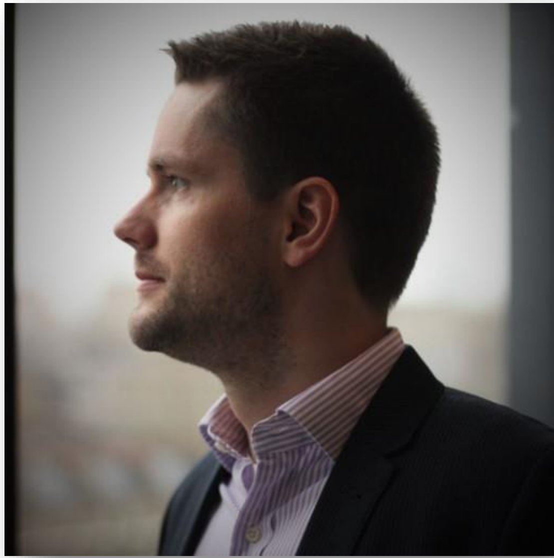
MECH0089
Control and Robotics:
Digital Control Systems

Prof. Helge Wurdemann



Professor of Robotics
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Research interests include the design and creation of:

- bio-inspired, soft and stiffness-controllable robotics
- innovative haptic interfaces
- sensor development
- robotic art

Feedback and Consultation Hours, MS Teams:
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Soft Robotics at UCL MechEng

Digital Control Systems

Aim of the course

The main aims of this course in digital control systems are:

- To give students an appreciation of the need for modern control systems.
- To explain and discuss the various hardware and software systems that make up modern control systems.
- To provide students with mathematical tools needed in the design and undertake performance analysis of digital control systems.

Assessment

Control exam (50%)
Robotics exam (50%)

Structure of the course

Continuous Systems and Transfer Function Revision

1. Introduction to Digital Control Systems

1.1 Review and Classification of Control Systems

1.2 Analogue and Digital Control Systems

1.3 Components of a Digital Control System

1.4 Sampling Theorem

2. Mathematics of Digital Control Engineering

2.1 Continuous Systems and Transfer Function Revision

2.2 Discrete Time Systems and Linear Difference Equations

2.3 z Transform

2.4 Transfer Function

2.5 Inverse z Transform

Structure of the course

3. Discrete Time Systems

3.1 z Domain Transfer Function

3.2 Stability Criteria

3.3 Time Domain Response

3.4 Frequency Response

4. Discrete Control Systems

4.1 Equivalent Continuous Time Design

4.2 Realization and Implementation

4.3 Discrete PID Controller Design

4.4 Digital Control Applications

Tutorial Sheets will be issued for chapters 2, 3 and 4.

Literature/reading material

[1] Digital Control Systems - B.C. Kuo
Saunders College Publishing

[2] Design of Feedback Control Systems
R.T. Stefani, B. Shahian, C.J. Savant Jr., and G.H. Hostetter
Oxford University Press

[3] Feedback Control of Dynamic Systems - G.F. Franklin, J.D. Powell, & Abbas Emami-Naeini Pearson
(Older versions are titled 'Digital Control of Dynamic Systems' – by Franklin & Powell)

[4] Real-Time Computer Control: An Introduction - S. Bennett
Prentice Hall International

[5] Control Systems Theory - O.I. Elgerd
McGraw-Hill

[6] The Art of Control Engineering - K. Dutton, S. Thompson, B. Barraclough
Addison-Wesley

[7] Control System Design and Simulation - J. Golten, A. Verwer
McGraw-Hill

Lecture style and expectations

- Each lecture will have a set of notes
  - There will be a few blanks for you to fill in (worked examples)
  - I will speak around the topic area – annotate your notes as you feel is appropriate.
  - Notes and overhead slides will be made available on moodle within 24 hours after each lecture.

- Tutorial Sheets
  - Should be attempted soon after the lectures, to reinforce the material and give you opportunities to ask questions.

Feel free to approach me with any feedback on teaching style, etc.

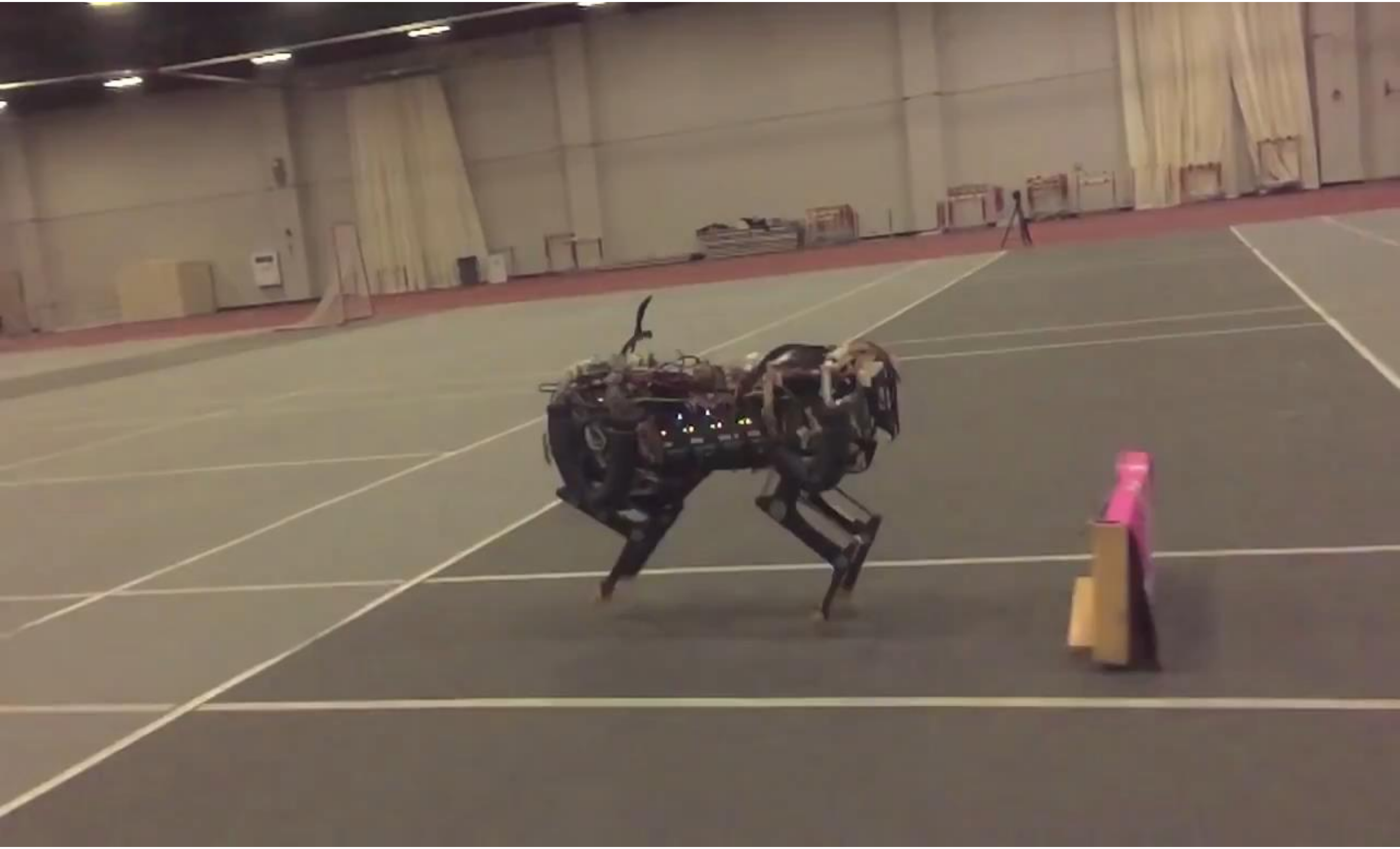
Control has a wide range of applications throughout engineering:

- Autopilot systems for aircraft and ships.
- Position control servomechanisms used in a variety of applications - radar tracking, machine tools, tv cameras etc.
- Machinery plant control.
- Robotics.

Control examples: Multi-legged robots

Control examples: Multi-legged robots

Control examples: Multi-legged robots



Control examples: Autonomous vehicles

Control examples: Drones

MECH0089
Control and Robotics:
Digital Control Systems

Lecture 1:
Continuous Systems and Transfer Function Revision (part 1)

A note about the notes

When you see something

In red like this…

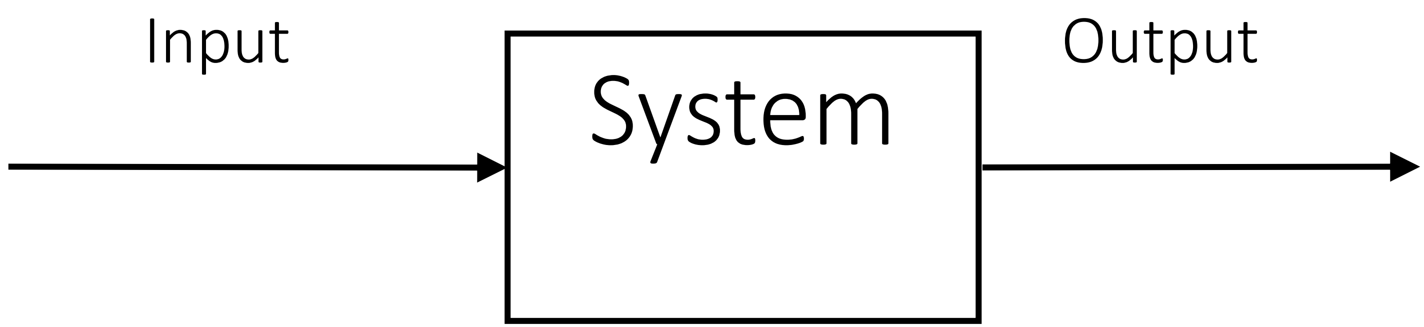
Its missing from your notes so write it down!

Continuous Systems and Transfer Function Revision

Continuous Systems and Transfer Function Revision

2.1.1 Continuous time signals

We want to derive an expression for y(t) in terms of x(t).



