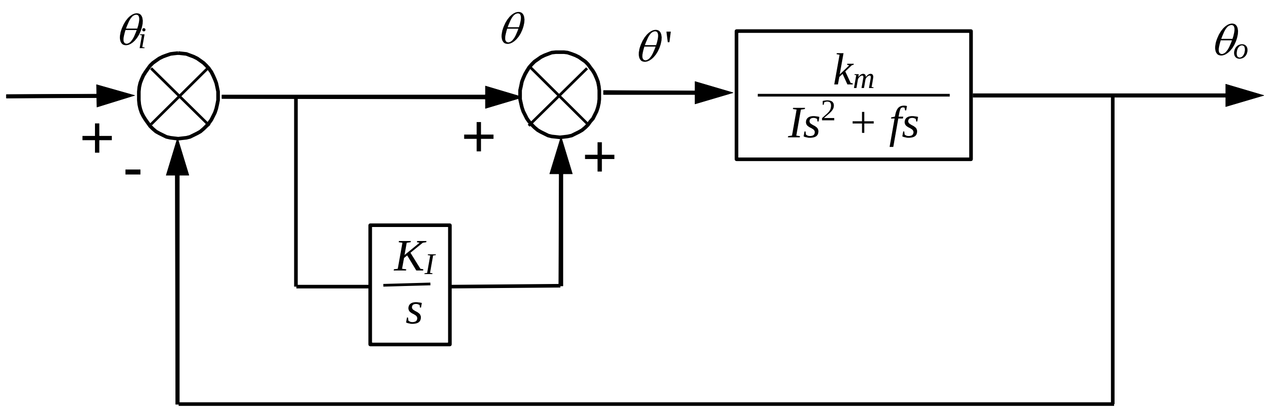
Further, if we consider the **integral** or total error over time:

This gradually increases over time, and can be used to magnify the control signal for small errors, and improve SSE

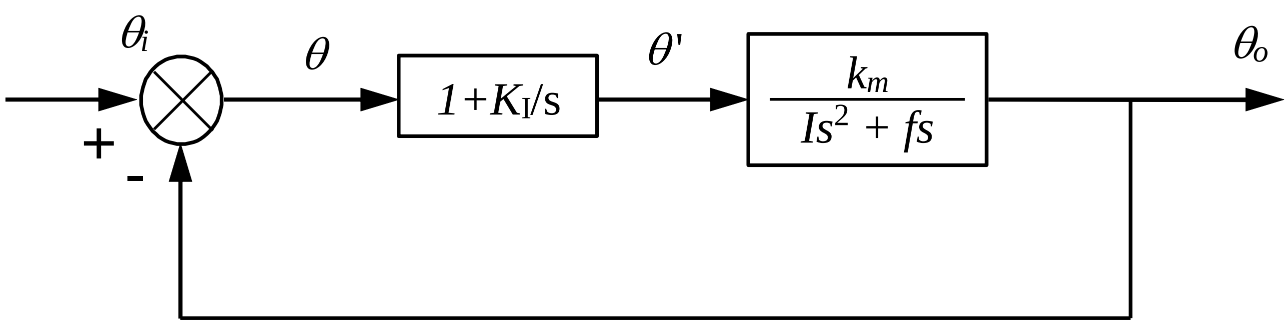


Continuous Systems and Transfer Function Revision: Integral Error

The integral of the error is used for correcting steady state errors, as it adds a pole at zero in the transfer function **Increases type**



$$\theta' = \theta + \frac{K_I \theta}{s} \qquad \frac{\theta'}{\theta} = 1 + \frac{K_I}{s}$$



Continuous Systems and Transfer Function Revision: Integral Error

Type 2 system

$$G' = \frac{\left(1+{K_I}/{s}\right)k_m}{Is^2 + fs} = \frac{\frac{1}{s}\left(s+K_I\right)k_m}{s\left(Is+f\right)} \qquad G' = \frac{k_m s + k_m K_I}{s^2\left(Is+f\right)}$$

With a close loop transfer function of

$$F(s) = \frac{G'(s)}{1+G'(s)} = \frac{k_m s + k_m K_I}{s^2\left(Is+f\right)+k_m s + k_m K_I}$$

$$F(s) = \frac{k_m s + k_m K_I}{Is^3 + fs^2 + k_m s + k_m K_I}$$

3rd Order transfer function

Continuous Systems and Transfer Function Revision: Integral Error

$$F(s) = \frac{k_m s + k_m K_I}{Is^3 + fs^2 + k_m s + k_m K_I}$$

3rd Order transfer function

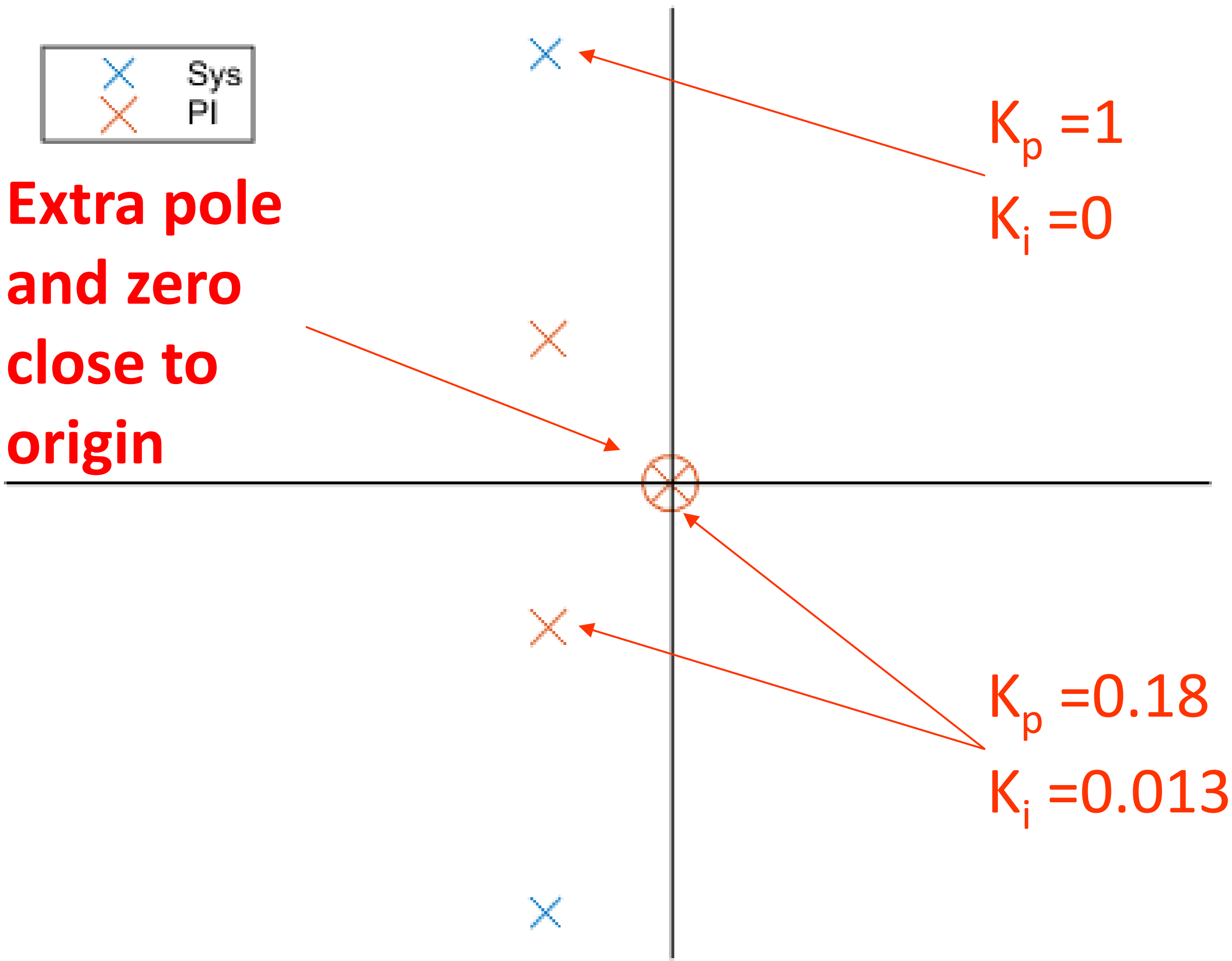
Increasing the system type improves the steady state response:

