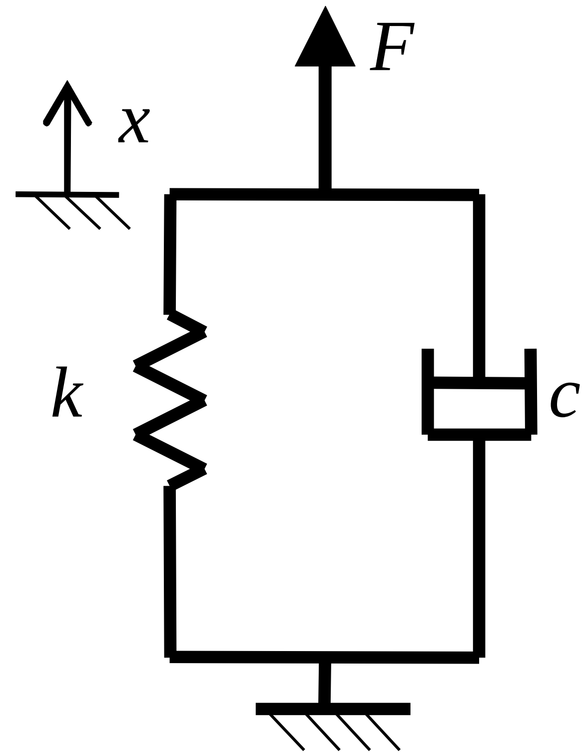
MECH0089
Control and Robotics:
Digital Control Systems

Lecture 2:
Continuous Systems and Transfer Function Revision (part 2)

First Order System Step Response

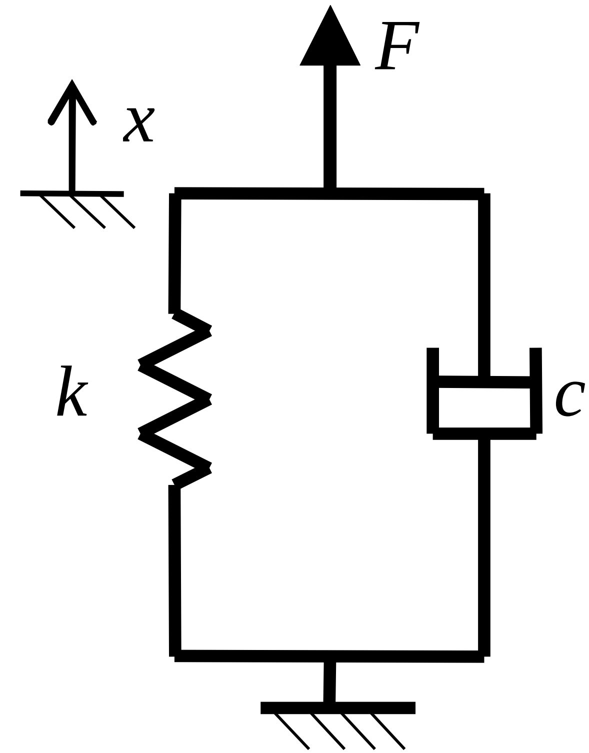
Continuous Systems and Transfer Function Revision: Parallel Spring & Damper



In a shock absorber in a car, damping added in parallel to spring suspension in vehicles to damp oscillations and absorb impulses

Transfer function desired:

$$G(s)=\frac{X(s)}{F(s)}$$



Balancing forces as function of time:

$$f(t)=f_{spring}(t)+f_{damper}(t)=kx(t)+c\frac{dx}{dt}$$

Rewriting as function of s

$$F(s)=kX(s)+csX(s)=X(s)(k+cs)$$

Transfer function is thus:

$$G(s)=\frac{X(s)}{F(s)}=\frac{1}{(k+cs)}$$

Continuous Systems and Transfer Function Revision: First Order Systems

All first order systems i.e. those with only $\frac{dx}{dt}$
Take the following "standard" forms

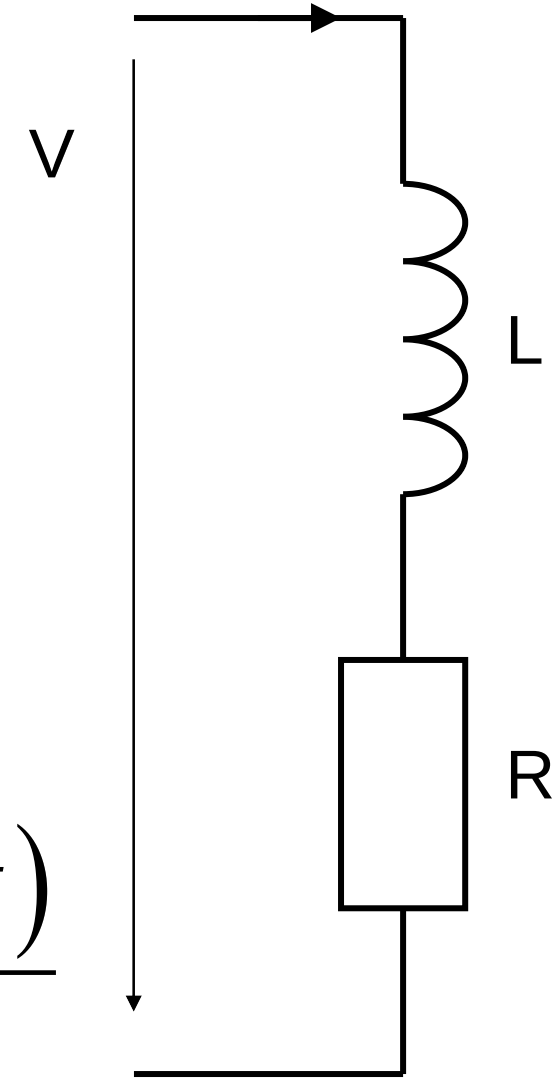
$$\frac{X(s)}{Y(s)} = \frac{\alpha}{(1+Ts)} = \frac{\gamma}{1+\tau s}$$

α, γ Gain

T, τ Time Constant

This function is commonly known as an exponential time delay, or lag.
This is an incredibly common function, they turn up everywhere!

Electromagnets are used in motors both rotary and linear, as well as in power transfer, and also magnetic levitation in trains



First obtain the transfer function $\dfrac{I(s)}{V(s)}$

Balancing voltages gives:

$$v(t) = v_R(t) + v_I(t)$$

where

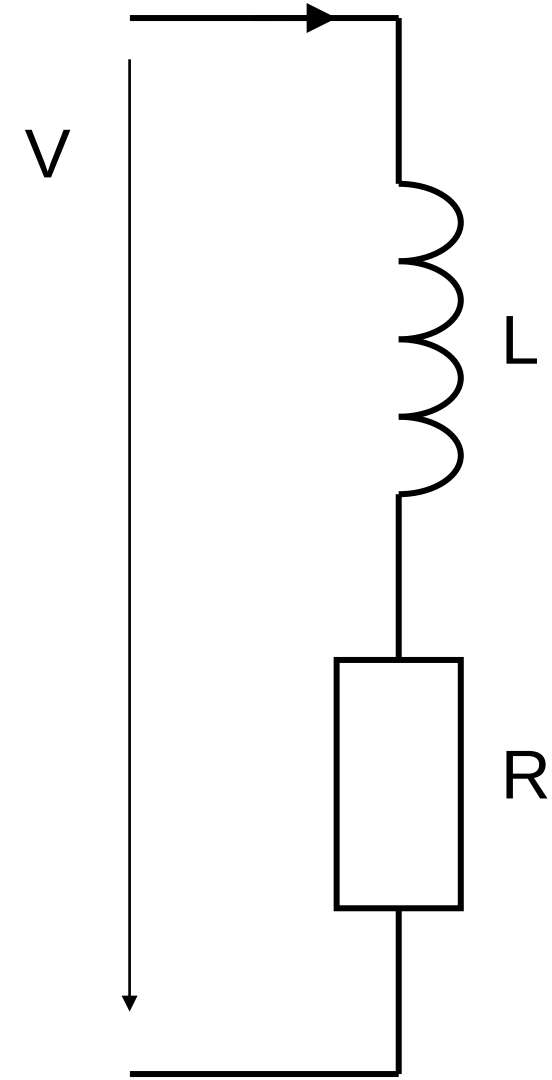
$$v_R(t) = i(t)R \qquad v_I(t) = L\frac{di(t)}{dt}$$

Substitution gives:

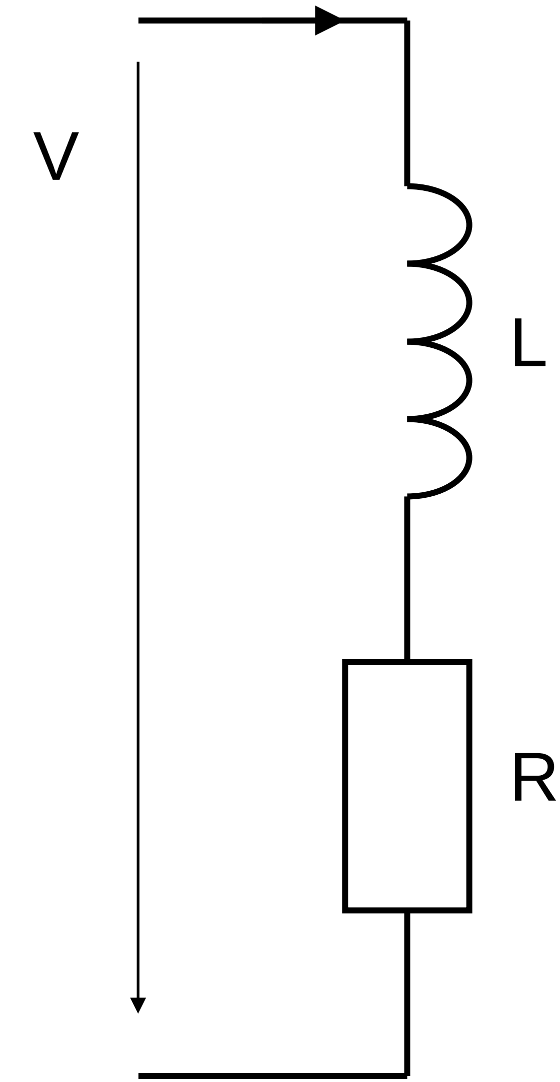
$$v(t) = Ri(t) + L\frac{di(t)}{dt}$$

In the Laplace domain:

$$V(s) = RI(s) + LsI(s)$$



For a **step** input of 1V (unit step) $V(s) = \frac{1}{s}$



Now we can look to the Laplace transform tables to find out I(t)

The first task is to split the expression up using partial fractions, into expressions that *are* given in the tables:

$$I(s) = \frac{1}{s(Ls+R)} = \frac{1/L}{s\left(s+R/L\right)}$$

$$= \frac{k_1}{s} + \frac{k_2}{\left(s+R/L\right)}$$

where $$k_1\left(s+R/L\right)+k_2 s = 1/L$$

Looking at $s = 0$ $\qquad k_1 \frac{R}{L} = \frac{1}{L} \longrightarrow k_1 = \frac{1}{R}$

Looking at $s = -\frac{R}{L}$ $\qquad -k_2 \frac{R}{L} = \frac{1}{L} \longrightarrow k_2 = -\frac{1}{R}$

Which yields: $$I(s) = \frac{1}{R}\left[\frac{1}{s} - \frac{1}{\left(s + \frac{R}{L}\right)}\right]$$

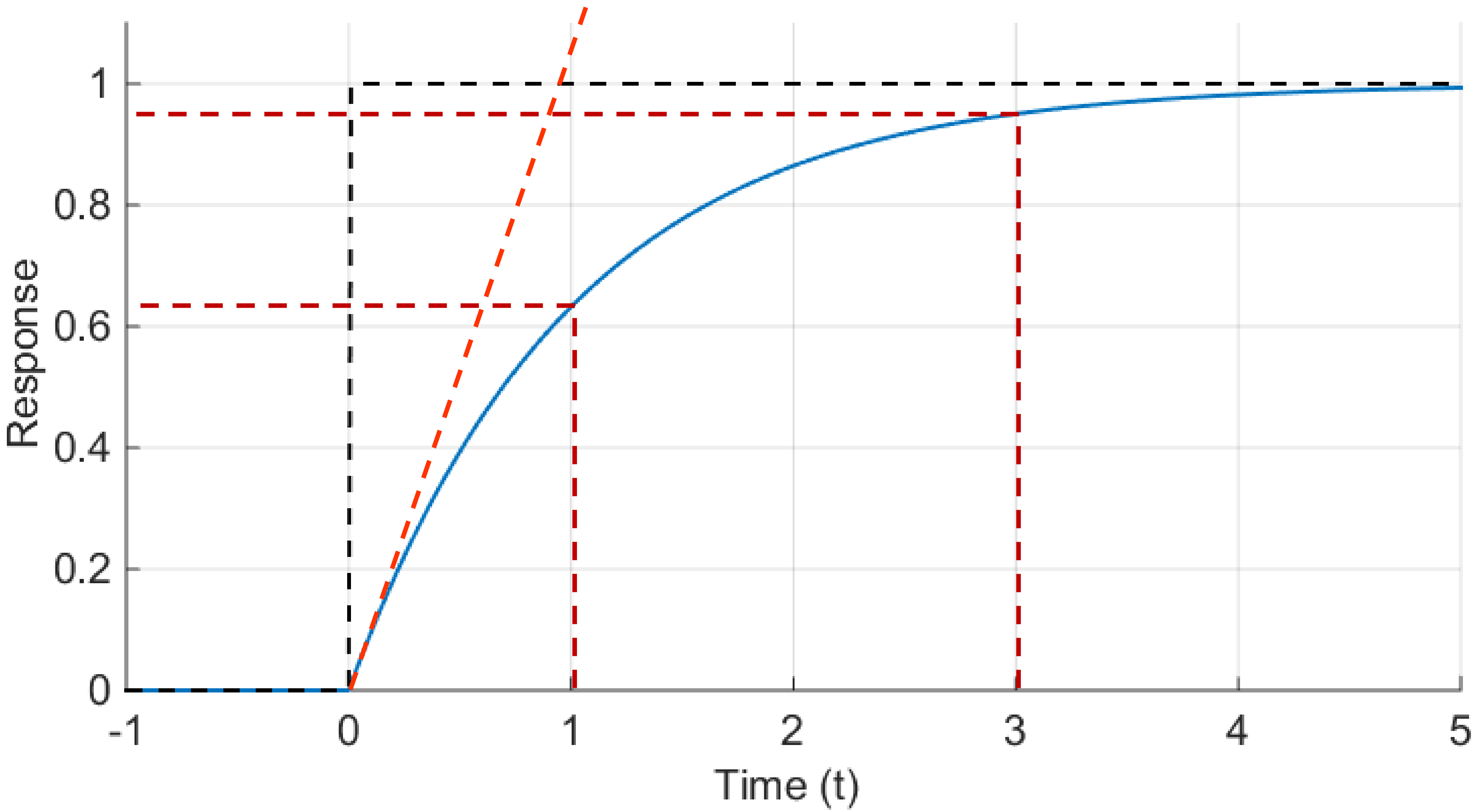
So looking at the Laplace tables, we can now use entries for $\frac{1}{s}$ and $\frac{1}{(s+a)}$

$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \quad s > 0$
$at \quad t \geq 0$	$\frac{a}{s^2}$
e^{-at}	$\frac{1}{s+a}$

$$I(t) = \frac{1}{R}\left[1 - e^{-\frac{R}{L}t}\right]$$

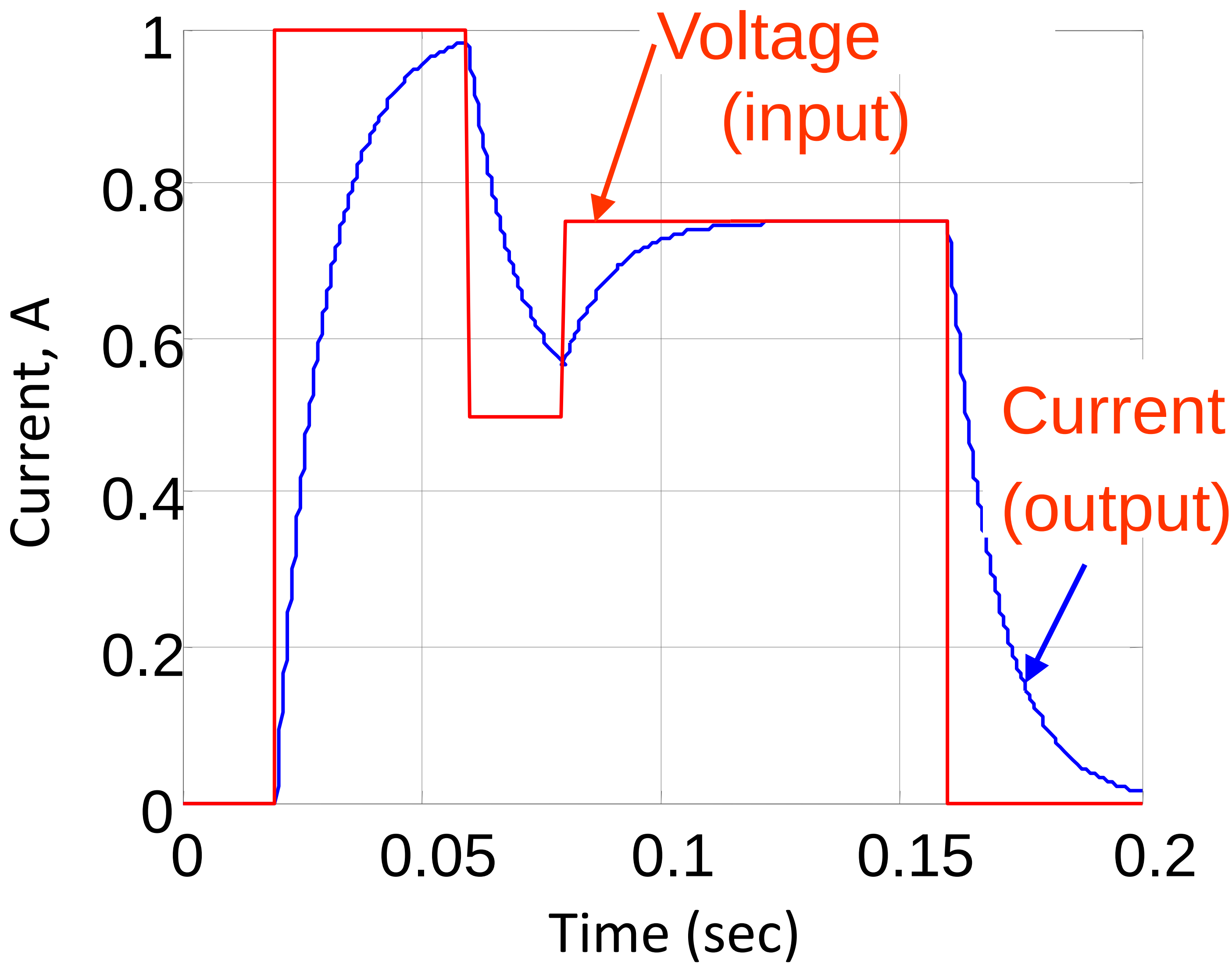
Which has the familiar form of a first order exponential rise, with a steady state gain of 1/R. Time constant τ = L/R

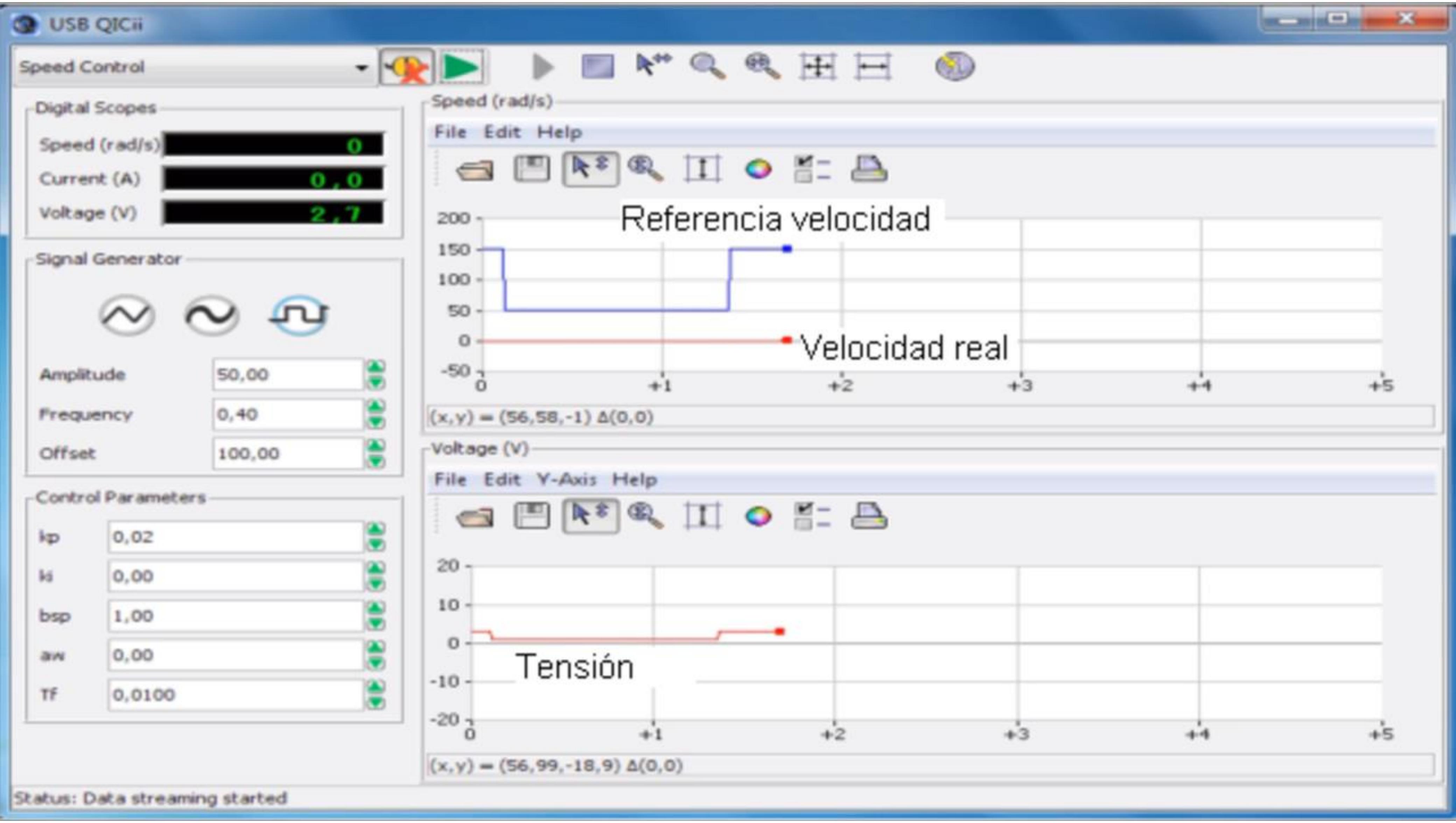
So (for now) assume L and R = 1, so gain and time constant =1



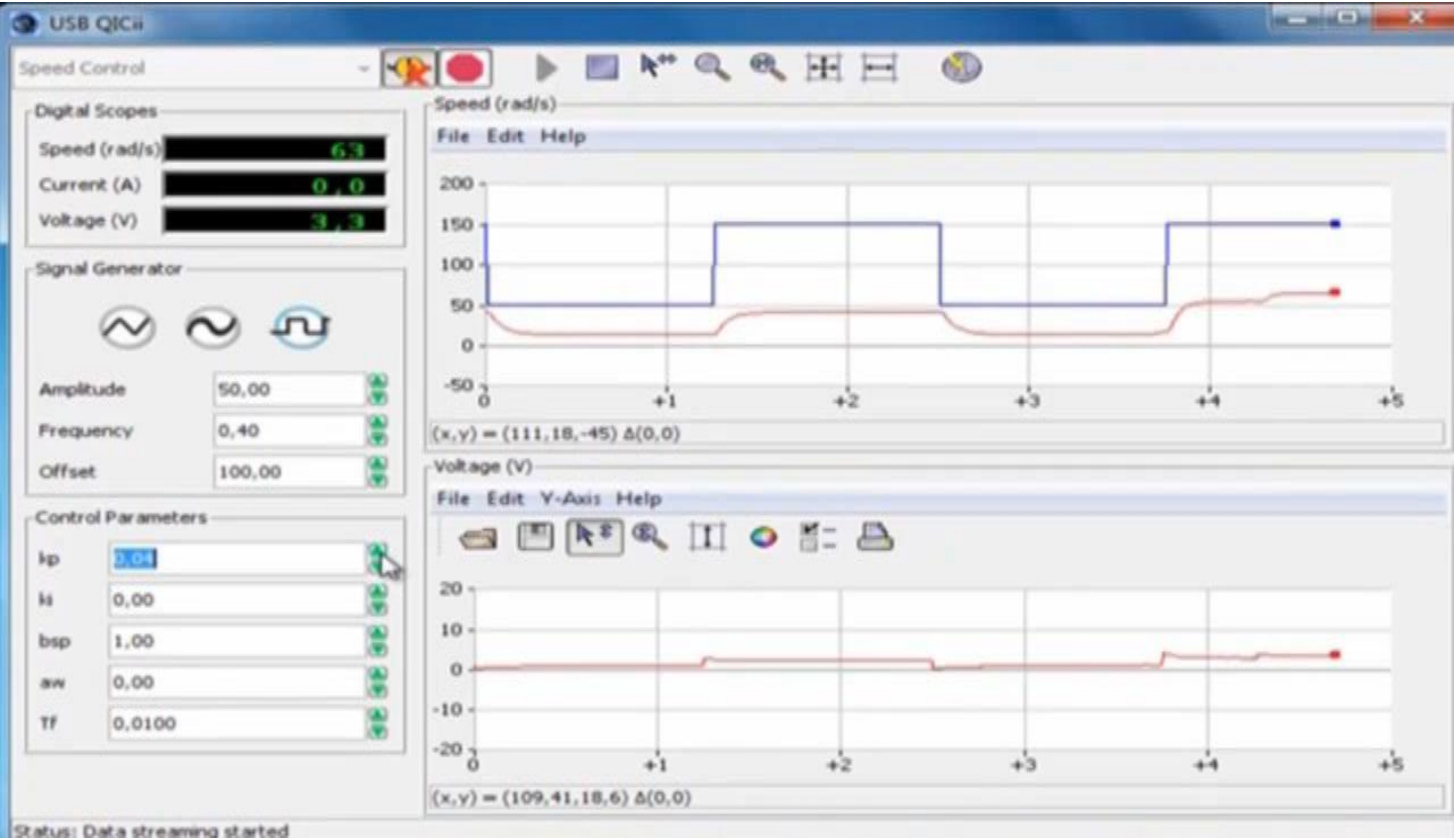
These "step responses" can be repeated for different values of V and for different starting values of *I(t)*

The response will always be behind the input for all time constants > 0, and thus these systems are referred to as *first order lags*





USB QICii
Speed Control
Digital Scopes
Speed (rad/s)
0
Current (A)
0,0
Voltage (V)
2,7
Signal Generator
Amplitude
50,00
Frequency
0,40
Offset
100,00
Control Parameters
kp
0,02
ki
0,00
bsp
1,00
aw
0,00
Tf
0,0100
Speed (rad/s)
File Edit Help
Referencia velocidad
Velocidad real
(x,y) = (56,58,-1) Δ(0,0)
Voltage (V)
File Edit Y-Axis Help
Tensión
(x,y) = (56,99,-18,9) Δ(0,0)
Status: Data streaming started

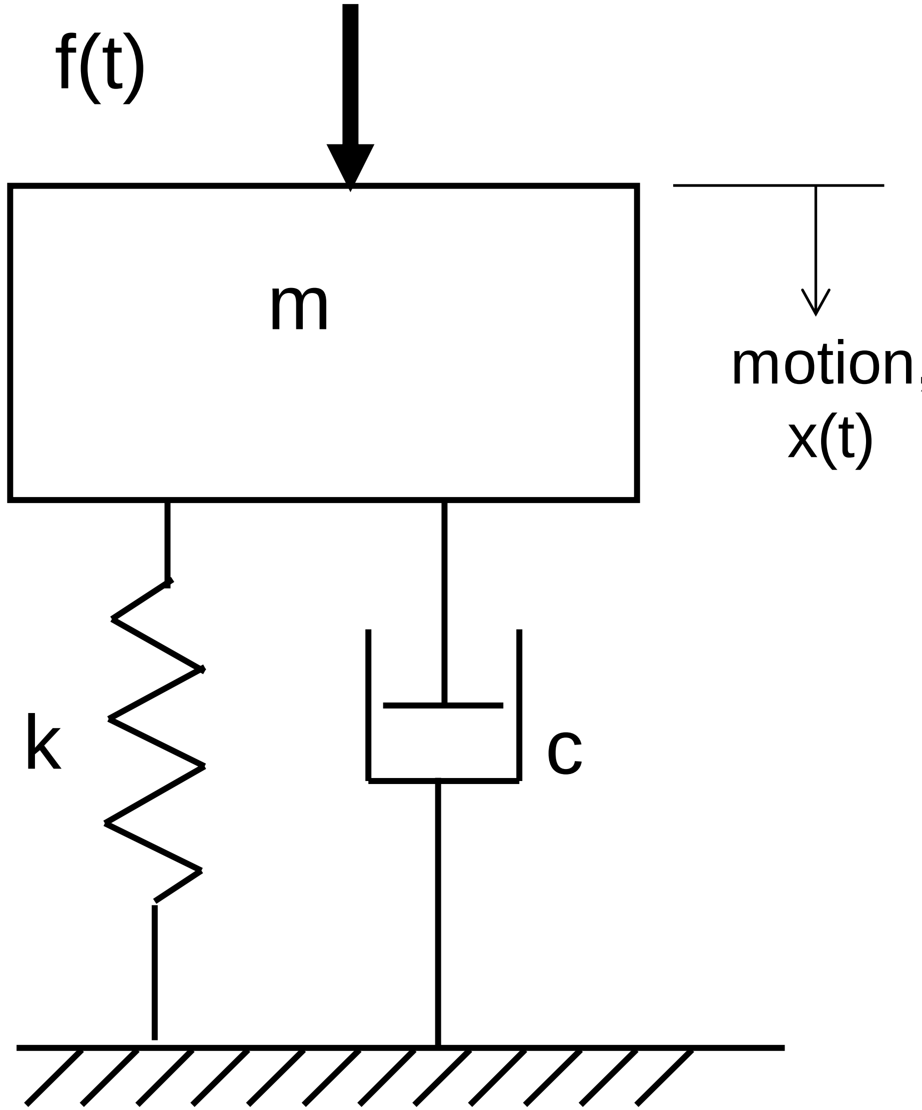

USB QICii
Speed Control
Digital Scopes
Speed (rad/s) 63
Current (A) 0,0
Voltage (V) 3,3
Signal Generator
Amplitude 50,00
Frequency 0,40
Offset 100,00
Control Parameters
kp 0,04
ki 0,00
bsp 1,00
aw 0,00
Tf 0,0100
Speed (rad/s)
File Edit Help
(x,y) = (111,18,-45) Δ(0,0)
Voltage (V)
File Edit Y-Axis Help
(x,y) = (109,41,18,6) Δ(0,0)
Status: Data streaming started

Second Order Systems

Continuous Systems and Transfer Function Revision: Mass Spring Damper

Consider a simple mechanical system Mass/Spring/Damper (MSD)

As before, we wish to relate the input force to the output displacement i.e. Transfer function desired:



Using Newton's Second Law Balancing forces as function of time:

$$f(t) = f_S(t) + f_D(t) + f_I(t)$$

Continuous Systems and Transfer Function Revision: Mass Spring Damper

$$f_S(t) = kx(t) \qquad f_D(t) = c\frac{dx(t)}{dt} \qquad f_I(t) = m\frac{d^2x(t)}{dt^2}$$

Combining yields the following time domain equations

$$f(t) = kx(t) + c\frac{dx(t)}{dt} + m\frac{d^2x(t)}{dt^2}$$

Rewriting as function of s

$$F(s) = kX(s) + csX(s) + ms^2X(s)$$

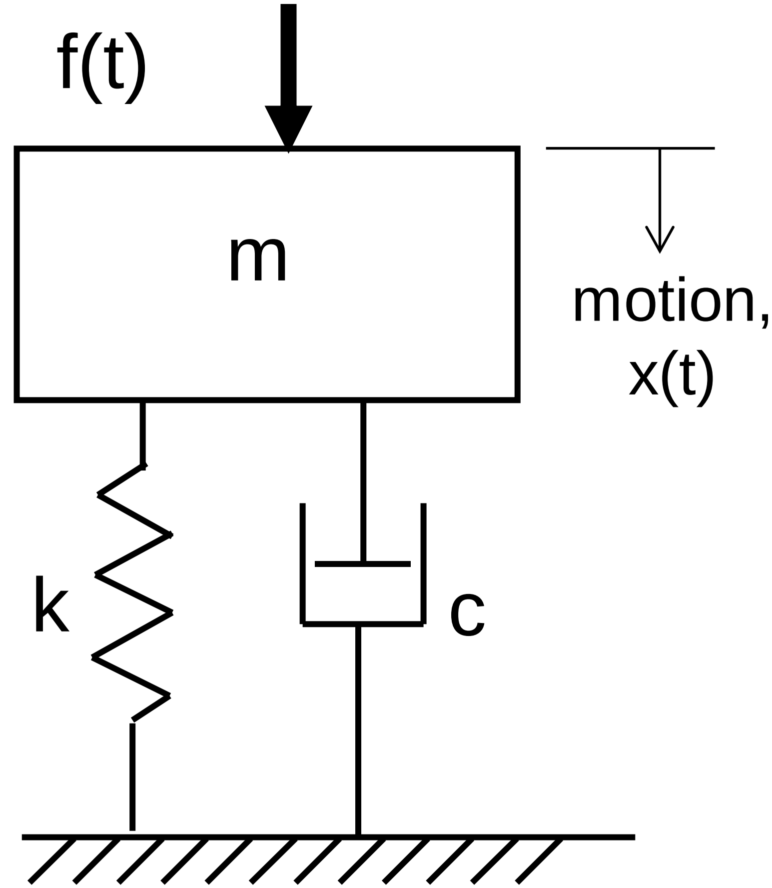
$$F(s) = X(s)\left(k + cs + ms^2\right)$$

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} = G(s)$$

$b-a$	$(s+a)(s+b)$		$(s+a)^2+\omega^2$
$\frac{1}{a^2}[1-e^{-at}(1+at)]$	$\frac{1}{s(s+a)^2}$	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$ $n=1,2,3..$	$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$ $s>a$	$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$ $s>a$	$1-e^{-at}$	$\frac{a}{s(s+a)}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	$\frac{1}{a^2}(at-1+e^{-at})$	$\frac{1}{s^2(s+a)}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$ $s>0$	$f(t-t_1)$	$e^{-t_1 s}F(s)$
$g(t)\cdot p(t)$	$G(s)\cdot P(s)$	$f_1(t)\pm f_2(t)$	$F_1(s)\pm F_2(s)$
$\int f(t)dt$	$\frac{F(s)}{s}+\frac{f^{-1}(0)}{s}$	$\delta(t)$ *unit impulse*	1 *all s*
$\frac{df}{dt}$	$sF(s)-f(0)$	$\frac{d^2f}{df^2}$	$s^2F(s)-sf(0)-f'(0)$
$\frac{d^nf}{dt^n}$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\cdots-f^{n-1}(0)$		

Continuous Systems and Transfer Function Revision: Mass Spring Damper

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$



However, we want to write it in a "standard form", not least because that's what it will look like in the Laplace transfer tables! This means the coefficient of the highest order of s on the denominator is 1.

$$\boxed{\frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}}$$

Continuous Systems and Transfer Function Revision: Mass Spring Damper

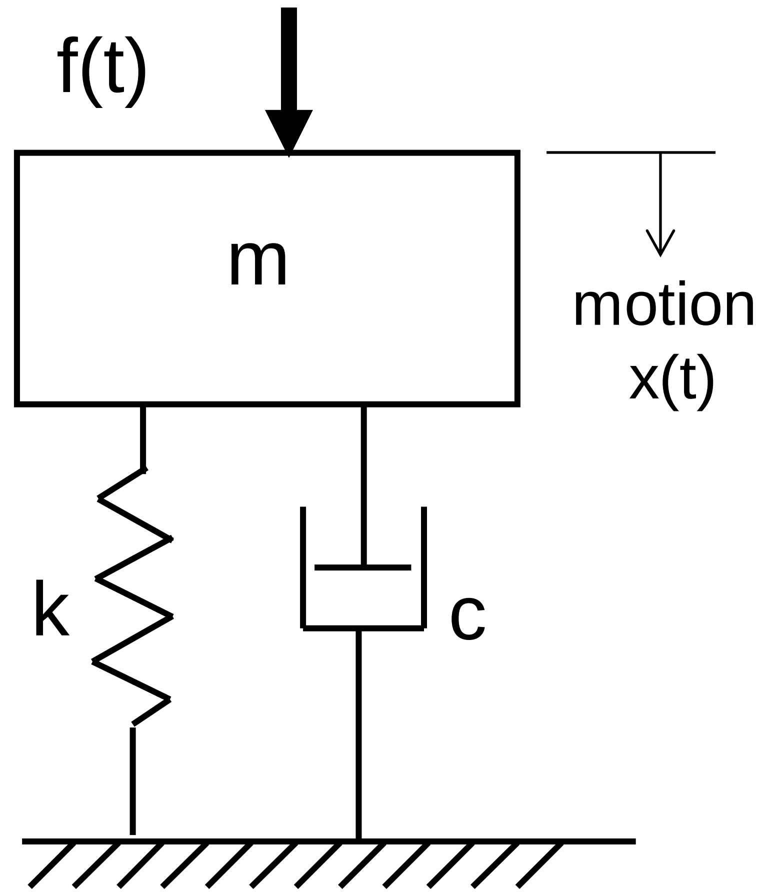
$$G(s) = \frac{k/m}{s^2 + c/m\, s + k/m} \frac{1}{k}$$

$$= \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2} \frac{1}{k}$$

Where,

$$\omega_n = \sqrt{\frac{k}{m}}$$ Natural frequency

$$\zeta = \frac{c}{2\sqrt{km}}$$ Damping ratio



The standard form for second order systems is shown below:

$$G(s) = \gamma \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

This function is known as a damped oscillator, in that it produces harmonic sinusoidal oscillations which decay over time. This type of system appears *everywhere* in physics, as well as in engineering. Even to the extent that some higher order systems are simplified to become second order, just because it is so well understood.