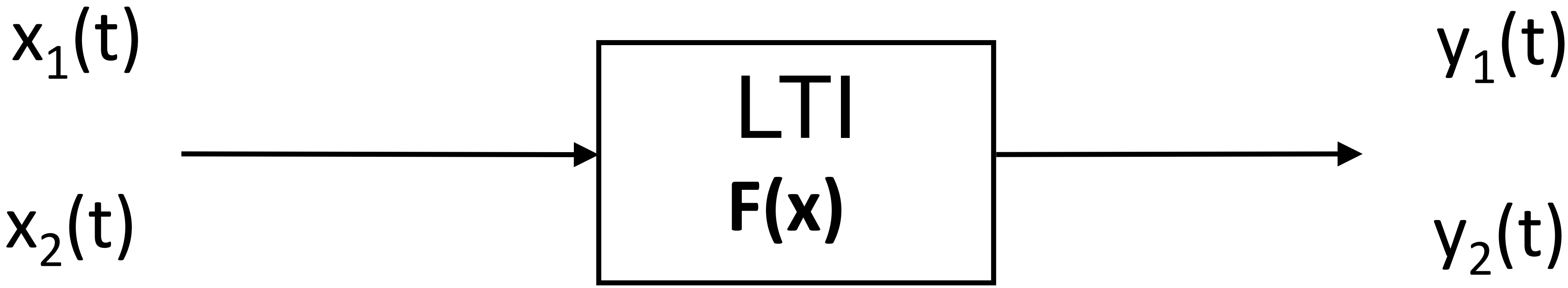
Now we will investigate how we model physical systems, and obtain the function block F(x) for electrical and mechanical components.

The focus of this course, and indeed much of control theory itself focuses on modelling physical systems as *linear* and *time invariant*.

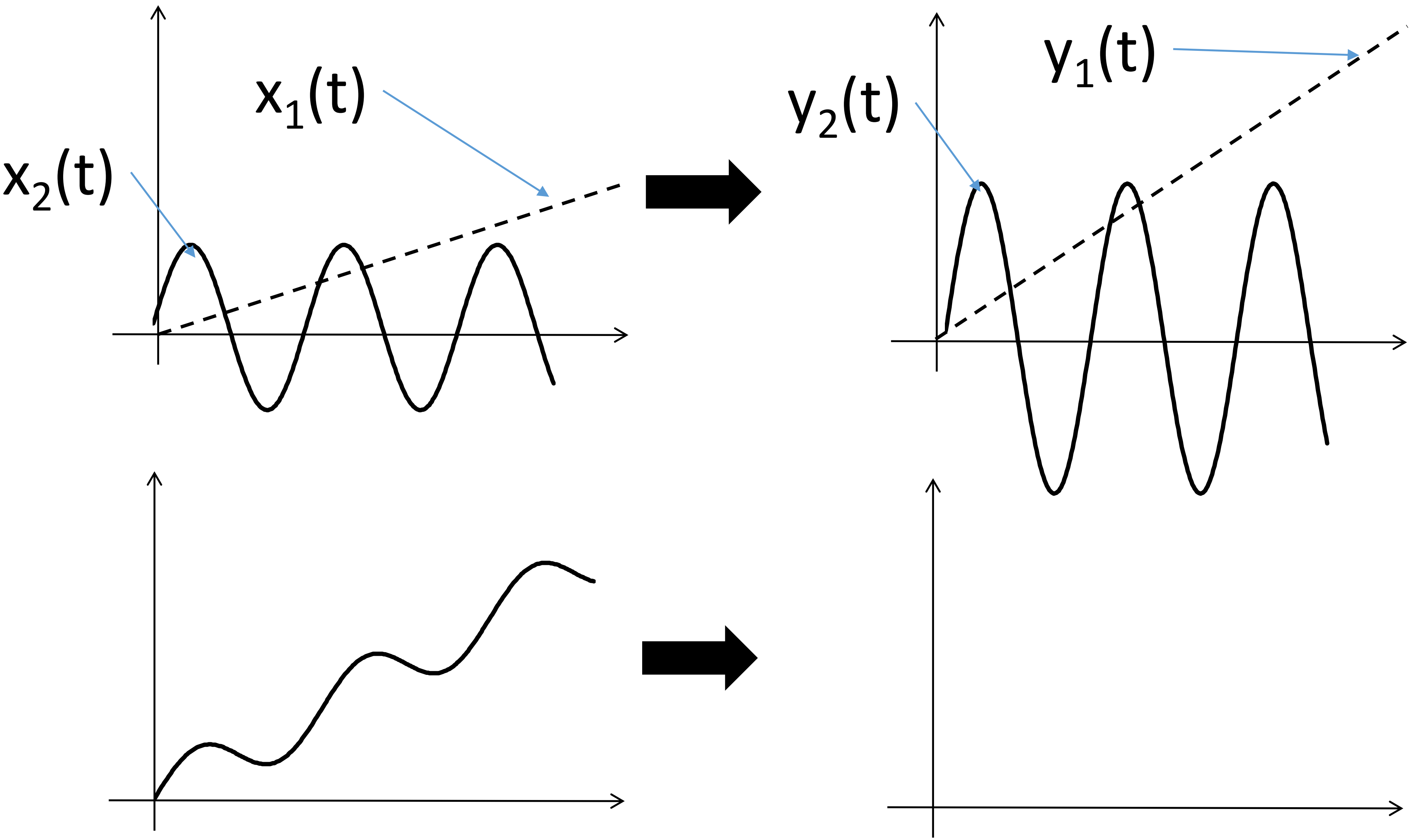
LTI systems have three key properties:

- Obey principle of superposition
- Homogeneity
- Time Invariance

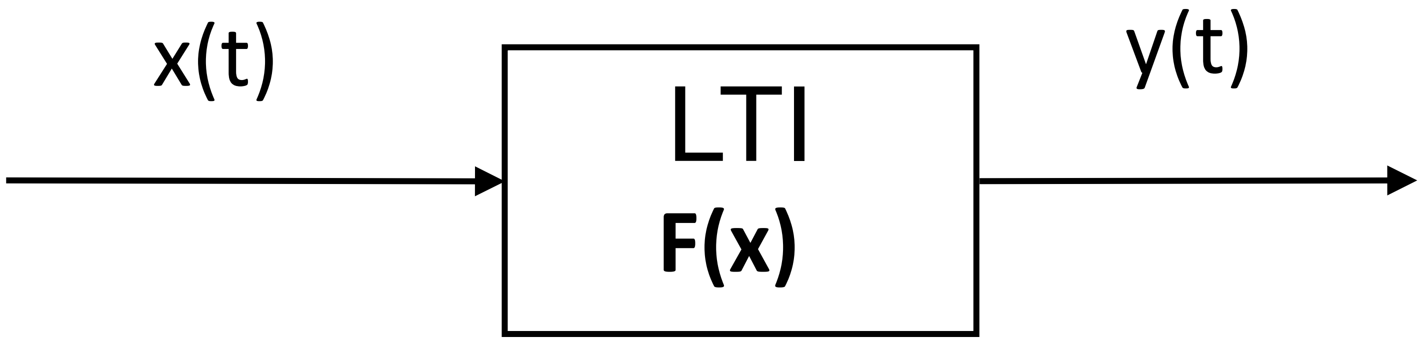
If input $x_1(t)$ produces output $y_1(t)$, and input $x_2(t)$ produces $y_2(t)$, then input $x_1(t) + x_2(t)$ produces output $y_1(t) + y_2(t)$.



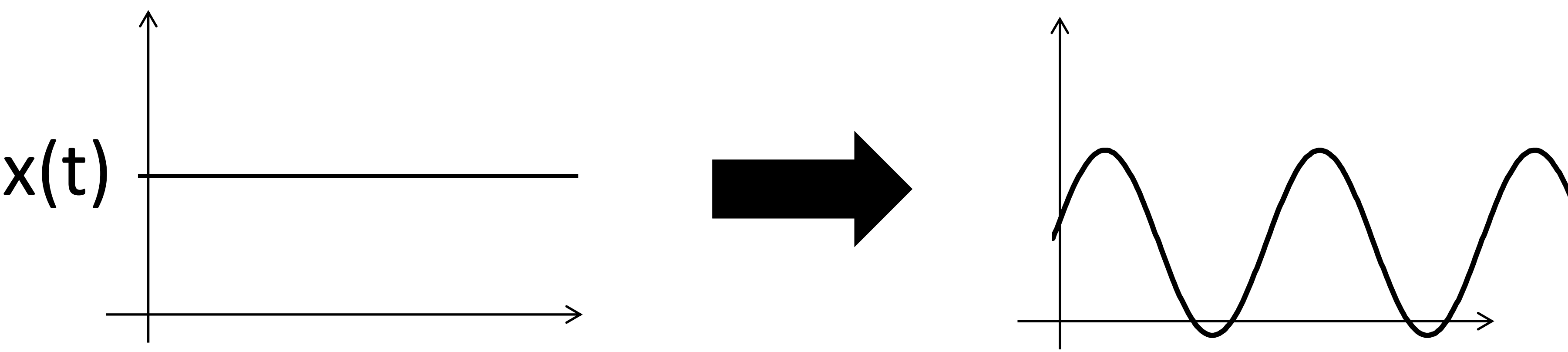
Say for a system which doubles the input F(x) =2x :



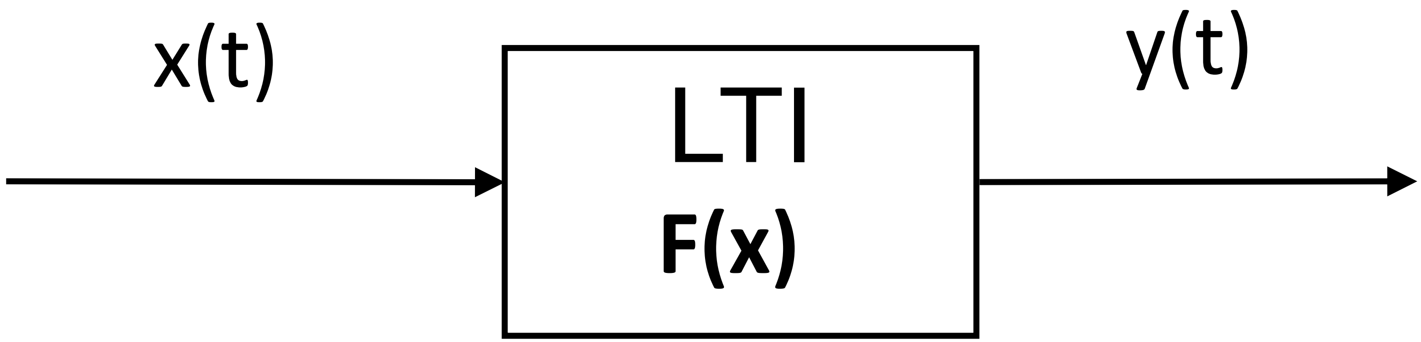
If the input to the system x(t) is scaled by a magnitude scale factor *A*, then the output y(t) is also scaled by the same factor.