$$k_v = \lim_{s \to 0} sG'(s) \qquad = \frac{s\left(k_m s + k_m K_I\right)}{s^2\left(Is + f\right)}$$

$$k_v = \frac{k_m K_I}{0} = \infty \qquad SSVL = \frac{a}{k_v} = 0$$

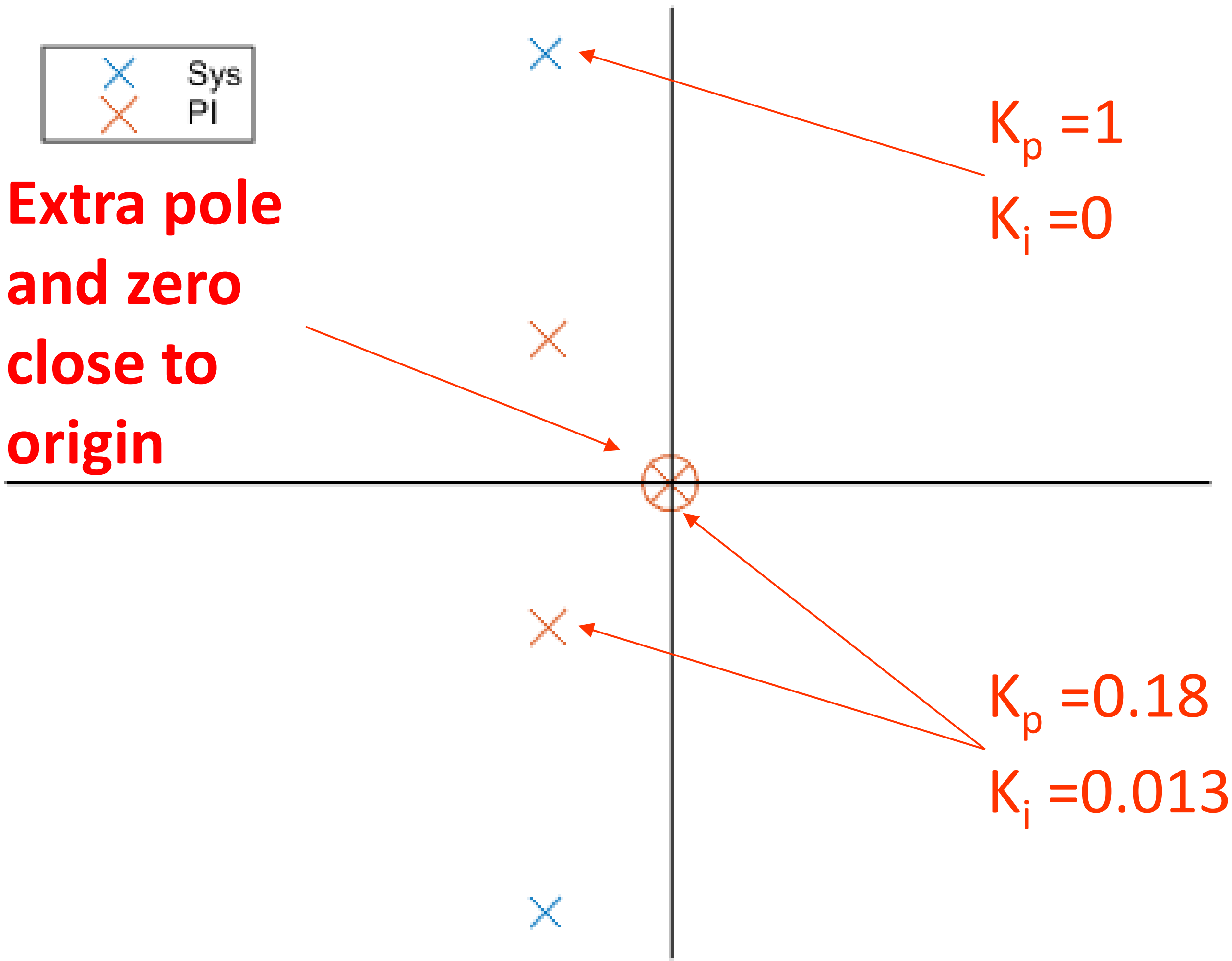
Continuous Systems and Transfer Function Revision: Integral Error

The effect of an extra pole close to origin – which would make our system *very slow* - is largely cancelled out by the nearby zero.



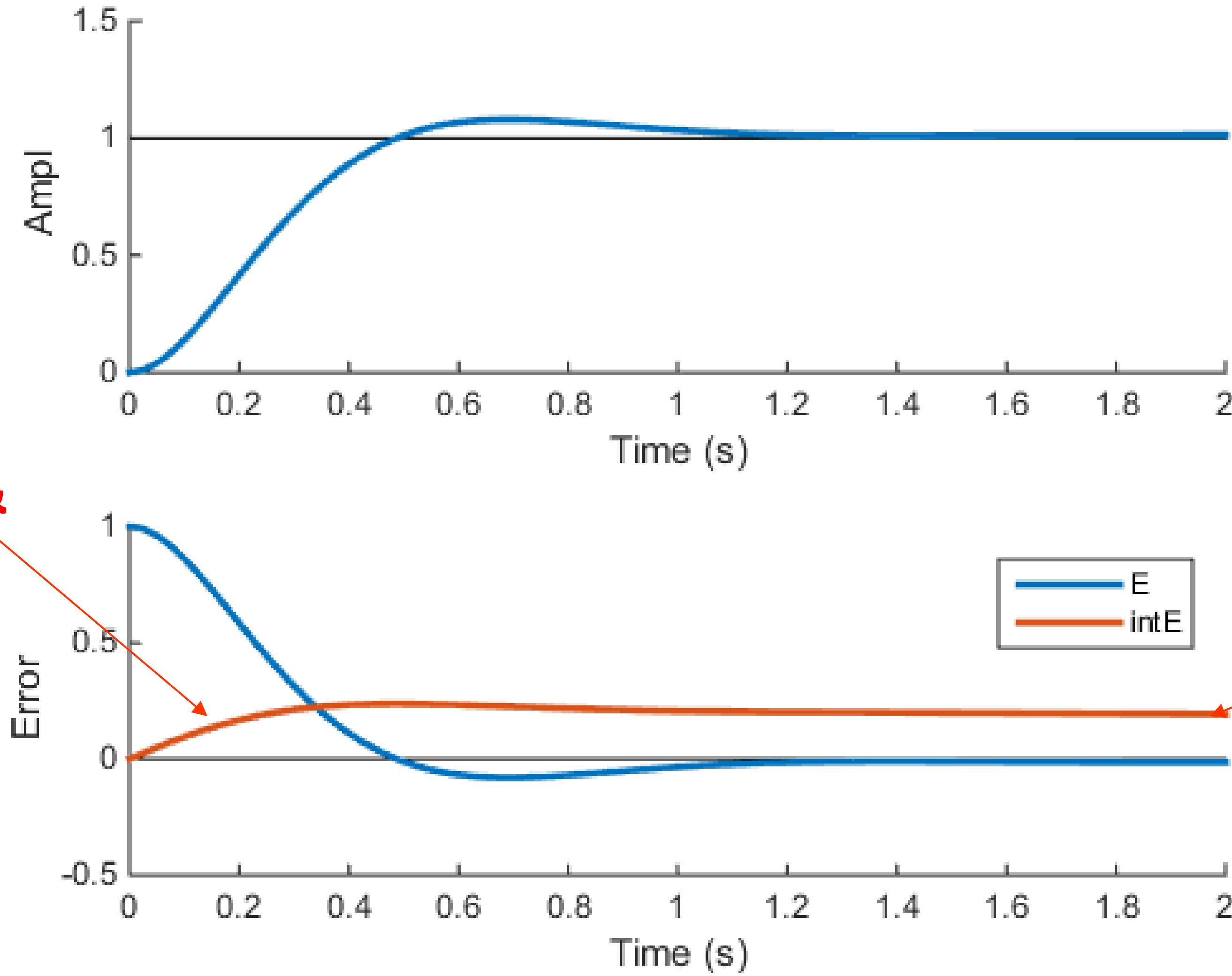
Continuous Systems and Transfer Function Revision: Integral Error