How do I go back to the time domain?

As before, we use the inverse Laplace transform to get the time domain response. To reiterate, the benefit of standard forms is that the transforms are given in the tables: *(e.g. from Dorf and Bishop)*

$f(t)$	$F(s)$
$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}t, \zeta < 1$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

γ not shown as it is unaffected by Laplace transform

$$\frac{x(t)}{x(0)} = \gamma \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2 t}\right)$$

$$\frac{x(t)}{x(0)} = \gamma \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2 t}\right)$$

This looks complicated, but consider one term at a time

$$\gamma \frac{\omega_n}{\sqrt{1-\zeta^2}}$$

γ ω_n ζ All constants, so whole term is just a number

$$e^{-\zeta\omega_n t}$$

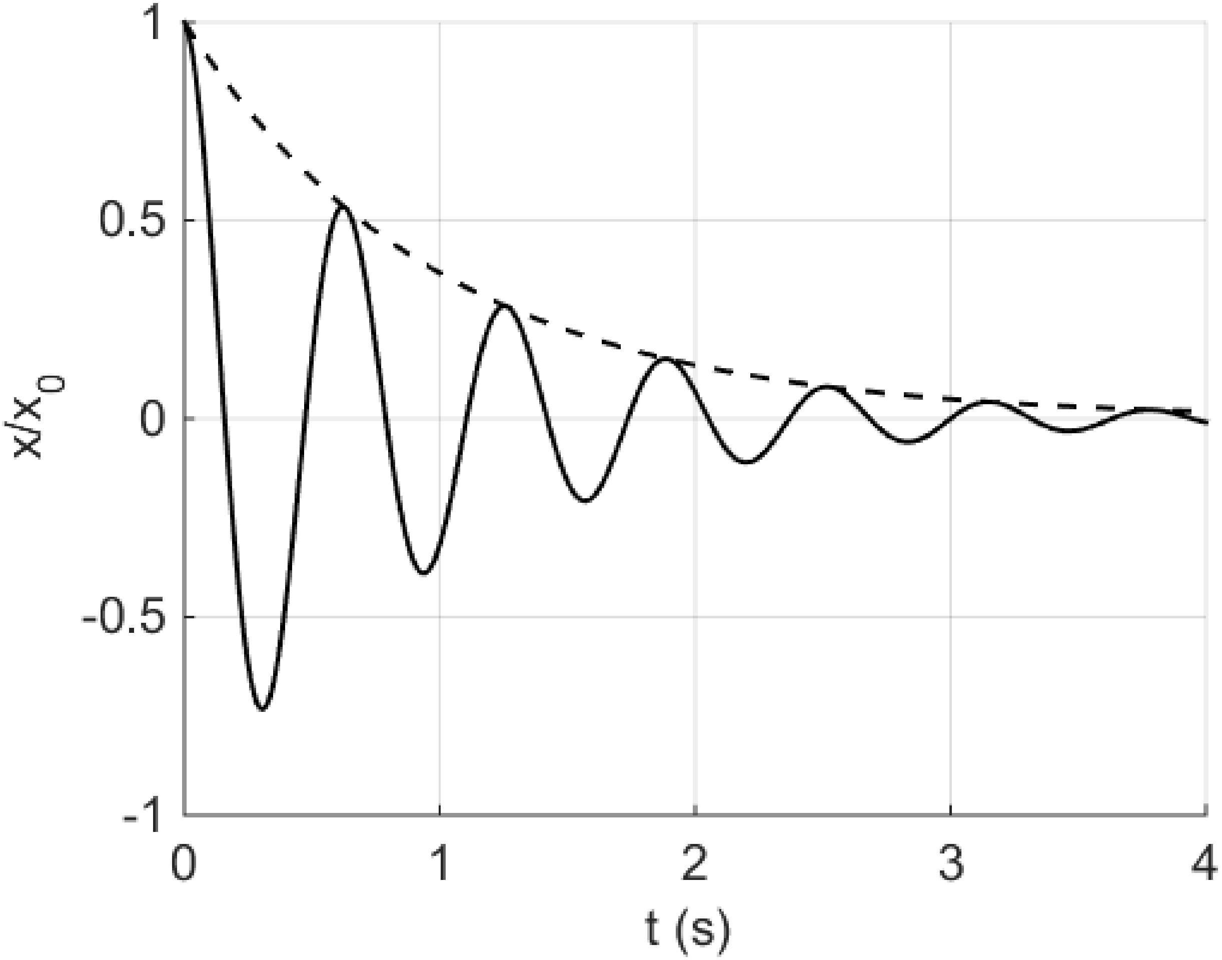
Is an exponential function, which depending on whether $-\zeta\omega_n$ is positive or negative, increases or decays

$$\sin\left(\omega_n \sqrt{1-\zeta^2 t}\right)$$

ω_n ζ Are just constants, so assuming ζ Less than 1, this is just a sine wave

$$\frac{x(t)}{x(0)} = \gamma \underline{\frac{\omega_n}{\sqrt{1-\zeta^2}}}\, \underline{e^{-\zeta\omega_n t}}\, \underline{\sin\left(\omega_n \sqrt{1-\zeta^2 t}\right)}$$

This is assuming $\zeta < 1$



This is an exponentially decaying sinusoidal oscillation, with

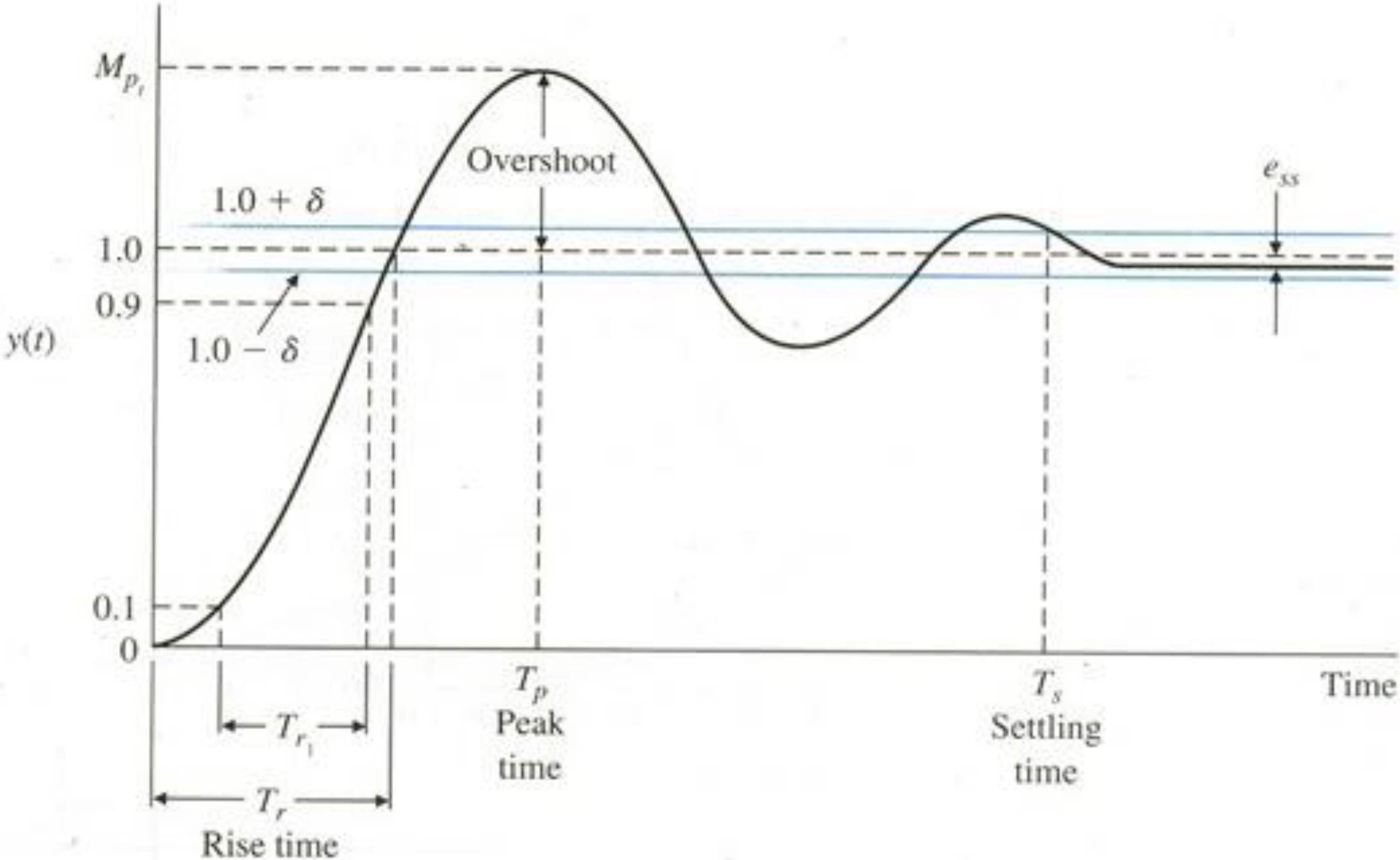
Frequency: $\omega_n\sqrt{1-\zeta^2 t}$

Decay: $e^{-\zeta\omega_n t}$

Gain: $\gamma\dfrac{\omega_n}{\sqrt{1-\zeta^2}}$

This is assuming $\zeta < 1$

Continuous Systems and Transfer Function Revision: Second Order Systems – Unit Step Response

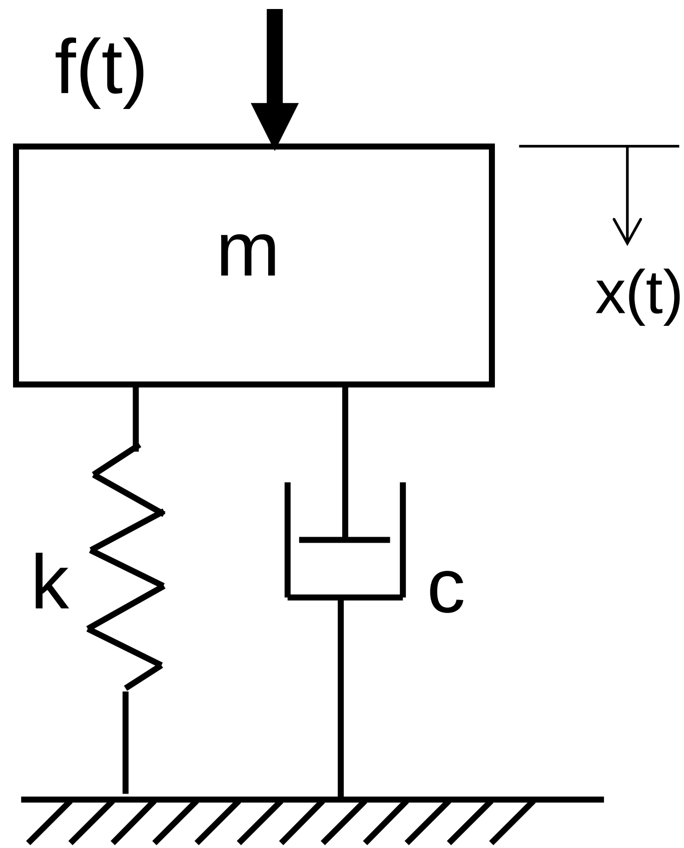


Effect of damping ratio on the transfer function of a 2nd order systems

Continuous Systems and Transfer Function Revision: 2nd Order Transfer Function

The standard form for second order systems is shown below:

$$G(s) = \gamma \frac{\omega_n^{\;2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\;2}}$$



γ **Gain**

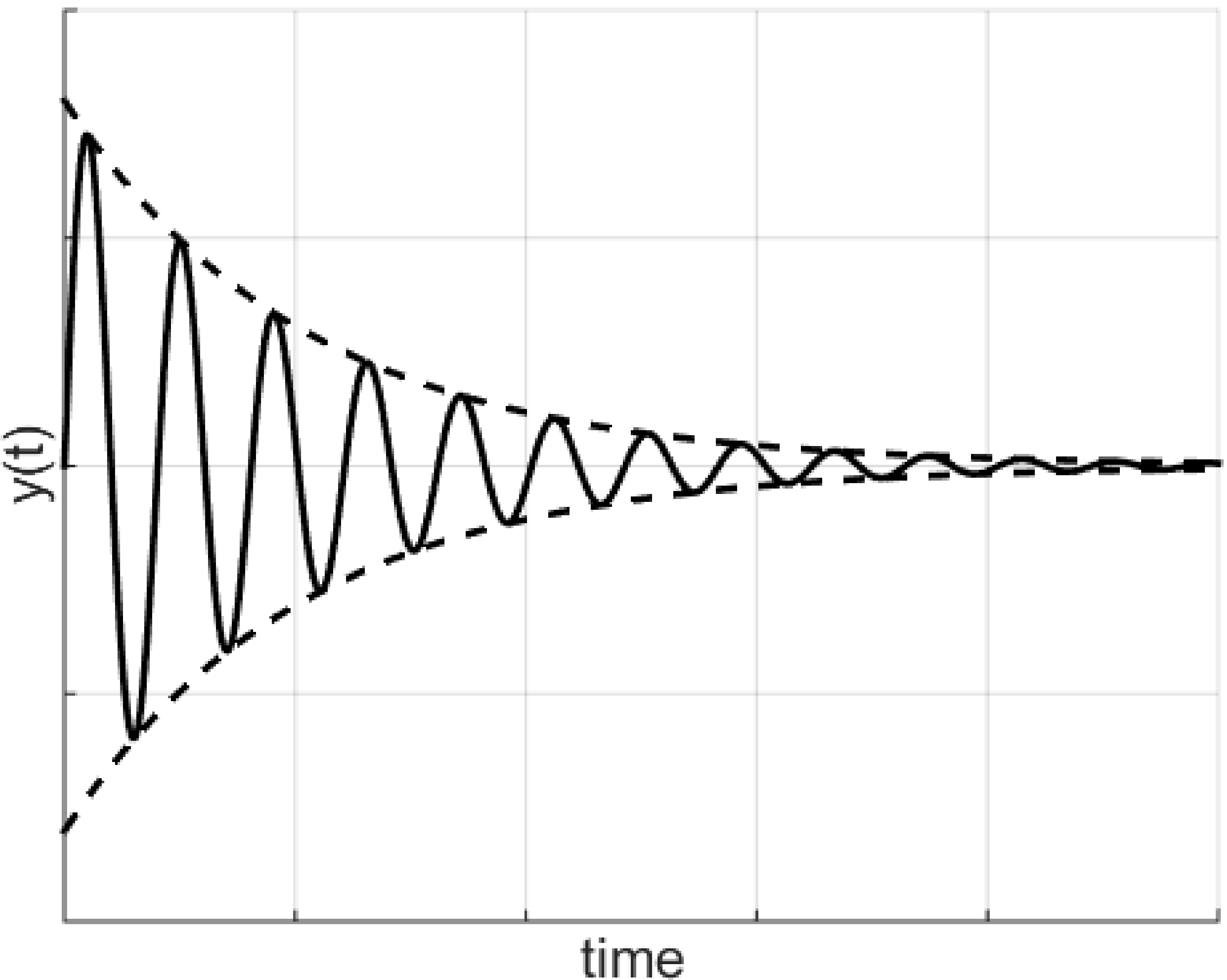
ω_n **Natural Frequency**

ζ **Damping Ratio**

Luckily, this equation is a "standard form" and the time domain result is given directly in Laplace Tables:

$$\gamma \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2} t\right)$$

Continuous Systems and Transfer Function Revision: 2^{nd} Order Transfer Function



But this is only for $\zeta < 1$

Given the damping ratio for a mass spring damper

$$\zeta = \frac{c}{2\sqrt{km}} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

Its clear that through choice of *m, k,* and *c* you could create a system with

$$\zeta \geq 1$$

$$G(s) = \gamma \frac{\omega_n^{\;2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\;2}}$$

The denominator determines the behaviour of the transfer function, as the numerator is a constant. This is known as the *characteristic equation*. It makes life much simpler if we first try to factor the denominator into first order terms like (s+a)(s+b)

So we require the roots of the equation $s^2 + 2\zeta\omega_n s + \omega_n^{\;2}$

Which can be determined from the quadratic equation:

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Thus it is clear that the roots of the characteristic equations *and thus the behaviour of the transfer function* is determined by the damping ratio

Overdamped	$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
Critically Damped	$s = -\omega_n$
Underdamped	$s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$
Undamped	$s = \pm j\omega_n$

Normally, only the underdamped case is given directly in the Laplace tables, so we need a bit of manipulation to get the others

Overdamped $\zeta > 1$

The characteristic equation has two real and negative roots:

$$s = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} = -\alpha,$$

$$s = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} = -\beta,$$

$$\frac{\omega_n^{\;2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\;2}} = \frac{\alpha\beta}{(s+\alpha)(s+\beta)}$$

Overdamped $\zeta > 1$

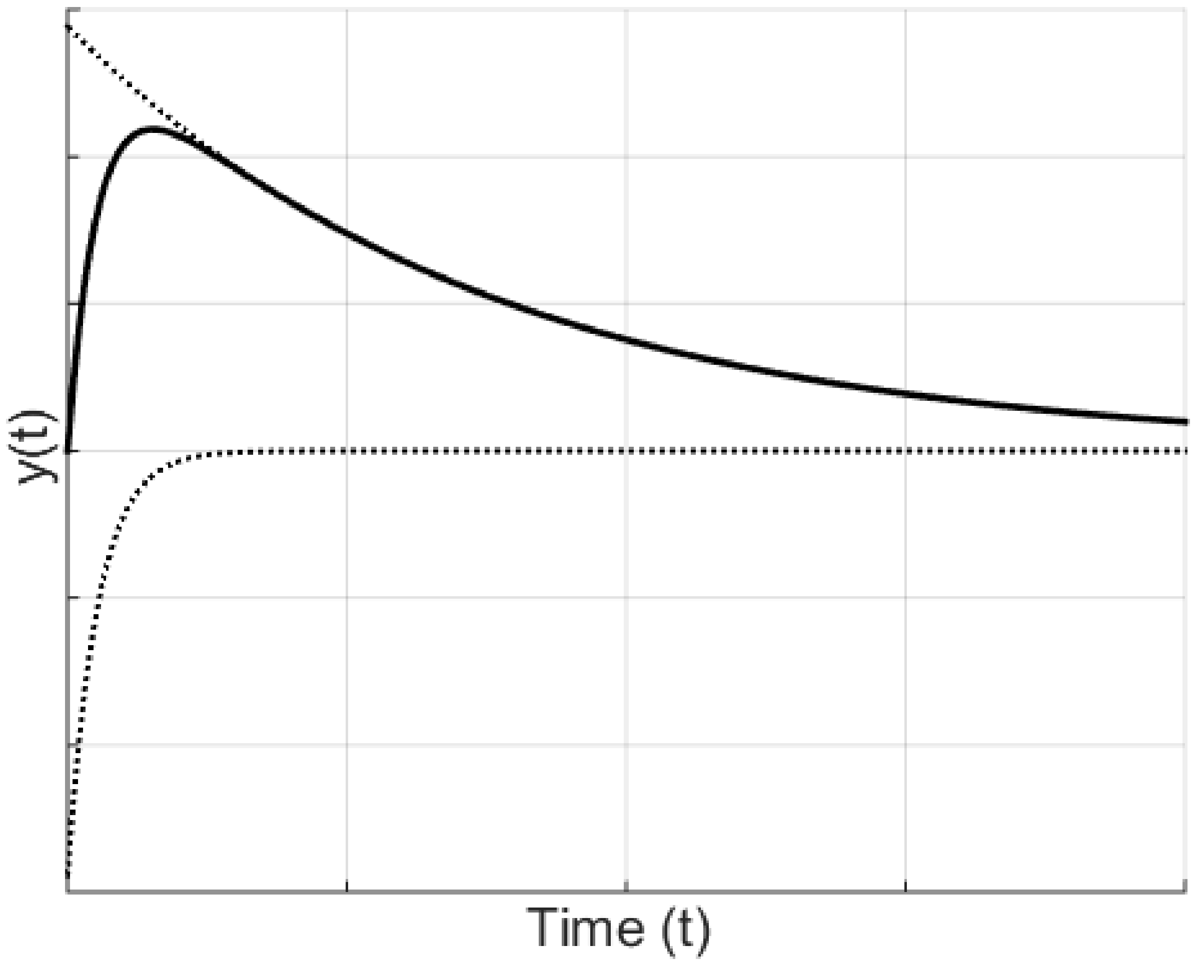
Using partial fractions we can split this into two separate first order systems:

$$\left(\frac{\alpha\beta}{\beta-\alpha}\right)\frac{1}{S+\alpha}+\left(\frac{\alpha\beta}{\alpha-\beta}\right)\frac{1}{S+\beta}$$

which gives the following in the time domain:

$$x(t)=\alpha\beta\left(\frac{e^{-\alpha t}}{\beta-\alpha}+\frac{e^{-\beta t}}{\alpha-\beta}\right)$$

Continuous Systems and Transfer Function Revision: Overdamped 2nd order system



There is *no* sinusoidal term anymore, the response is only a combination of exponential decays, a more complex but similar response to a first order system

Critically damped $\zeta = 1$

Special case with a single repeated root $s = -\omega_n$

$$\frac{\omega_n^{\ 2}}{s^2 + 2\omega_n s + \omega_n^{\ 2}} = \frac{\omega_n^{\ 2}}{\left(s + \omega_n\right)^2}$$

To get the time response, we first take the numerator outside:

$$L^{-1}\left[\frac{\omega_n^{\ 2}}{\left(s + \omega_n\right)^2}\right] = \omega_n^2 L^{-1}\left[\frac{1}{\left(s + \omega_n\right)^2}\right]$$

Critically damped $\zeta = 1$

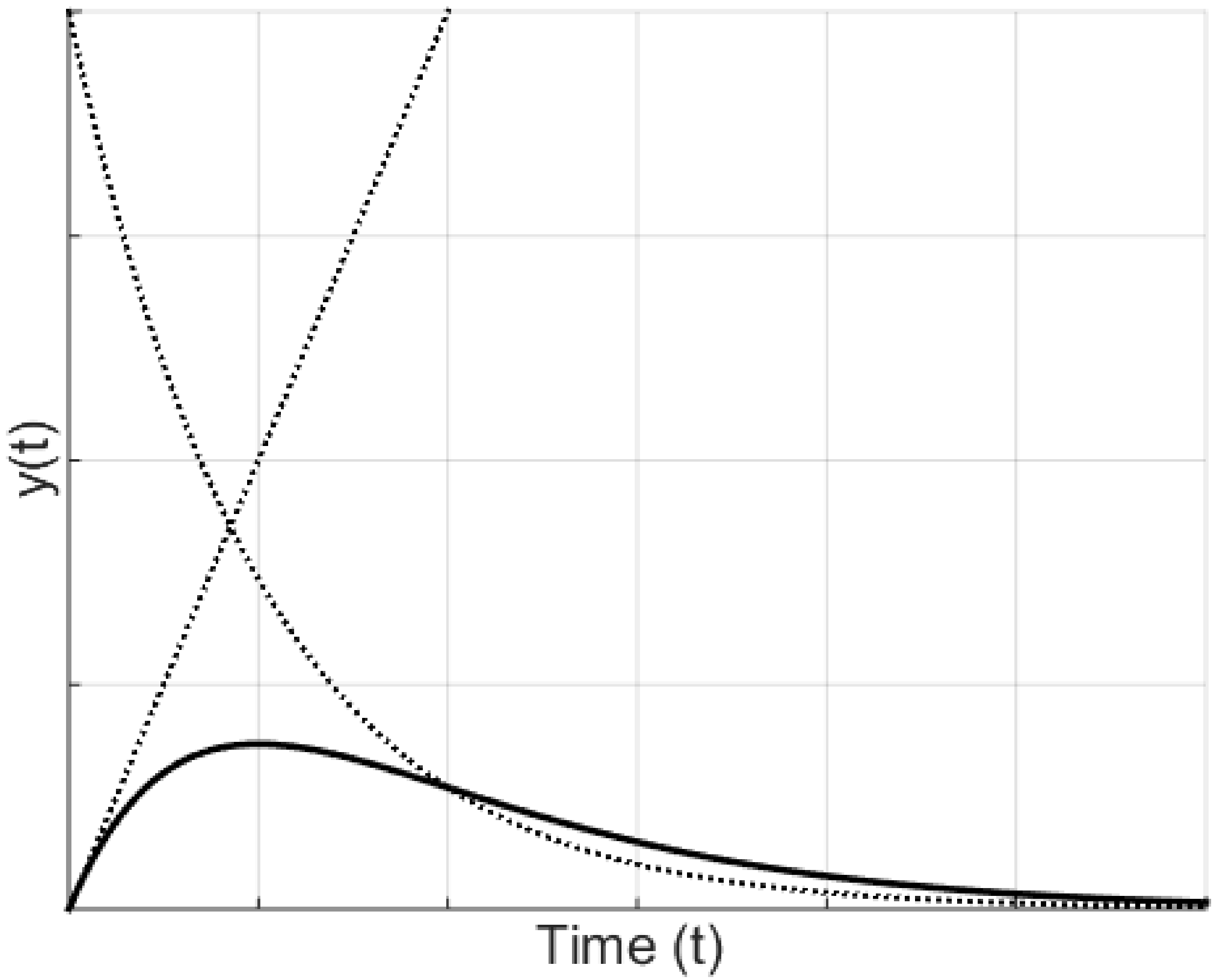
We can then use the inverse Laplace rule

if $L^{-1}\left[F(s)\right] = f(t)$ then $L^{-1}\left[F(s-a)\right] = e^{at} f(t)$

$$\omega_n^2 L^{-1}\left[\frac{1}{(s+\omega_n)^2}\right] = \omega_n^2 e^{-\omega_n t} L^{-1}\left[\frac{1}{s^2}\right]$$

$\frac{1}{s^2}$ is the standard form of a ramp, or more generally from the Laplace tables:

Critically damped $\zeta = 1$



Result is an exponential decay, multiplied by a ramp $= \omega_n^2 e^{-\omega_n t} t$

Undamped $\zeta = 0$

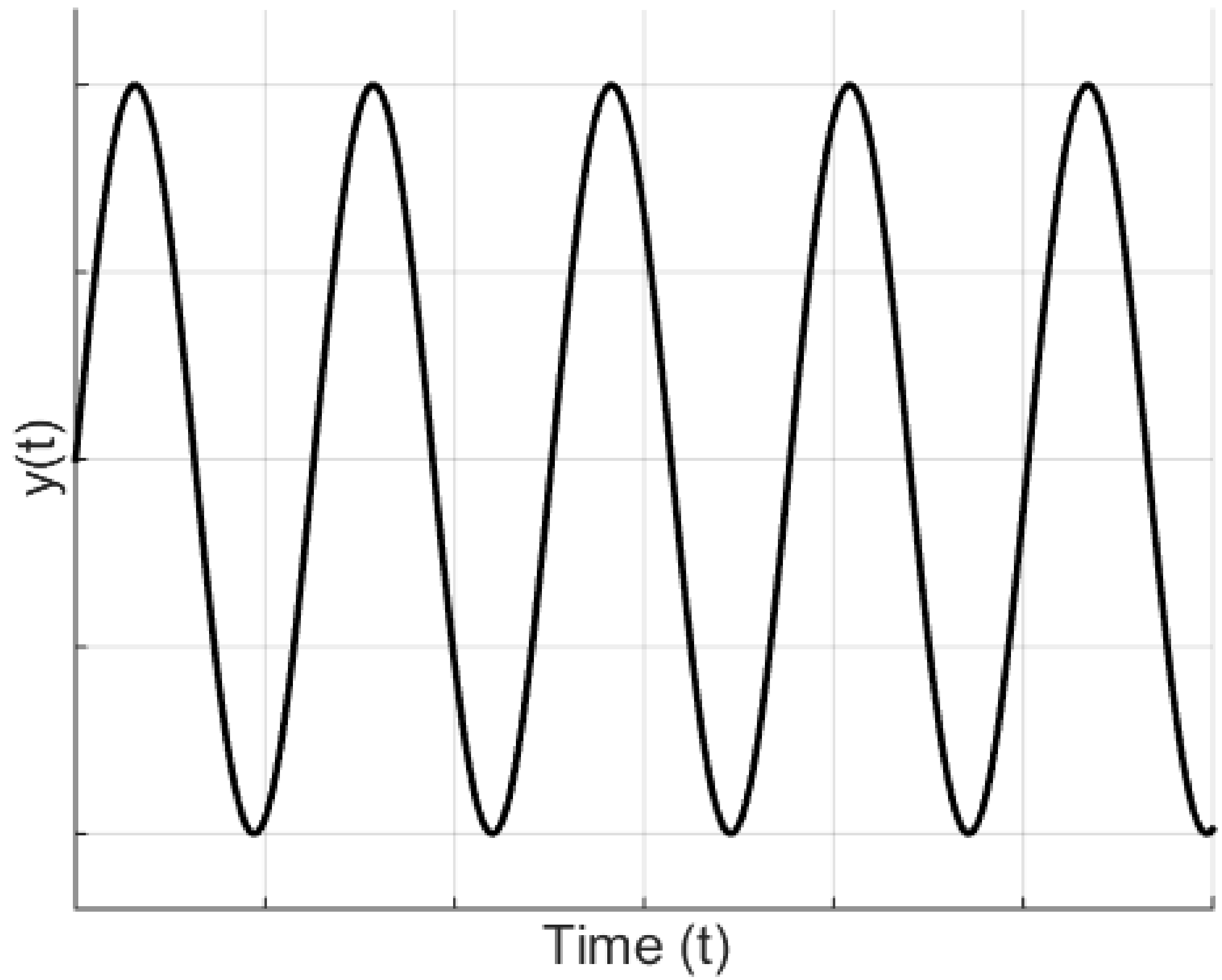
Another special case with two imaginary roots $s = \pm j\omega_n$

$$\frac{\omega_n^{\;2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\;2}} = \frac{\omega_n^{\;2}}{s^2 + \omega_n^{\;2}}$$

Which, using the standard form for an oscillator is

$$\omega_n L^{-1}\left[\frac{\omega_n}{s^2 + \omega_n^{\;2}}\right] = \omega_n \sin(\omega_n t)$$

Undamped $\zeta = 0$ $\omega_n \sin(\omega_n t)$



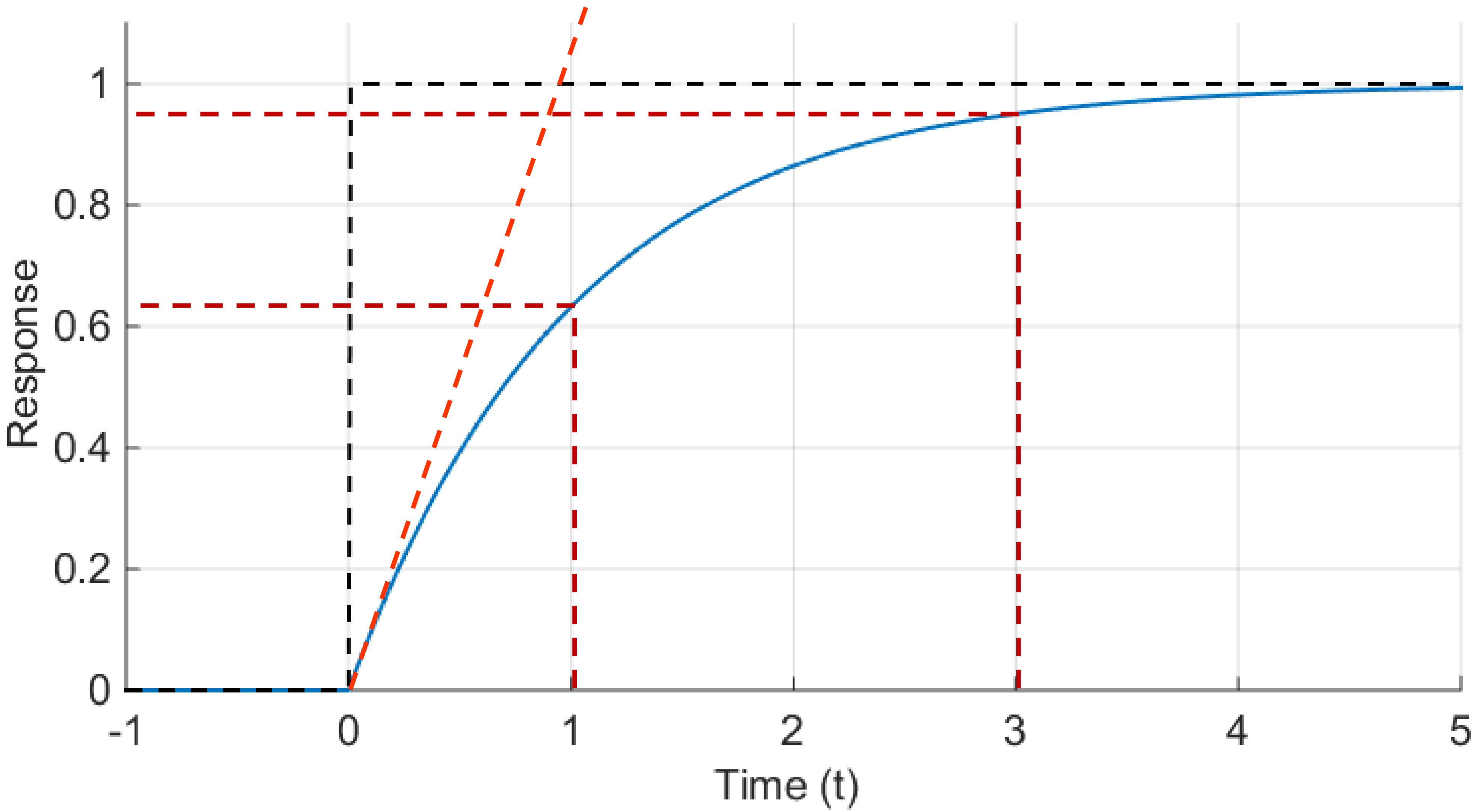
So the response is now just a sinusoid, with no exponential components

Unlike the first order system, the response of the second order system can vary considerably depending upon the parameters:

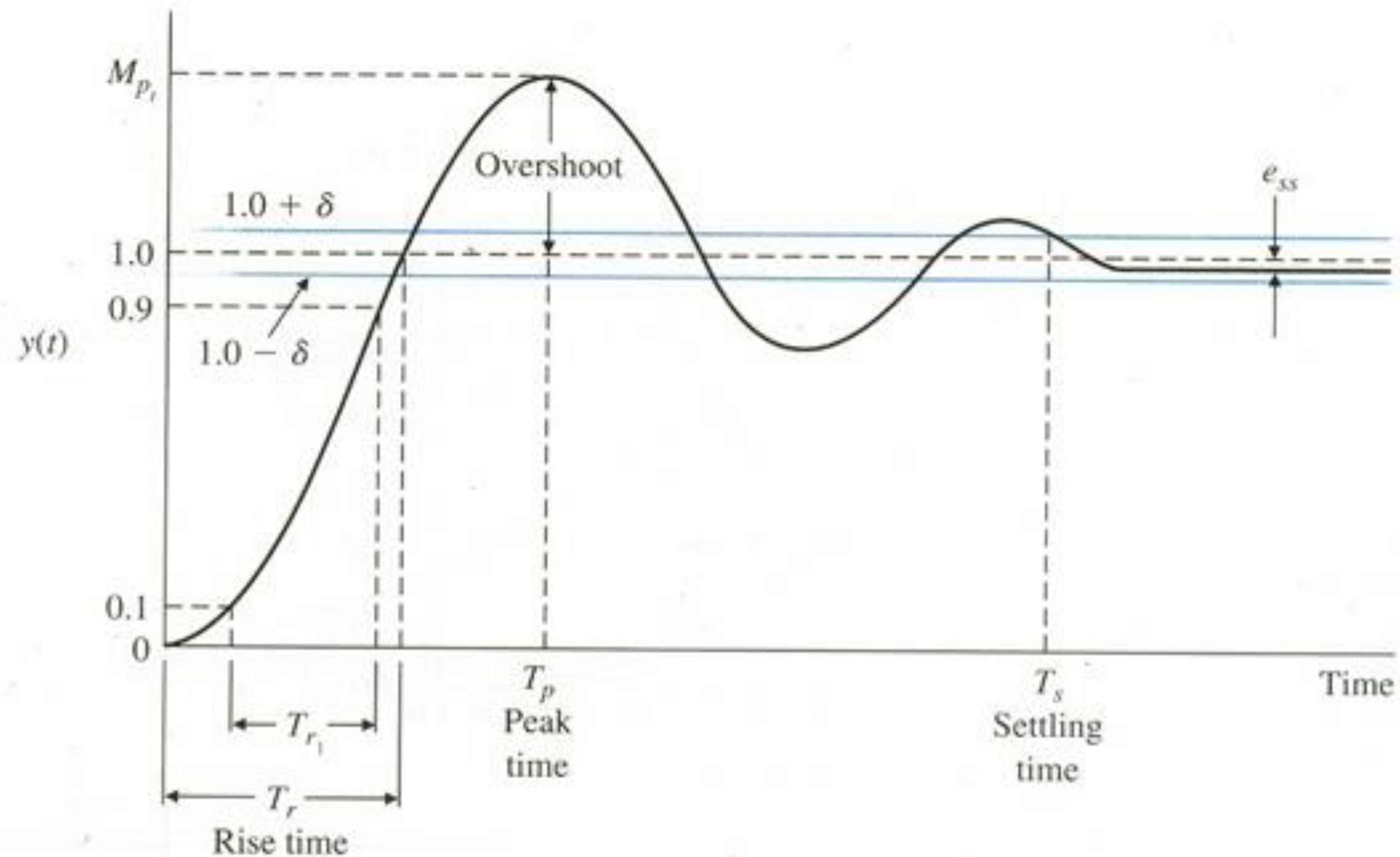
Overdamped	Addition of two exp. decays
Critically Damped	Ramp multiplied by exp. decay
Underdamped	Exp. Decaying sinusoid
Undamped	Sinusoid