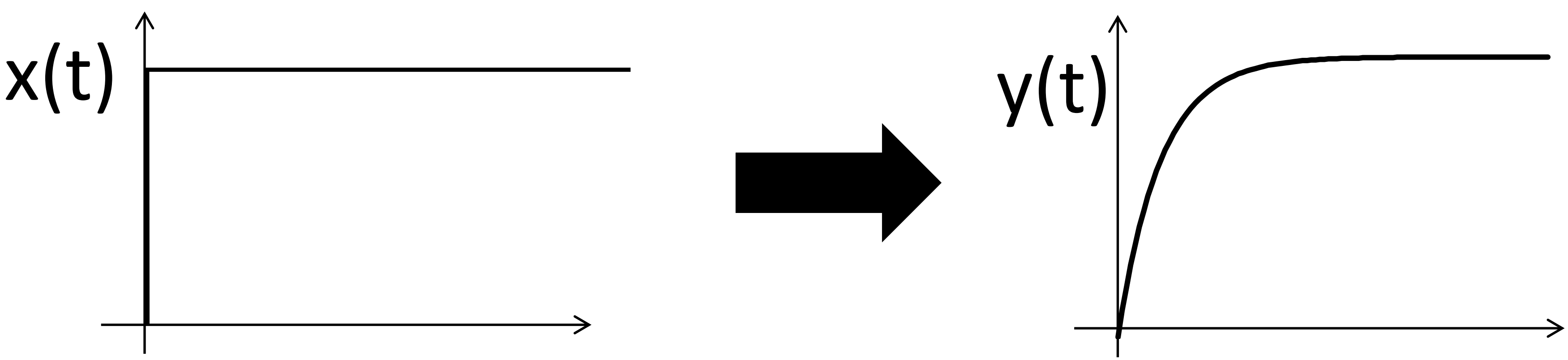


For example, consider a system which generates a sine wave at a given amplitude, with a set frequency:



If input is applied at time t=0 or *T* seconds from now, the output is identical with the exception of a delay of *T* seconds.





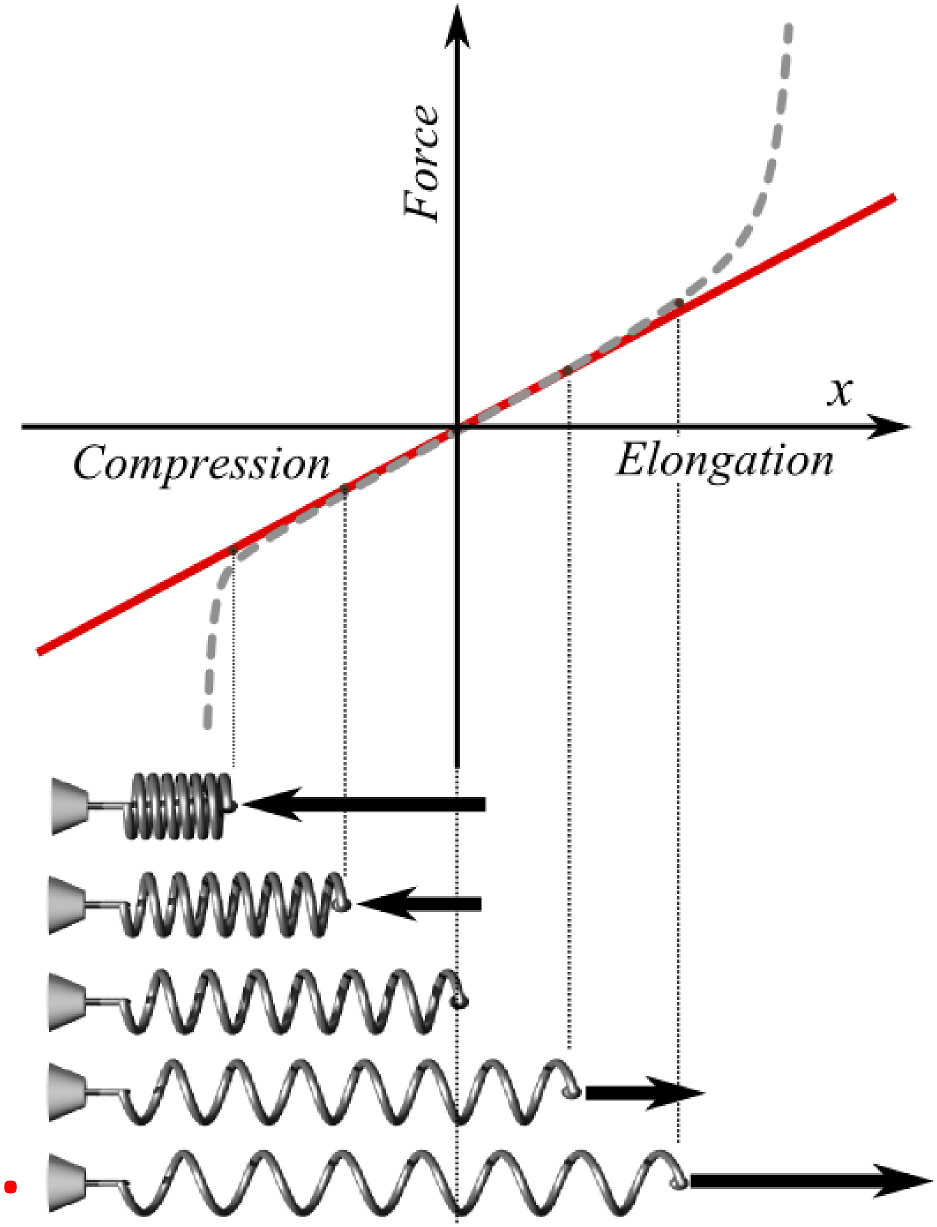
Continuous Systems and Transfer Function Revision: LTI systems example

For a simple system such as a spring, across all possible compressions or extensions the response is non linear:

$$F = -kX$$

Hookes law is only a linear approximation of the true response

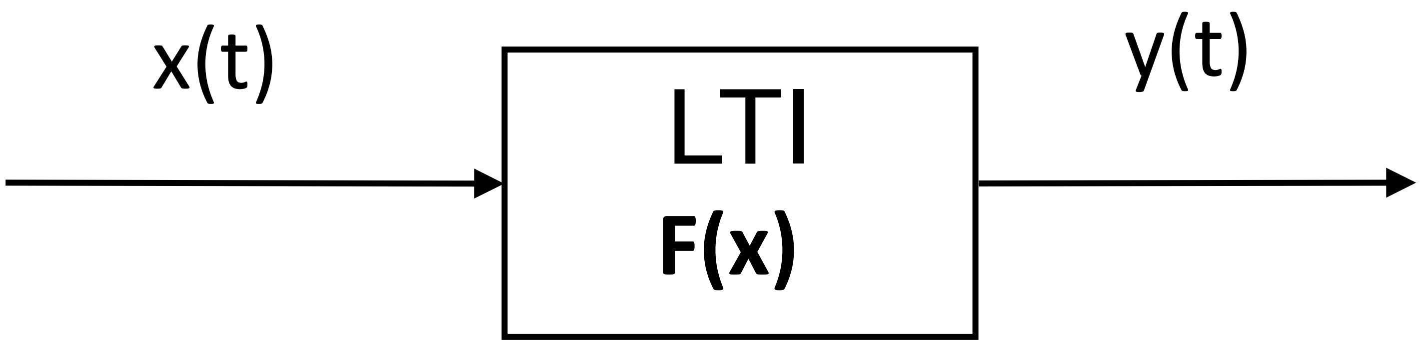
However, if we choose the operating range of the spring correctly, the response is within the linear region



And the approximation is valid.

Continuous Systems and Transfer Function Revision

As we are interested in describing something that *changes* with time, it is useful to express the function block of the system F(t) as an ordinary differential equation (ODE).



$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_o = bx$$

x is input function or forcing function

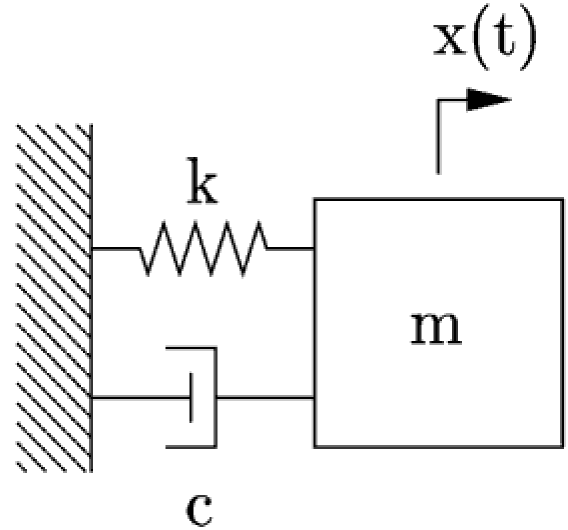
y is output

n is *order* of the ODE

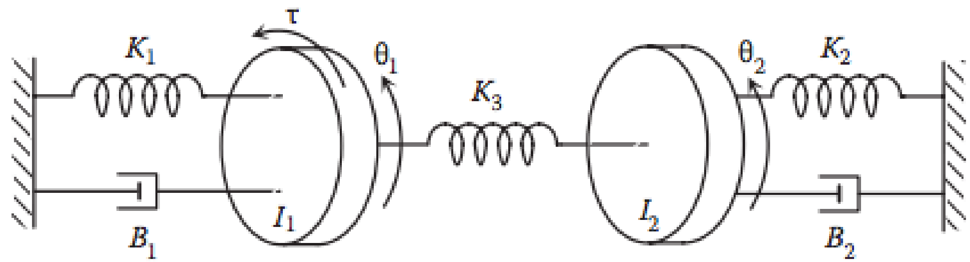
a_0… are coefficients. These *completely characterise the system*

Mechanical systems consist of three basic types of elements:

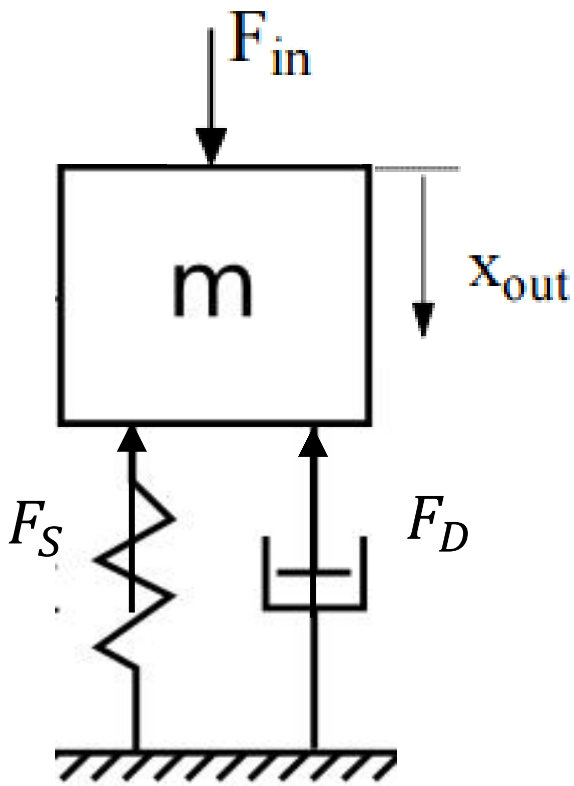
- Inertia
  Examples: mass, moment of inertia
- Spring
  Examples: translational/rotational spring
- Damper
  Examples: dashpot, friction, wind drag