This means the system is broadly similar to a proportional controller, with the exception of improved steady state performance



Continuous Systems and Transfer Function Revision: Integral Error

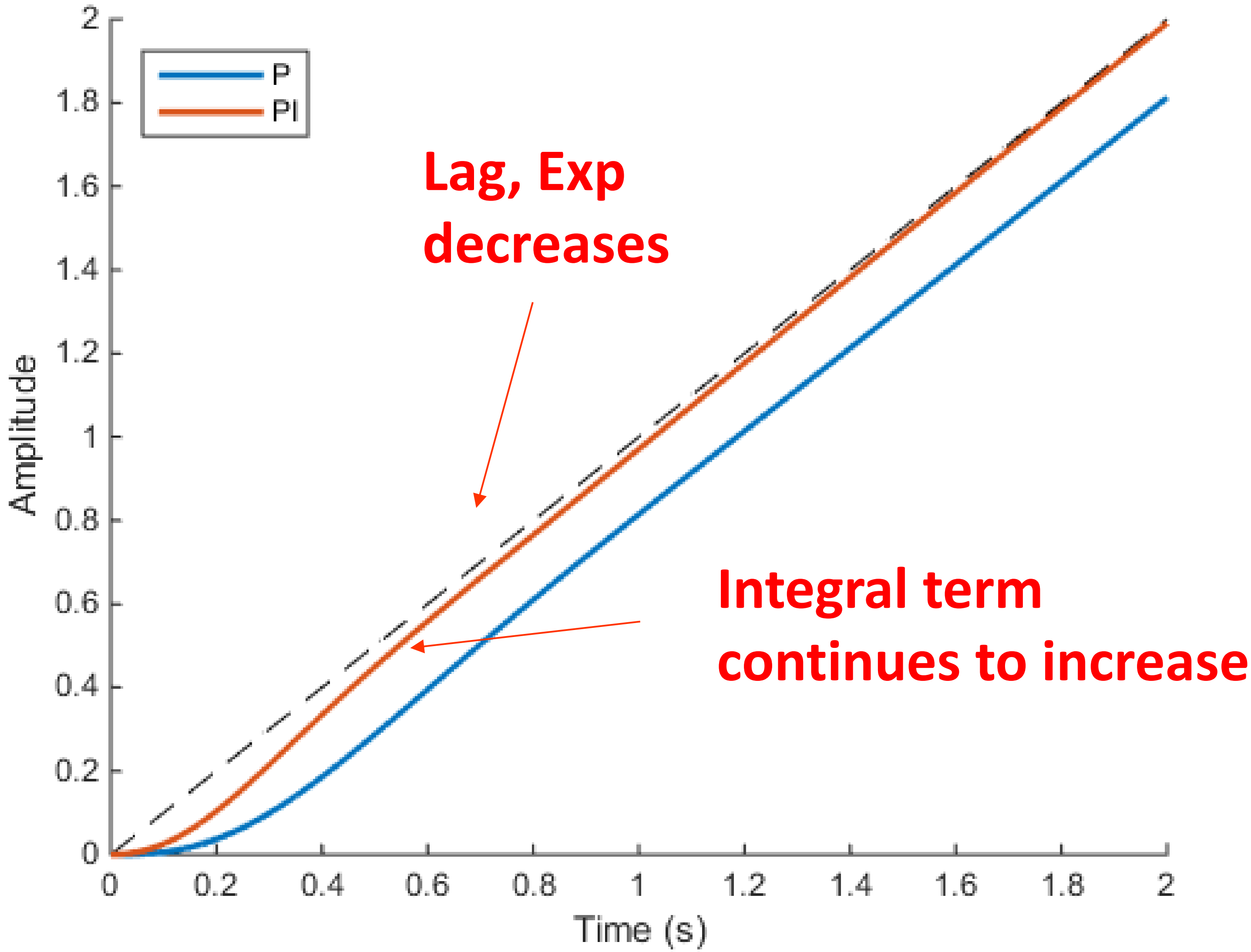
Gradually increases & Reinforces motion



Non Zero after settling

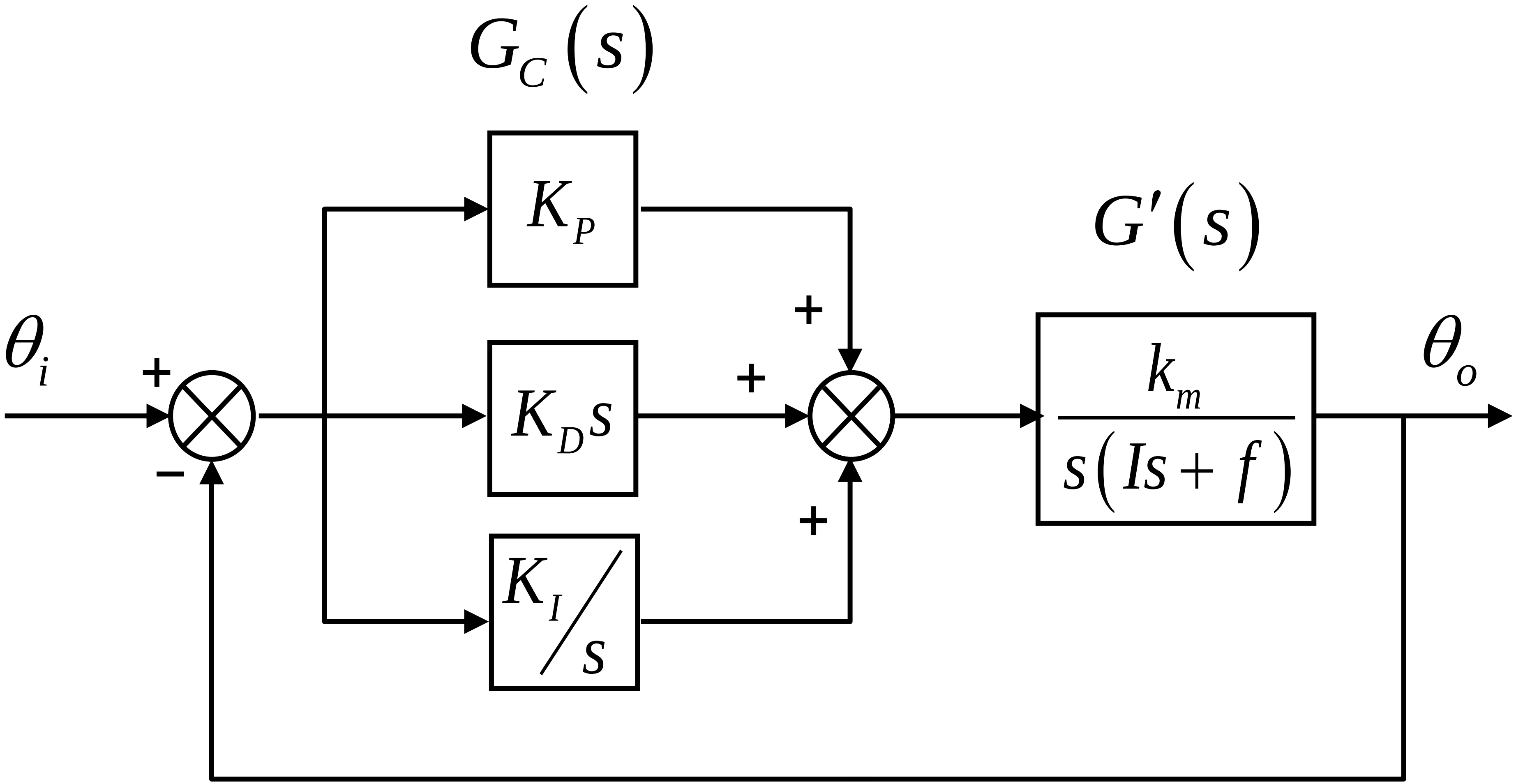
Continuous Systems and Transfer Function Revision: Integral Error