Applying a step input to a 2^{nd} order systems - Damping

So (for now) assume L and R = 1, so gain and time constant =1



Continuous Systems and Transfer Function Revision: Second Order Systems – Unit Step Response



$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Thus it is clear that the roots of the characteristic equations *and thus the behaviour of the transfer function* is determined by the damping ratio

Overdamped	$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
Critically Damped	$s = -\omega_n$
Underdamped	$s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$
Undamped	$s = \pm j\omega_n$

Normally, only the underdamped case is given directly in the Laplace tables, so we need a bit of manipulation to get the others

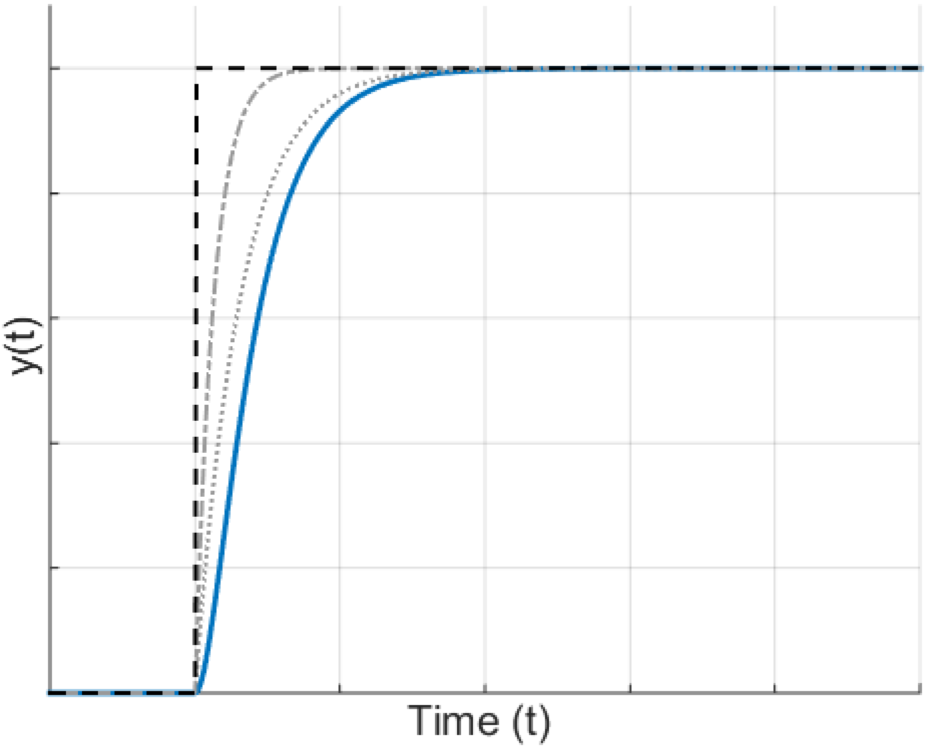
Previously we saw for a step of *A* the output can be calculated from

$$Y(s) = \frac{1}{s} \cdot \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

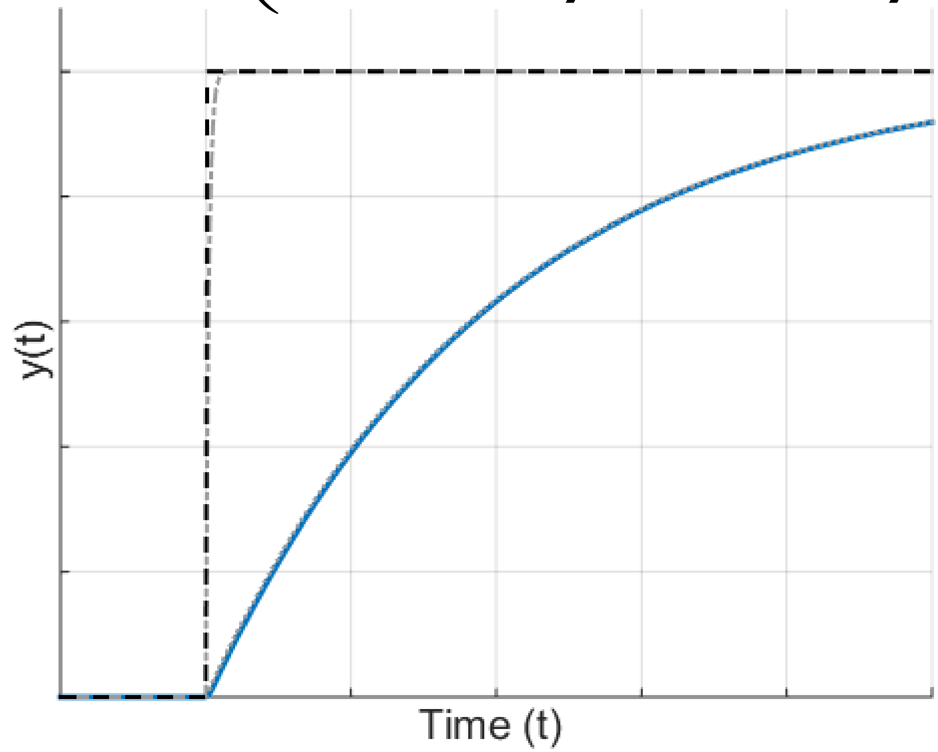
All the following can be found through partial fraction decomposition, and the standard Laplace tables.

Overdamped

$$y(t) = \left(1 + \frac{\beta e^{-\alpha t}}{\alpha - \beta} - \frac{\alpha e^{-\beta t}}{\alpha - \beta}\right)$$



Low ζ

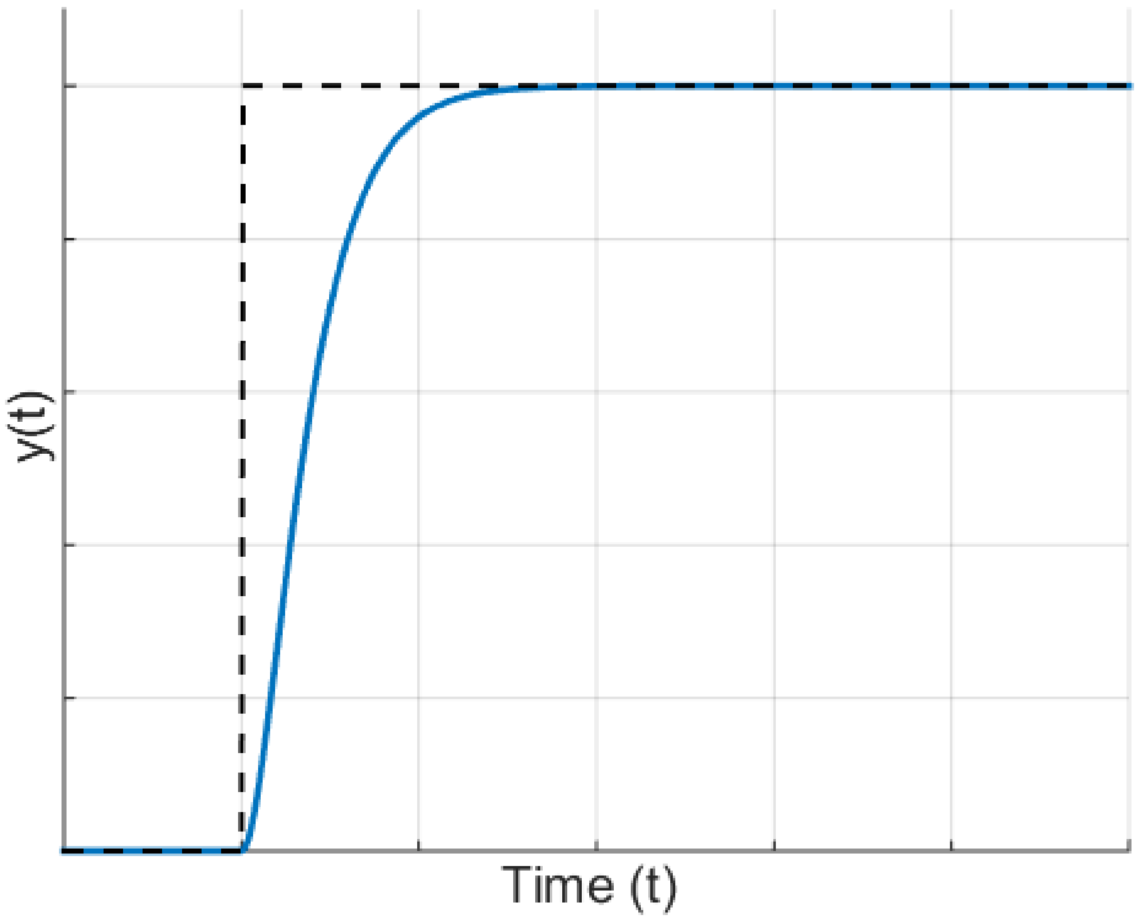


High ζ

The slowest rise from the β exponential dominates, as ζ increases the α term becomes negligible, and system becomes first order

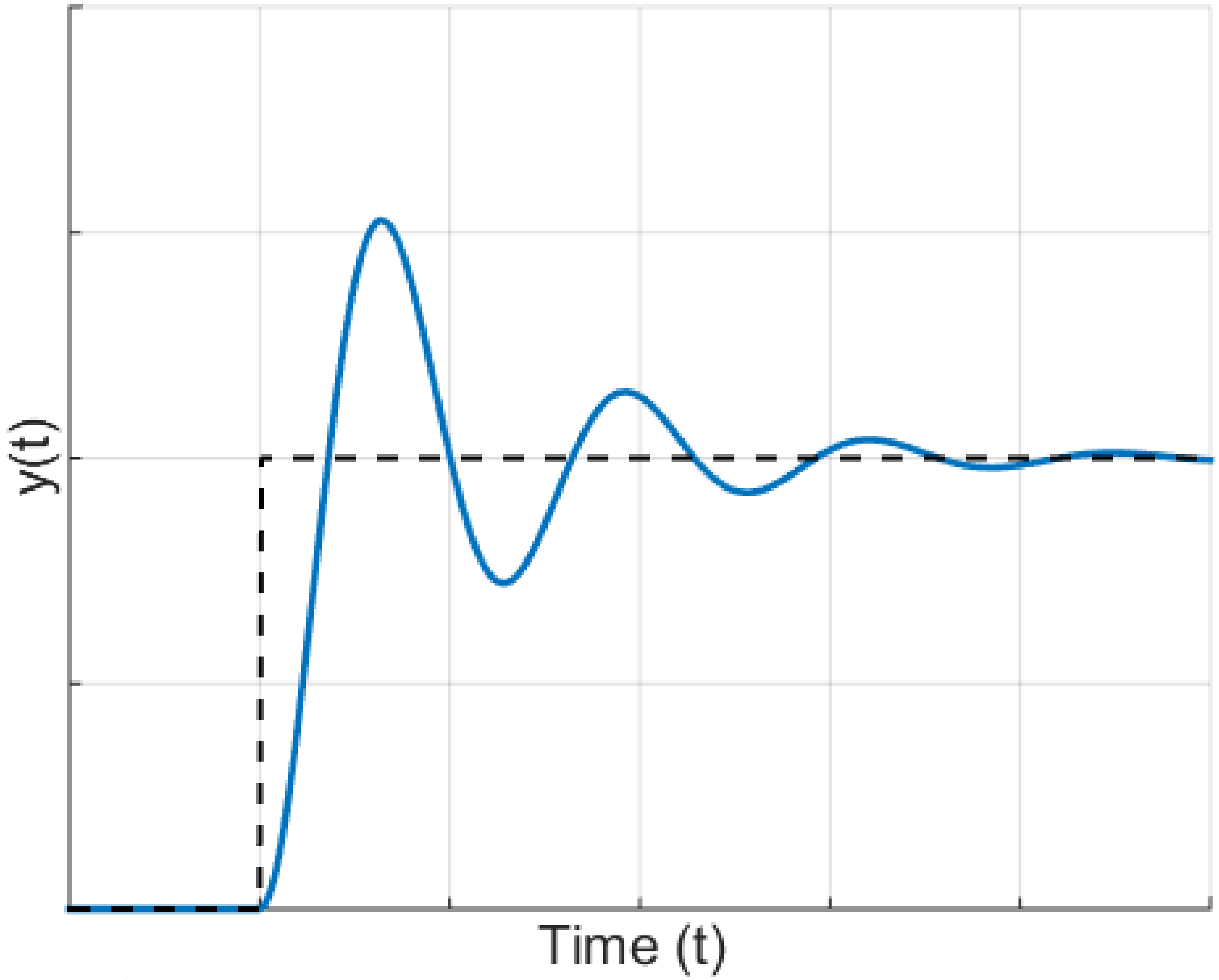
Critically Damped

$$y(t) = \left(1 - e^{-\omega_n t} - \omega_n e^{-\omega_n t} t\right)$$



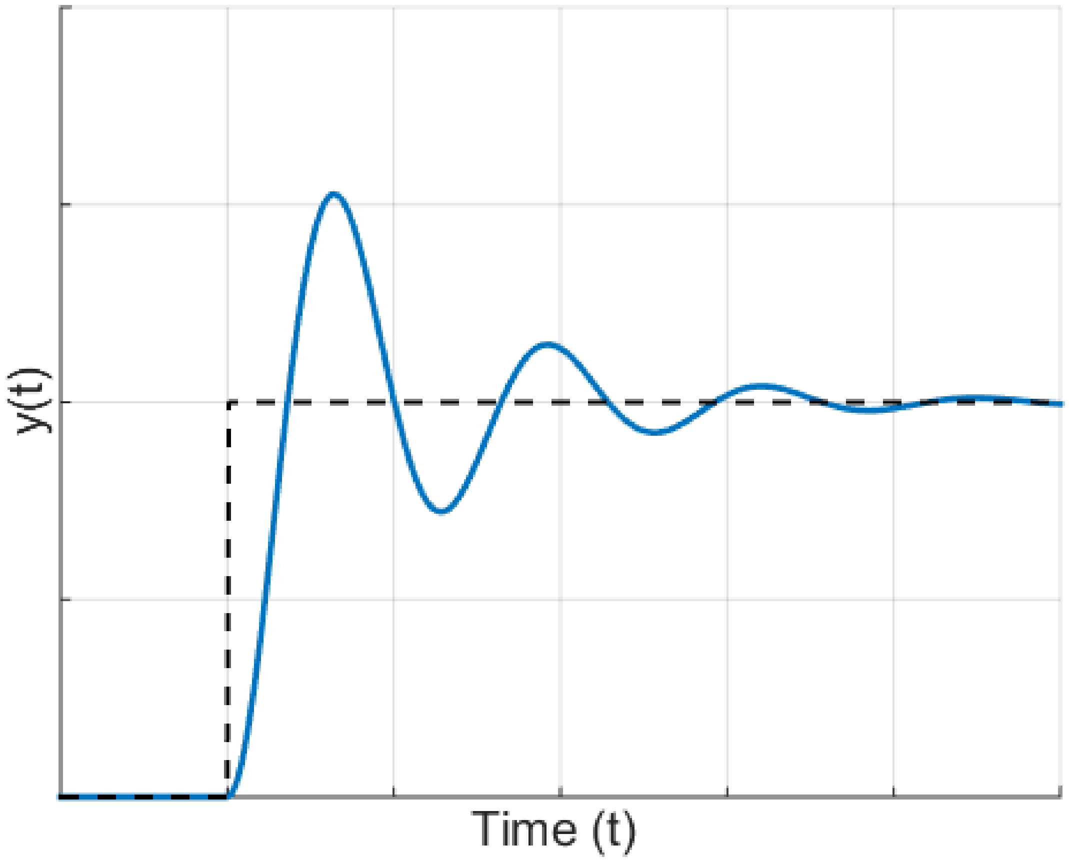
Fastest possible rise without oscillating

Underdamped



$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \cos^{-1}(\zeta)\right)$$

Underdamped



Or is sometimes written in terms of the *damped natural frequency*

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_d t + \cos^{-1}(\zeta)\right)$$

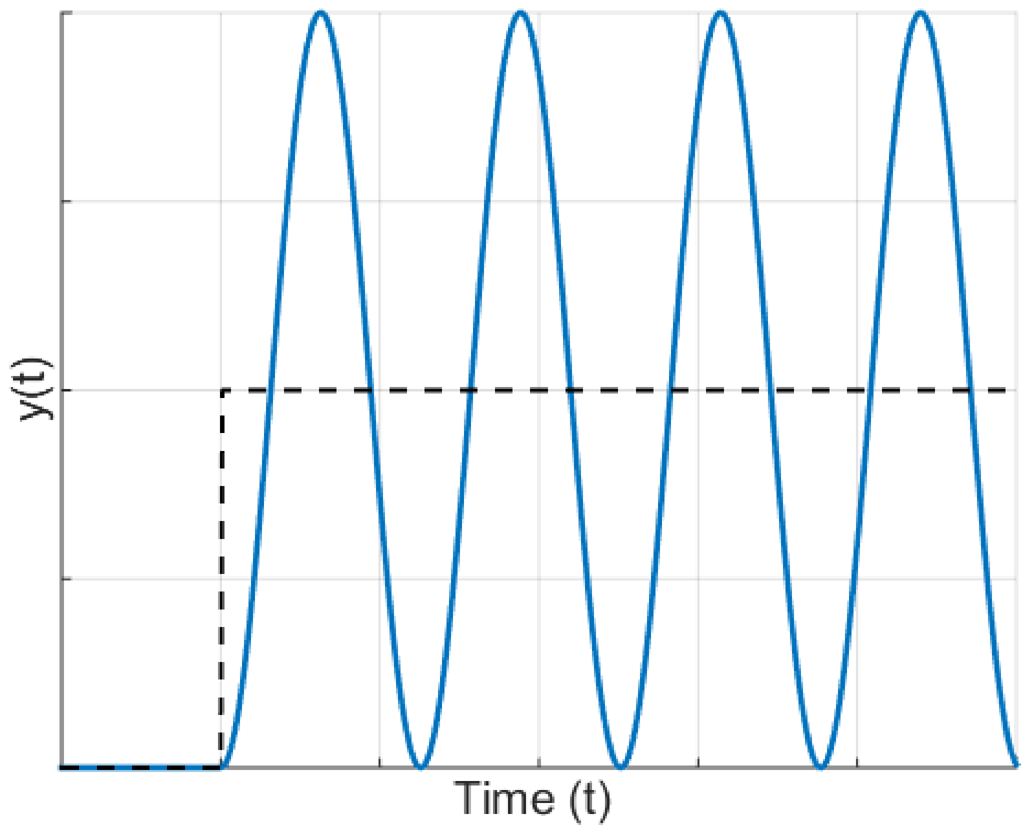
Undamped

$$y(t) = \left(1 - \cos\left(\omega_n t\right)\right)$$

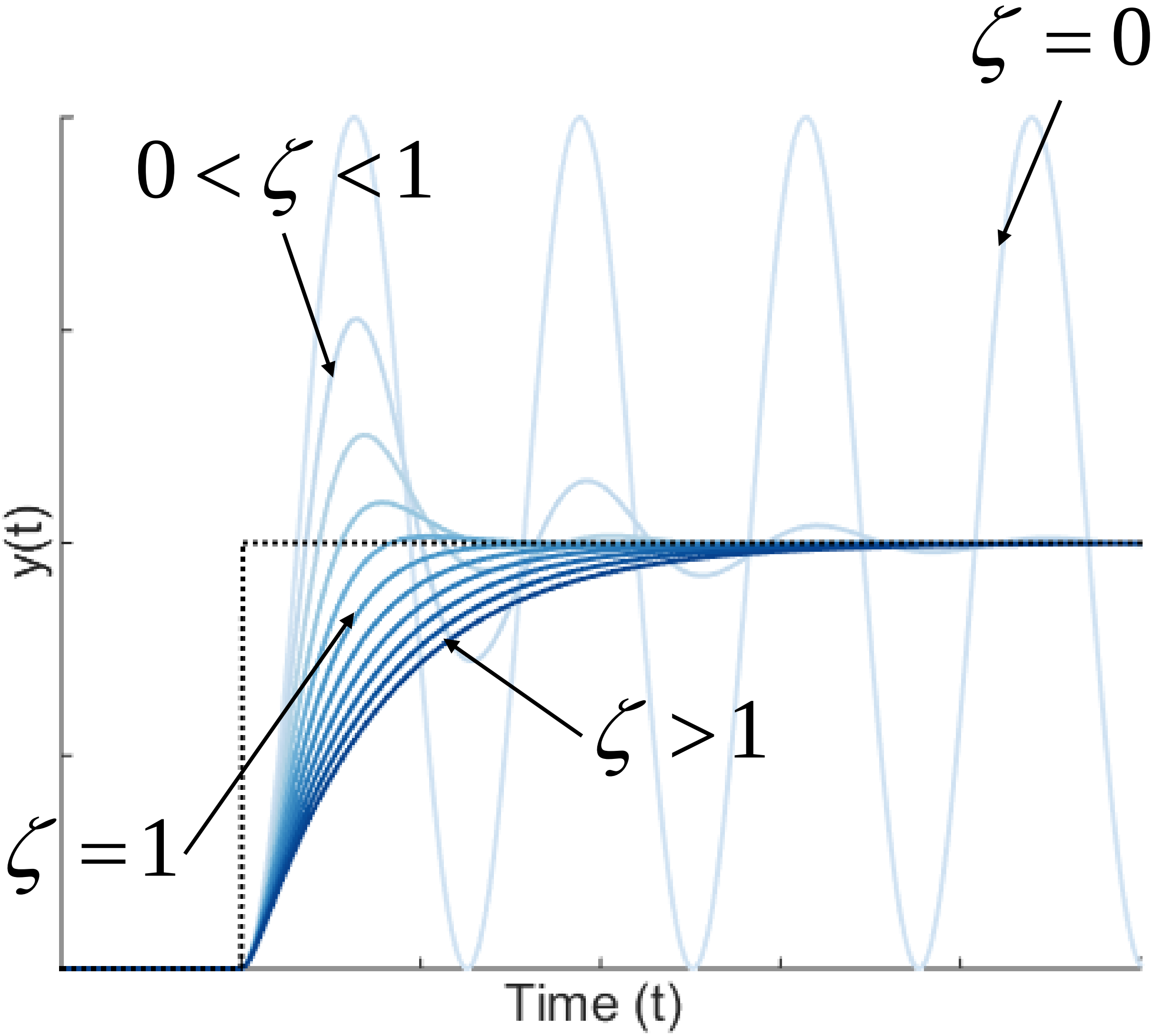
As damping ratio reaches zero, the oscillations no longer decay.

$$\omega_d = \omega_n \sqrt{1-0} = \omega_n$$

So oscillations at natural frequency with pi phase shift



Summary: Step input response of 2nd order systems



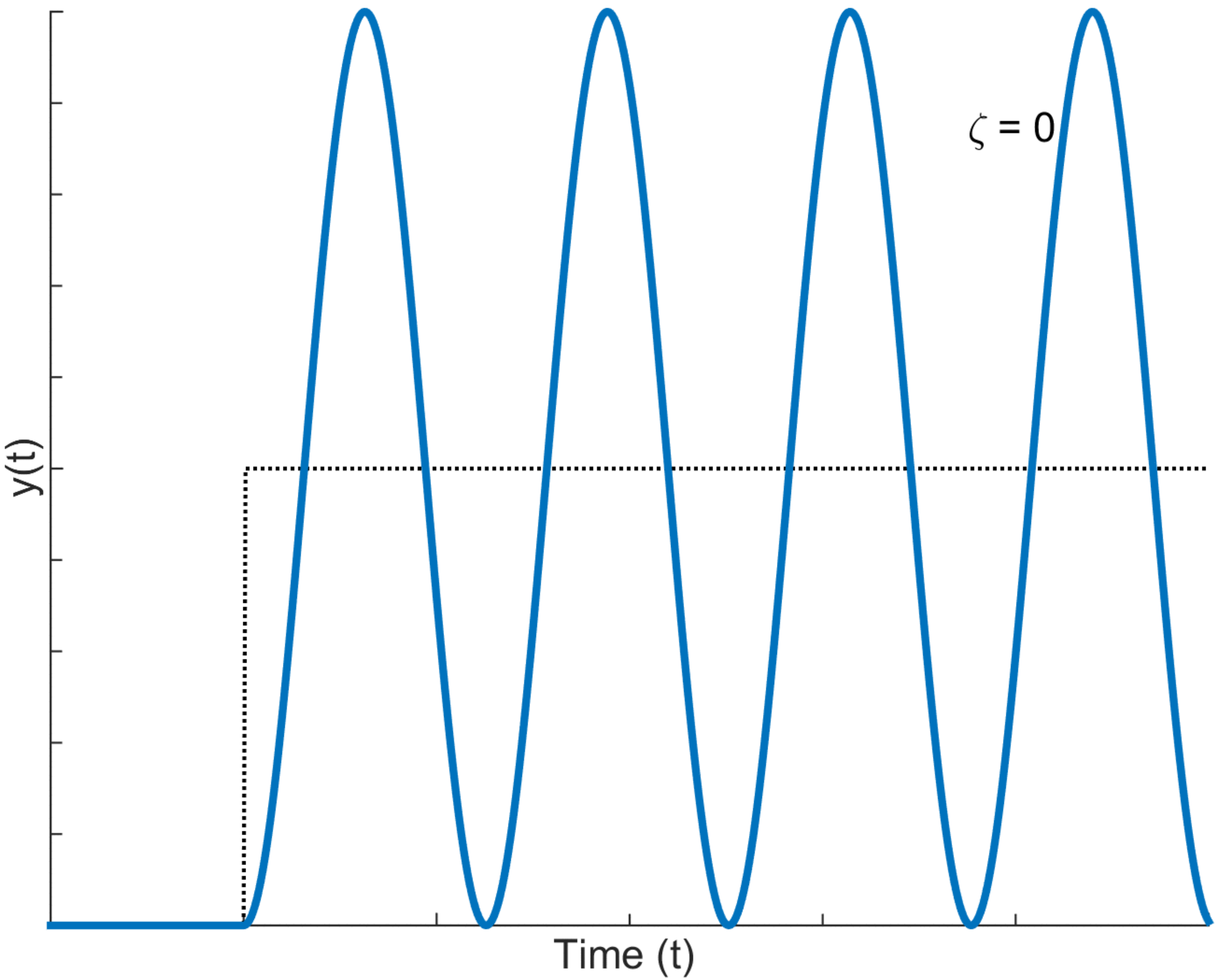
ζ=0 : undamped – Sustained oscillations

ζ<1 : under-damped – Exp Decaying Oscillatory response

ζ=1 : critically damped – Fastest response with no overshoot

ζ>1 : over-damped – cannot overshoot

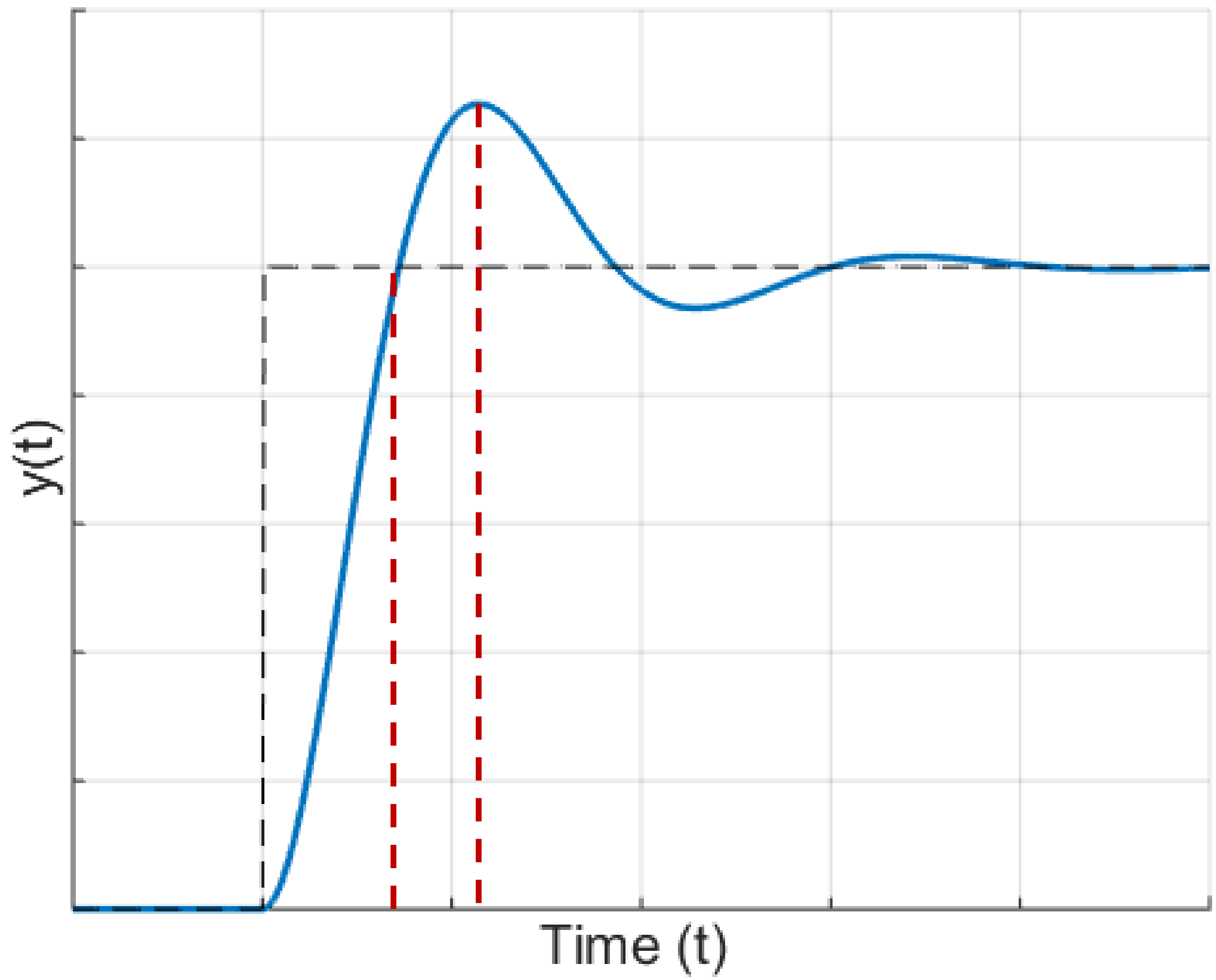
Continuous Systems and Transfer Function Revision:
Summary: Step input responses of a 2^{nd} order systems



Underdamped step input responses

Rise Time
Time to reach steady state value (for first time)

Peak Time
Time to initial overshoot

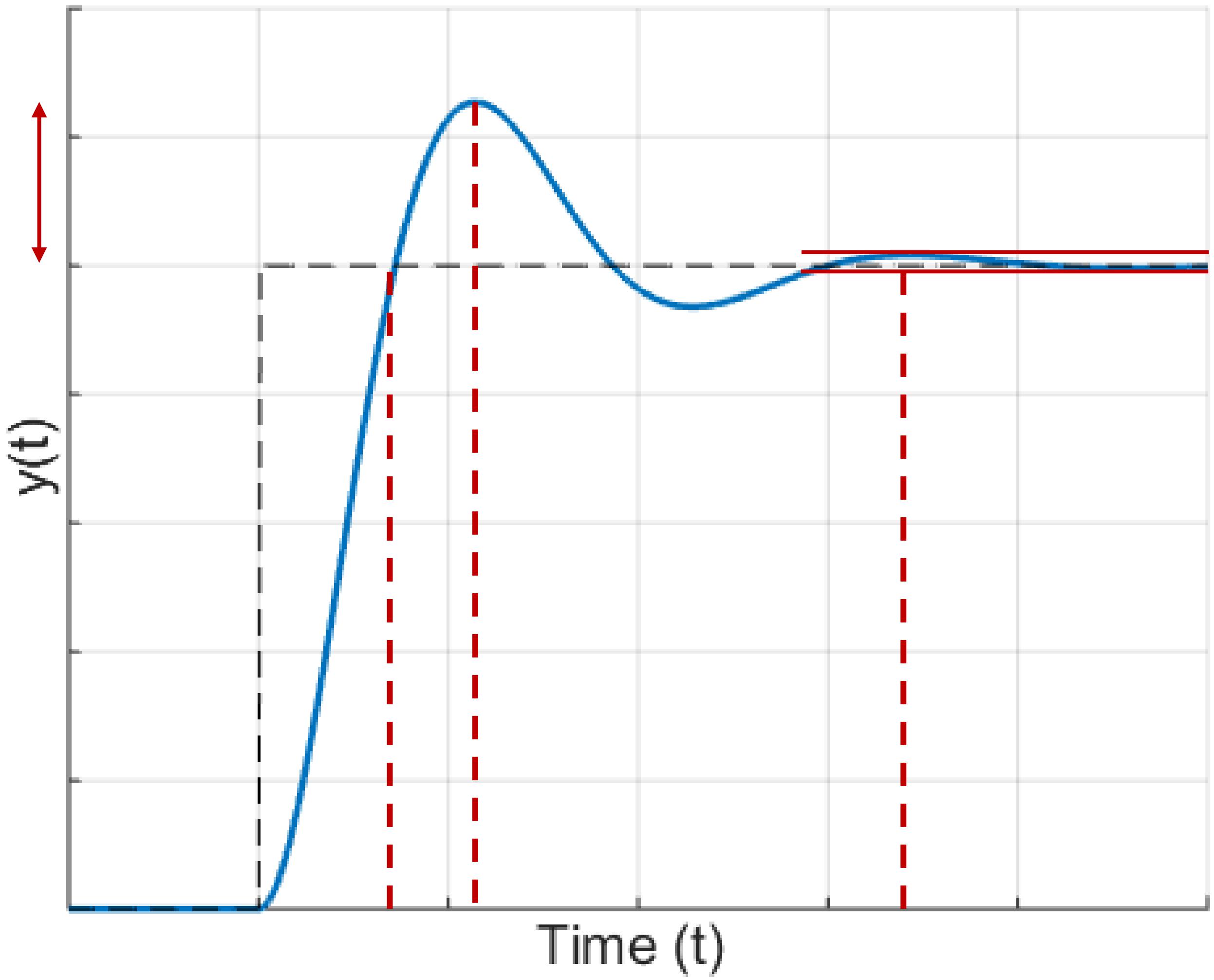


Peak overshoot

Value of overshoot above steady state

Settling Time

Time for response to reach and maintain a certain ratio of the steady state value. Commonly ±5% or ± 2%



These criteria can be read from the graph, but having equations for them allows for system parameters ζ and ω_n to be estimated. It is also useful for quantifying the effect of ζ on the response.

Rise Time

This can be found by setting the response *y* to 1 and then rearranging:

$$t_r = \frac{1}{\omega_d}\left(\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

So a higher damping ratio gives a slower response, but lower damping ratios give a more oscillatory response. Also, that for *overdamped* systems where ζ > 1, the rise time is infinite.