	Translational
Spring	$F = kx$
Dashpot	$F = c\, dx/dt$
Mass	$F = m\, d^2x/dt^2$
	Rotational
Spring	$T = k\theta$
Damper	$T = c\, d\theta/dt$
Moment of inertia	$T = J\, d^2\theta/dt^2$



Continuous Systems and Transfer Function Revision: Mass/spring/damper system



Inertia: $F = ma = m\dfrac{d^2x}{dt^2}$

Damping: $F_D = Dv = D\dfrac{dx}{dt}$

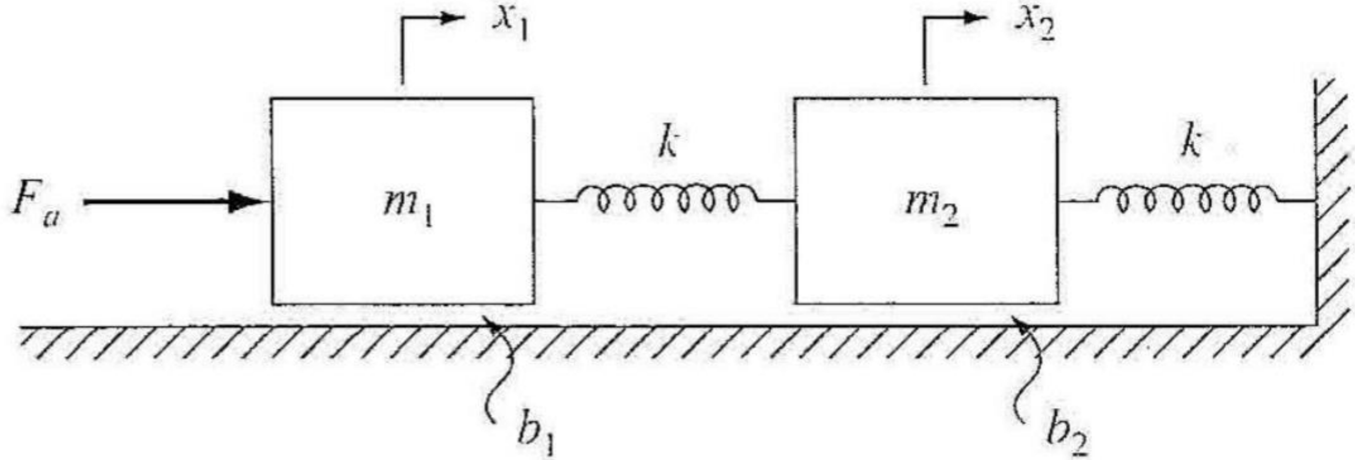
Spring: $F_S = kx$

$$m\frac{d^2x_{out}}{dt^2} = -kx - D\frac{dx}{dt} + F_{in}$$

$$m\frac{d^2x_{out}}{dt^2} + D\frac{dx}{dt} + kx = F_{in}$$

Continuous Systems and Transfer Function Revision: Mass/spring systems

Derive the equation of motion for x_2 as a function of F_a.



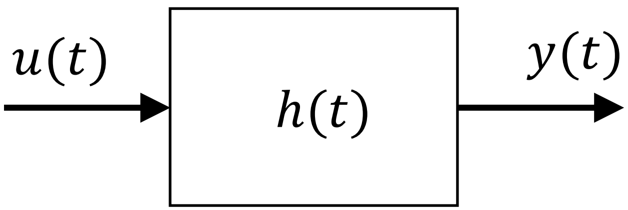
Continuous Systems and Transfer Function Revision: Mass/spring systems

Time vs frequency domain
Transfer function

Continuous Systems and Transfer Function Revision: Time domain

2.1.4 Convolution Approach

Time domain



- u is the impulse function to the system
- h is called the impulse response of the system

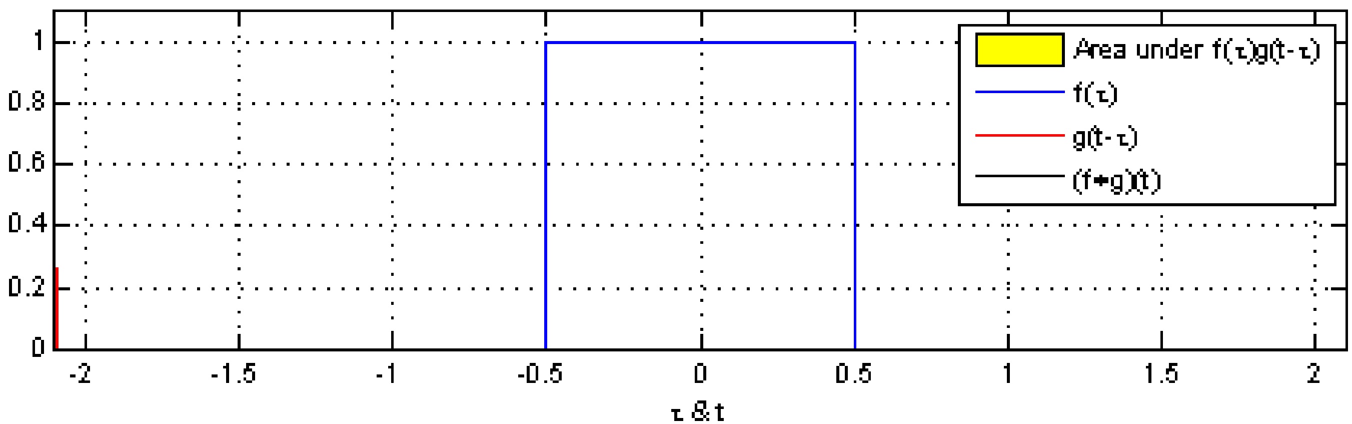
$$y(t) = \int_0^\infty h(\tau)u(t-\tau)d\tau = \int_0^\infty h(\tau - t)u(\tau)d\tau$$

with $0 \leq \tau \leq t$

Convolution

$$y(t) = h(t) * u(t)$$

Essentially, the steps for convolving two signals are to first reflect the signal *g*, then offset the reflected signal. Then calculate the area under the graph for every offset, by sliding –*g.* The convolution at each time point is equal to the area under the intersection of functions. For two pulses, the result is a triangle wave:



However, the calculations to obtain this result in the time domain are complicated, but are only multiplication in the Laplace domain.

Continuous Systems and Transfer Function Revision: Time vs frequency domain example

The impulse response of a certain system is given by ***g(t)*** below. Use the convolution integral to determine the response ***y(t)*** due to a ramp input ***x(t)*** also below:

g(t) = 0 for t< 0 and by
g(t) = e^{-2t} for t $\geq$ 0

x(t) = 0 for t< 0 and by
x(t) = 4t for t $\geq$ 0

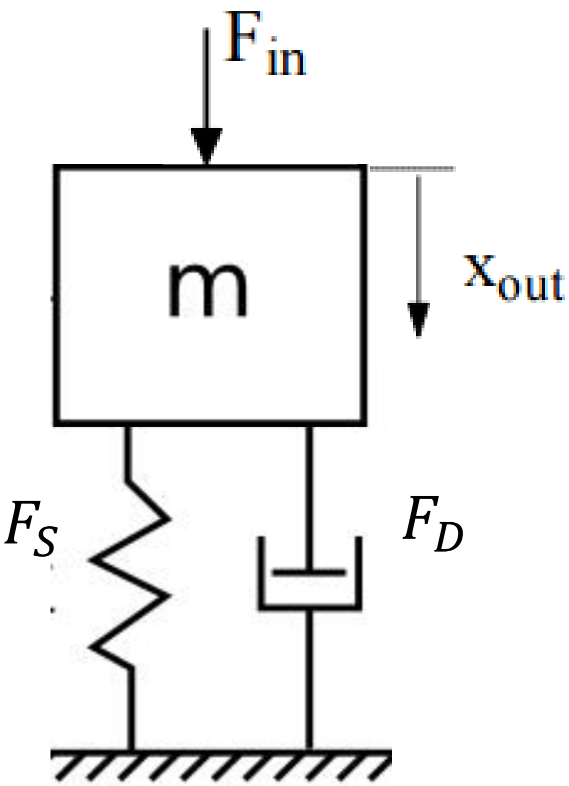
$$y(t) = x(t) * g(t) = \int_{-\infty}^{\infty} x(t-\tau)g(\tau)d\tau$$

$$= \int_{-\infty}^{0} x(t-\tau)\cancel{g(\tau)}d\tau + \int_{0}^{t} x(t-\tau)g(\tau)d\tau + \int_{t}^{\infty} \cancel{x(t-\tau)}g(\tau)d\tau$$

$$= \int_{0}^{t} 4(t-\tau)e^{-2\tau}d\tau = 4t\int_{0}^{t} e^{-2\tau}d\tau - 4\int_{0}^{t} \tau e^{-2\tau}d\tau$$

$= e^{-2t} + (2t-1)$ for t$\geqq$0

Continuous Systems and Transfer Function Revision: Mass/spring/damper system



Inertia: $F = ma = m\frac{d^2x}{dt^2}$

Damping: $F_D = Dv = D\frac{dx}{dt}$

Spring: $F_S = kx$

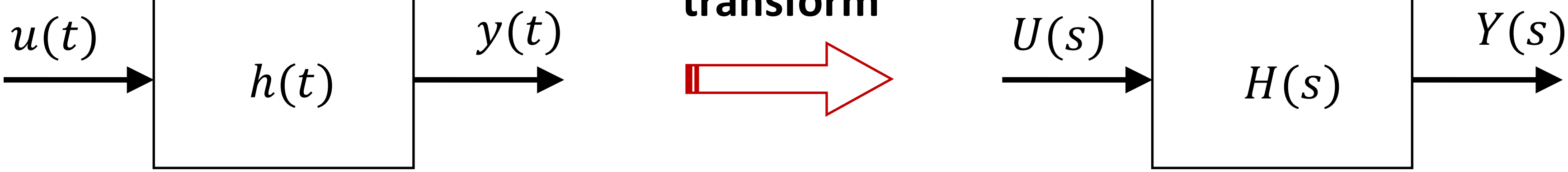
$$m\frac{d^2x_{out}}{dt^2} = -kx - D\frac{dx}{dt} + F_{in}$$

$$\boxed{m\frac{d^2x_{out}}{dt^2} + D\frac{dx}{dt} + kx = F_{in}}$$

Continuous Systems and Transfer Function Revision: Time vs frequency domain

2.1.4 Convolution Approach

Time domain | **Laplace transform** | **s-domain**



- u is the impulse function to the system
- h is called the impulse response of the system