We can see that the integral term has enabled proper tracking of a ramp input



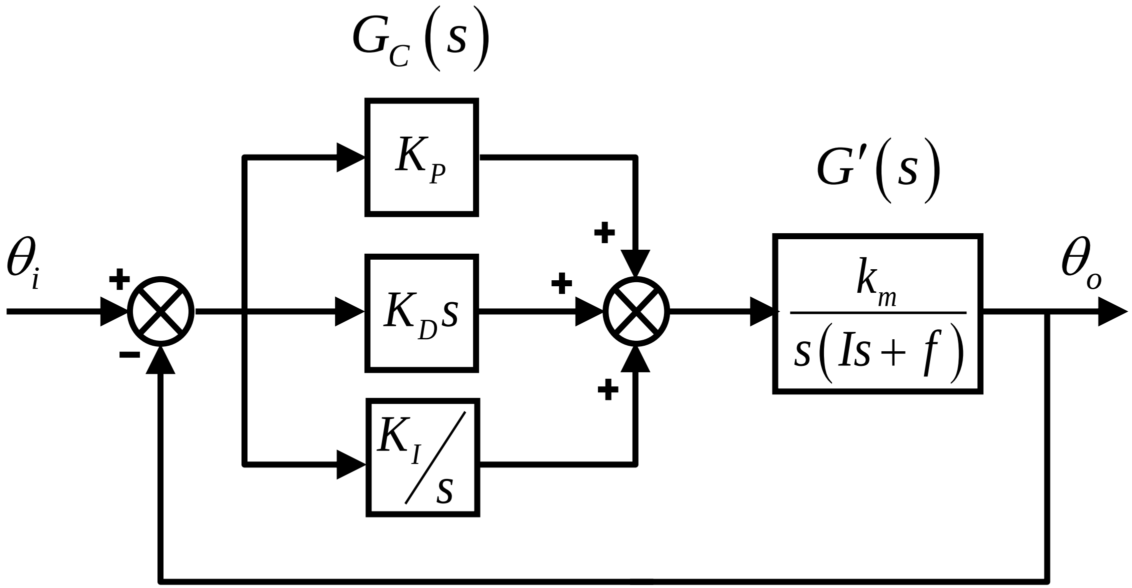
Continuous Systems and Transfer Function Revision: PID control

Putting all three of these controllers together gives the complete PID



$$G_c(s) = K_P + K_D s + \frac{K_I}{s} = \frac{K_D s^2 + K_P s + K_I}{s}$$

Continuous Systems and Transfer Function Revision: PID control



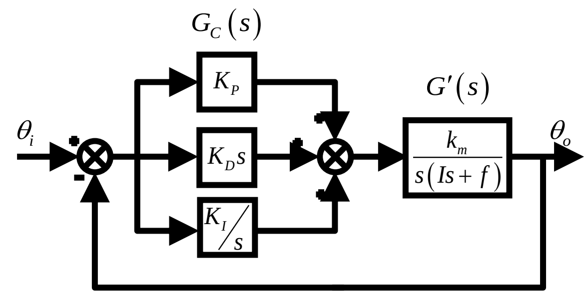
Open Loop gain

$$G'(s) = \frac{k_m\left(K_D s^2 + K_P s + K_I\right)}{s^2\left(Is + f\right)}$$

The closed loop transfer function is then

$$F(s) = \frac{k_m\left(K_D s^2 + K_P s + K_I\right)}{s^2\left(Is + f\right) + k_m\left(K_D s^2 + K_P s + K_I\right)}$$

Continuous Systems and Transfer Function Revision: PID control



$$F(s)=\frac{k_mK_Ds^2+k_mK_Ps+k_mK_I}{Is^3+fs^2+k_mK_Ds^2+k_mK_Ps+k_mK_I}$$

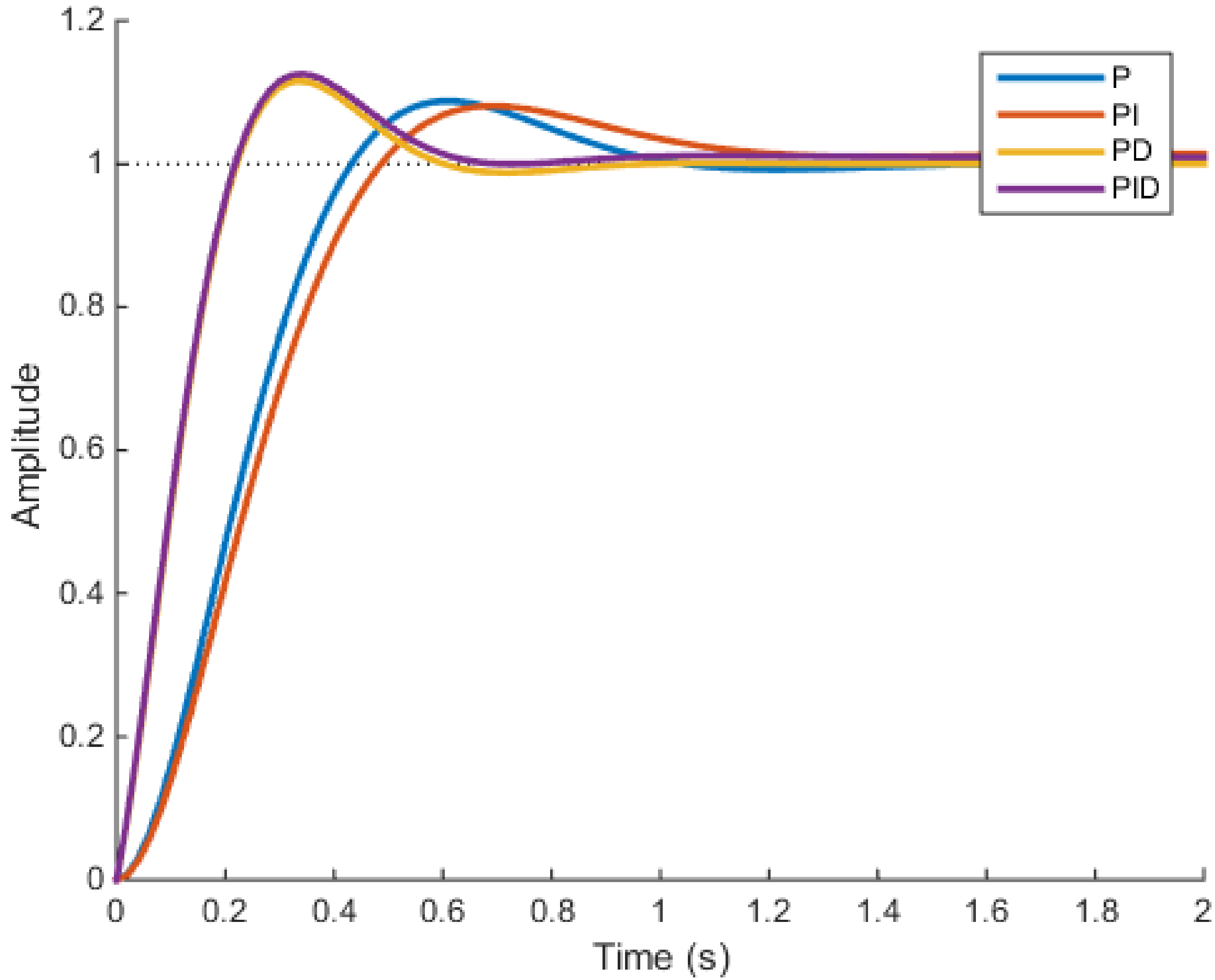
$$F(s)=\frac{k_mK_Ds^2+k_mK_Ps+k_mK_I}{Is^3+\left(f+k_mK_D\right)s^2+k_mK_Ps+k_mK_I}$$