Peak Time

Peak time occurs at half a period of oscillation, or can be found from finding first minima by setting derivative to zero

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Inserting the above expression into the underdamped response :

$$1+A=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\pi/\sqrt{1-\zeta^2}}\sin\left(\pi+\cos^{-1}(\zeta)\right)$$

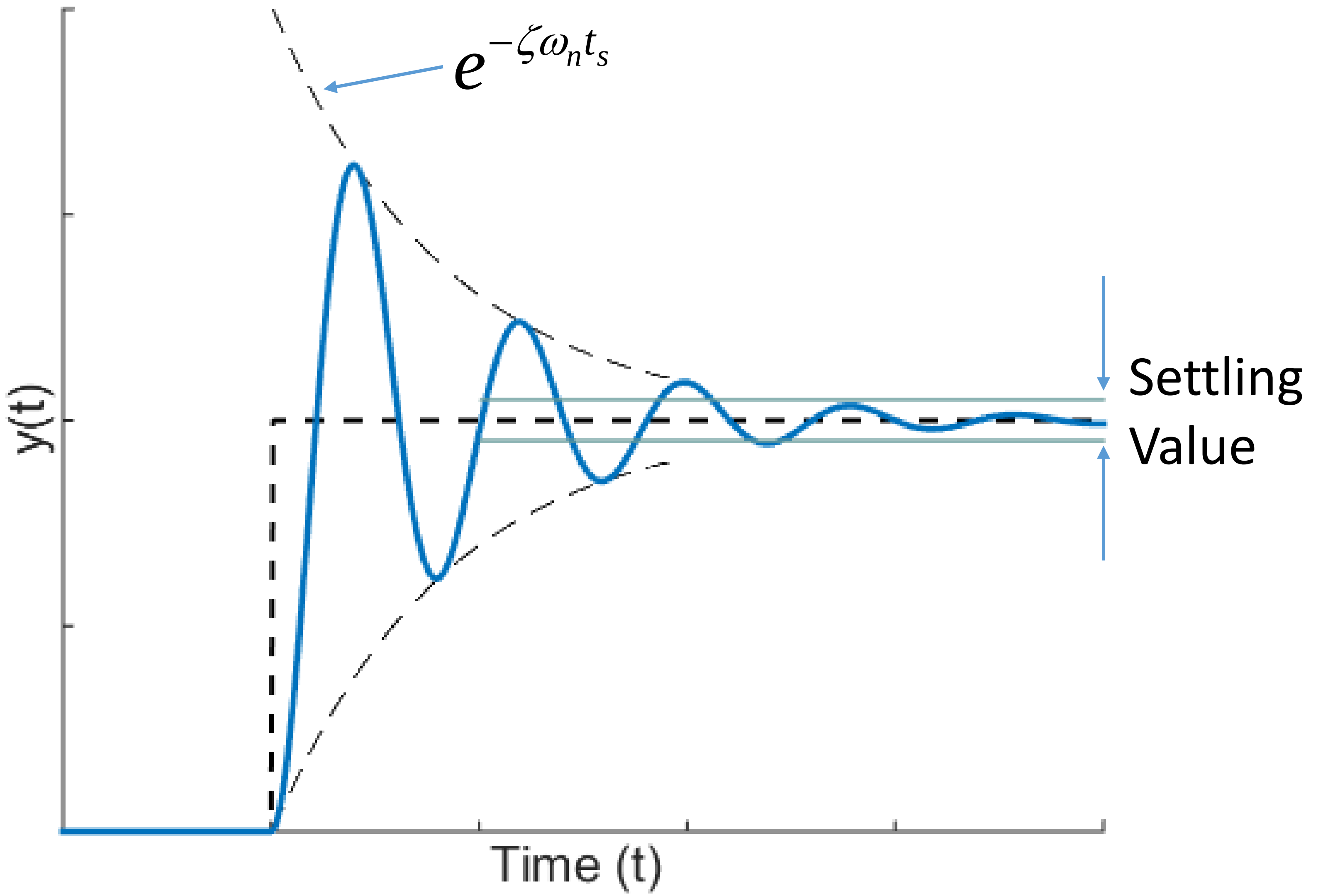
$$A=\frac{\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}}e^{-\zeta\pi/\sqrt{1-\zeta^2}}=e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

Often this is given as a percentage:

This can be rearranged to find the required damping ratio for a required overshoot

$$\zeta=\sqrt{\frac{\left(\ln\left(A/100\right)\right)^2}{\pi^2+\left(\ln\left(A/100\right)\right)^2}}$$

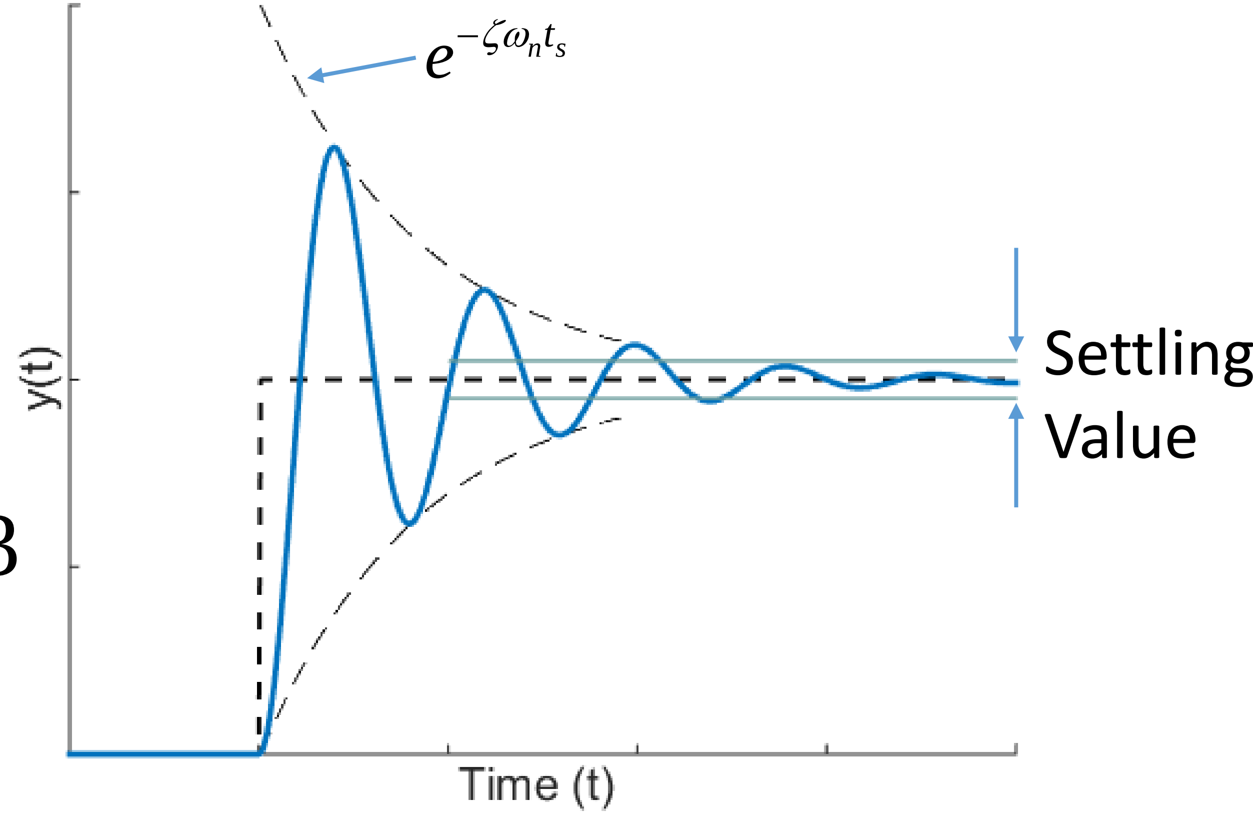
The envelope of the oscillation is described by the decaying exponential term, so to find the settling time, we can just consider this term alone



So for example, the common target is 5%:

$$e^{-\zeta\omega_n t_s} = 0.05$$

$$-\zeta\omega_n t_s = \ln(0.05) = -3$$



$$t_s(5\%) = \frac{3}{\zeta\omega_n}$$

Or similarly

So settling time increases with a *decreasing* damping ratio, for underdamped systems only

Understanding Poles and Zeros

Continuous Systems and Transfer Function Revision: System Poles and Zeros

- As defined, the transfer function is a rational function in the complex variable $s = \sigma + j\omega$, that is

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

- It is often convenient to factor the polynomials in the numerator and denominator, and to write the transfer function in terms of those factors:

$$G(s) = \frac{N(s)}{D(s)} = K \frac{(s - z_1)(s - z_2) \ldots (s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2) \ldots (s - p_{n-1})(s - p_n)}$$

- where the numerator and denominator polynomials, $N(s)$ and $D(s)$, have real coefficients defined by the system's differential equation and $K = \frac{b_m}{a_n}$.

Continuous Systems and Transfer Function Revision: System Poles and Zeros

- Poles and zeros are found by

$$N(s) = 0 \text{ and } D(s) = 0$$

- All of the coefficients of polynomials $N(s)$ and $D(s)$ are real, therefore the poles and zeros must be either purely real, or appear in complex conjugate pairs.

The poles and zeros are properties of the transfer function, and therefore of the differential equation describing the input-output system dynamics. Together with the gain constant, they completely characterize the differential equation, and provide a complete description of the system.

System Poles and Zeros - Example

- A linear system is described by the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 3\frac{du}{dt} + 12u$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y\right\} = s^2Y(s) + 2sY(s) + 5Y(s) \qquad \mathcal{L}\left\{3\frac{du}{dt} + 12u\right\} = 3sU(s) + 12U(s)$$

$$N(s) = 3s + 12 = 0 \Rightarrow z_1 = -4$$

$$ax^2 + bx + c = 0$$

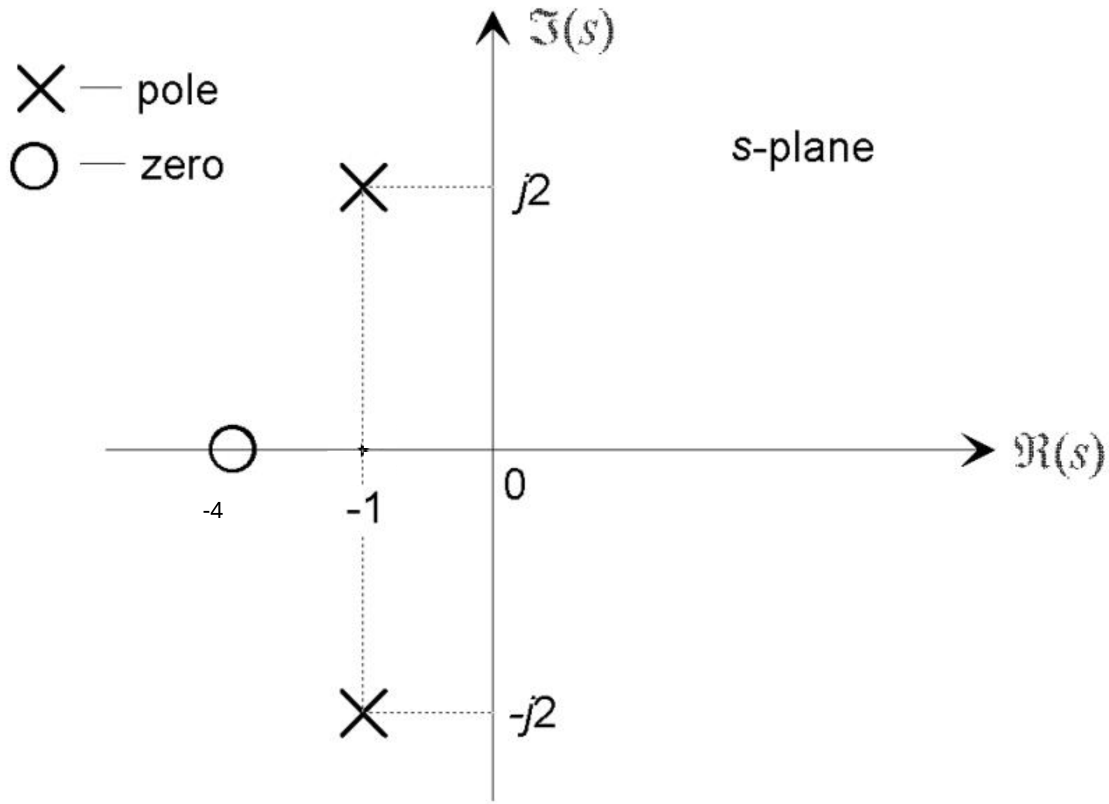
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D(s) = s^2 + 2s + 5 \Rightarrow p_{1/2} = \frac{-2 \pm \sqrt{2^2 - 4 \times 5}}{2} = -1 \pm j2$$

$$G(s) = \frac{N(s)}{D(s)} = 3\frac{(s-(-4))}{(s-(-1+j2))(s-(-1-j2))}$$

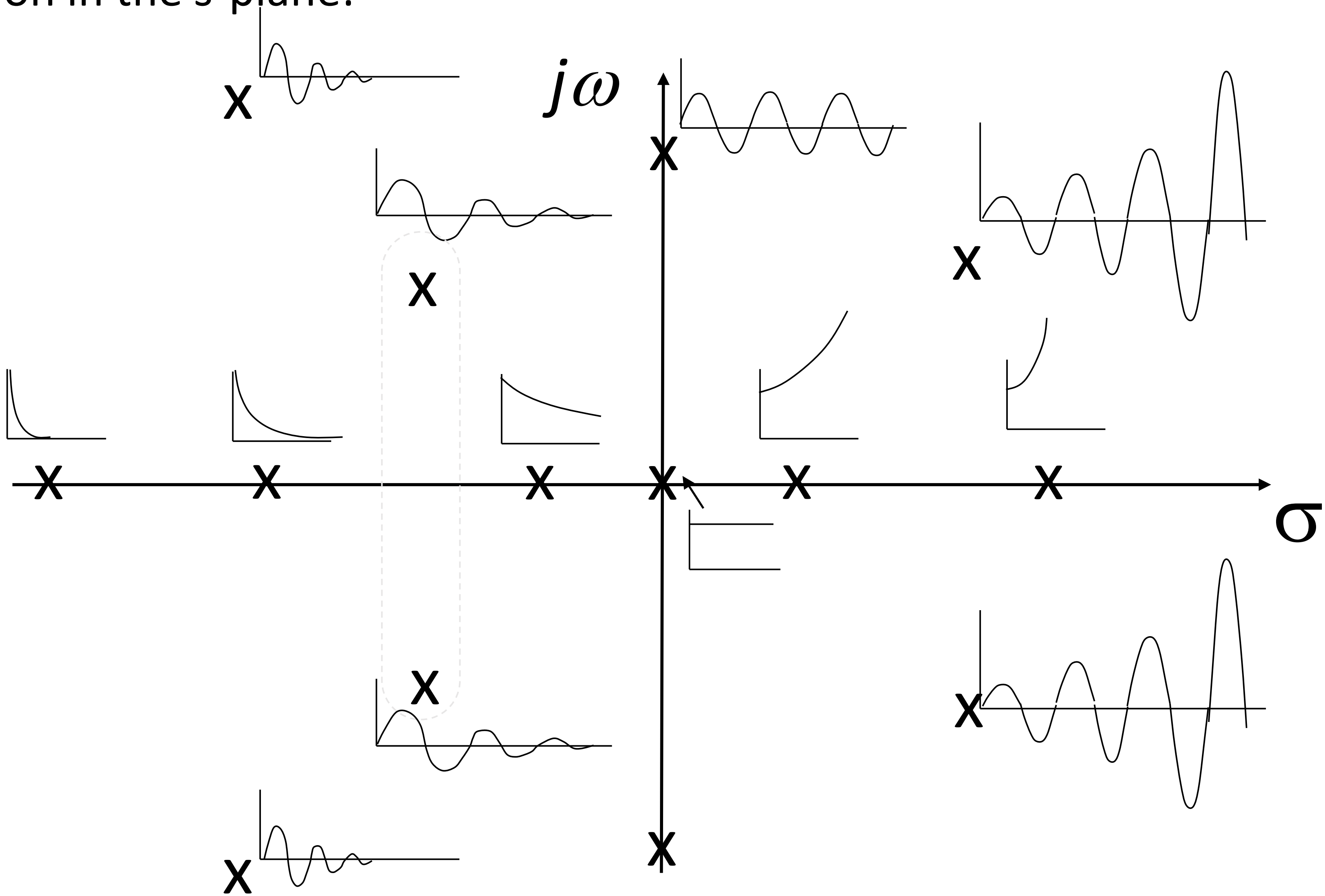
System Poles and Zeros - Example

$$G(s) = \frac{N(s)}{D(s)} = 3\,\frac{(s-(-4))}{(s-(-1+j2\,))(s-(-1-j2\,))}$$



Summary: Transfer functions in the s-plane - Zeros

The impulse response of each pole in the transfer function depends upon its location in the s-plane:



Effects of the damping ratio on poles

As we have seen, the roots of the transfer function of a second order system depends upon the damping coefficient:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

We can plot the poles of the characteristic equation as with any other transfer function:

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

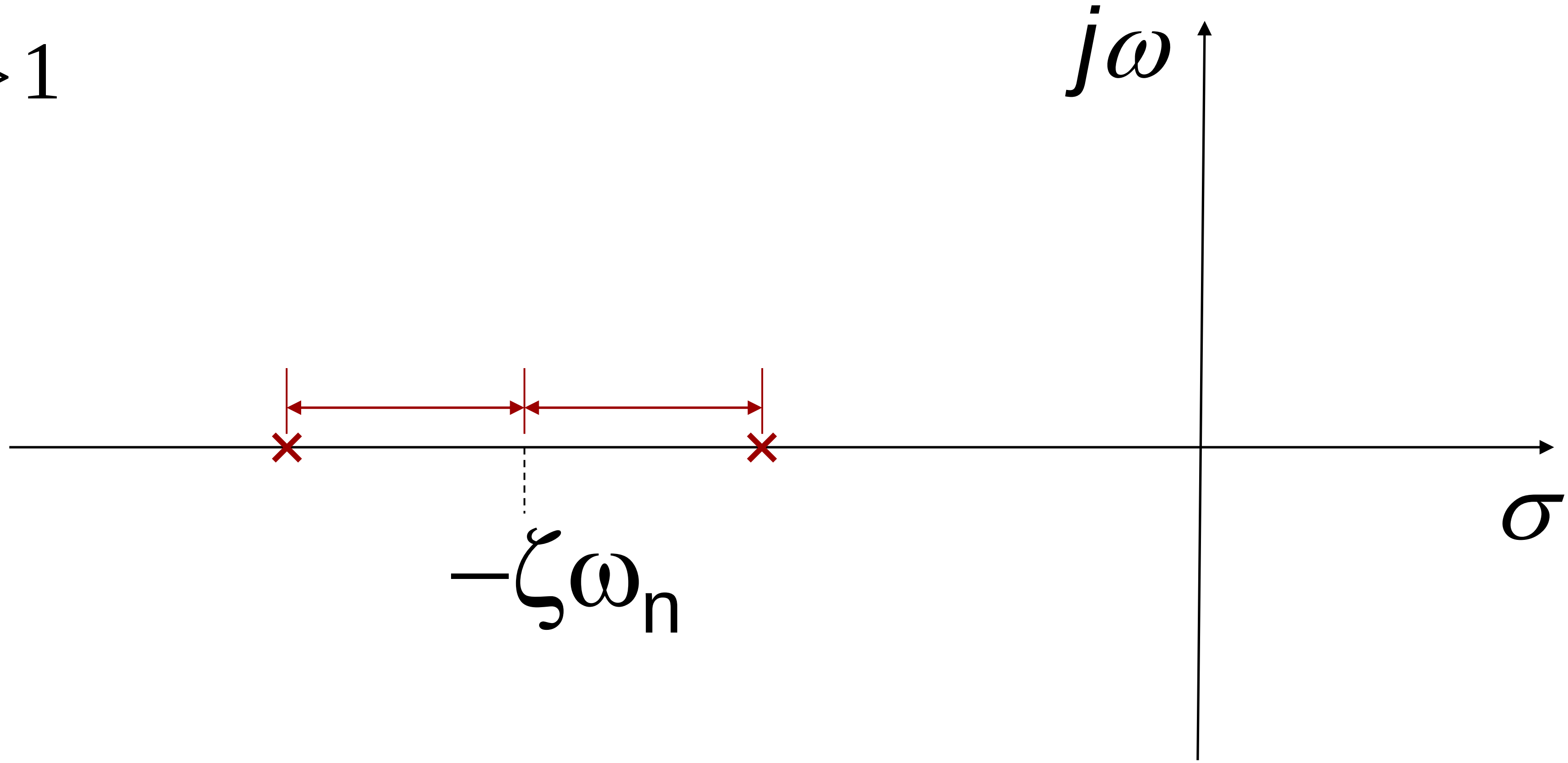
Both roots are real, and thus are on real axis

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

No oscillations in step response

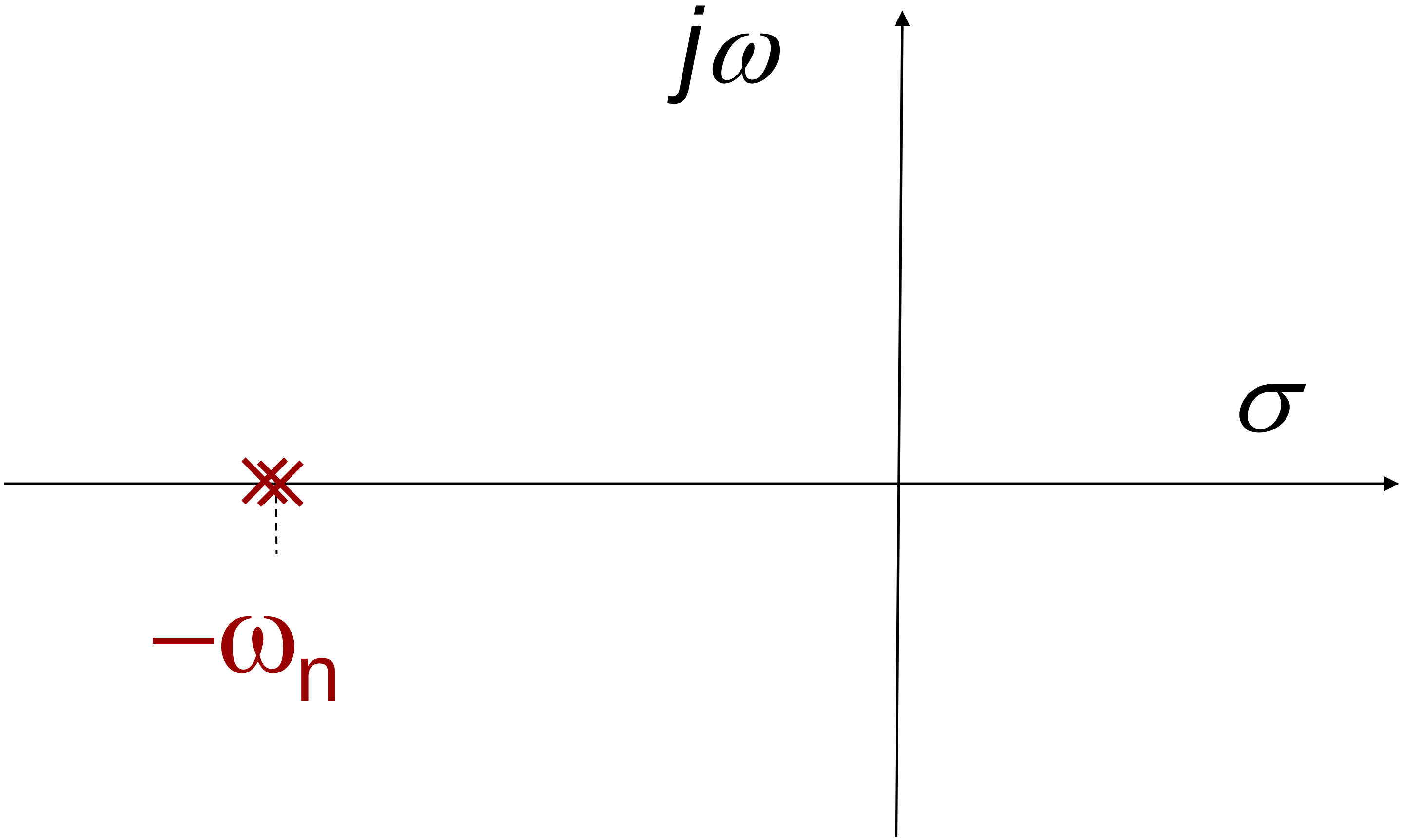
$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$\zeta > 1$



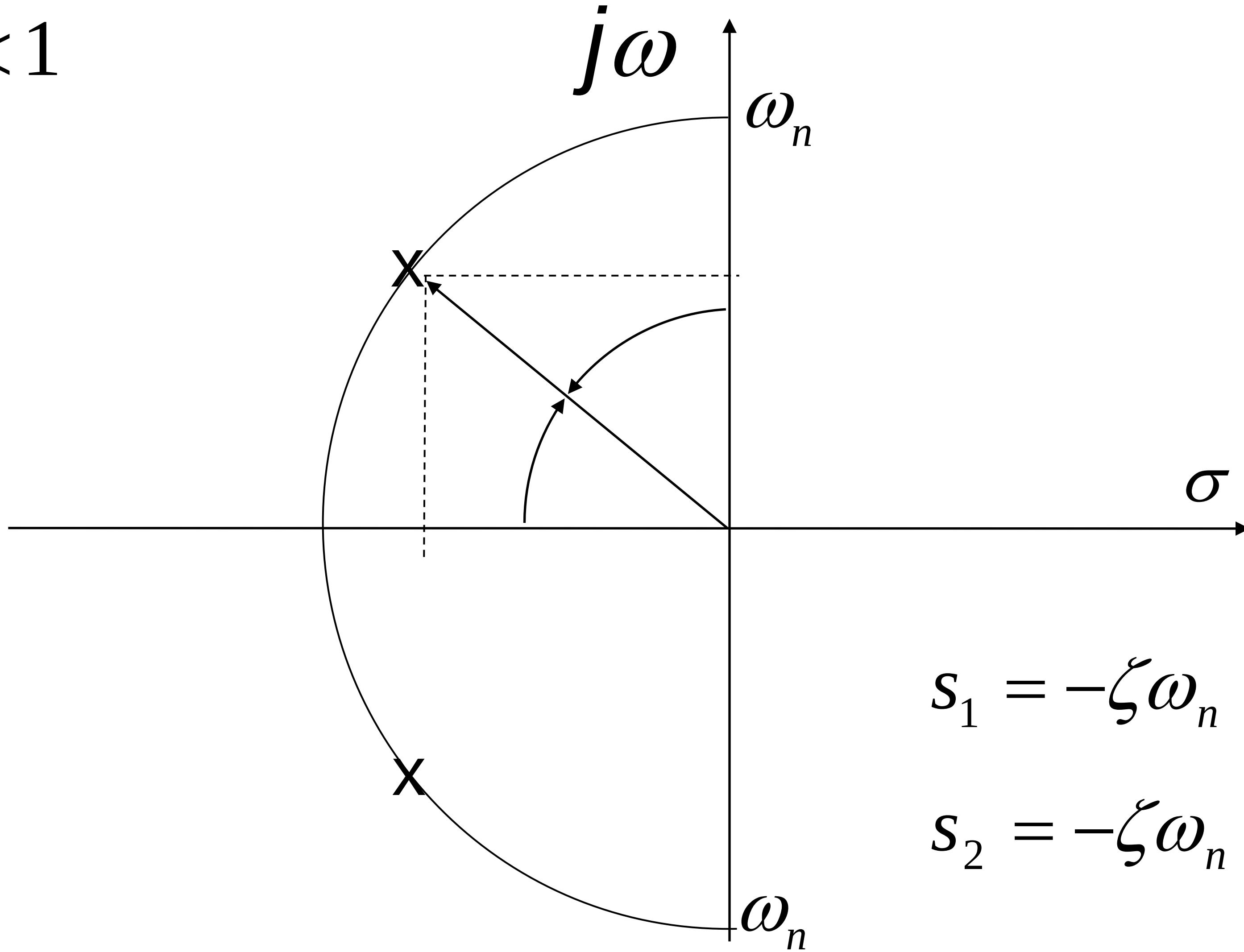
$\zeta = 1$

$s = -\omega_n$



The roots of the equations coincide, and the system is a product of two equal first order lags.

$\zeta < 1$



$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

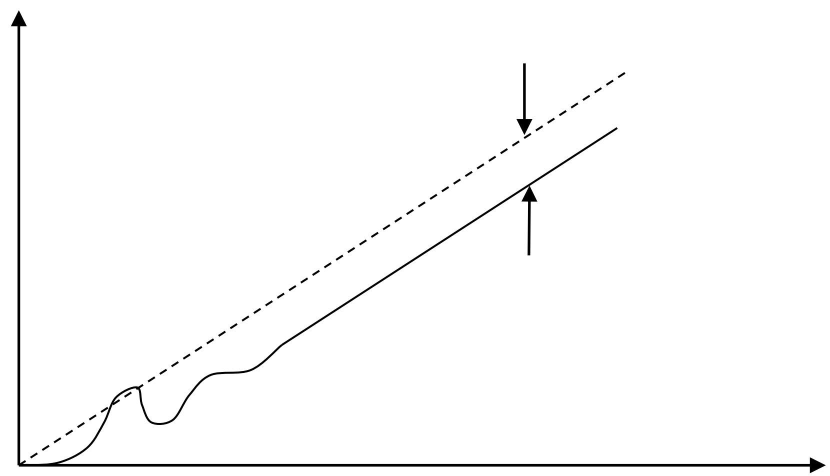
Both roots complex, and form conjugate pair. System is a product of exponential function and oscillatory component.

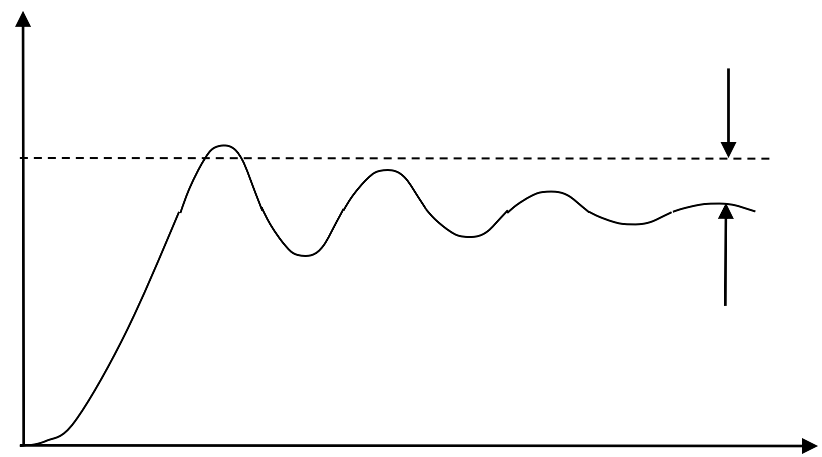
Steady State Performance

Previously we have looked at performance criteria for the transient response of a system: *settling time, peak time, overshoot etc.*

In addition to this it is important to know how accurately the control system tracks the demand once it has settled down, i.e. it is in the *steady state.*

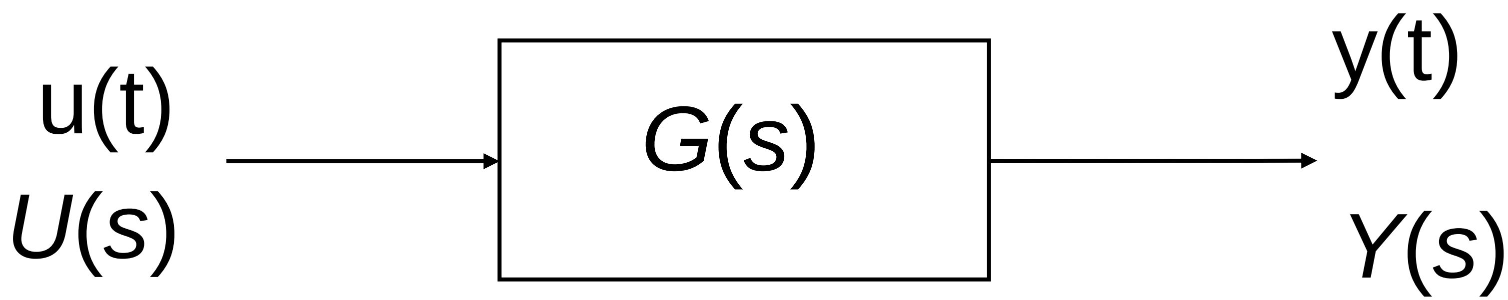
For example, in the case of a ramp or step input:





Steady state doesn't imply system is not in motion! – e.g. can be moving at constant velocity, or oscillating.

For this we can use the Final Value Theorem, which takes advantage of some handy properties of the Laplace transform



For a constant input, the output will (for some systems) settle down to a steady state value that is a multiple of the input.

The final value theorem gives the final value reached:

EXTRA S TERM

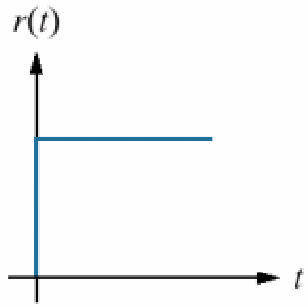
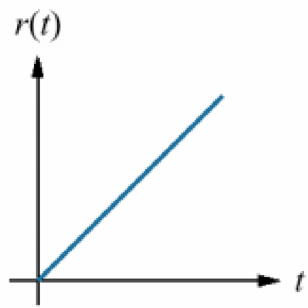
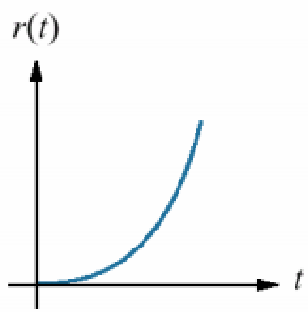
$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)$$

So, the final value is found by setting s to zero in the Laplace representation of the output, and multiplying by s.

There are two checks performed in Control theory which confirm valid results for the Final Value Theorem:

1) Y(s) should have no poles in the right half of the complex plane.
2) Y(s) should have no poles on the imaginary axis, except at most one pole at s=0.

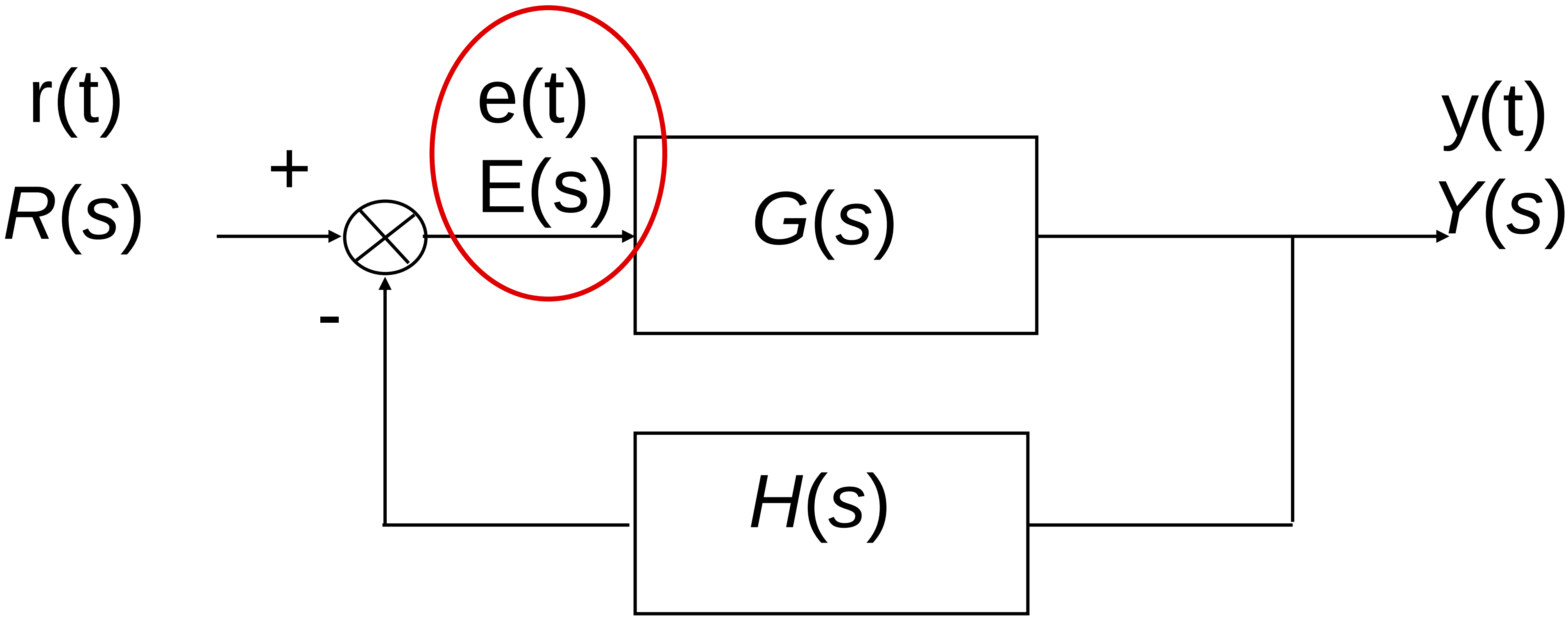
Continuous Systems and Transfer Function Revision: Final Value Theorem (FVT)

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

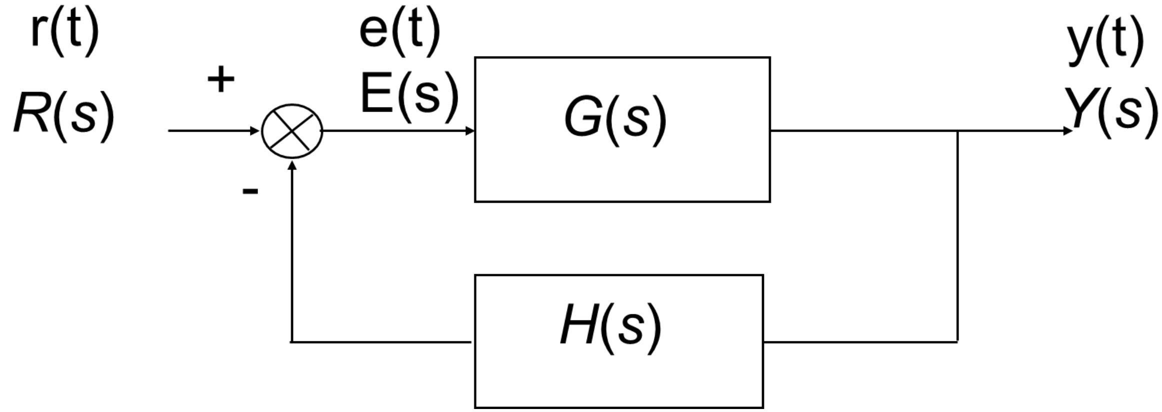
Continuous Systems and Transfer Function Revision: Steady State Error

We are normally more interested in the final value of the *error* rather than the output, as the goal of the controller is to drive the error as close to zero.

Further for some inputs such as a ramp or sinusoidal input, the final output isn't really meaningful.