- H is called the transfer function (TF) of the system

$$Y(s) = H(s) \cdot U(s)$$

Multiplication

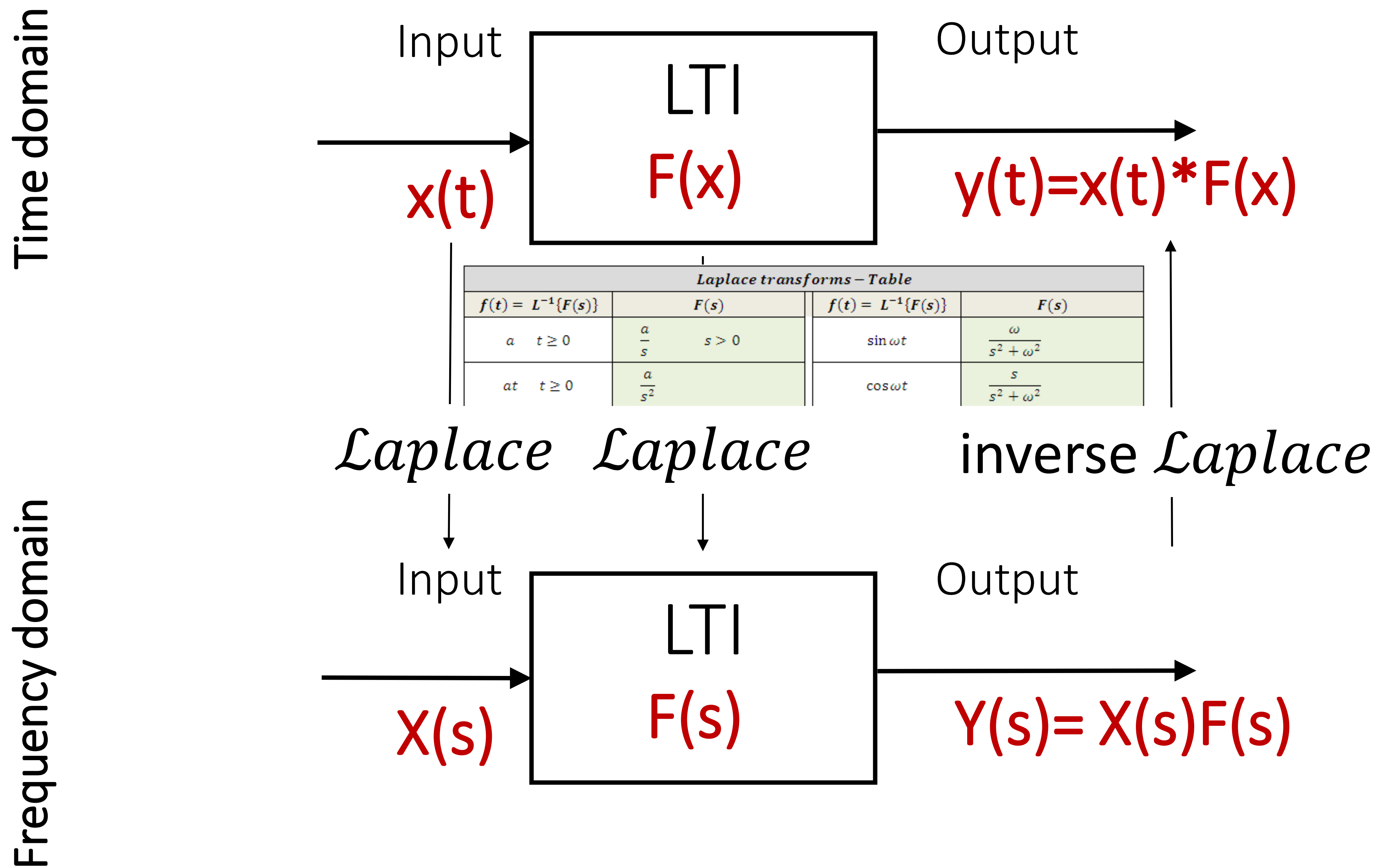
$$y(t) = \int_0^\infty h(\tau)u(t-\tau)d\tau = \int_0^\infty h(\tau - t)u(\tau)d\tau$$

with $0 \leq \tau \leq t$

Convolution

$$y(t) = h(t) * u(t)$$

Continuous Systems and Transfer Function Revision: Time vs frequency domain



$Laplace\ transforms - Table$			
$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \quad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$

2.1.5 Laplace Transforms

Because of our **linear assumptions** we can use Laplace transforms to simplify solving the ODEs.

The Laplace transform of a signal (function) x is the Function $X = \mathcal{L}(x)$ defined by

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

for those $s \in \mathbf{C}$ for which the integral makes sense.

- X is a complex-valued function of complex numbers.
- s is called the (complex) *frequency variable*, with units $\sec^{-1}$, t is called the *time variable* (in sec); st is unitless.
- $s = \sigma + j\omega$

Continuous Systems and Transfer Function Revision: Laplace transform table

Laplace transforms – Table			
$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \qquad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
e^{-at}	$\frac{1}{s+a}$	$\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\cos(\omega t + \theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$\frac{1}{2}t^2 e^{-at}$	$\frac{1}{(s+a)^3}$	$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s+a)^n}$	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
e^{at}	$\frac{1}{s-a} \qquad s > a$	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2} \qquad s > \lvert\omega\rvert$
te^{at}	$\frac{1}{(s-a)^2}$	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2} \qquad s > \lvert\omega\rvert$
$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$\frac{1}{a^2}[1 - e^{-at}(1 + at)]$	$\frac{1}{s(s+a)^2}$	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
t^n	$\frac{n!}{s^{n+1}} \qquad n = 1,2,3..$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}} \qquad s > a$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$
$t^n e^{-at}$	$n!$	$1 - e^{-at}$	a

Continuous Systems and Transfer Function Revision: Laplace transforms

The Laplace variable, s, can be considered to represent the differential operator (VERY useful for control engineering):

$$\boxed{s \equiv \frac{d}{dt}}$$

$$\boxed{\frac{1}{s} \equiv \int_{0^-}^{\infty} dt}$$

Continuous Systems and Transfer Function Revision: Laplace transforms

Laplace transform of time derivative dx/dt:

$$L\left\{\frac{dx}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx}{dt} e^{-st}\, dt$$

Integrating by parts:

$$L\left\{\frac{dx}{dt}\right\} = s\int_{0^-}^{\infty} x(t)e^{-st}\, dt + \left[x(t)e^{-st}\right]_{0^-}^{\infty}$$

The initial condition $x(0^-)$ is often zero in practice

$$\boxed{L\left\{\frac{dx}{dt}\right\} = sX(s) - x(0^-)}$$

We can substitute this result to solve higher order derivatives:

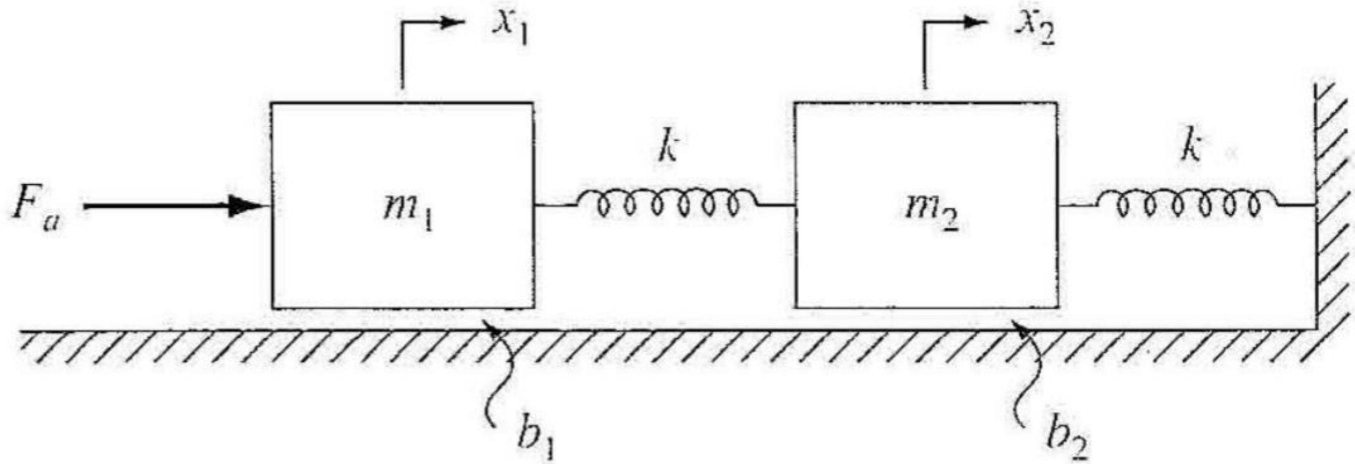
$$L\left\{\frac{d^2x}{dt^2}\right\} = sL\left\{\frac{dx}{dt}\right\} - \frac{dx}{dt}(0^-)$$

$$\boxed{L\left\{\frac{d^2x}{dt^2}\right\} = s^2X(s) - sx(0^-) - \frac{dx}{dt}(0^-)}$$

So more generally, with all initial conditions set to zero:

$$L\left\{\frac{d^nx}{dt^n}\right\} = s^nX(s)$$

Continuous Systems and Transfer Function Revision: Mass/spring systems



Continuous Systems and Transfer Function Revision: Mass/spring systems

Continuous Systems and Transfer Function Revision: Modelling electrical systems

- The voltage across the capacitor.