Thus by choosing the appropriate values of K_P K_D K_I

It is possible to design a controller with improved transient response **and** decreased/no steady state error

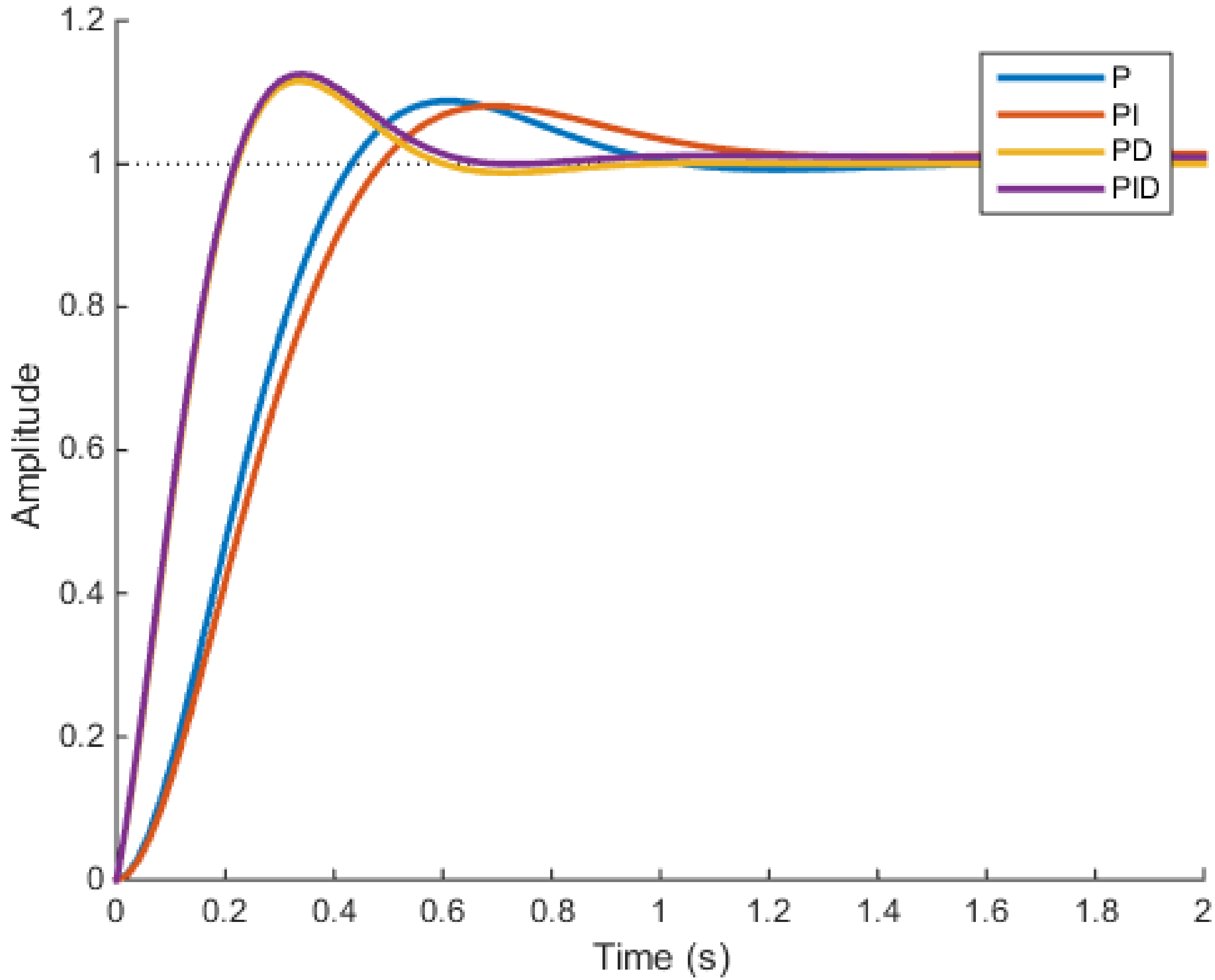
Continuous Systems and Transfer Function Revision: PID control

There is no unique solution for the settings for the three gains, and so the values are dependent upon the specific system and the application.



Continuous Systems and Transfer Function Revision: PID control

Selecting these parameters is known as **tuning**, and it was (and still is in machine learning circles) a very active area of research.