Continuous Systems and Transfer Function Revision: Steady State Error

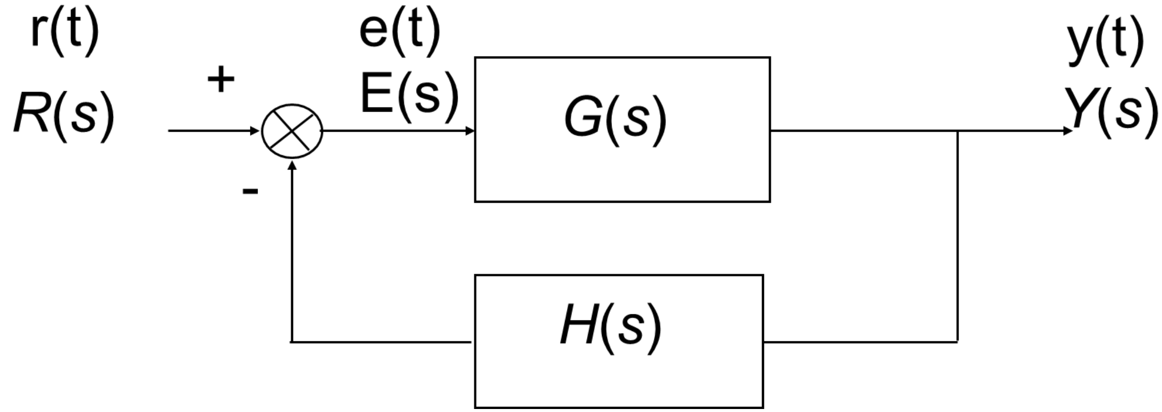


So, we first need to write the error signal with respect to the input:

$$E(s) = R(s) - H(s)Y(s) \qquad Y(s) = G(s)E(s)$$

So:

Continuous Systems and Transfer Function Revision: Steady State Error



Therefore the final value of the error signal e(t) would be:

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

with

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

so

$$\lim_{s \to 0} sE(s) =$$

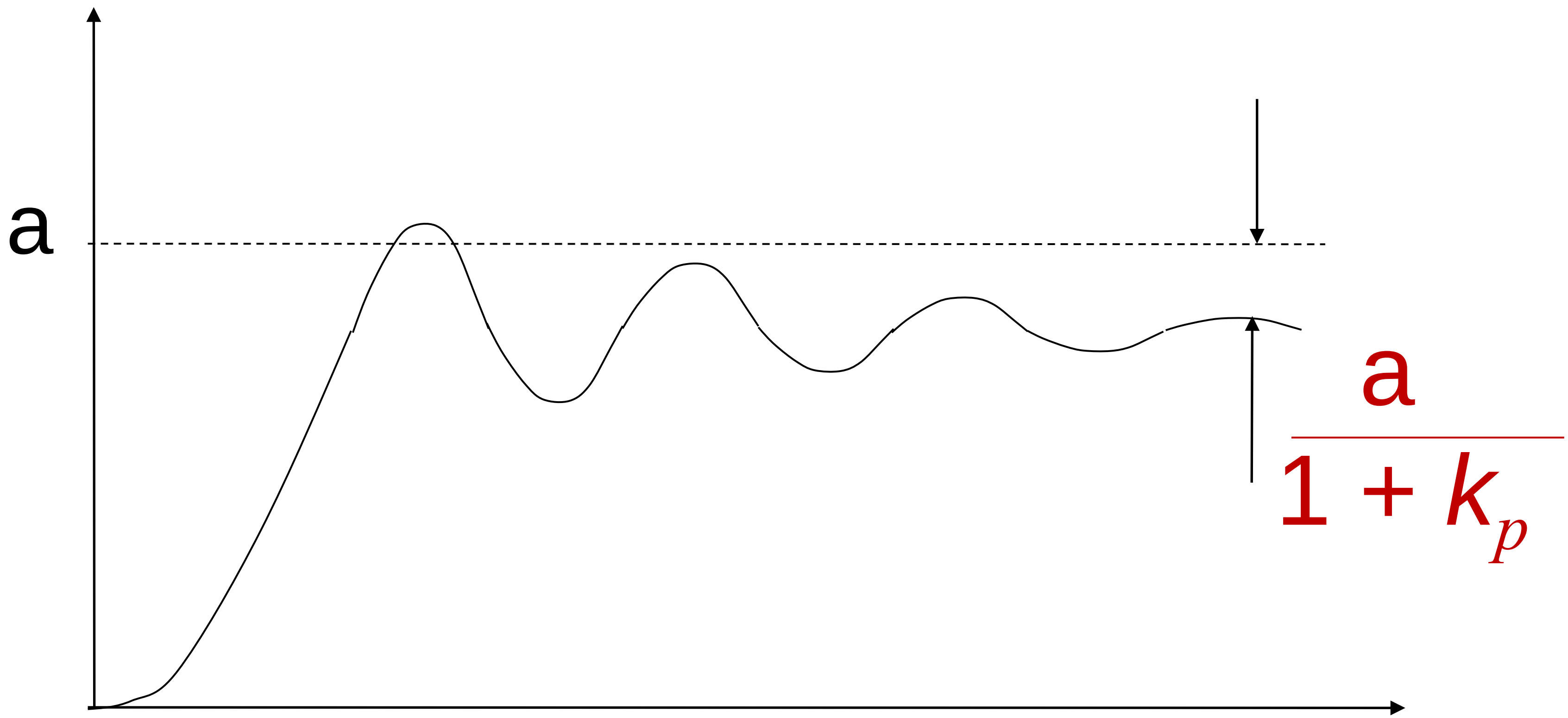
$$R(s) = \frac{A}{s} \qquad E(s) = \frac{R(s)}{1+G(s)H(s)}$$

Thus the final error is:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{s\,{}^{a}\!/_{s}}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{a}{1 + G(s)H(s)} = \frac{a}{1 + k_p}$$

where Is the **position error constant**



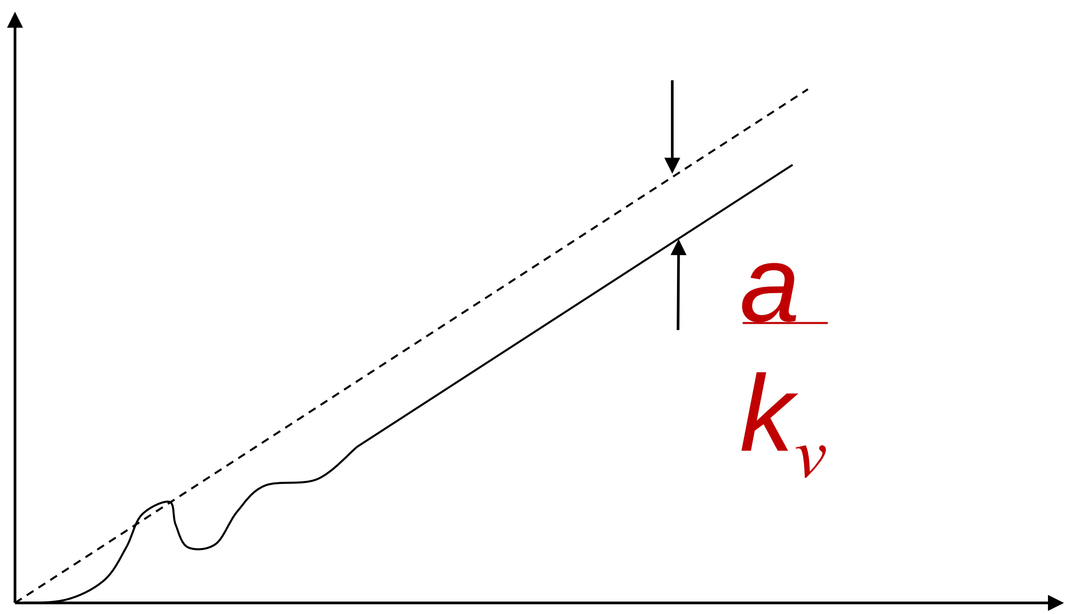
$$R(s) = \frac{A}{s^2} \qquad E(s) = \frac{R(s)}{1+G(s)H(s)}$$

Thus the final error is:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{s\, {}^{a}\!/\!{}_{s^2}}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{a}{s + sG(s)H(s)} = \frac{a}{k_v}$$

where Is the **velocity error constant**



Once again, lets return to the servo example to see how this system would perform given these inputs.

System types

Continuous Systems and Transfer Function Revision: System Type

Let's look more generally at the transfer function of our system and introduce the concept of system **type.** By factoring out any s terms from the denominator, we can write the transfer function in the following form:

$$G(s)H(s) = \frac{(s-z_1)(s-z_2)(s-z_3)\ldots}{s^p(s-\sigma_1)(s-\sigma_2)(s-\alpha_k+j\omega_k)(s-\alpha_k-j\omega_k)\ldots}$$

So, if there are *p poles at the origin*, the system is said to be a 'type *p*' system.

For example, a servo motor can be expressed as:

$$\frac{\Theta_o(s)}{\Theta_i(s)} = \frac{k}{s(Is+f)}$$

Continuous Systems and Transfer Function Revision: System Type

Or an electro magnet system

$$\frac{I(s)}{V(s)} = \frac{1}{(Ls+R)}$$

Or a mass spring damper system

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2+cs+k}$$

Essentially, for a zero error we want k_p and k_v to be infinite, or at least be a constant for a finite error, depending on our requirements.

$$SSE = \frac{a}{1+k_p}$$

$$SSVL = \frac{a}{k_v}$$

$$k_p = \lim_{s \to 0} G(s)H(s)$$

$$k_v = \lim_{s \to 0} sG(s)H(s)$$

The **position error constant** k_p for a *type p* system is given by:

p > 0 $k_p = \lim_{s \to 0} G(s) = \infty$ no steady-state position error

p = 0 finite position error

The **velocity error constant** for a *type p* system is given by:

p > 1 $k_v = \lim_{s \to 0} sG(s) = \infty$ no velocity error

p = 1 steady state velocity lag

p = 0 Infinite lag (completely fails to track)

Continuous Systems and Transfer Function Revision: System Type for Unit Feedback Control Systems – Calculating error

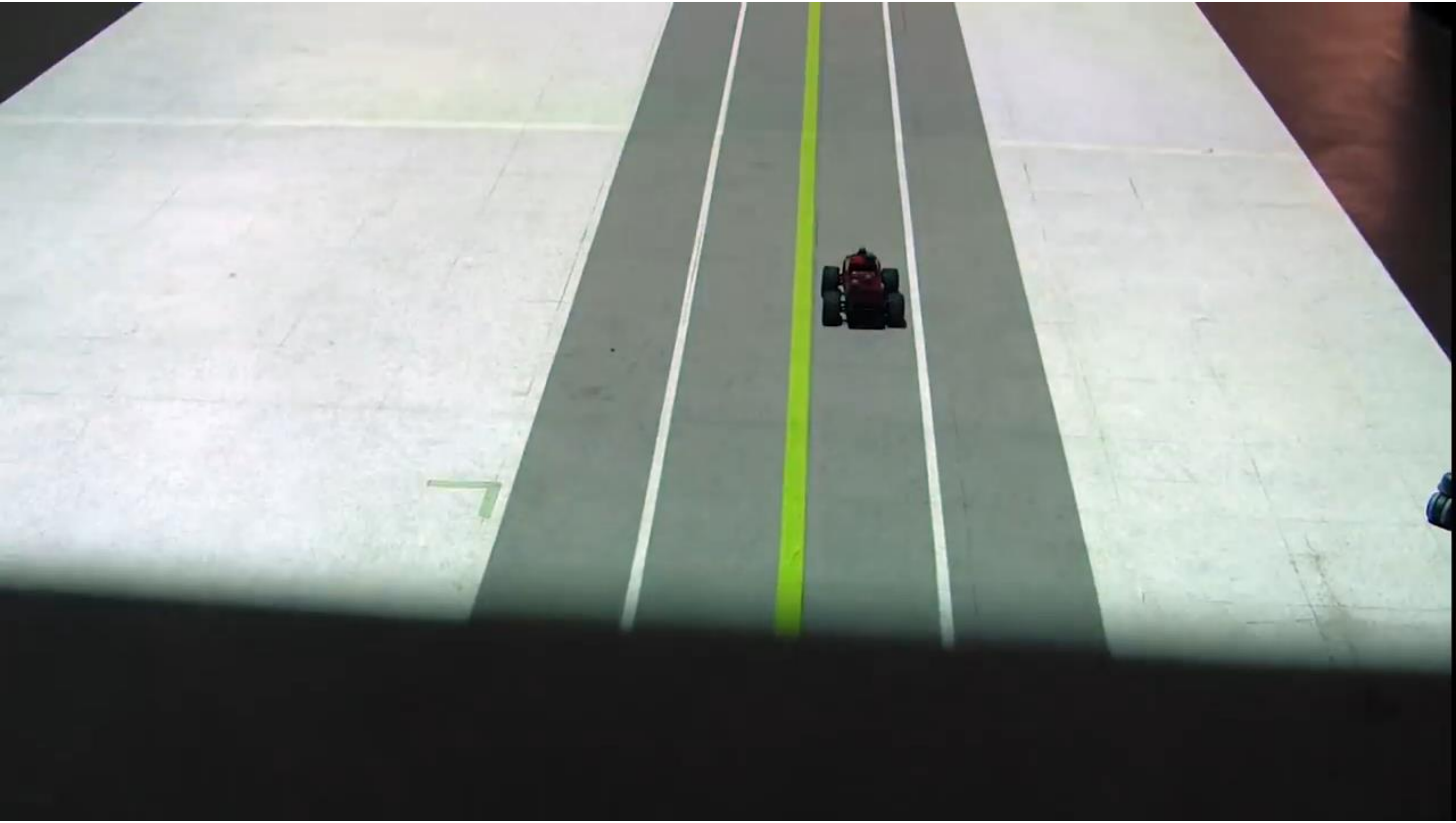
No. Integrators in denominator = system TYPE	Input type		
	Step $r(t) = a$ $R(s) = a/s$	Ramp $r(t) = at$ $R(s) = a/s^2$	Acceleration $r(t) = at^2/2$ $R(s) = a/s^3$
0	$e_{ss} = a/(1+k_p)$	$e_{ss} = \infty$	$e_{ss} = \infty$
1	$e_{ss} = 0$	$e_{ss} = a/k_v$	$e_{ss} = \infty$
2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = a/k_a$

The more poles there are at the origin of the open-loop system, the better the steady-state tracking performance.

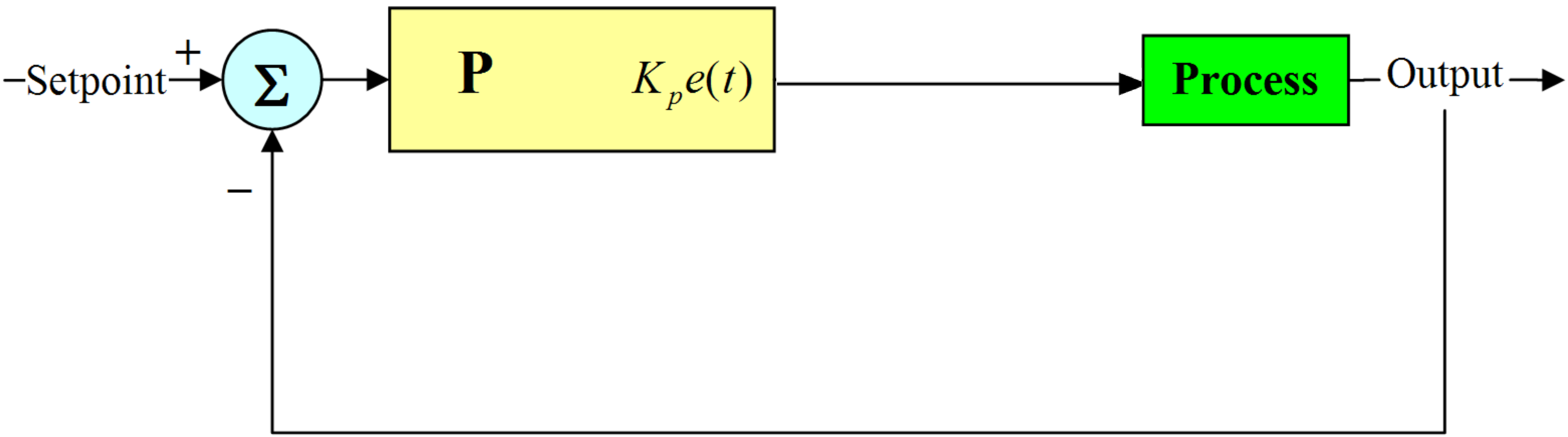
However, pure integrations have a highly destabilizing effect on the control system!

PID Control

Continuous Systems and Transfer Function Revision: Line following

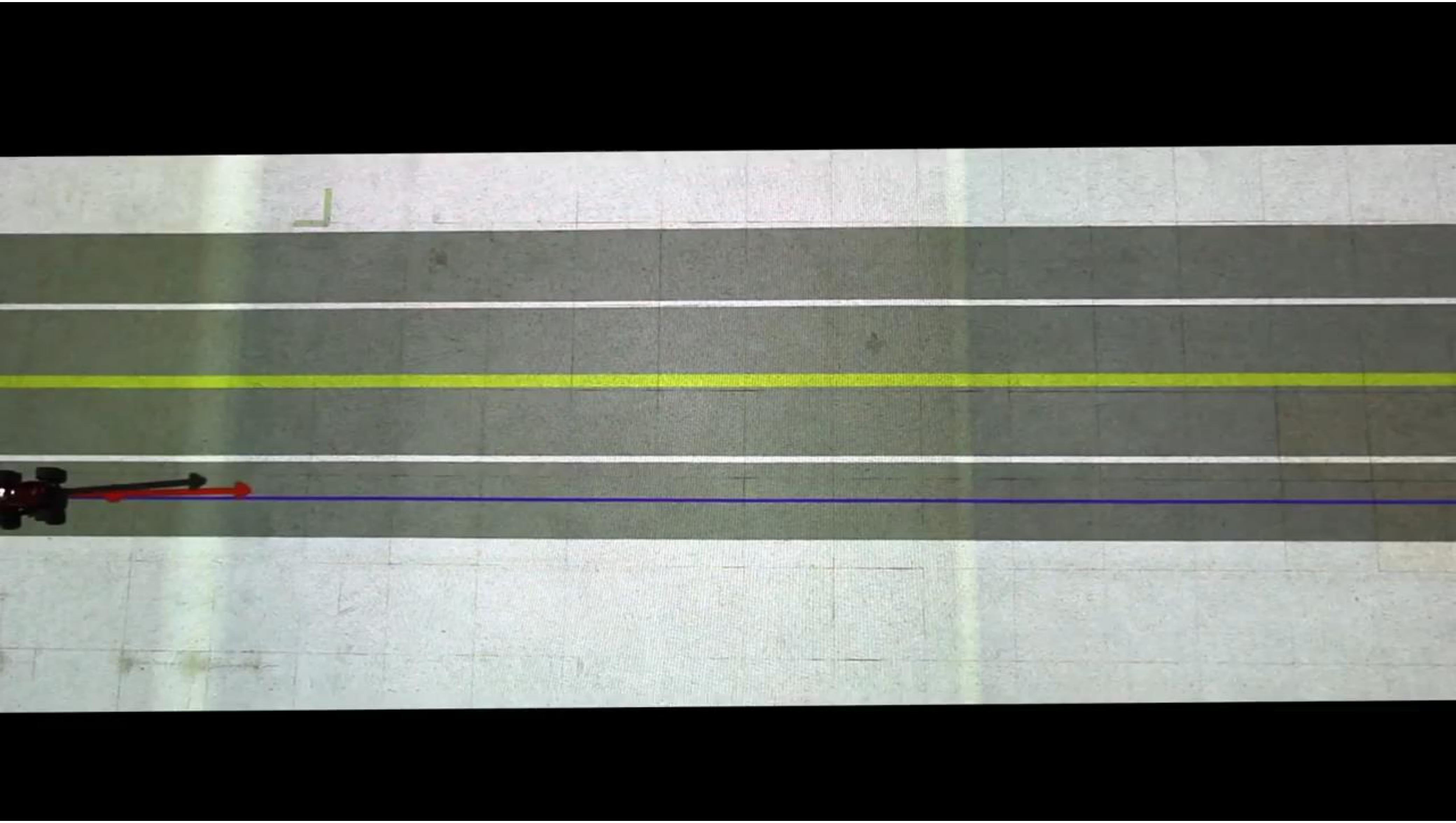


Continuous Systems and Transfer Function Revision: Proportional Control

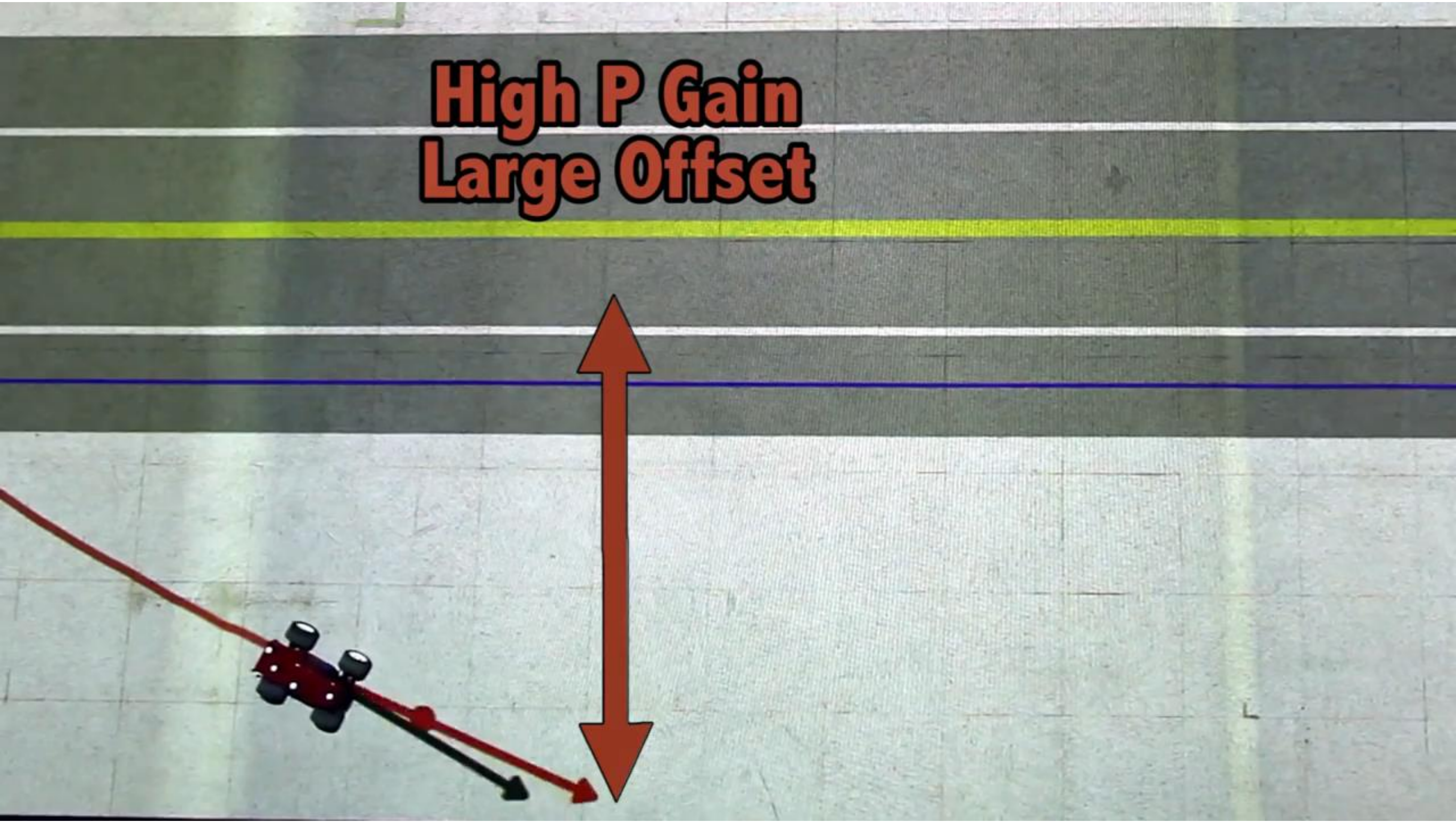


Continuous Systems and Transfer Function Revision: Proportional Error

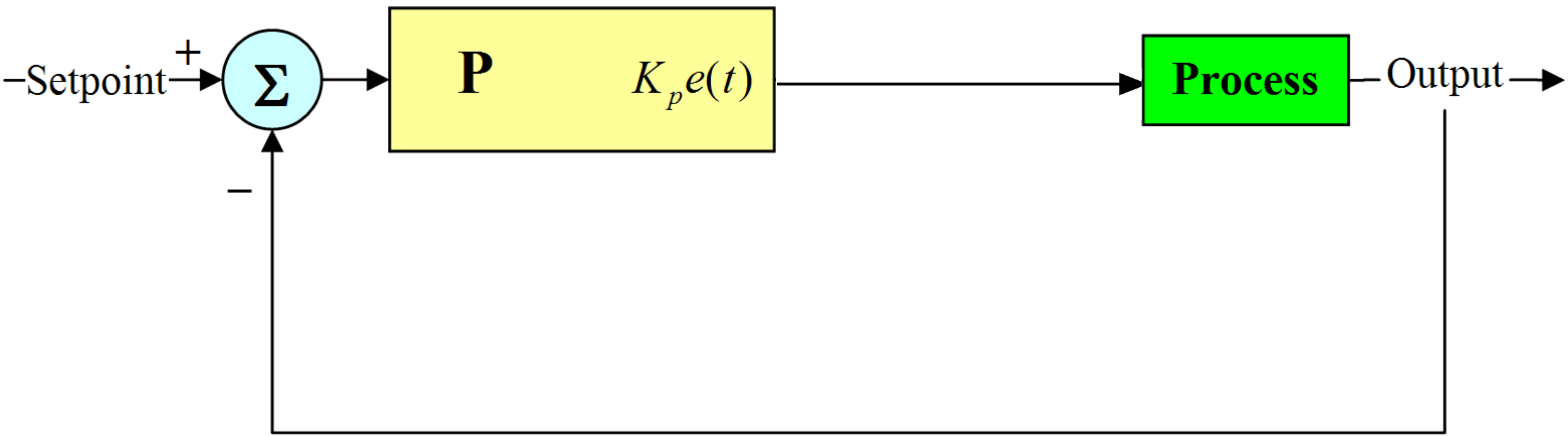
Continuous Systems and Transfer Function Revision: Trade-off: Proportional Error



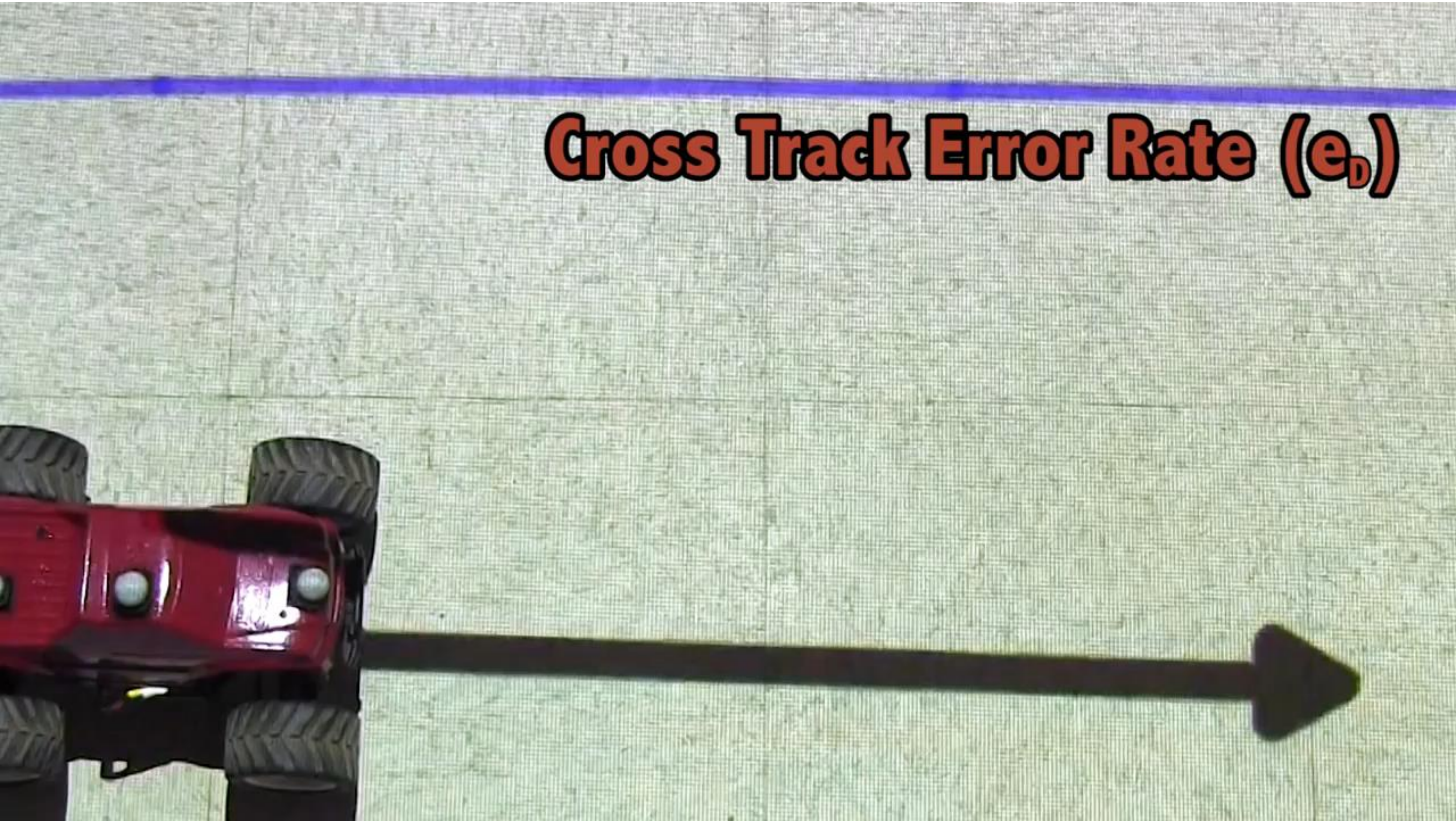
Continuous Systems and Transfer Function Revision: Trade-off: Proportional Error



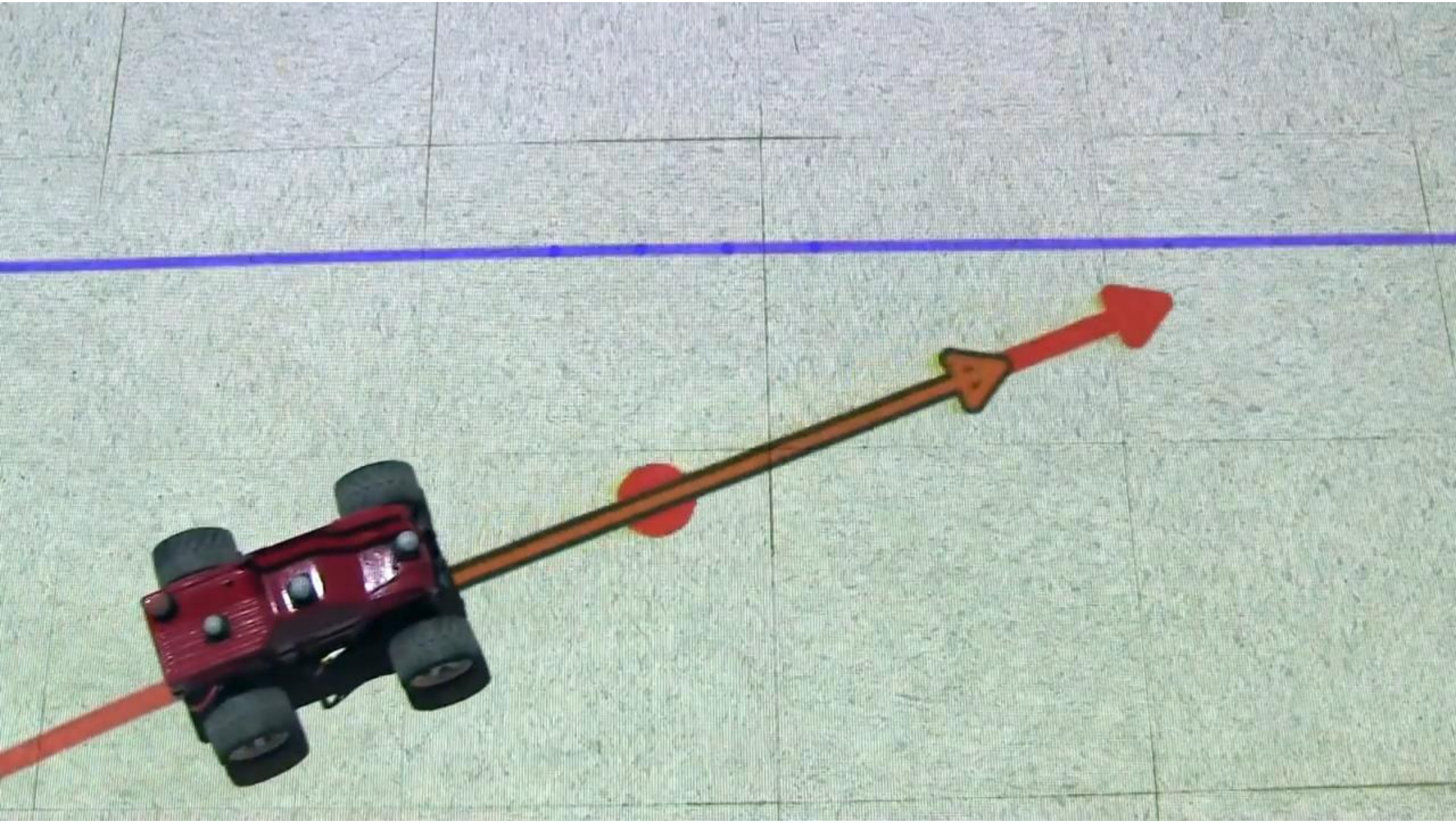
Continuous Systems and Transfer Function Revision: PID Control