$$v(t) = \frac{1}{C}\int i(t)dt$$

- For the current, it yields:

$$i(t) = C\frac{dv(t)}{dt}$$

- The voltage drop $v(t)$ across the inductor is:

$$v(t) = \frac{d\Phi_B(t)}{dt} = \mathrm{L}\frac{di(t)}{dt}$$

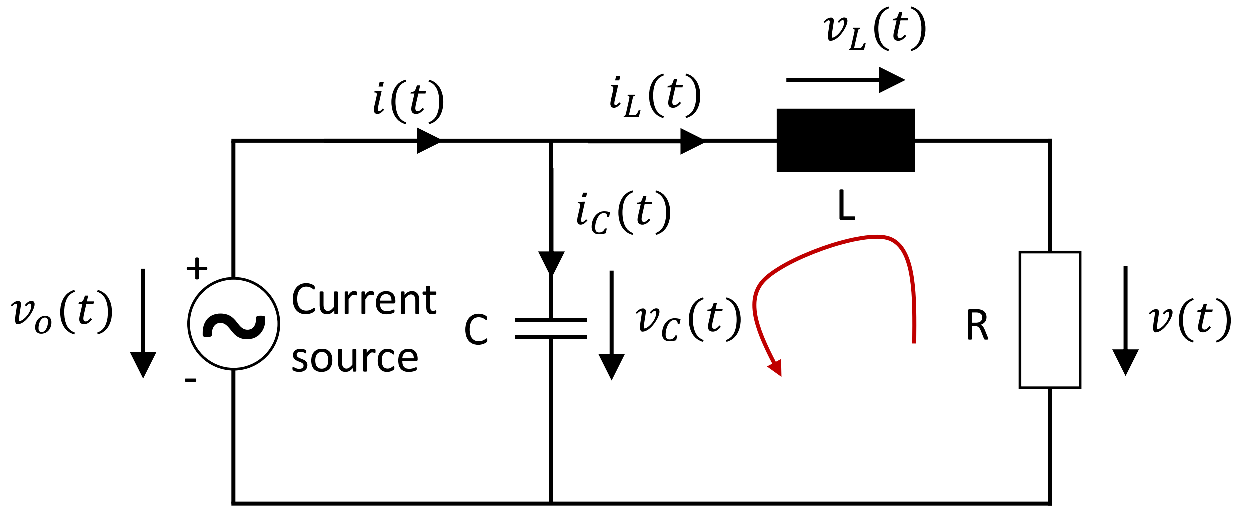
- Hence, the current yields:

$$i(t) = \frac{1}{\mathrm{L}}\int v(t)dt$$

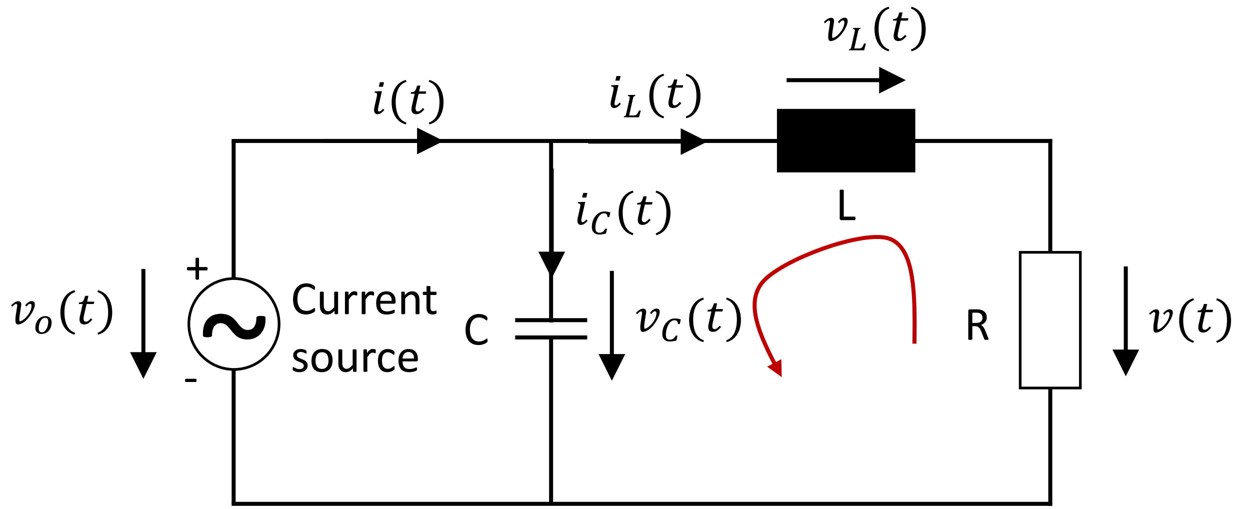
Revision: Solutions to differential equations

$$v_o(t) = v_C(t)$$

KCL and KVL:



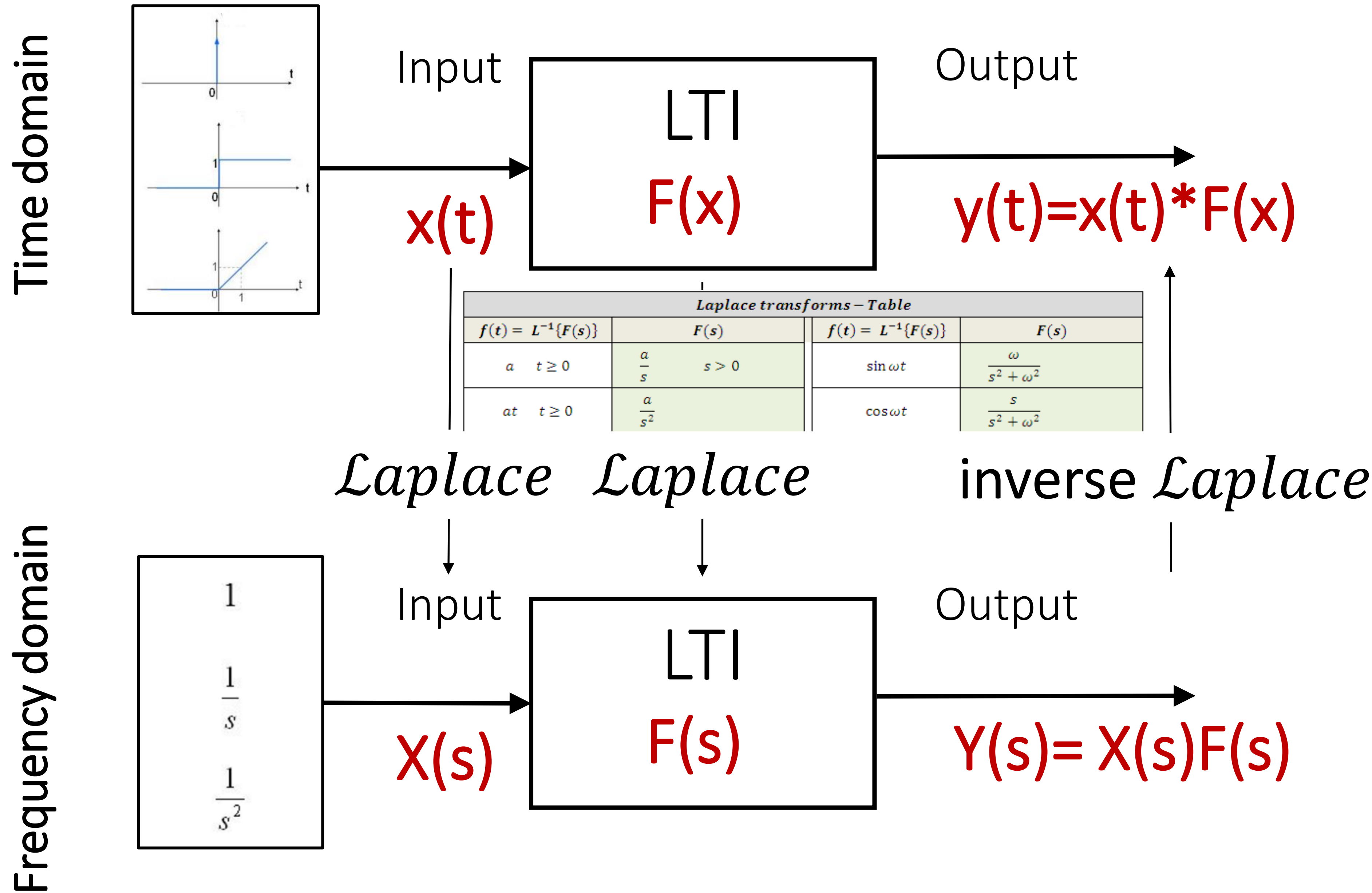
Revision: Solutions to differential equations



Revision: Solutions to differential equations - Laplace

Input functions:
Impulse – Step - Ramp

Continuous Systems and Transfer Function Revision: Time vs frequency domain



Laplace transforms – Table			
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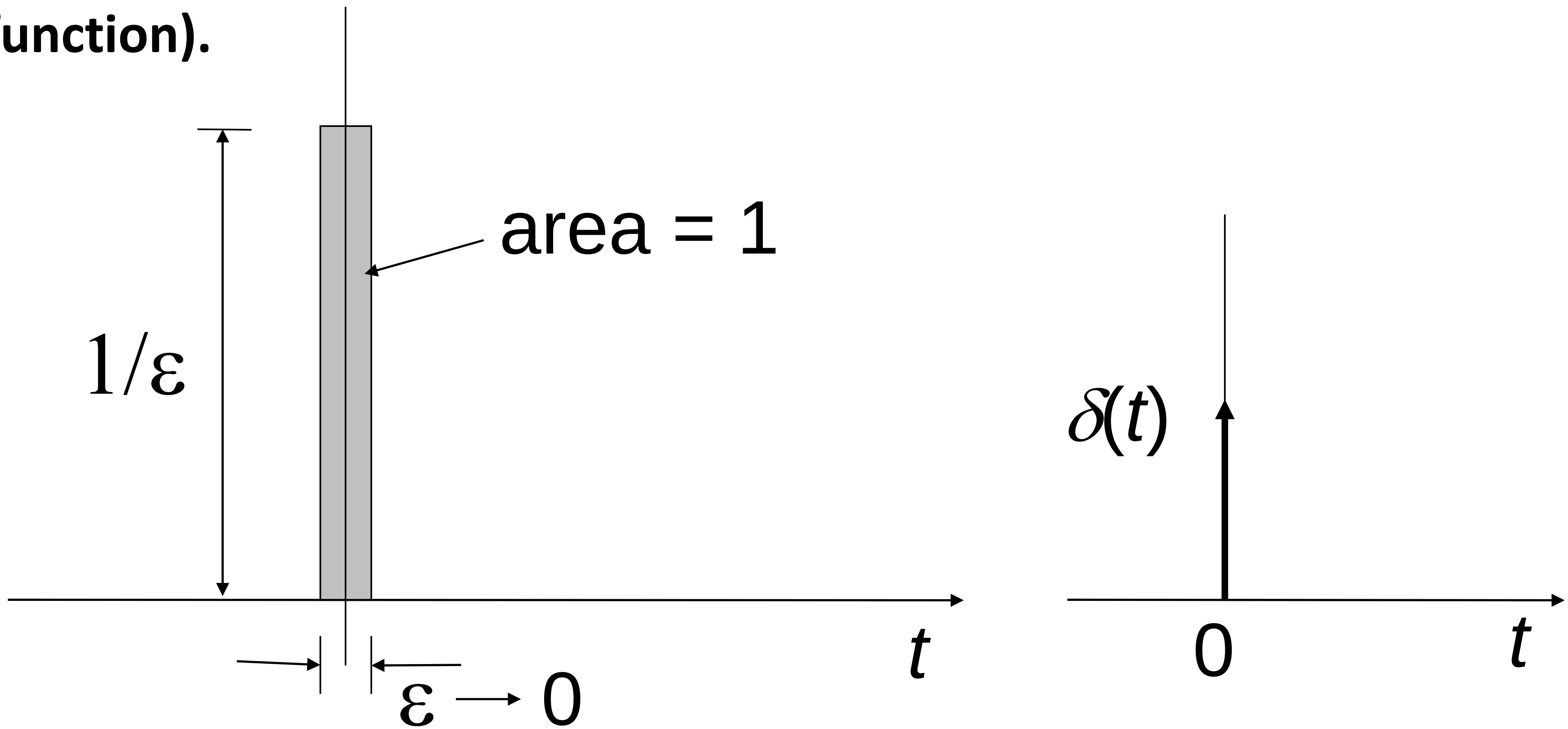
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$

Now we will consider some standard inputs and look at the response of first and second order systems:

- Impulse (In practice, we cannot create an input of $\boldsymbol{x(t) = \delta(t)}$ to characterise a system.)
- Step
- Ramp

There are many others, particularly sinusoidal inputs or other discontinuous inputs, which are important in control loops, but we will focus on the two classic examples.