Continuous Systems and Transfer Function Revision: PID control

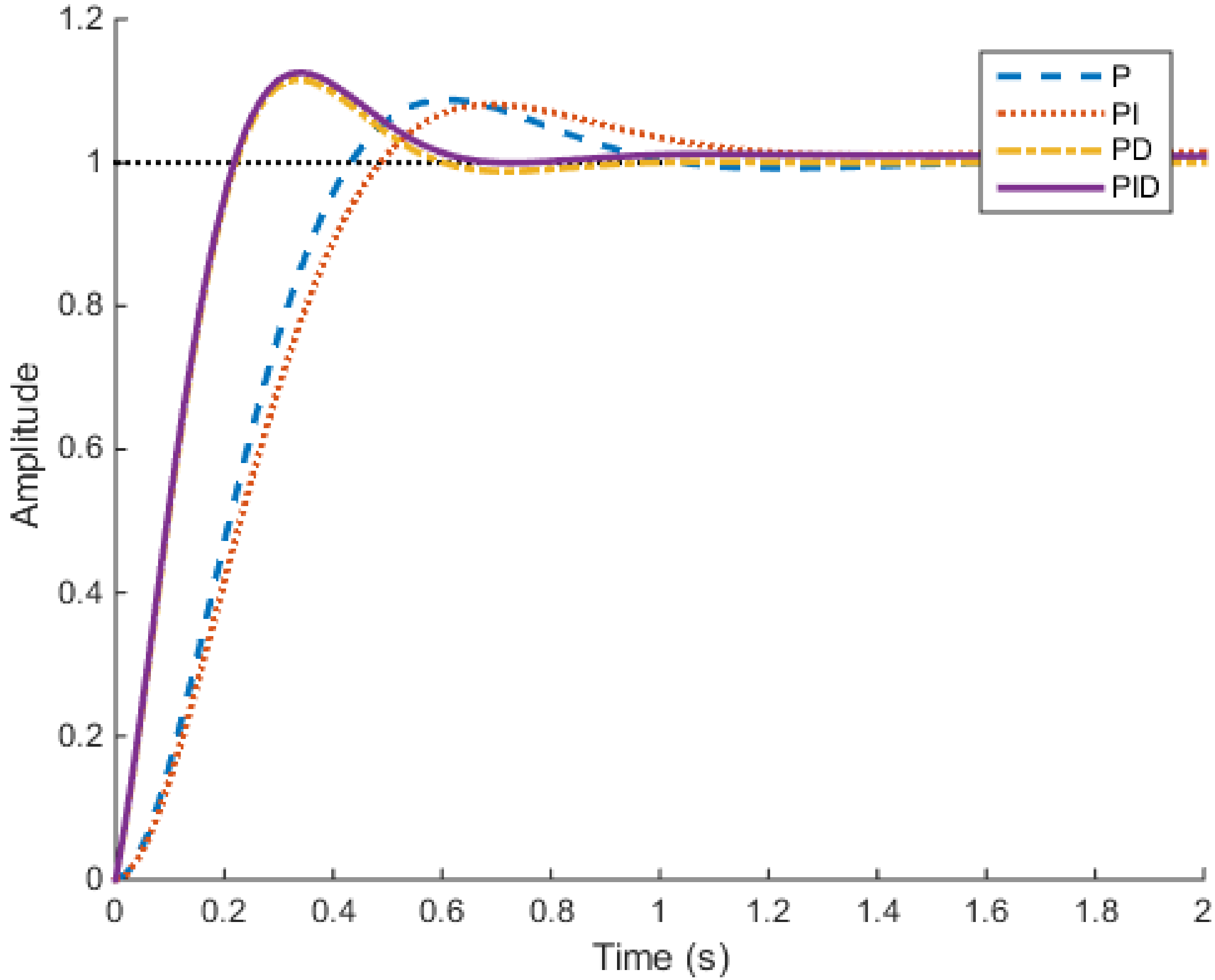
If we set one of the gains to zero, then we remove that term from the controller, e.g.,

$K_I \to 0$ PID becomes PD controller

Qualitatively, the three terms can be thought of as follows:

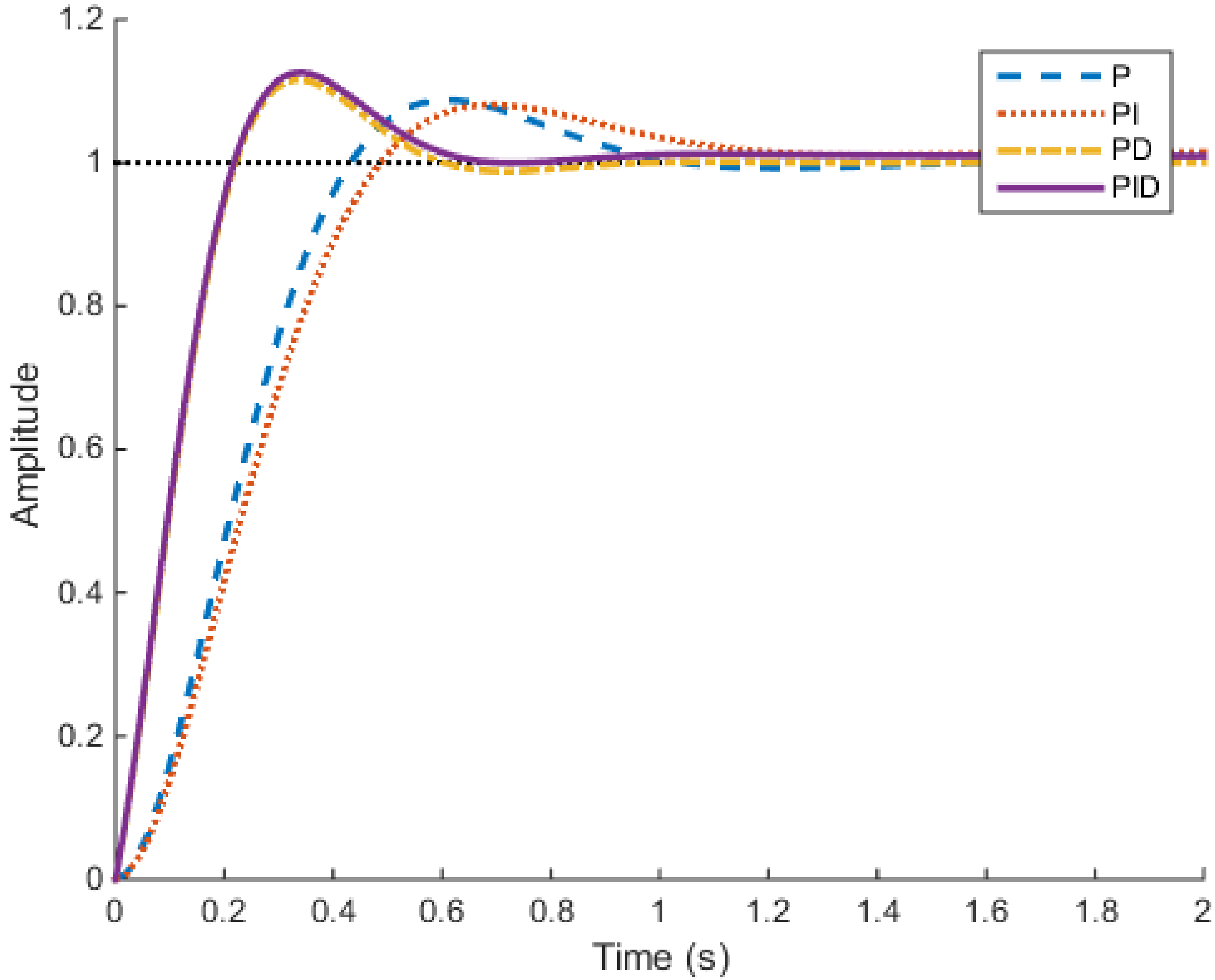
Proportional – Tries to reach target as soon as possible
Derivative – resists overshooting
Integral – Corrects for steady state errors