Continuous Systems and Transfer Function Revision: Derivative Error

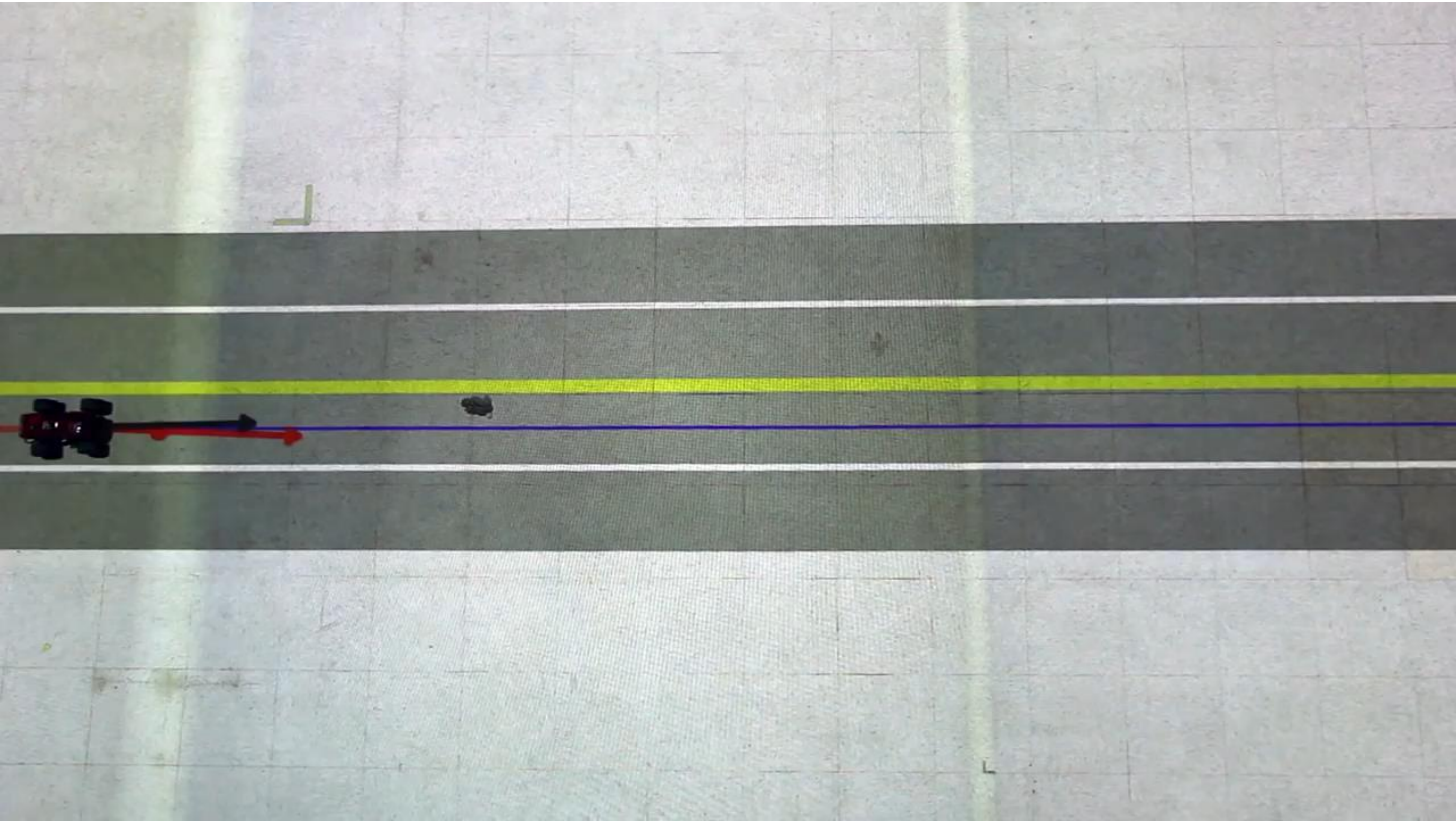


Continuous Systems and Transfer Function Revision: Derivative Error

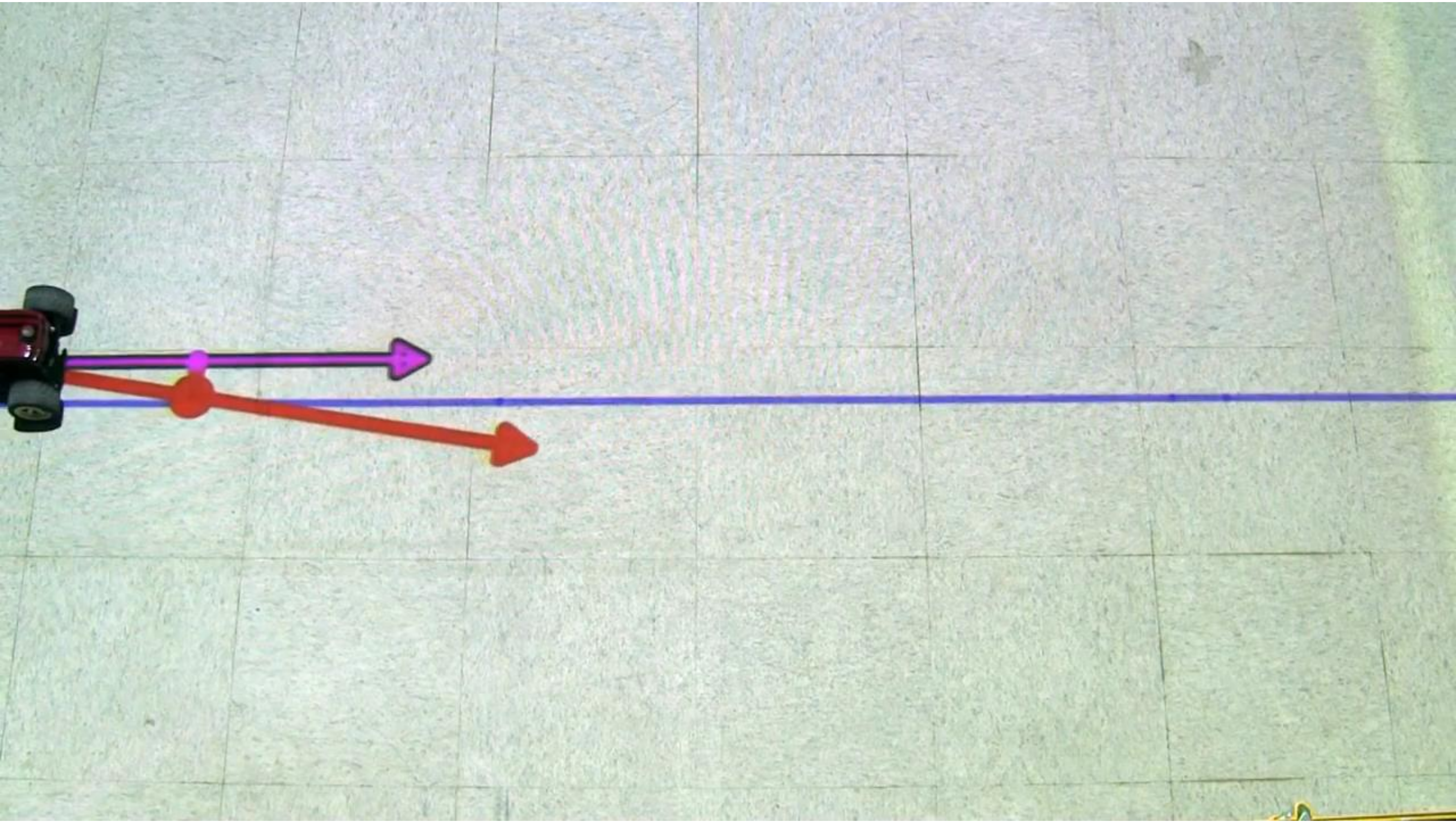


Continuous Systems and Transfer Function Revision: Integral Error

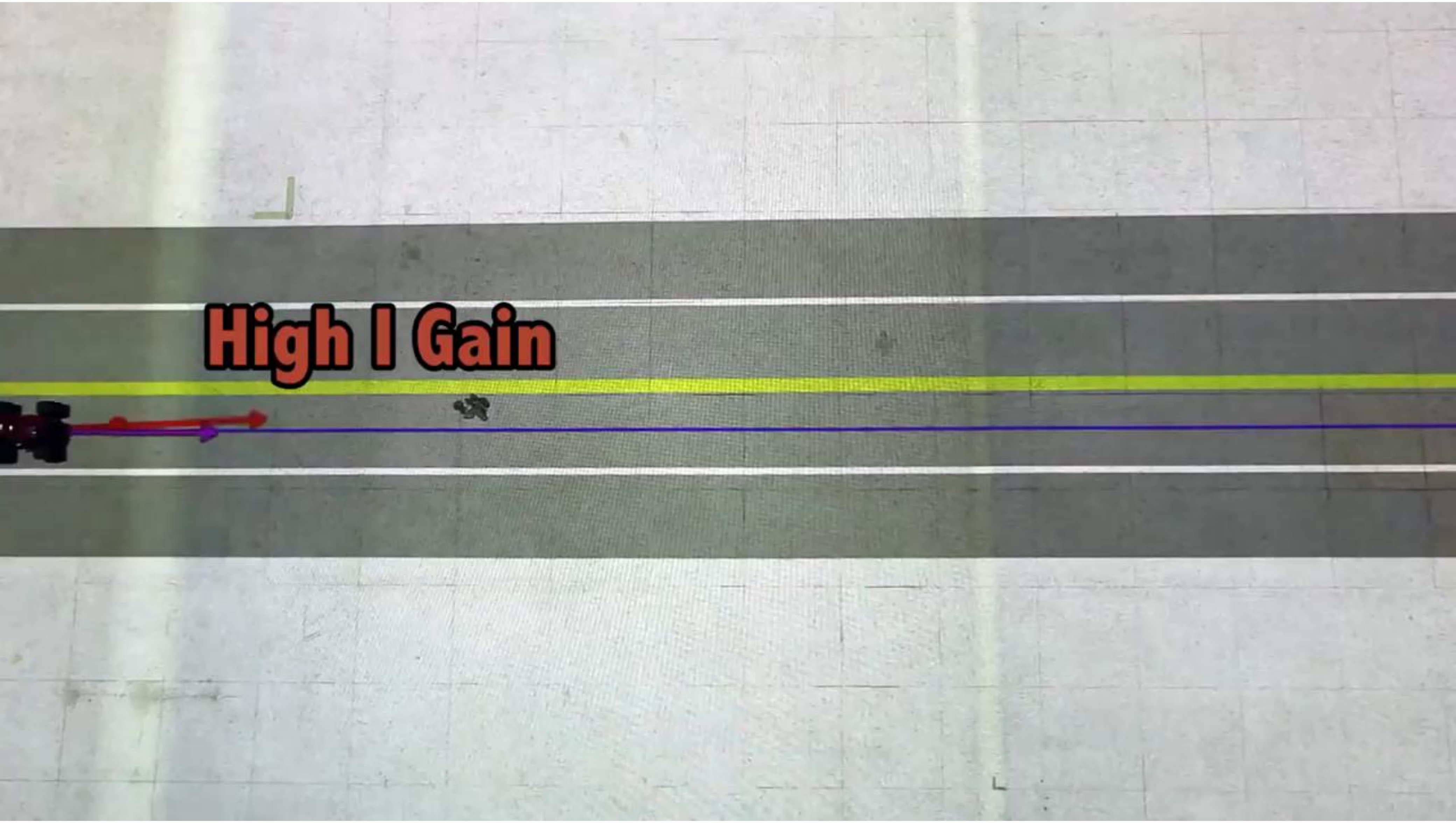
Continuous Systems and Transfer Function Revision: Integral Error



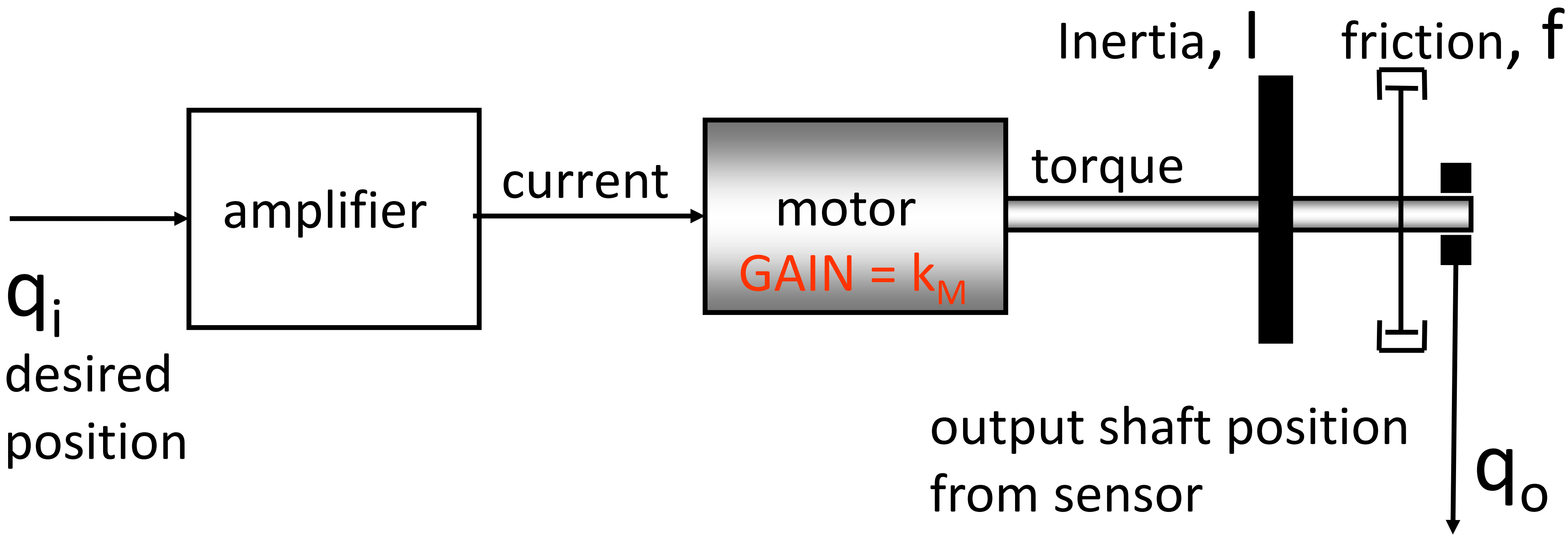
2.1 Continuous Systems and Transfer Function Revision: Integral Error



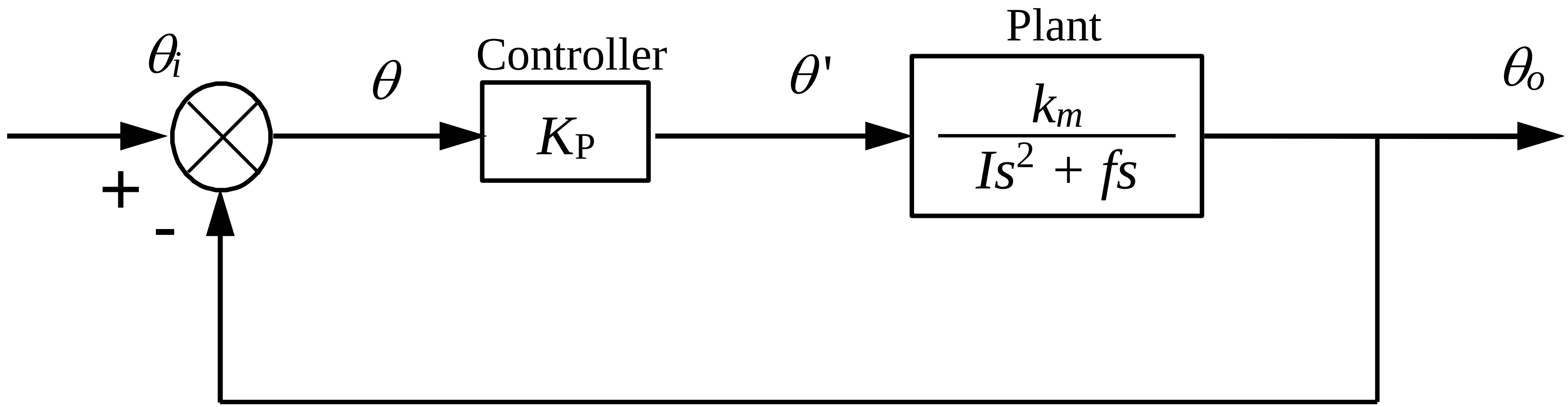
Continuous Systems and Transfer Function Revision: Integral Error



These can be modelled as shown below, with a DC motor rotating an inertial load, with a friction or damping resistance.

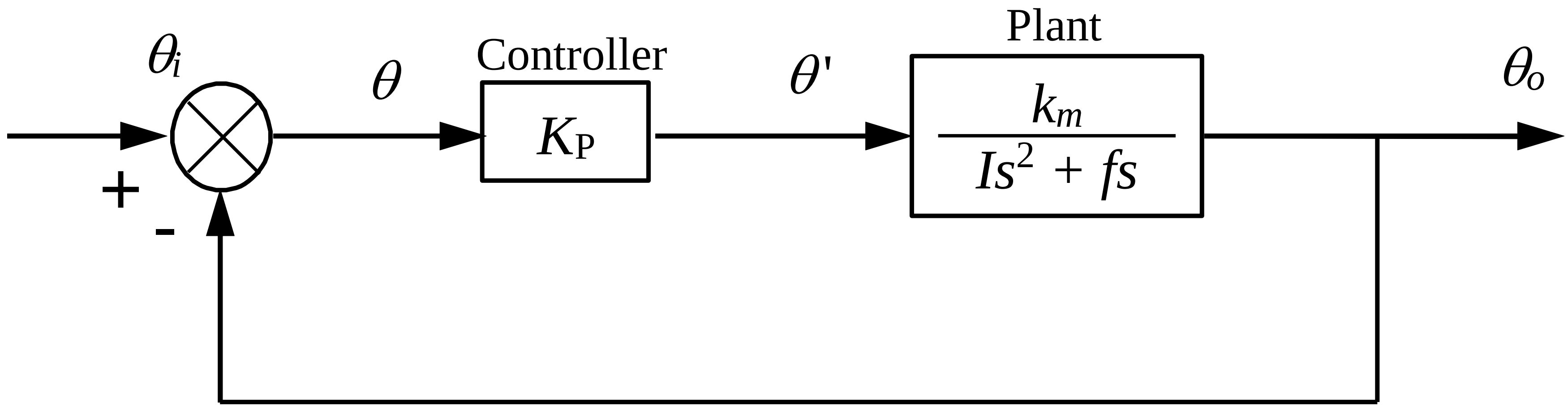


Continuous Systems and Transfer Function Revision: Proportional Error



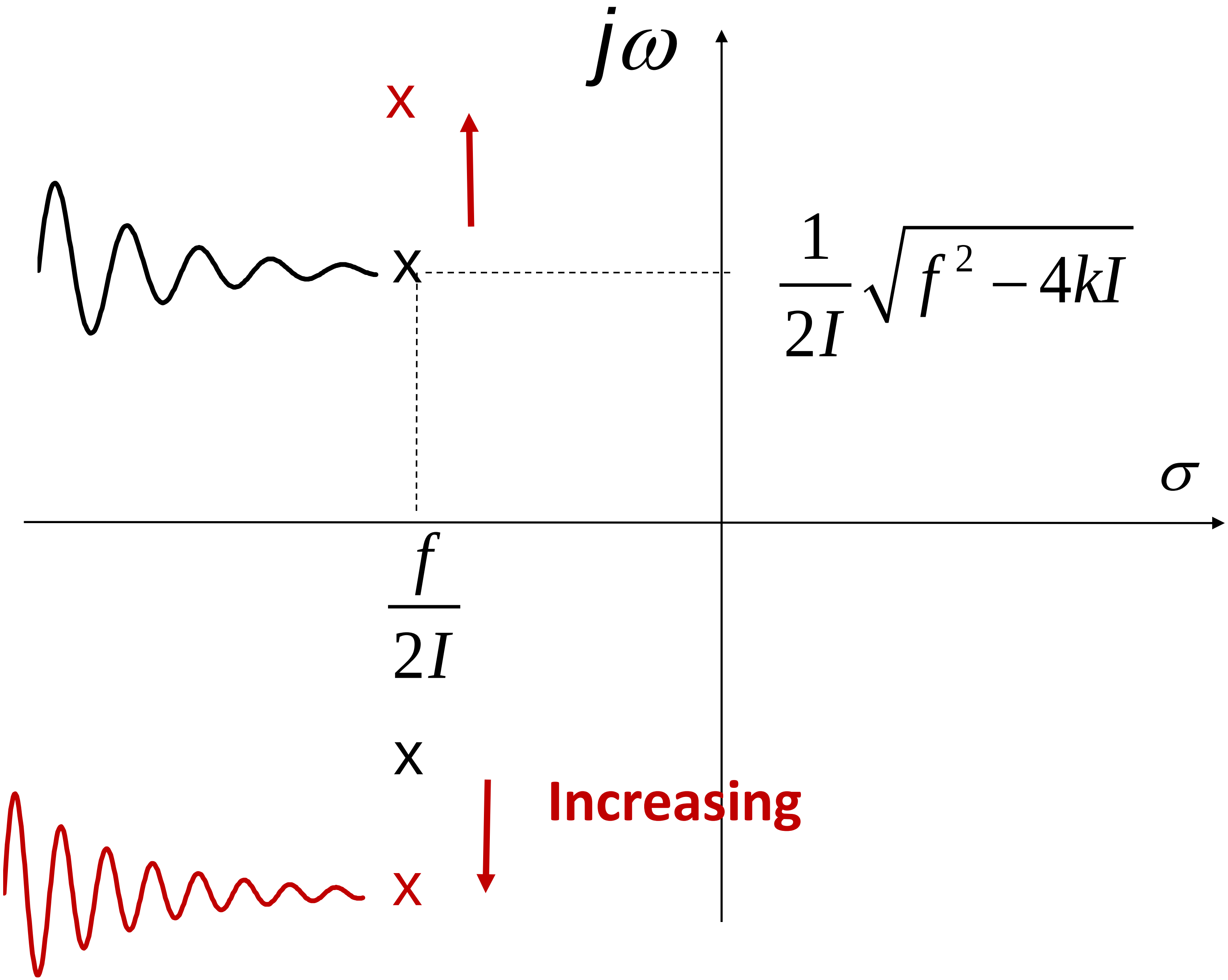
$$F(s) = \frac{G(s)}{1+G(s)} = \frac{K_P k_m}{Is^2 + fs + K_P k_m}$$

Continuous Systems and Transfer Function Revision: Proportional Error



$$F\left(s\right) = \frac{G\left(s\right)}{1+G\left(s\right)} = \frac{K_P k_m}{Is^2 + fs + K_P k_m}$$

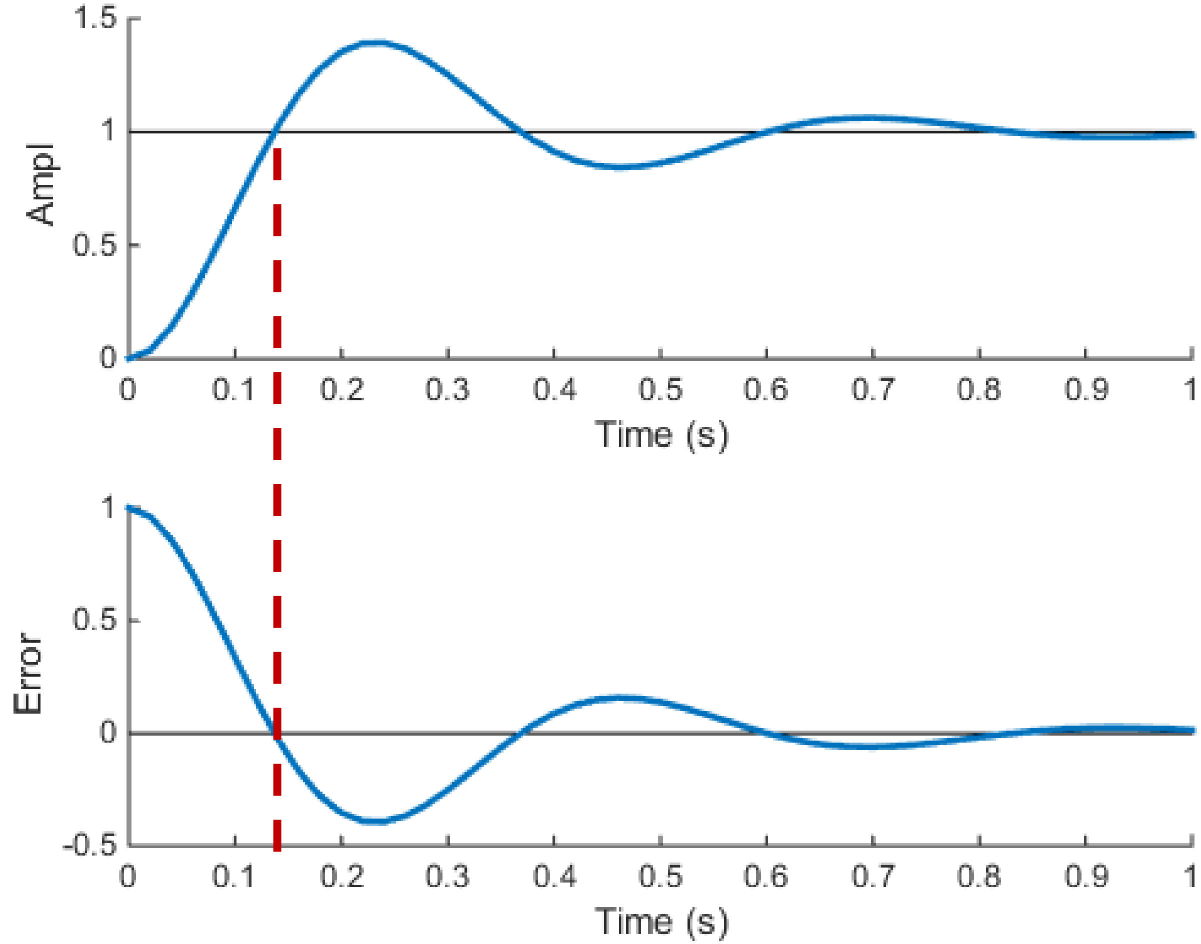
Low values of K_p give stable but slow responses, and high SSVL.
High values reduce SSVL but response overshoots considerably.



$j\omega$
x
x
$\frac{1}{2I}\sqrt{f^2 - 4kI}$
σ
$\frac{f}{2I}$
x
Increasing
x

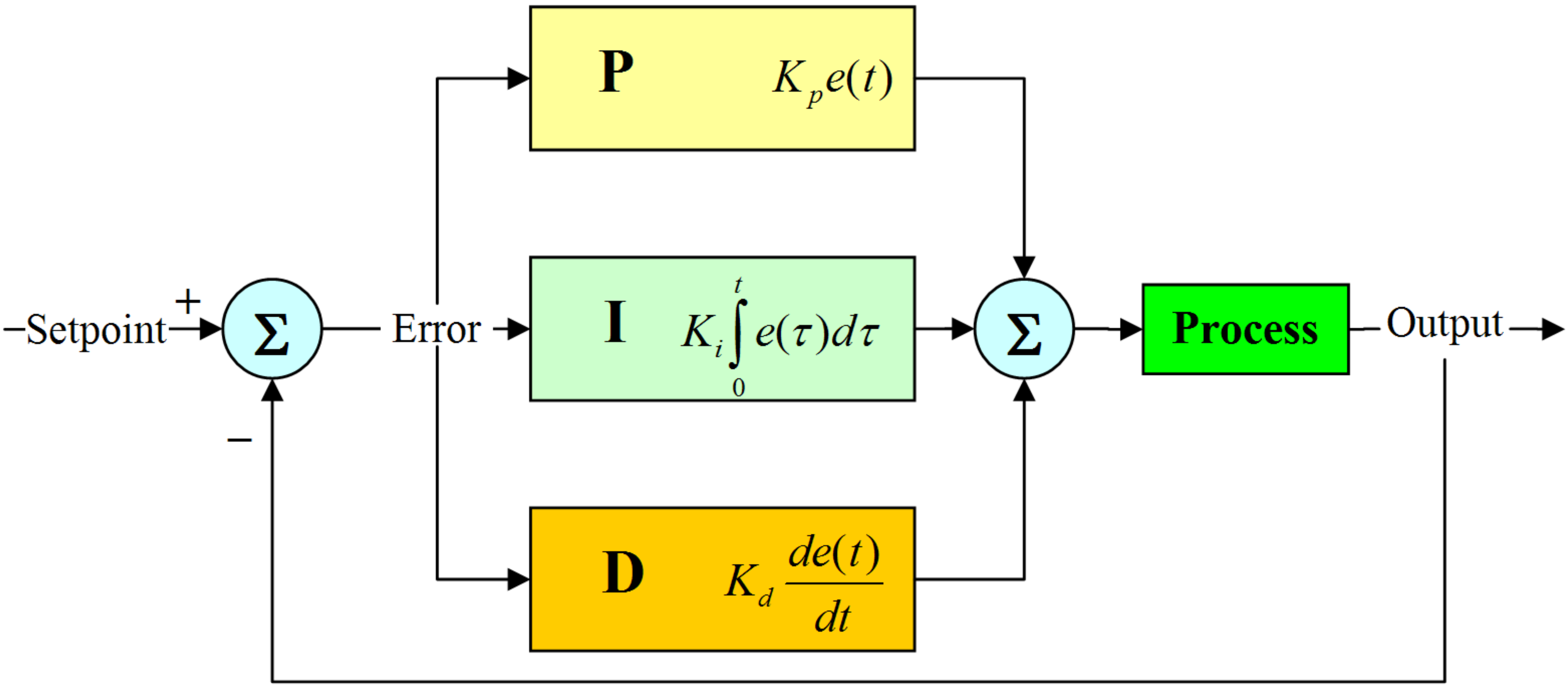
Continuous Systems and Transfer Function Revision: Proportional Error

With proportional control the error *and thus the control signal* does not reach zero until the motor is at the desired position.



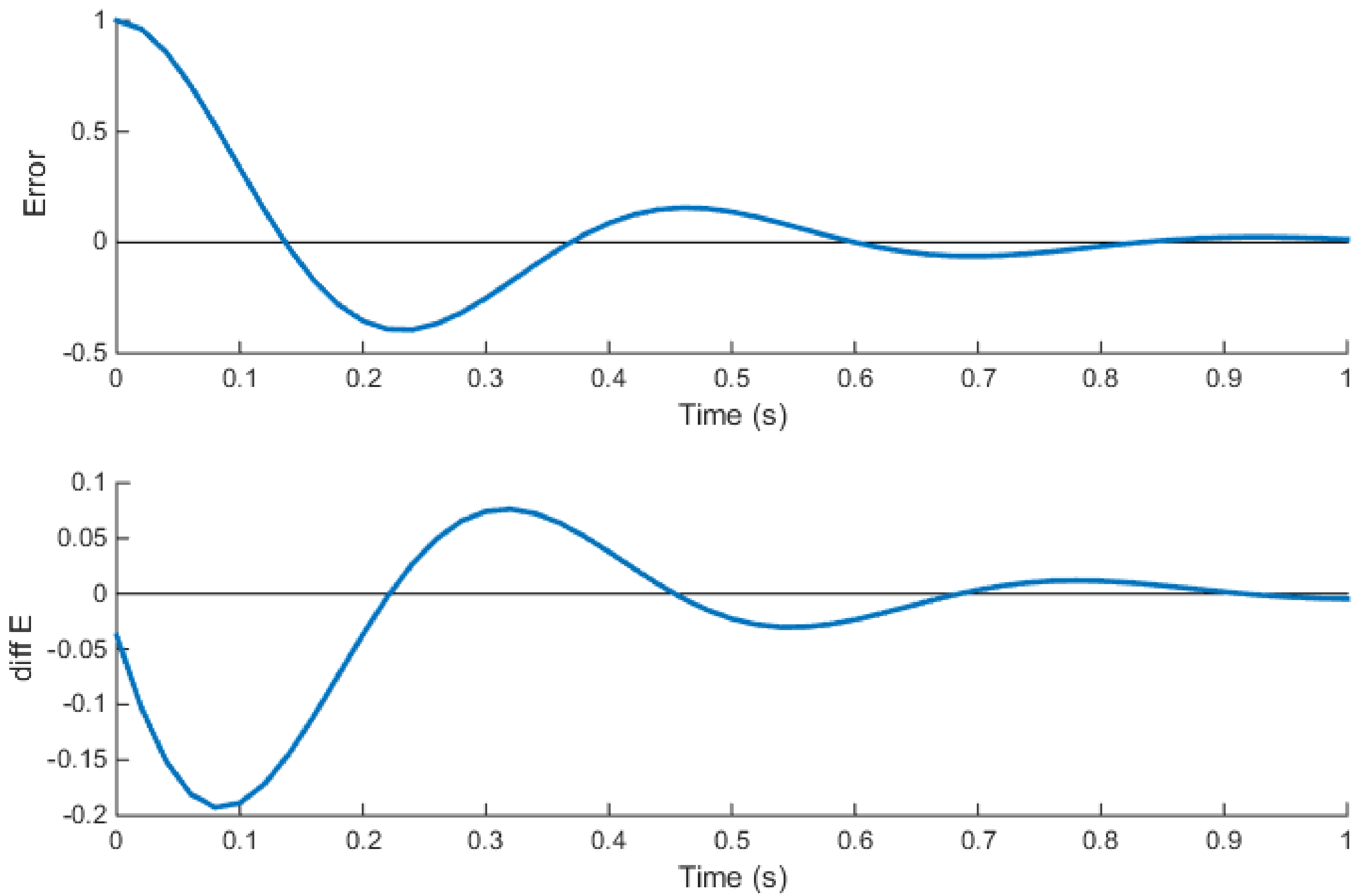
However, due to the inertia for the system, the motor continues to move even without a control signal, resulting in an overshoot.

Continuous Systems and Transfer Function Revision: PID Control



Continuous Systems and Transfer Function Revision: Derivative Error

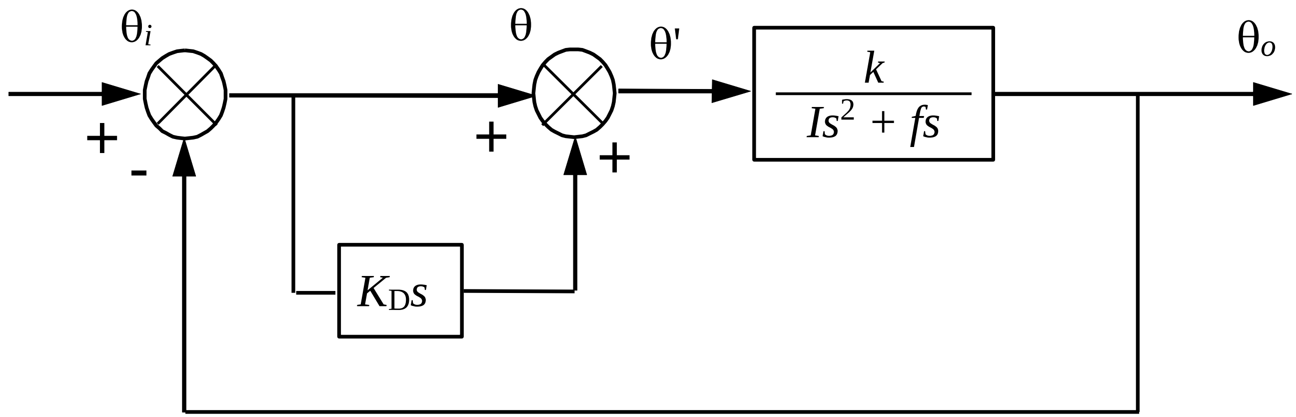
We can replicate this by considering the **derivative** of the error.



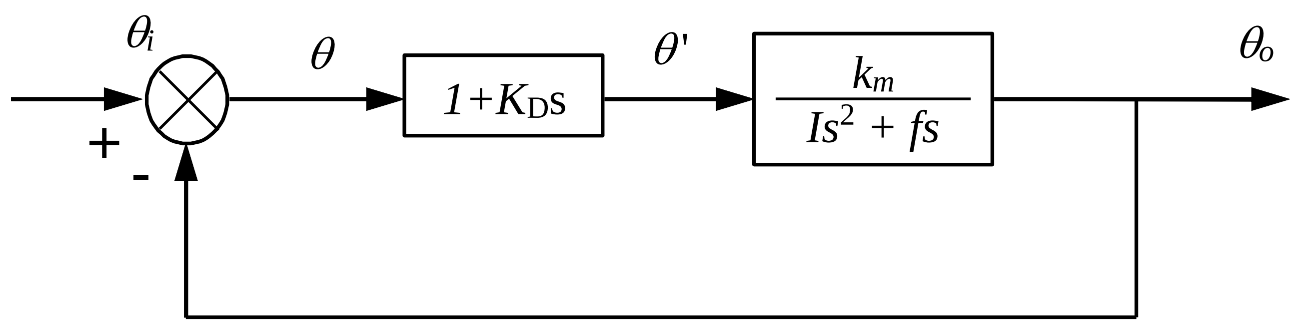
This is negative as the servo approaches the target, so offers a way of restraining the servos forward motion.

Continuous Systems and Transfer Function Revision: Derivative Error

Let’s consider a derivative controller with unity proportional gain



$$\theta' = \theta + K_D s\theta \qquad \theta = \theta_i - \theta_o$$



The open loop transfer function now becomes:

$$G' = \frac{(1 + K_D s)k_m}{Is^2 + fs}$$

Continuous Systems and Transfer Function Revision: Derivative Error

The open loop transfer function now becomes:

$$G' = \frac{\left(1 + K_D s\right) k_m}{Is^2 + fs}$$

Giving a *closed loop transfer function* of

$$F\left(s\right) = \frac{G'\left(s\right)}{1 + G'\left(s\right)} = \frac{\left(1 + K_D s\right) k_m}{Is^2 + fs + \left(1 + K_D s\right) k_m}$$

$$\boxed{F\left(s\right) = \frac{k_m K_D s + k_m}{Is^2 + \left(f + k_m K_D\right) s + k_m}}$$

Continuous Systems and Transfer Function Revision: Derivative Error

$$\boxed{F(s) = \frac{k_m K_D s + k_m}{Is^2 + \left(f + k_m K_D\right)s + k_m}}$$

System is still second order, but now there is a zero in the numerator. The effect of this is subtle compared to the change in the poles.

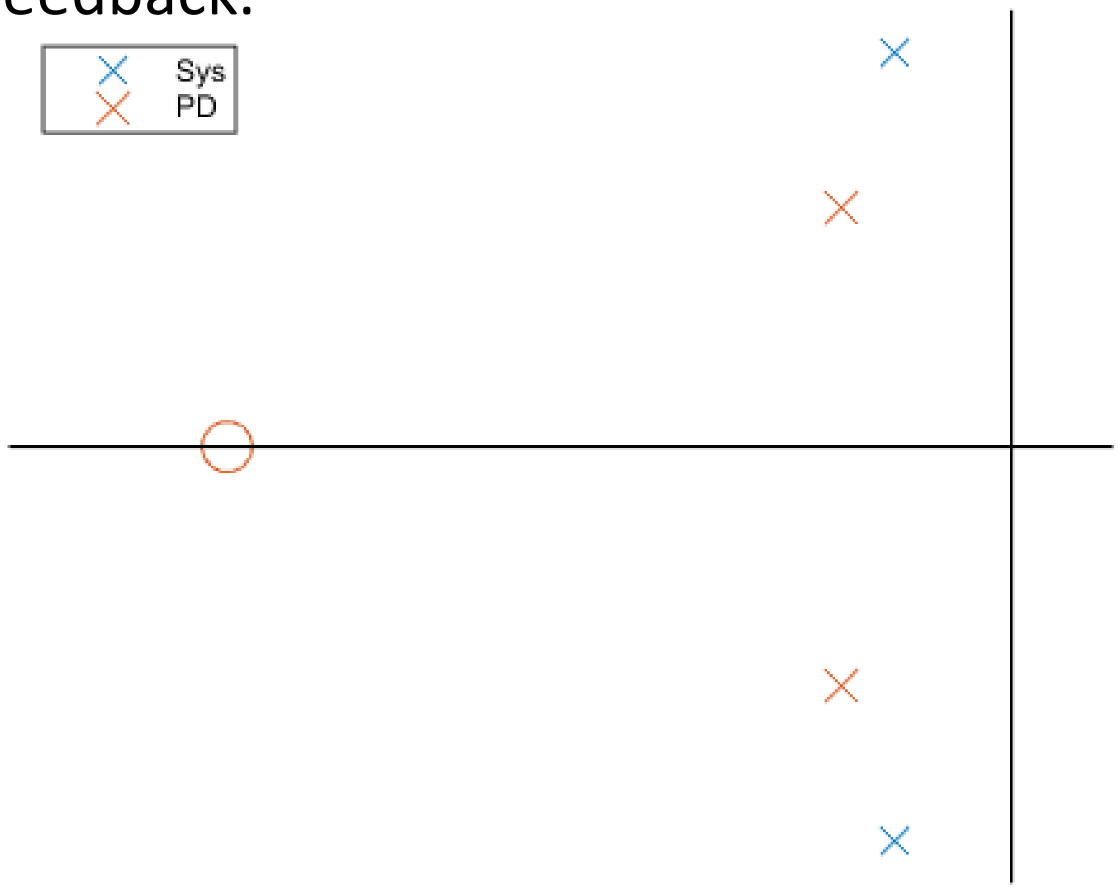
More importantly, lets consider the SSVL of this system:

$$k_v = \lim_{s \to 0} sG'(s) = \frac{s\left(1 + K_D s\right)k_m}{s\left(Is + f\right)}$$

So unlike velocity feedback, the SSVL is unchanged by derivative error. So we can improve transient response without compromising steady state error.