A useful tool in analysing the transient response of a system is the impulse signal, a unit (amplitude=1) pulse infinitesimally small, with area =1. Formally this is known as the **Dirac delta function (impulse function).**



The Dirac delta function is a non-physical, singularity function with the following definition

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0 \\ \text{undefined} & \text{for } t = 0 \end{cases}$$

but with the requirement that

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

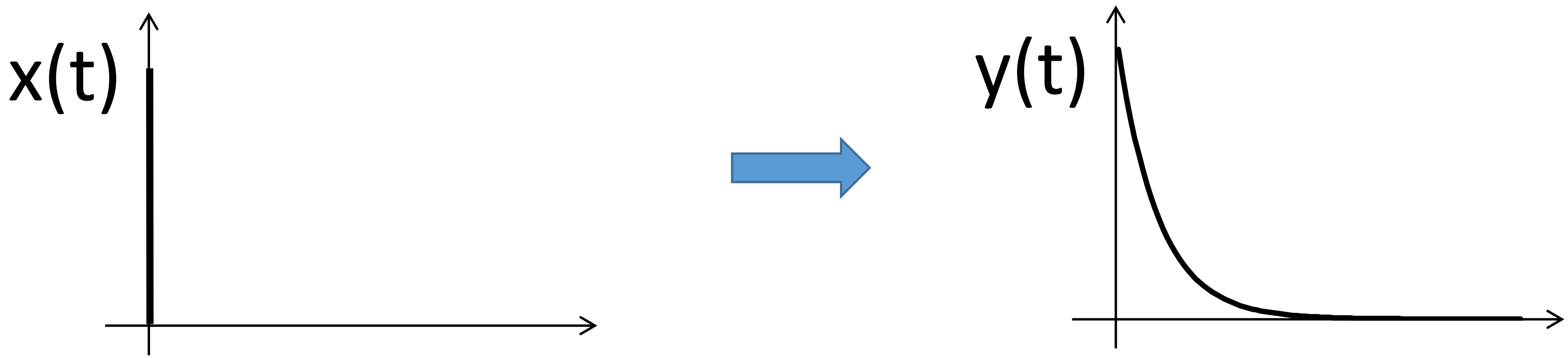
so taking the Laplace transform of this is also just 1

$$\mathcal{L}(\delta(t)) = \int_{0-}^{\infty} \delta(t)e^{-st}dt = 1$$

Thus the impulse response of the system is equal to the transfer function, and from this it can be shown that *any* arbitrary signal can be described as a summation of impulse responses.

Continuous Systems and Transfer Function Revision: Impulse response on a first order system

Take a first order response for example, the transfer function *and thus the impulse response* looks like this:



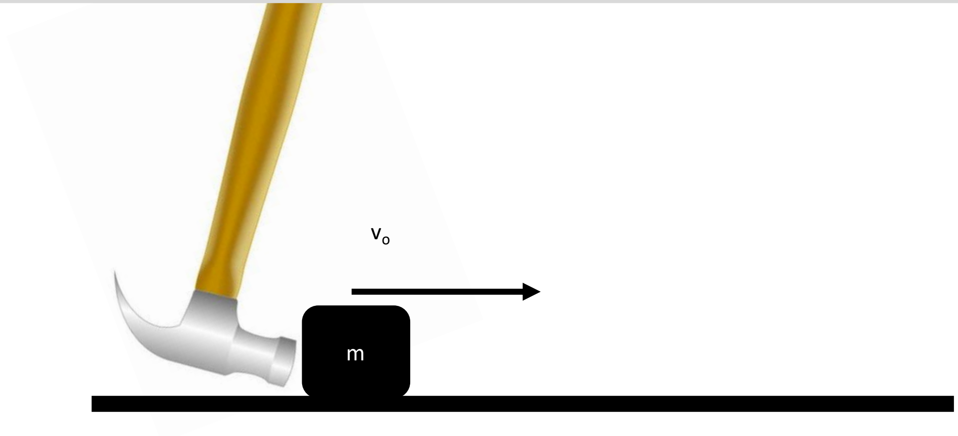
Input

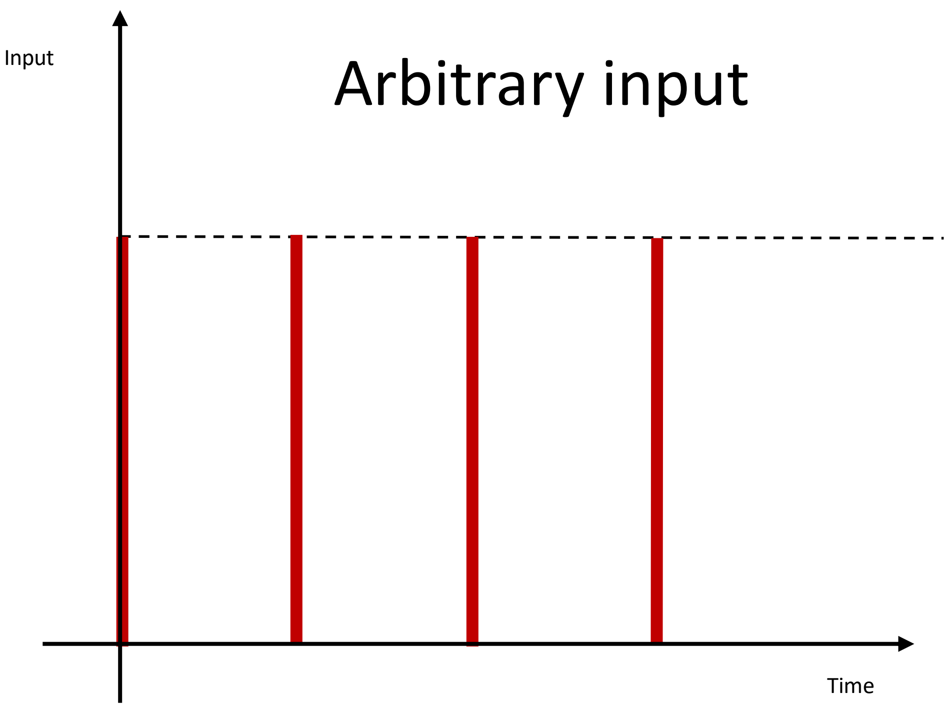
Output

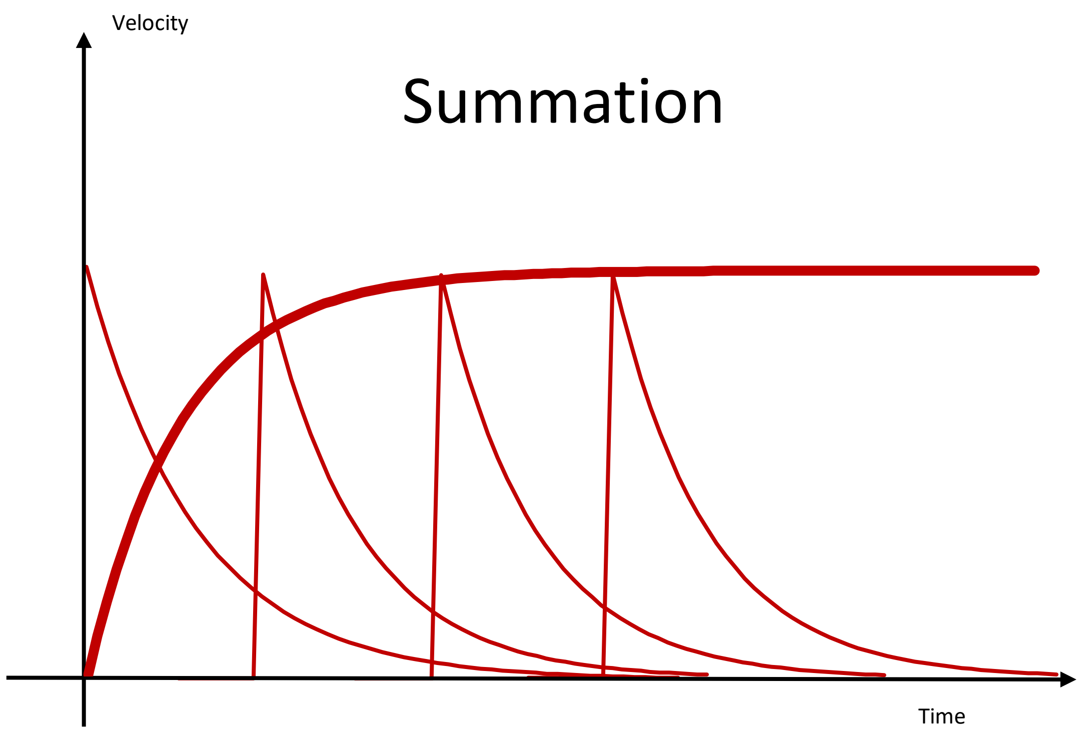
Due to our LTI assumptions:

- Scaling the input scales the output,
- Superposition of inputs equals superposition of outputs
- Time invariance