Continuous Systems and Transfer Function Revision: PID control



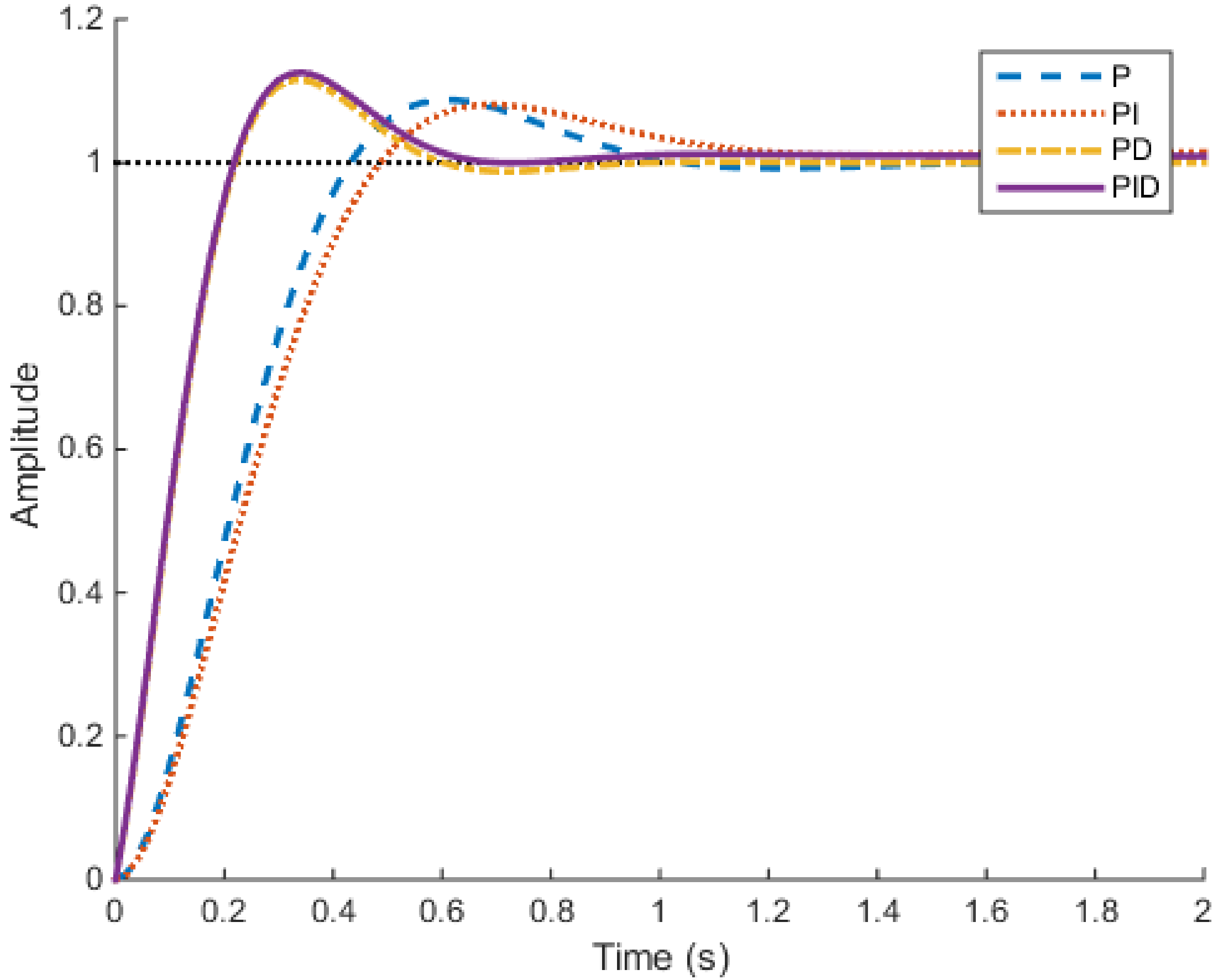
Consider our servo model, the each controller changes the step response in the following ways:

Continuous Systems and Transfer Function Revision: PID control



P – Initially chosen to give ζ close to 0.7 for acceptable transient response.

Continuous Systems and Transfer Function Revision: PID control



PI – I term has little effect on the transient response for this **TYPE 1** system. Steady state error technically 0, but may improve in practice.

Continuous Systems and Transfer Function Revision: PID control

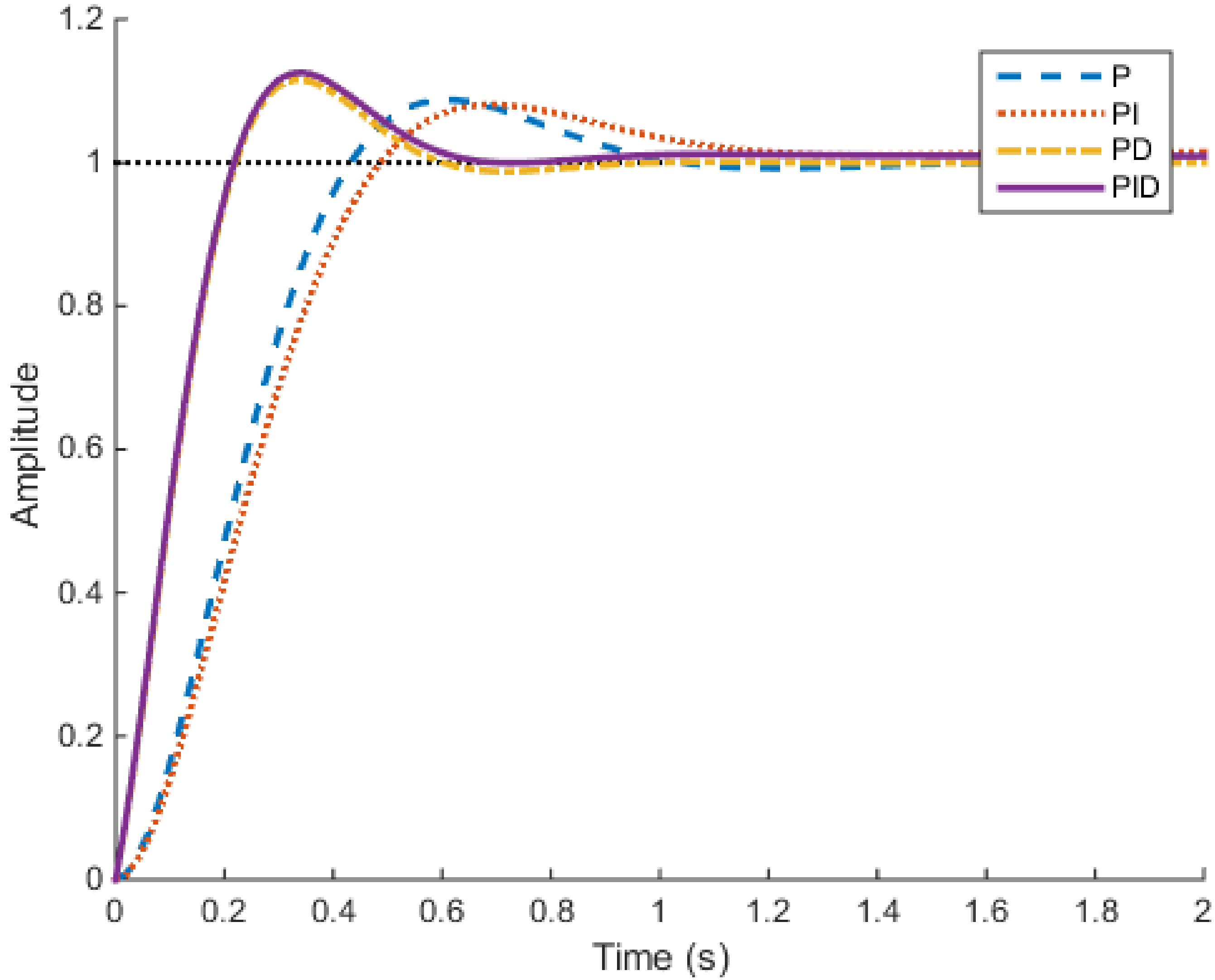
Open loop transfer function:

$$G(s)H(s) = \frac{(s - z_1)(s - z_2)(s - z_3)\ldots}{s^p(s - \sigma_1)(s - \sigma_2)(s - \alpha_k + j\omega_k)(s - \alpha_k - j\omega_k)\ldots}$$

No. Integrators in denominator = system TYPE	Input type		
	Step $r(t) = a$ $R(s) = a/s$	Ramp $r(t) = at$ $R(s) = a/s^2$	Acceleration $r(t) = at^2/2$ $R(s) = a/s^3$
0	$e_{ss} = a/(1+k_p)$	$e_{ss} = \infty$	$e_{ss} = \infty$
1	$e_{ss} = 0$	$e_{ss} = a/k_v$	$e_{ss} = \infty$
2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = a/k_a$

PI – I term has little effect on the transient response for this **TYPE 1** system. Steady state error technically 0, but may improve in practice.