Continuous Systems and Transfer Function Revision: Derivative Error

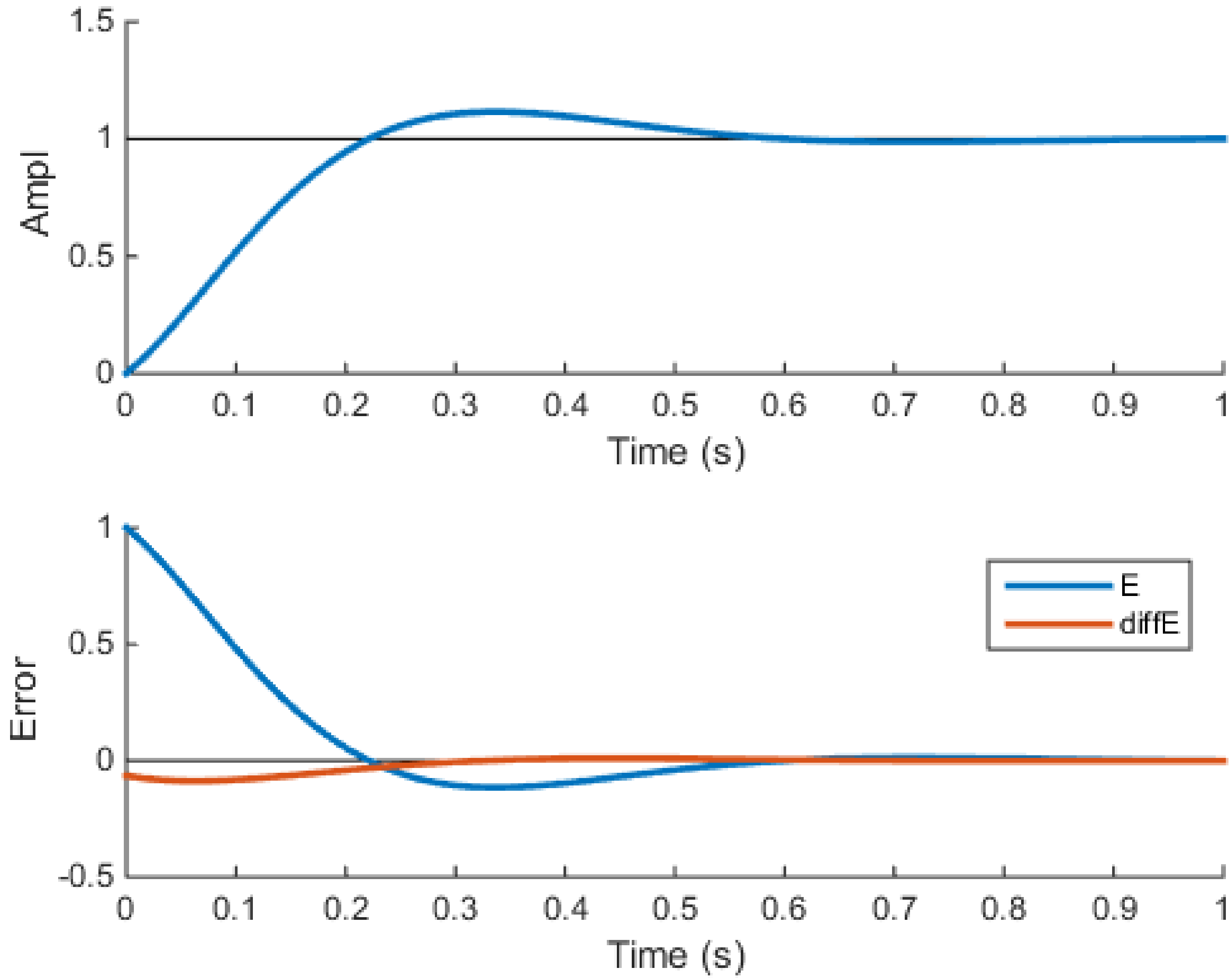
The effect on the step response is similar to that of the minor loop velocity feedback.



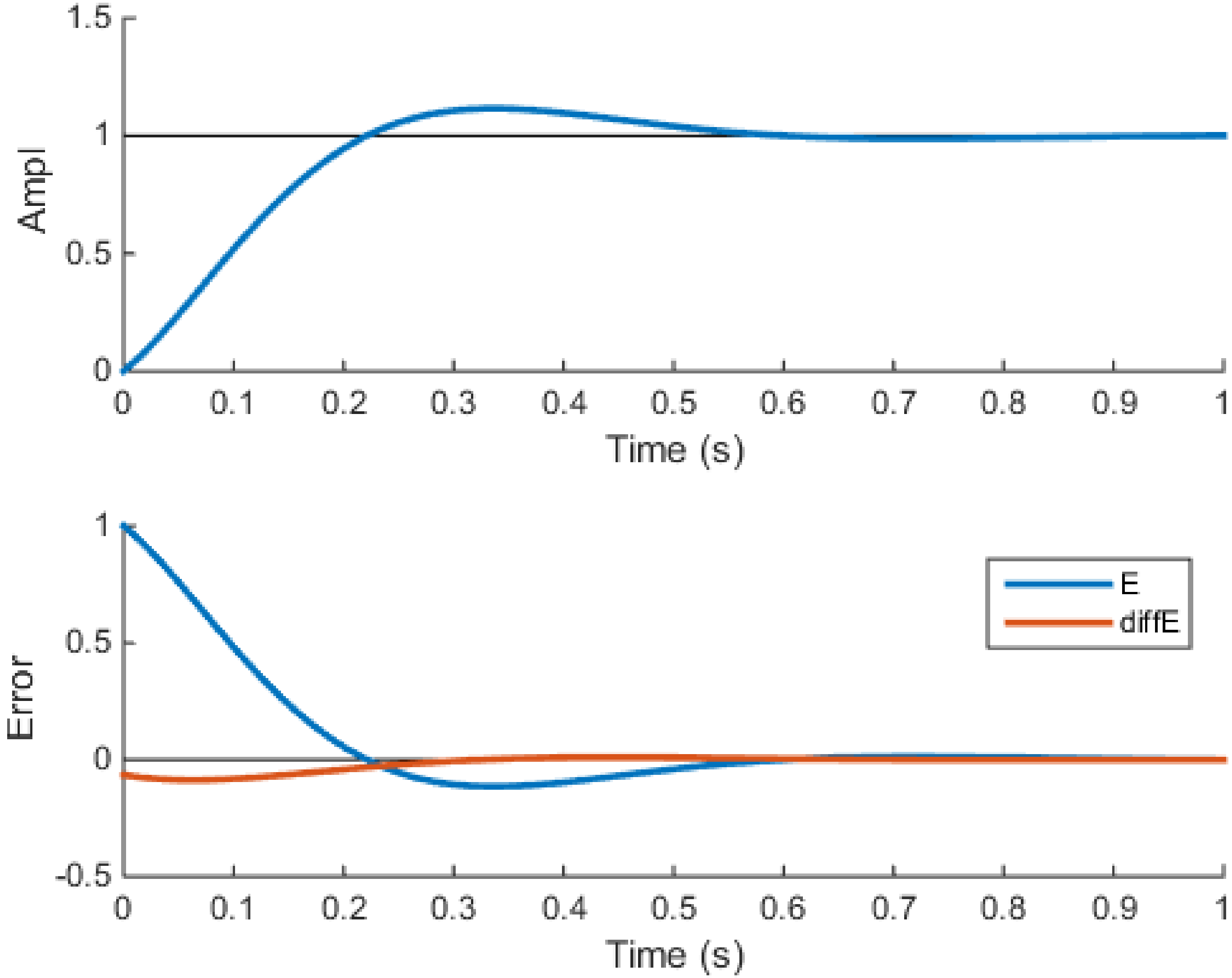
The poles of the system become less oscillatory and decay quicker. This combined with a reduced proportional gain, gives an improved transient response.

Continuous Systems and Transfer Function Revision: Derivative Error

This is clear when looking at the step response, and the relative contributions of the two error terms.

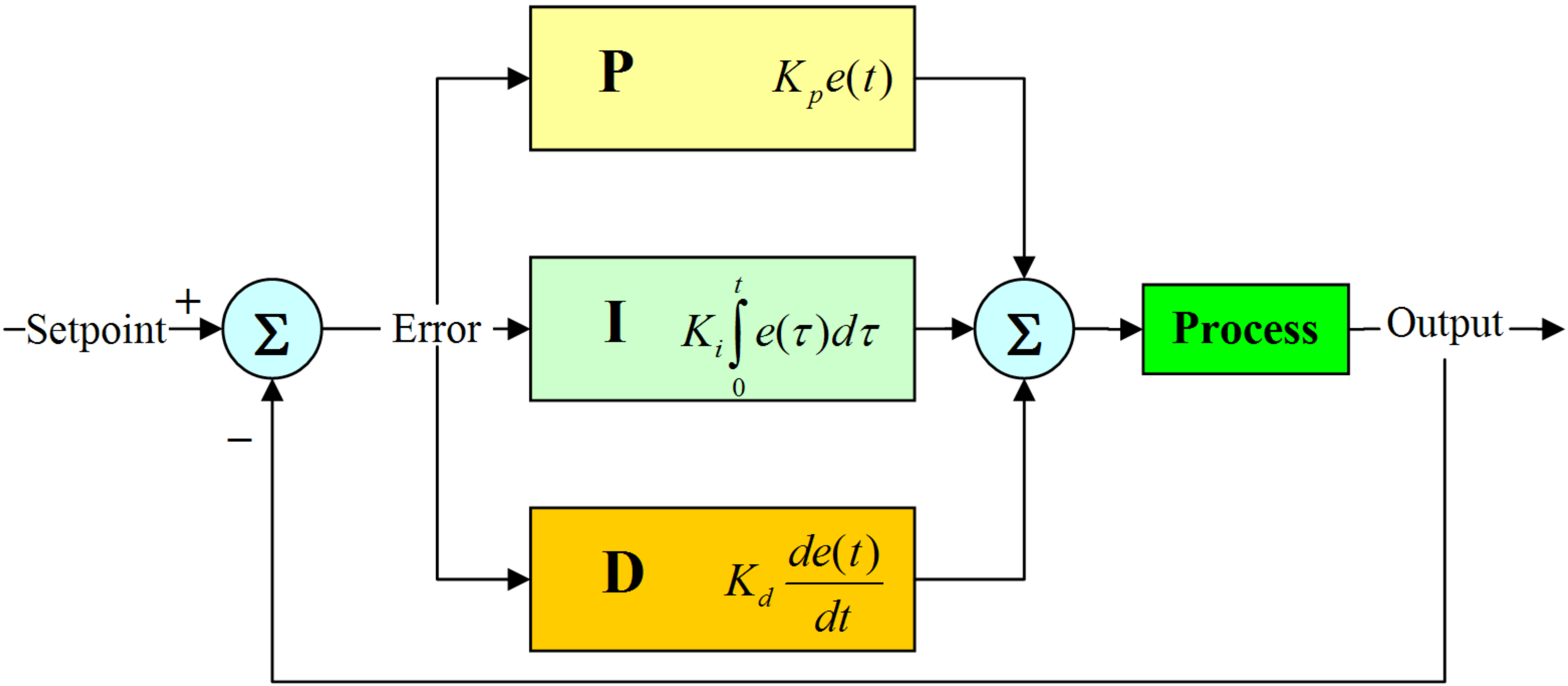


Continuous Systems and Transfer Function Revision: Derivative Error



At the start, the derivative term is significant *and in the opposite direction to* the proportional error term, but becomes negligible as the system settles.

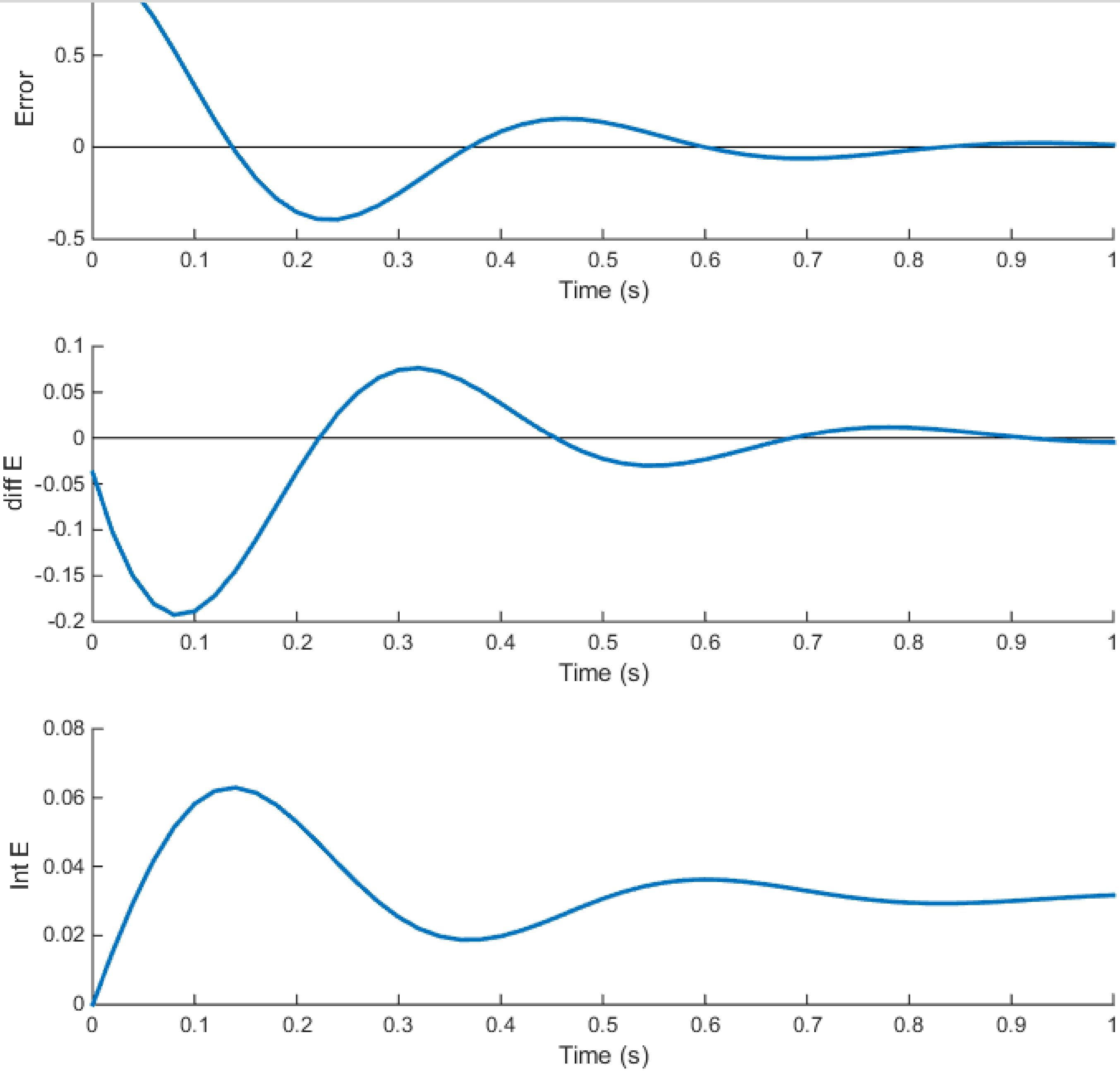
Continuous Systems and Transfer Function Revision: PID Control



Continuous Systems and Transfer Function Revision: Integral Error

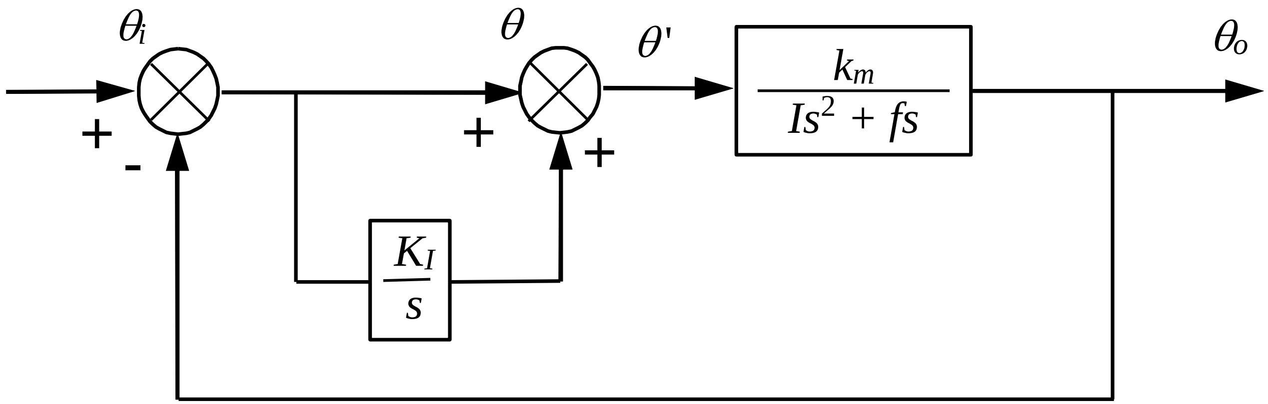
Further, if we consider the **integral** or total error over time:

This gradually increases over time, and can be used to magnify the control signal for small errors, and improve SSE

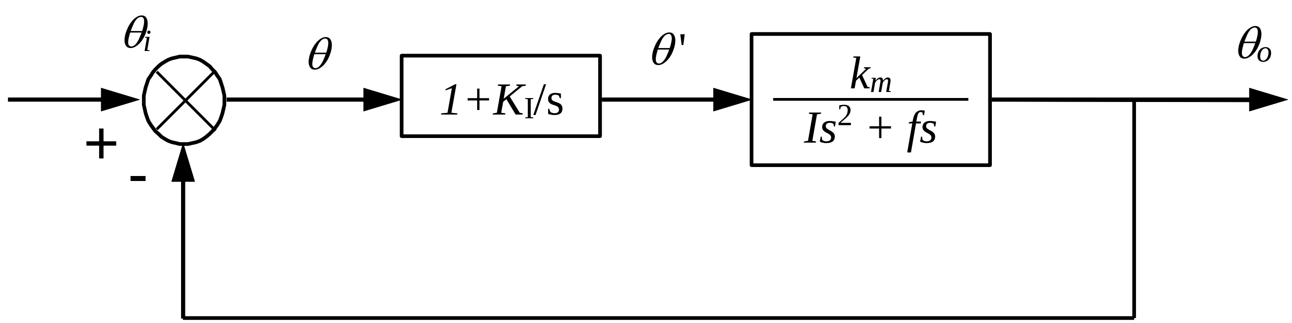


Continuous Systems and Transfer Function Revision: Integral Error

The integral of the error is used for correcting steady state errors, as it adds a pole at zero in the transfer function



$$\theta' = \theta + \frac{K_I \theta}{s} \qquad \frac{\theta'}{\theta} = 1 + \frac{K_I}{s}$$



Continuous Systems and Transfer Function Revision: Integral Error

$$G' = \frac{\left(1+{K_I}/{s}\right)k_m}{Is^2 + fs} = \frac{\frac{1}{s}\left(s+K_I\right)k_m}{s\left(Is+f\right)} \quad G' = \frac{k_m s + k_m K_I}{s^2\left(Is+f\right)}$$

With a close loop transfer function of

$$F(s) = \frac{G'(s)}{1+G'(s)} = \frac{k_m s + k_m K_I}{s^2\left(Is+f\right)+k_m s + k_m K_I}$$

$$F(s) = \frac{k_m s + k_m K_I}{Is^3 + fs^2 + k_m s + k_m K_I}$$

Continuous Systems and Transfer Function Revision: Integral Error

$$F(s) = \frac{k_m s + k_m K_I}{Is^3 + fs^2 + k_m s + k_m K_I}$$

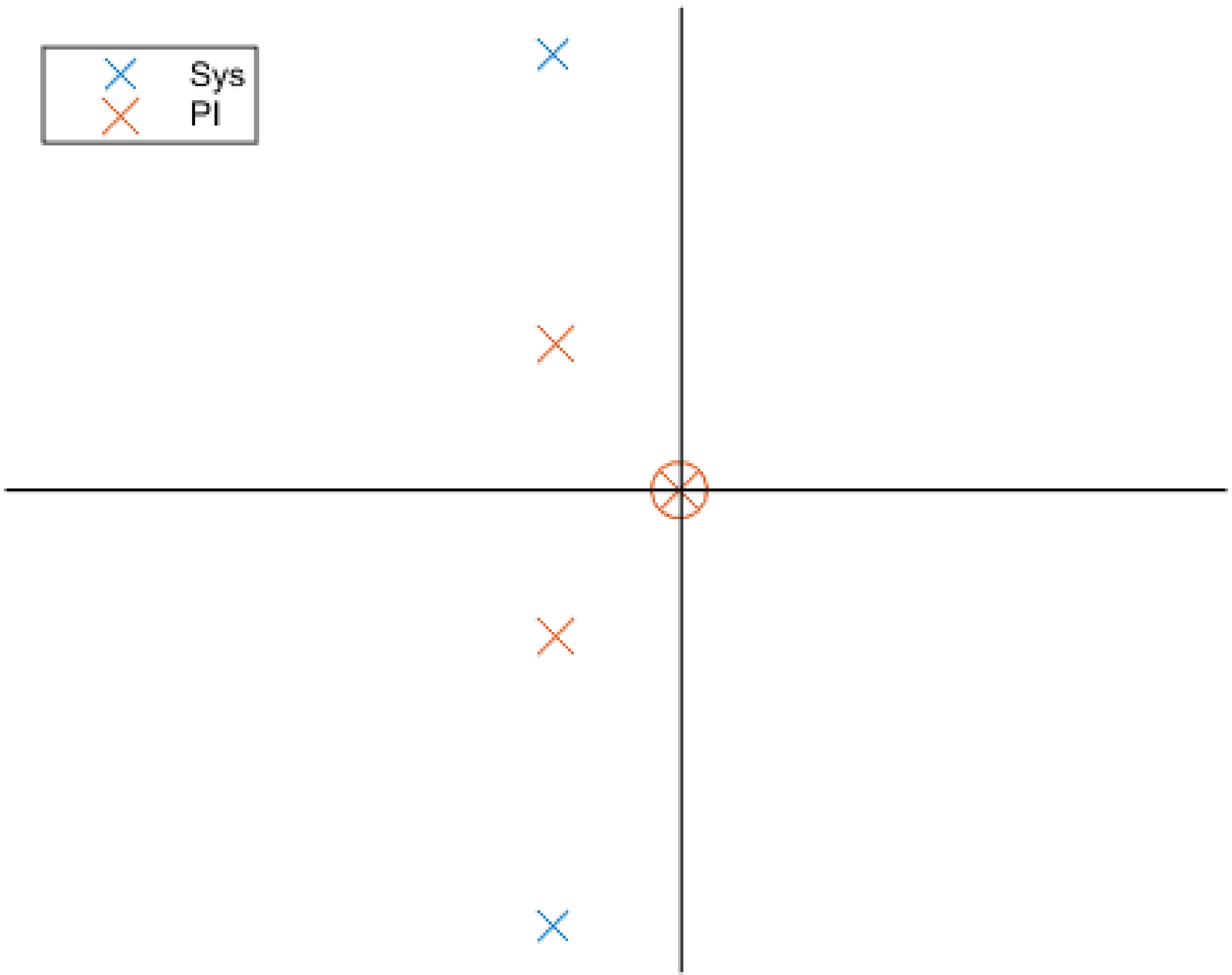
Increasing the system type improves the steady state response:

$$k_v = \lim_{s \to 0} sG'(s)$$

$$k_v = \frac{k_m K_I}{0} = \infty$$

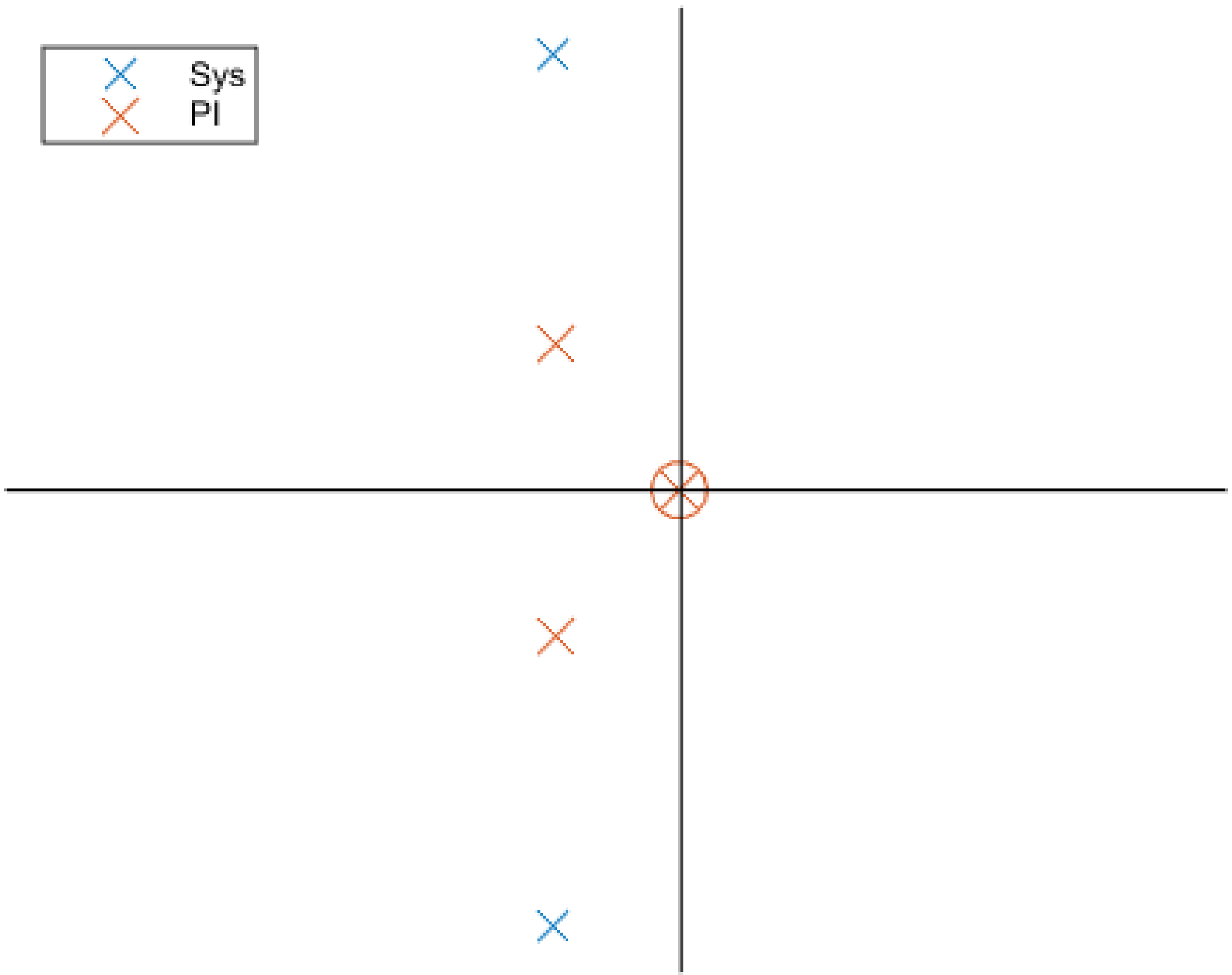
Continuous Systems and Transfer Function Revision: Integral Error

The effect of an extra pole close to origin – which would make our system *very slow* - is largely cancelled out by the nearby zero.

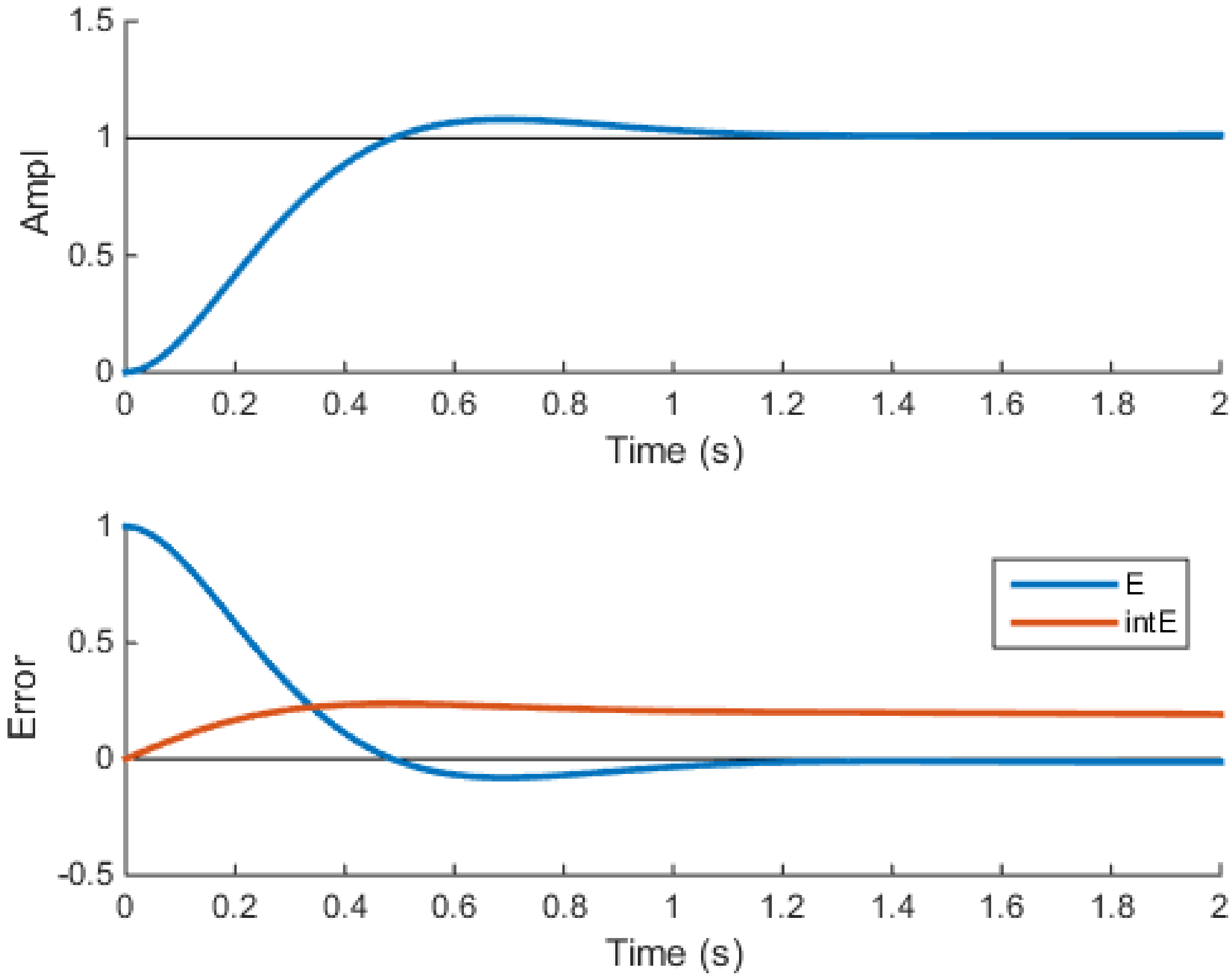


Continuous Systems and Transfer Function Revision: Integral Error

This means the system is broadly similar to a proportional controller, with the exception of improved steady state performance

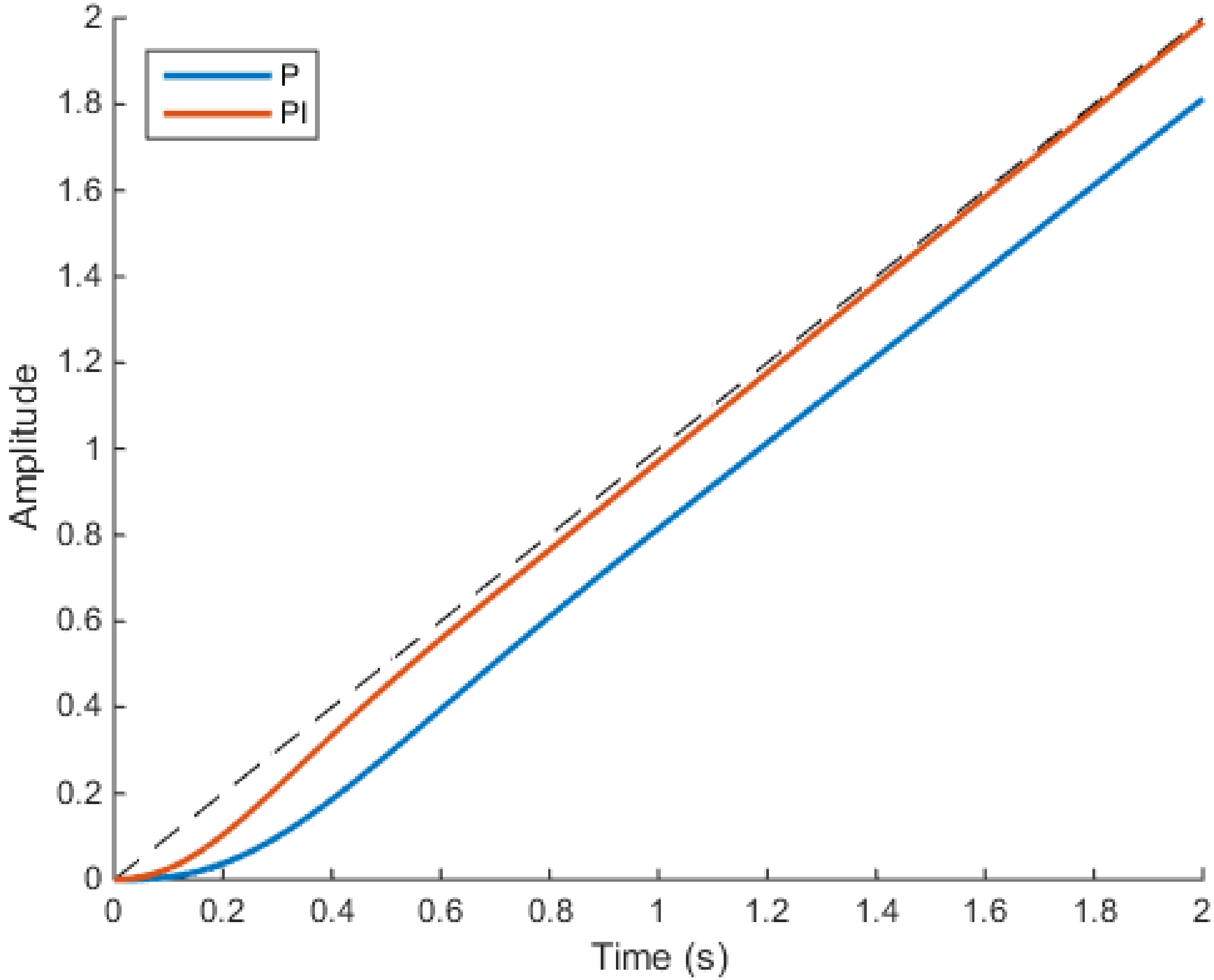


Continuous Systems and Transfer Function Revision: Integral Error



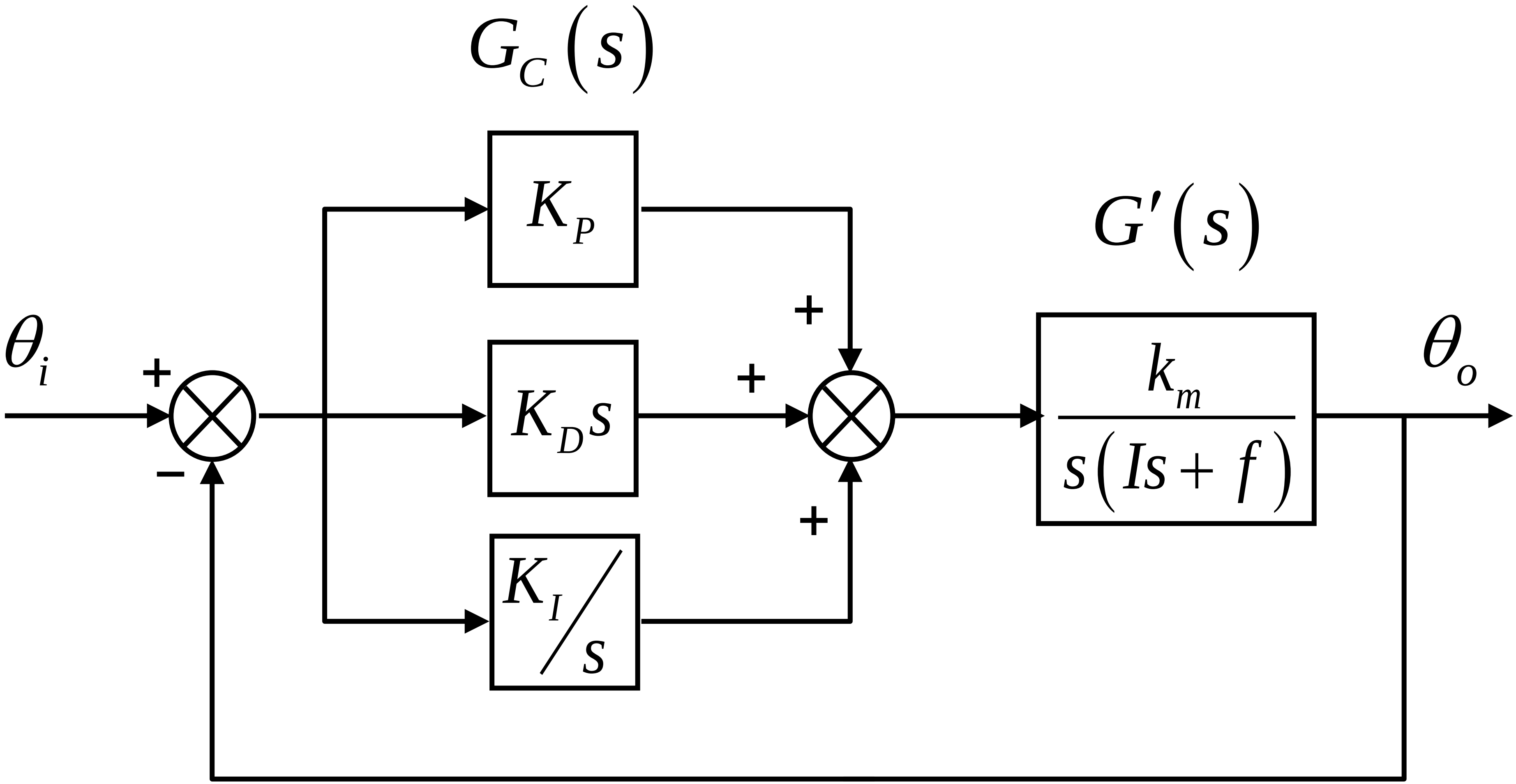
Continuous Systems and Transfer Function Revision: Integral Error