Continuous Systems and Transfer Function Revision: Impulse response of a system

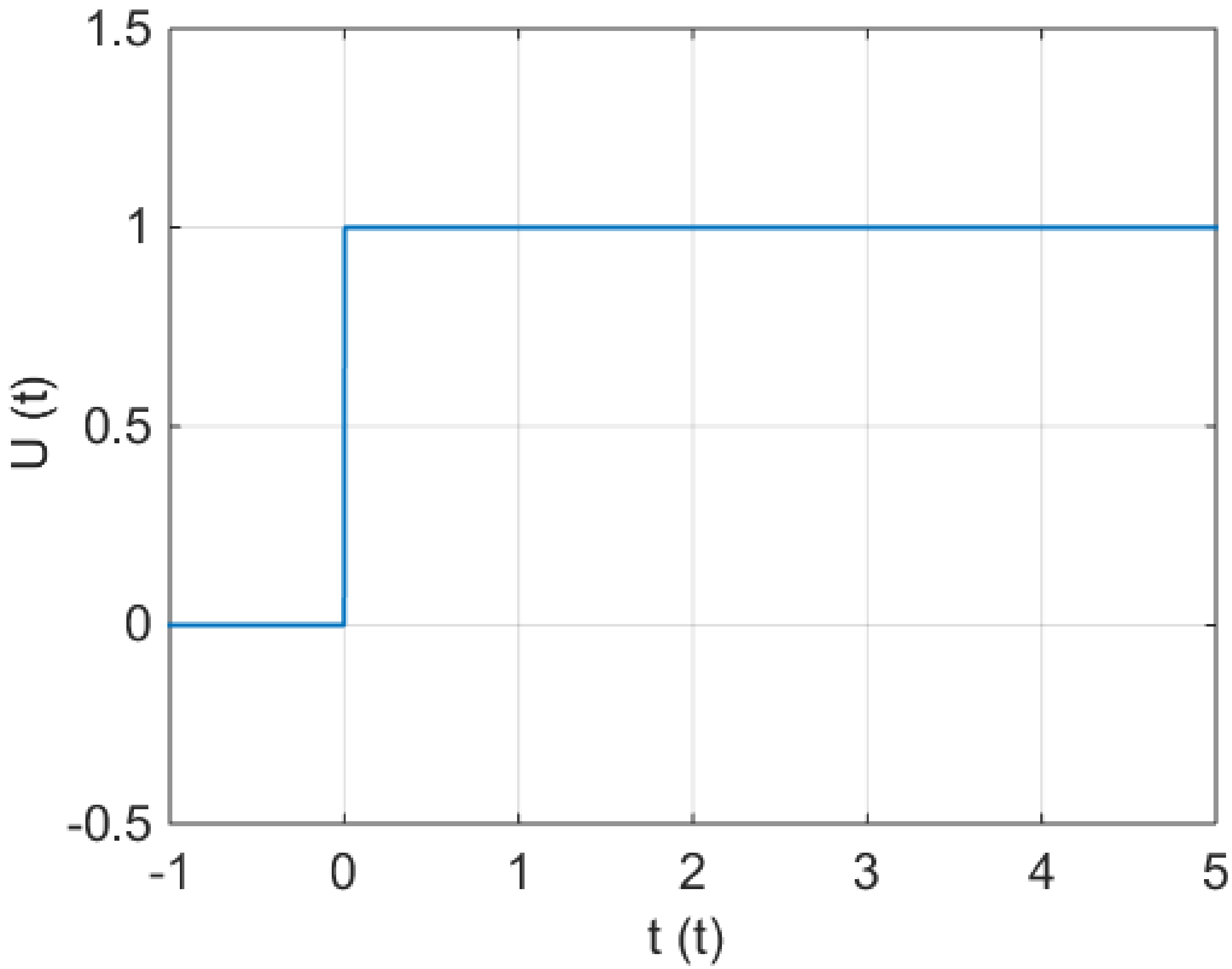






Continuous Systems and Transfer Function Revision: Step Input

A step input is a discontinuous function, which is zero for all negative values of *t* and 1 for all positive values



Continuous Systems and Transfer Function Revision: Step Input – Laplace transform

$$x(t) = U(t)$$

$$L\{U(t)\} = \int_{0^-}^{\infty} e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_{0^-}^{\infty}$$

Or for a gain of *A*

$$x(t) = AU(t) \quad L\{AU(t)\}$$

Continuous Systems and Transfer Function Revision: Step Input – Applications

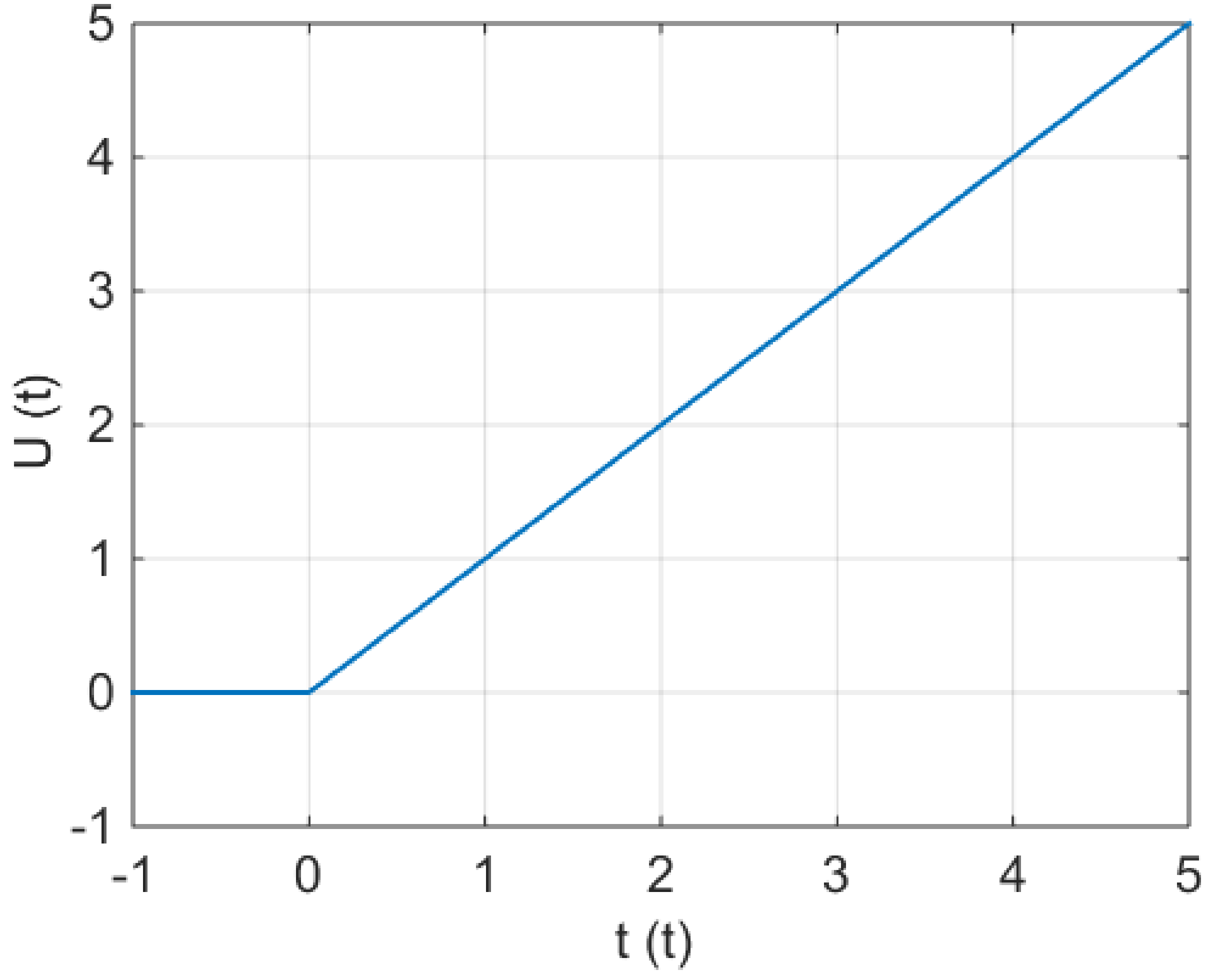
The step response is extremely useful in control theory for describing the behaviour of the system. In part because it incorporates the "transient" behaviour – from the sudden change from zero to one, as well as the "steady state" behaviour as the system settles down to a single value.

It also replicates many real world control applications such as

- Position control – Move to a X=10mm position and stay
- Speed control – go to 33.333 RPM
- Temperature – Heat element on 3D printer to 230°C

Continuous Systems and Transfer Function Revision: Ramp Input

A Ramp input has a value of *t* for all *t* value above zero and zero elsewhere, often is scaled by a gain *A*



Continuous Systems and Transfer Function Revision: Ramp Input – Laplace transform

$$x(t) = at$$

$$L\{at\} = \int_{0^-}^{\infty} ate^{-st}\,dt = -a\left[\frac{t}{s}e^{-st}\right]_0^{\infty} + a\int_0^{\infty}\frac{1}{s}e^{-st}\,dt$$

$$= a\left[-\frac{1}{s^2}e^{-st}\right]_0^{\infty}$$

Continuous Systems and Transfer Function Revision: Ramp Input – Applications

Ramp inputs are useful in understanding the steady state behaviour of a system i.e. when *t* goes to infinity

Practical examples of control applications using ramp inputs are

- Servo motors – Shaft *position* rather than speed
- Ovens for PCB manufacturing etc. – strict linear *profile* of temperature required as opposed to “get to the this temperature quickly”
- CNC milling machine, move in X direction and constant rate

Continuous Systems and Transfer Function Revision: Summary of input functions

Impulse

Step

For *unit* response
A=1

Ramp

We can apply these inputs to the LTI system by multiplying the transfer function by the input, both in terms of *s*

Continuous Systems and Transfer Function Revision: Laplace transforms example

The input-output relationship of a certain system is described by a differential equation shown below. Find the response when ***x(t)*** is a step function of 10 units applied at ***t = 0*** and when the initial conditions are ***y(0) = 2, y'(0) = -10***.

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = x$$

Apply Laplace transforms to both sides of the equation and then solve using partial fraction expansion.

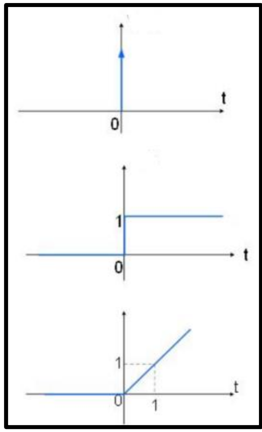
Steps:

- Apply Laplace transforms to both sides of the equation
- Rearrange to solve for $L[y(t)]$
- Use partial fractions to break the expression into components to which we can easily apply the inverse Laplace transform.

To account for initial conditions: $L[x'] = sX - x(0)$ $\quad L[x''] = s^2X - sx(0) - x'(0)$

Continuous Systems and Transfer Function Revision: Time vs frequency domain

Time domain



Input → **LTI** $F(x)$ → Output

$x(t)$ $\qquad$ $y(t)=x(t)*F(x)$

Laplace transforms – Table			
$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \qquad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$

$\mathcal{L}aplace$ $\qquad$ $\mathcal{L}aplace$ $\qquad$ inverse $\mathcal{L}aplace$

Frequency domain

1

$\frac{1}{s}$

$\frac{1}{s^2}$

Input → **LTI** $F(s)$ → Output

$X(s)$ $\qquad$ $Y(s)= X(s)F(s)$

1st order: $\frac{X(s)}{Y(s)} = \frac{\alpha}{(1+Ts)} = \frac{\gamma}{1+\tau s}$

2nd order: $G(s) = \gamma \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$