Continuous Systems and Transfer Function Revision: PID control



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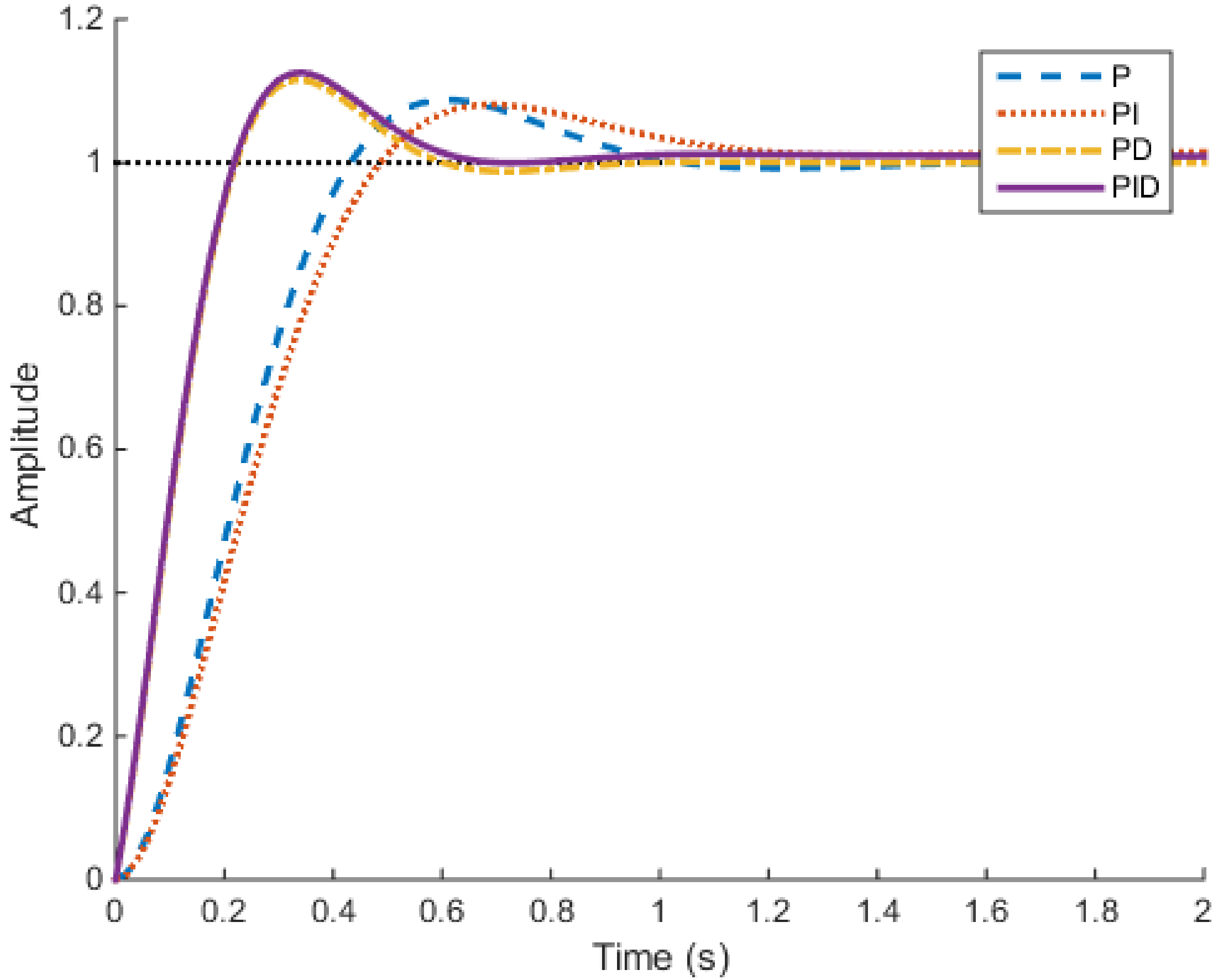
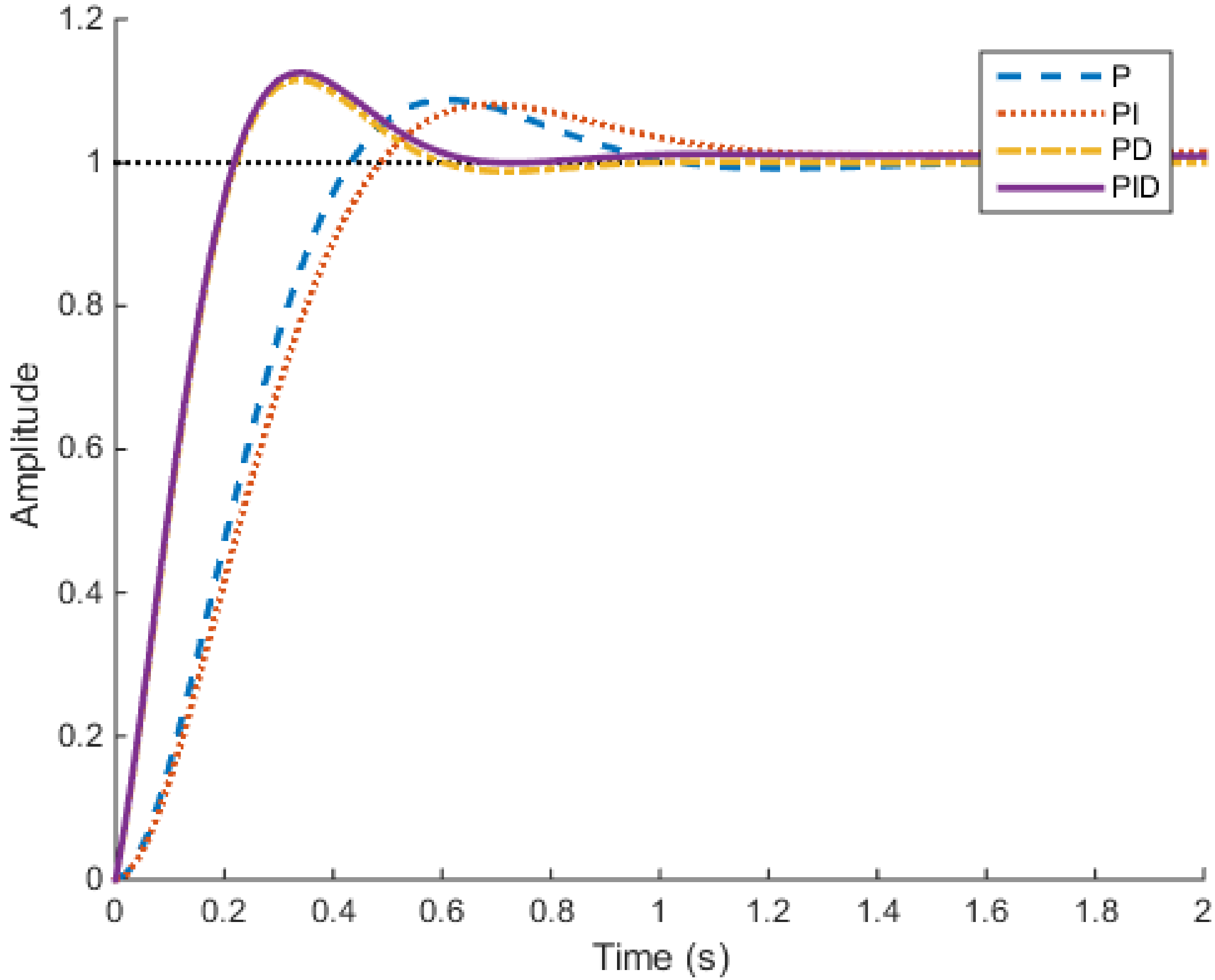
Continuous Systems and Transfer Function Revision: PID control



PD – D term improves transient response as enabling higher P weighting to be used, but with a reduced overshoot. Steady State unchanged.

Continuous Systems and Transfer Function Revision: PID control



PID – Again, transient response largely unchanged, but potential benefits in steady state.