We can see that the integral term has enabled proper tracking of a ramp input



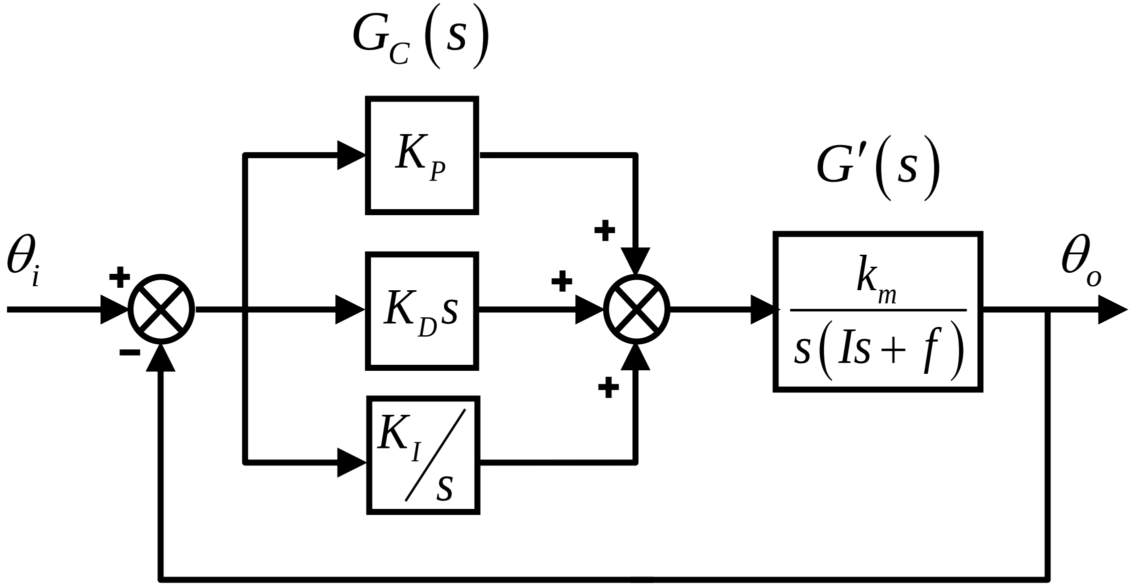
Continuous Systems and Transfer Function Revision: PID control

Putting all three of these controllers together gives the complete PID



$$G_c(s) = K_P + K_D s + \frac{K_I}{s}$$

Continuous Systems and Transfer Function Revision: PID control



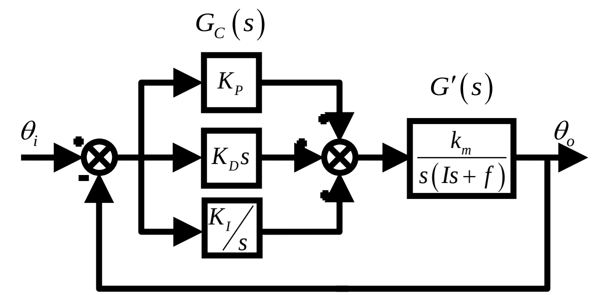
Open Loop gain

$$G'(s) = \frac{k_m\left(K_D s^2 + K_P s + K_I\right)}{s^2\left(Is + f\right)}$$

The closed loop transfer function is then

$$F(s) = \frac{k_m\left(K_D s^2 + K_P s + K_I\right)}{s^2\left(Is + f\right) + k_m\left(K_D s^2 + K_P s + K_I\right)}$$

Continuous Systems and Transfer Function Revision: PID control



$$F(s) = \frac{k_m K_D s^2 + k_m K_P s + k_m K_I}{Is^3 + fs^2 + k_m K_D s^2 + k_m K_P s + k_m K_I}$$

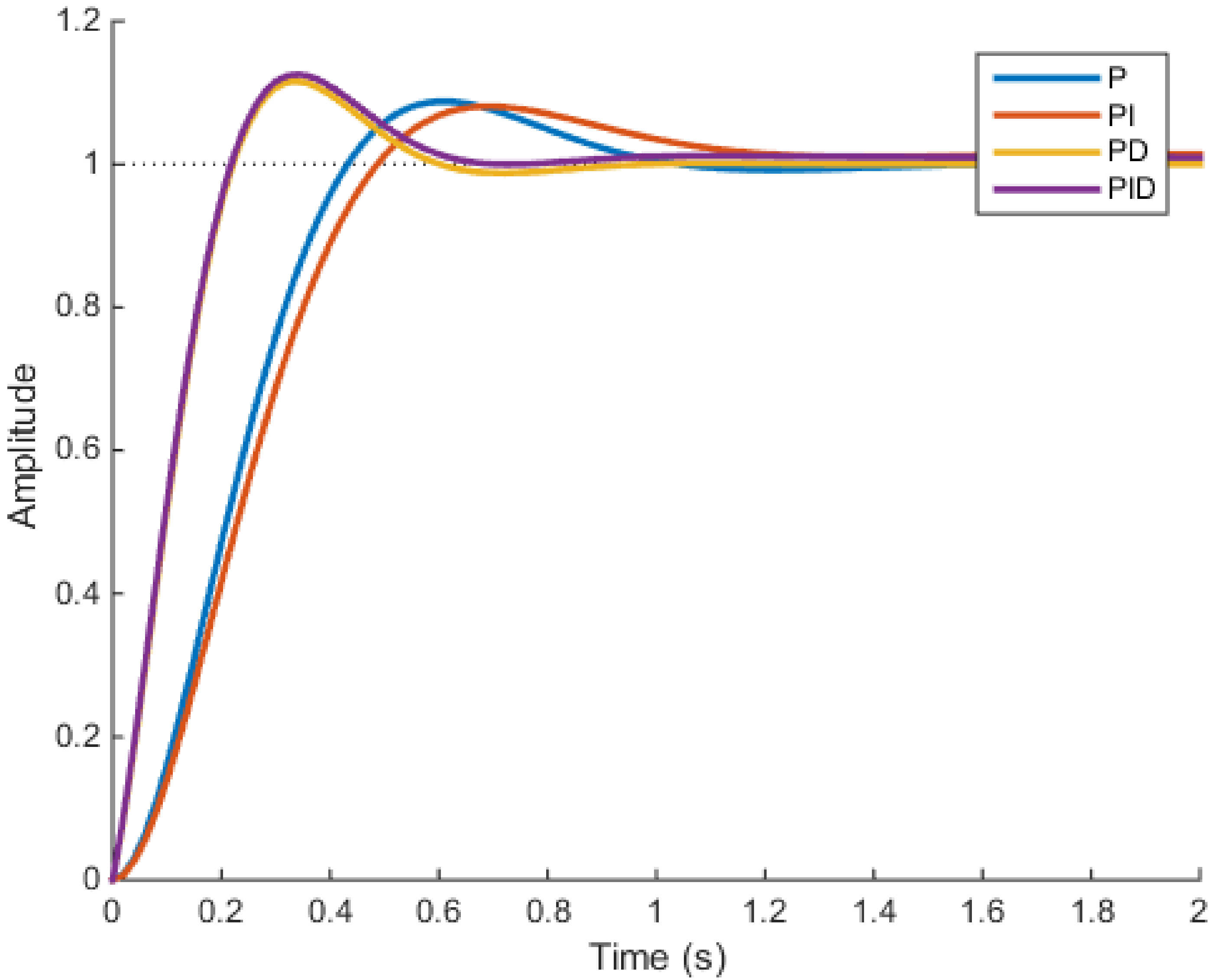
$$F(s) = \frac{k_m K_D s^2 + k_m K_P s + k_m K_I}{Is^3 + (f + k_m K_D) s^2 + k_m K_P s + k_m K_I}$$

Thus by choosing the appropriate values of

It is possible to design a controller with improved transient response **and** decreased/no steady state error

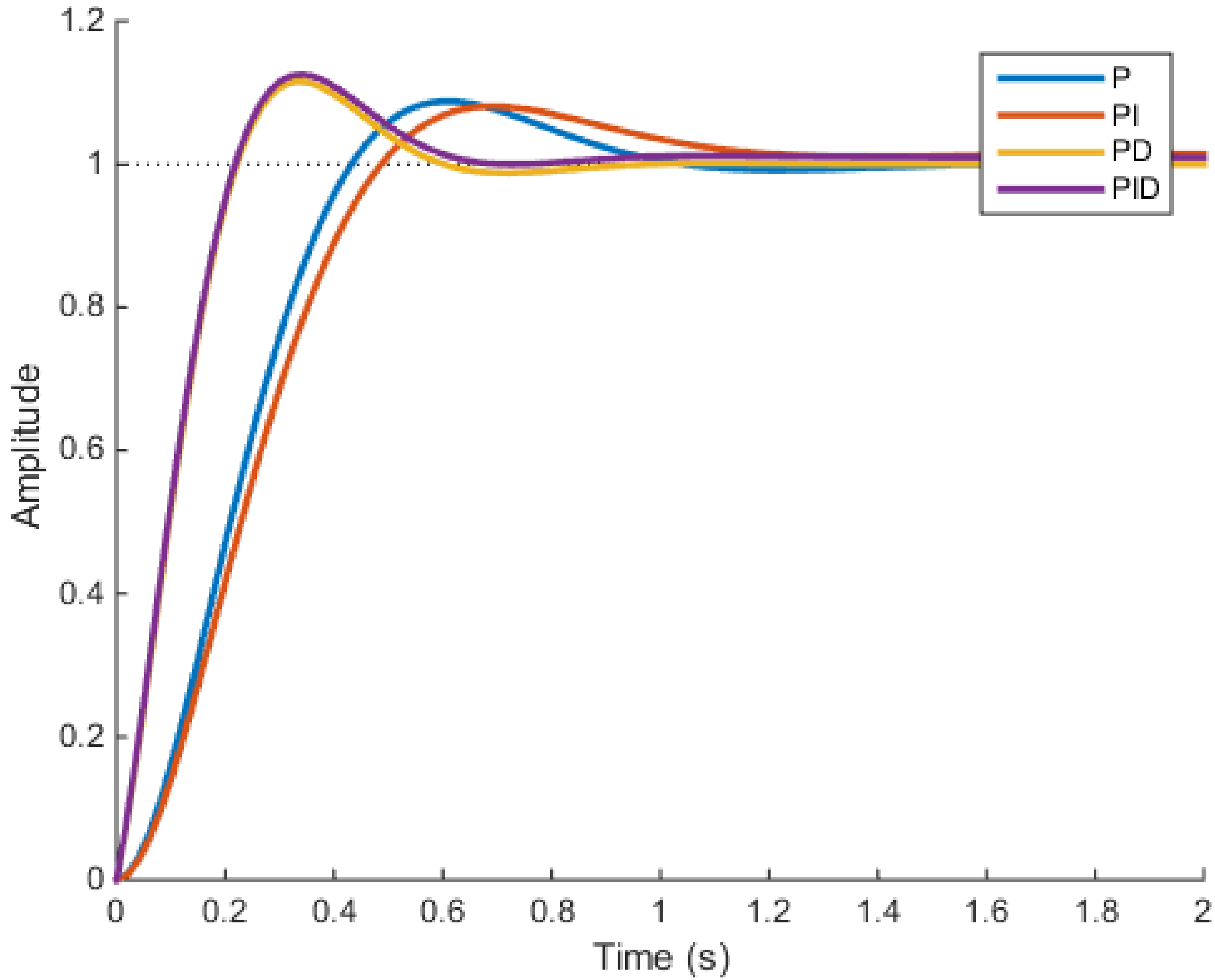
Continuous Systems and Transfer Function Revision: PID control

There is no unique solution for the settings for the three gains, and so the values are dependent upon the specific system and the application.



Continuous Systems and Transfer Function Revision: PID control

Selecting these parameters is known as **tuning**, and it was (and still is in machine learning circles) a very active area of research.



Continuous Systems and Transfer Function Revision: PID control

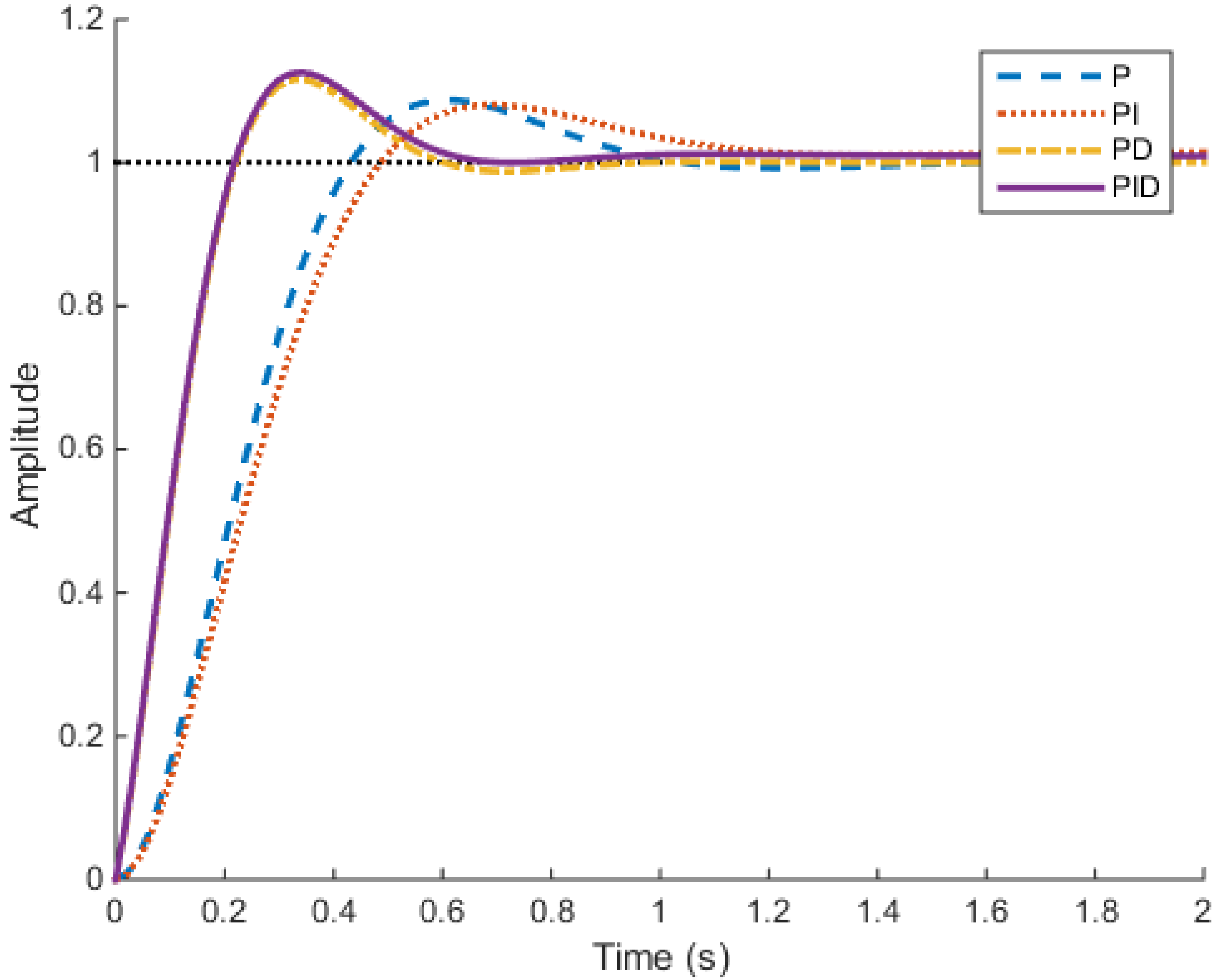
If we set one of the gains to zero, then we remove that term from the controller, e.g.,

$K_I \to 0$ PID becomes PD controller

Qualitatively, the three terms can be thought of as follows:

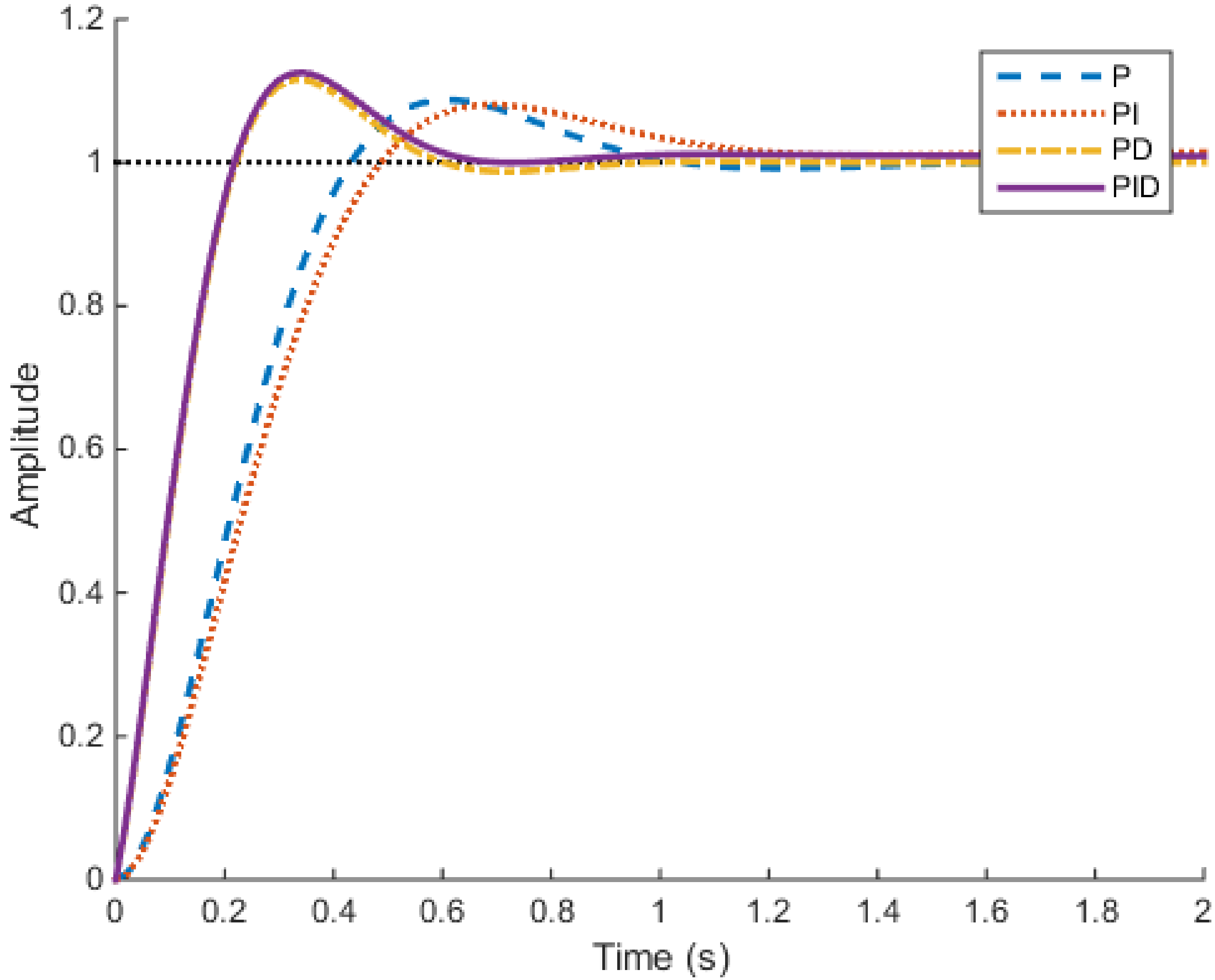
Proportional – Tries to reach target as soon as possible
Derivative – resists overshooting
Integral – Corrects for steady state errors

Continuous Systems and Transfer Function Revision: PID control



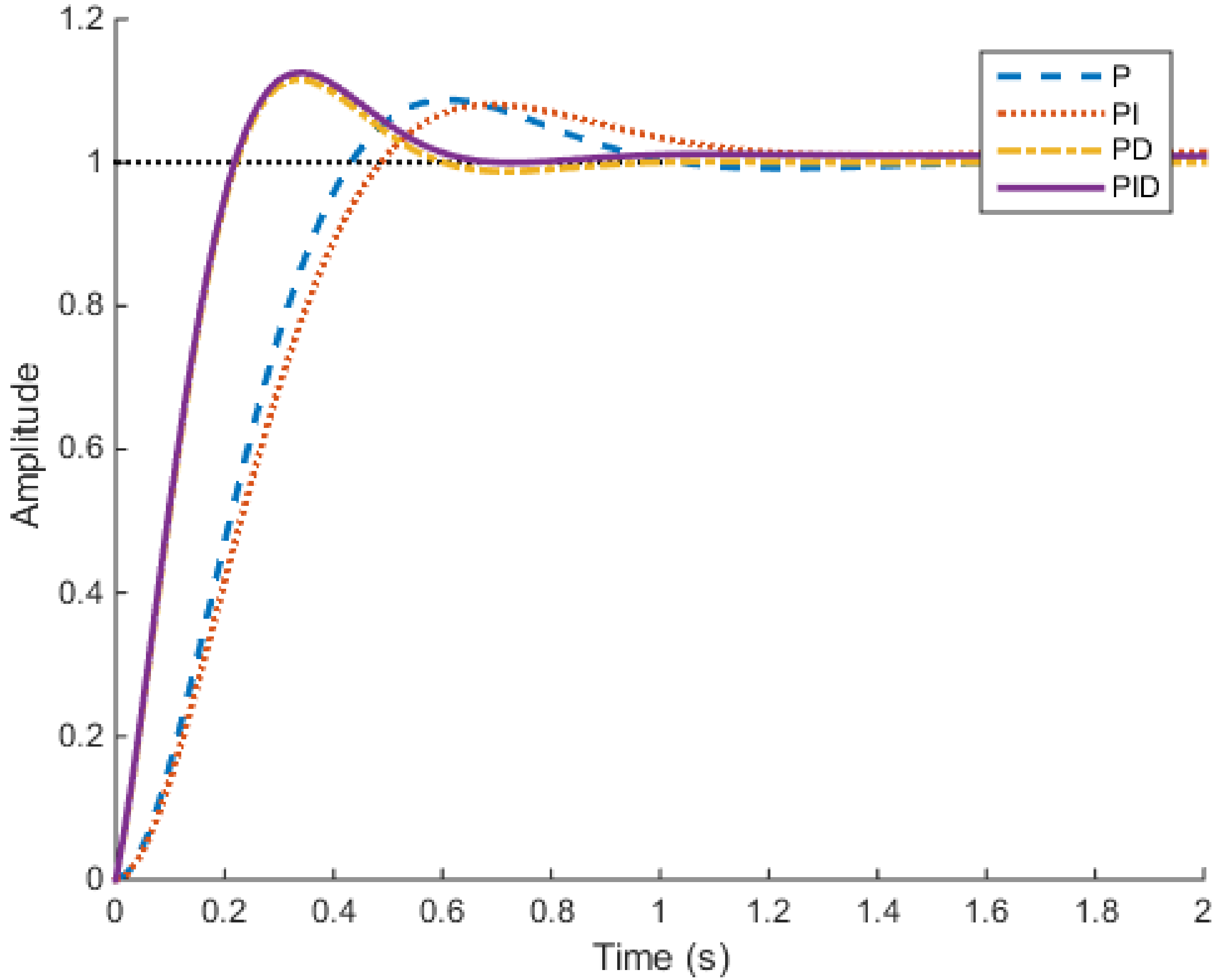
Consider our servo model, the each controller changes the step response in the following ways:

Continuous Systems and Transfer Function Revision: PID control



P – Initially chosen to give ζ close to 0.7 for acceptable transient response.

Continuous Systems and Transfer Function Revision: PID control



PI – I term has little effect on the transient response for this **TYPE 1** system. Steady state error technically 0, but may improve in practice.

Continuous Systems and Transfer Function Revision: PID control

Open loop transfer function:

$$G(s)H(s) = \frac{(s-z_1)(s-z_2)(s-z_3)\ldots}{s^p(s-\sigma_1)(s-\sigma_2)(s-\alpha_k+j\omega_k)(s-\alpha_k-j\omega_k)\ldots}$$

No. Integrators in denominator = system TYPE	Input type		
	Step $r(t) = a$ $R(s) = a/s$	Ramp $r(t) = at$ $R(s) = a/s^2$	Acceleration $r(t) = at^2/2$ $R(s) = a/s^3$
0	$e_{ss} = a/(1+k_p)$	$e_{ss} = \infty$	$e_{ss} = \infty$
1	$e_{ss} = 0$	$e_{ss} = a/k_v$	$e_{ss} = \infty$
2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = a/k_a$

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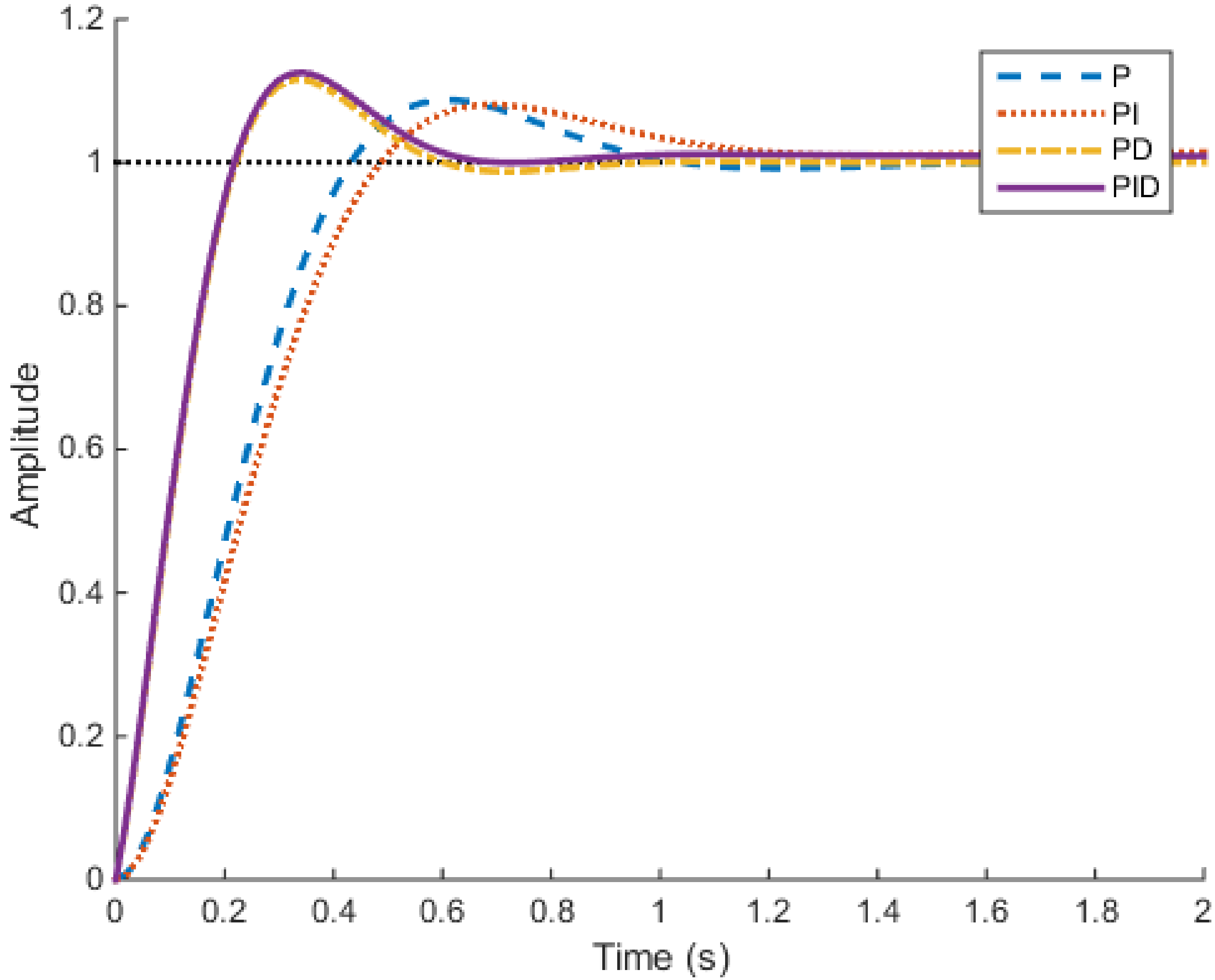
Continuous Systems and Transfer Function Revision: PID control



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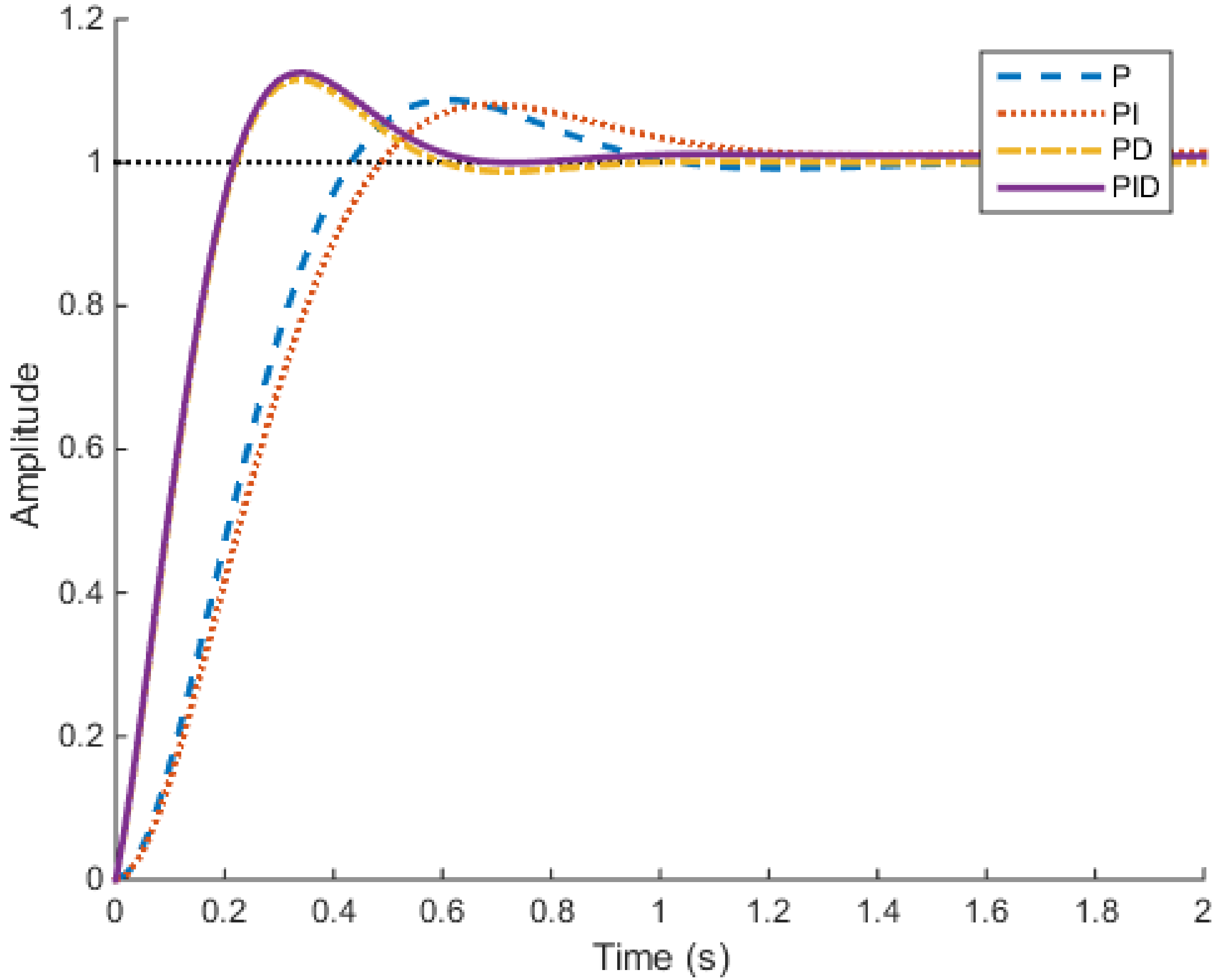
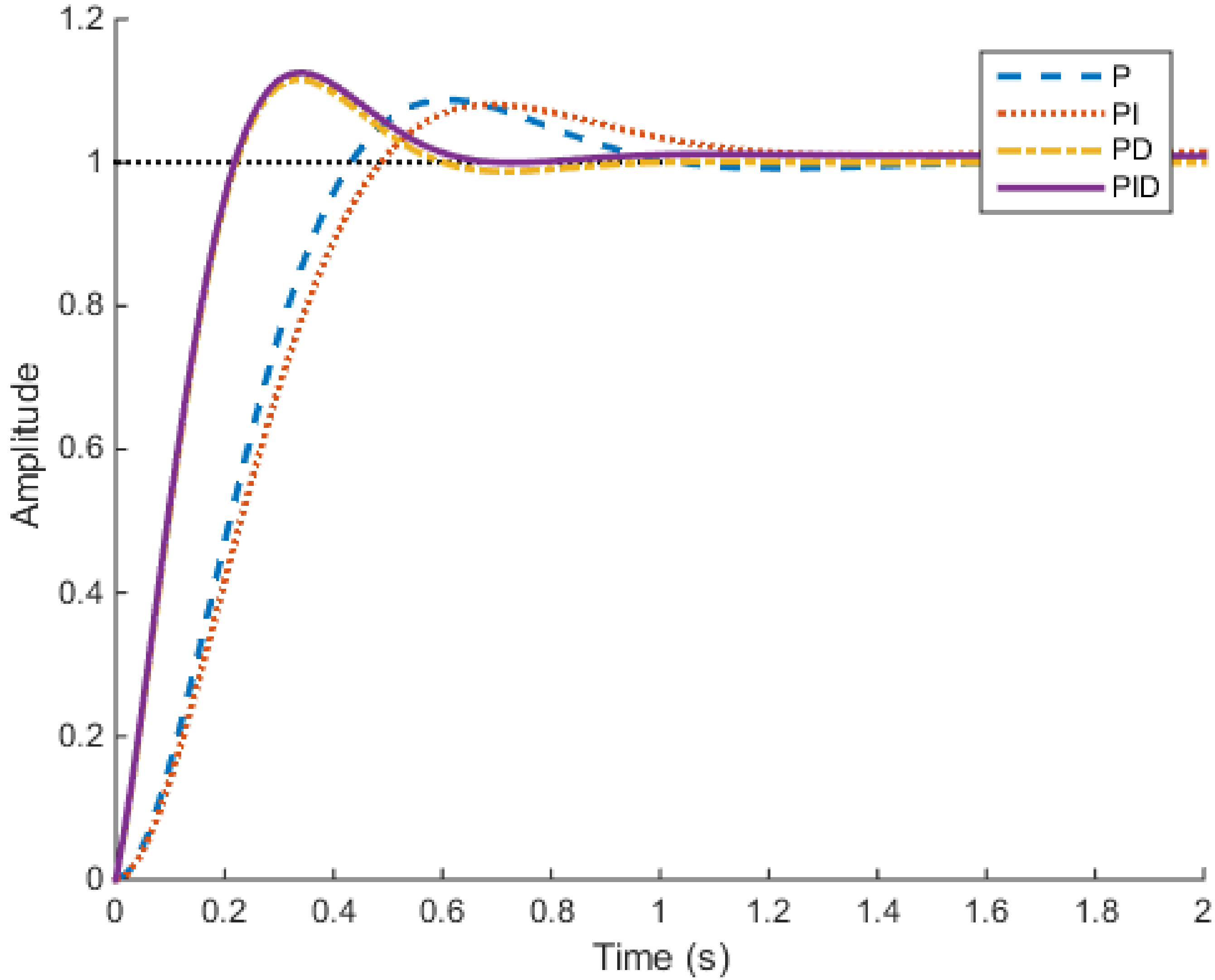
Continuous Systems and Transfer Function Revision: PID control



PD – D term improves transient response as enabling higher P weighting to be used, but with a reduced overshoot. Steady State unchanged.

Continuous Systems and Transfer Function Revision: PID control



PID – Again, transient response largely unchanged, but potential benefits in steady state.