Graph clustering and ranking for lead-lag detection in equity markets

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Signed graph clustering

Financial time series clustering & cluster portfolios

Directed graph clustering

Lead-lag detection in financial multivariate time series

Overview
MetaCluster Lead-lag Portfolios
GlobalRank Lead-lag Portfolios
ClusterRank Lead-lag Portfolios

Main motivation: build a network from time series data, leverage structural properties of the network to inform downstream time series tasks (eg prediction).

Network of financial assets

- Mel MacMahon and Diego Garlaschelli. Phys.Rev.X5, 2015. *Community Detection for Correlation Matrices*.

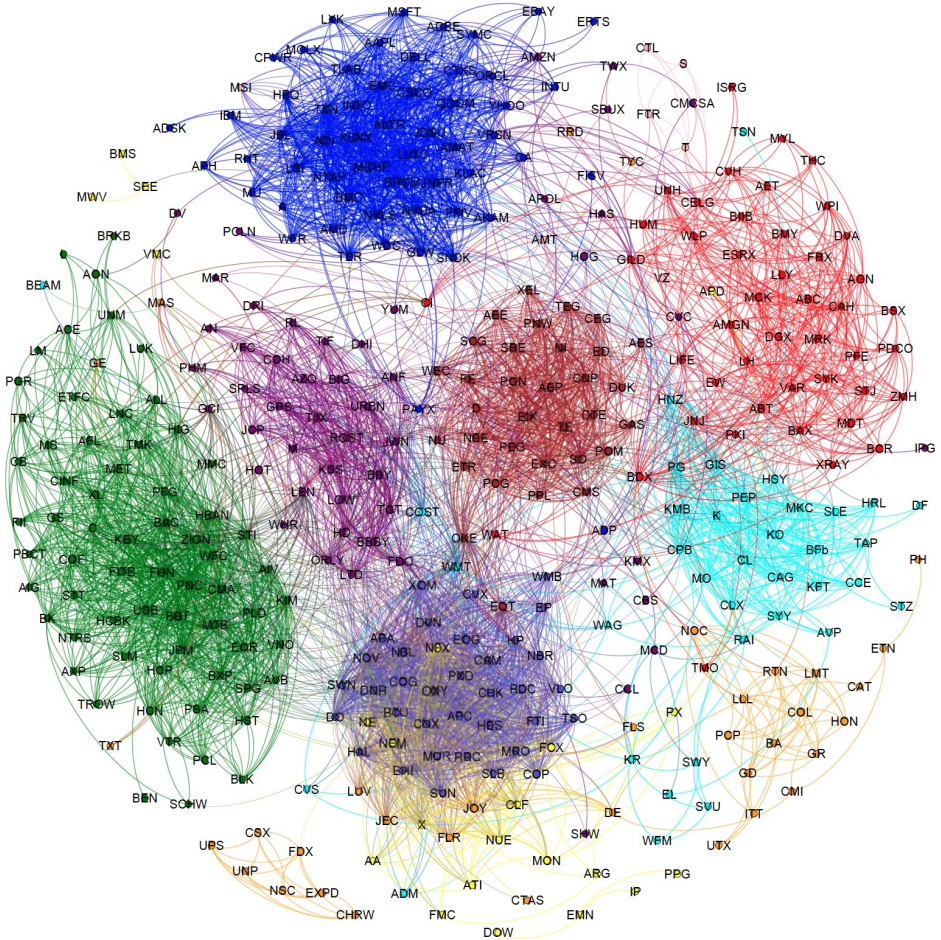

Figure: Asset correlation matrix after thresholding. The color of each node represents the industry sector to which that stock belongs. The force-based layout clearly indicates the existence of strong connections between stocks of the same industry sector.

Clustering graphs

- Consider an undirected graph $G = (V, E)$ with n vertices
- Each $\{i, j\} \in E$ has an associated **positive** weight $w_{ij} > 0$ (similarity between vertices)
- Potentially constructed via a kernel K_ϵ such that

$$w_{ij} = K_\epsilon(||x_i - x_j||), \quad (ij) \in E(G) \tag{1}$$

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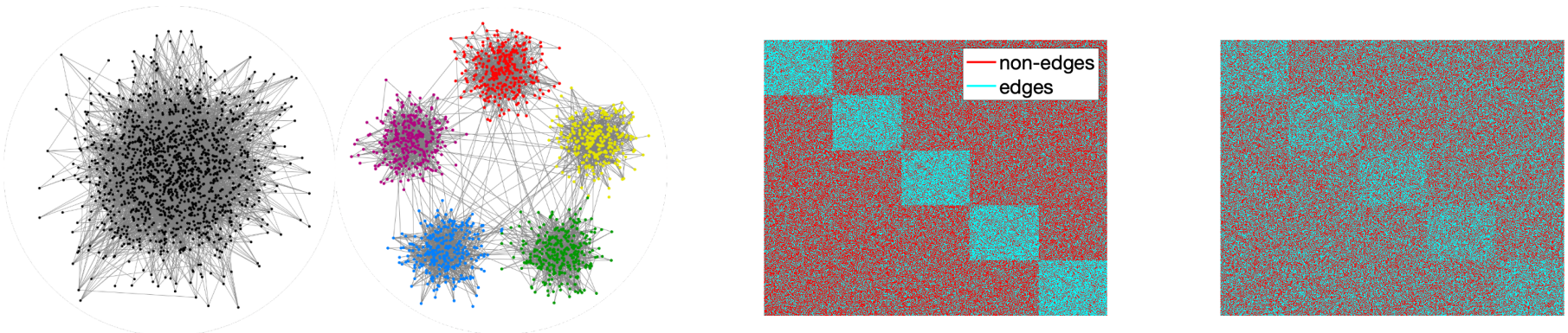

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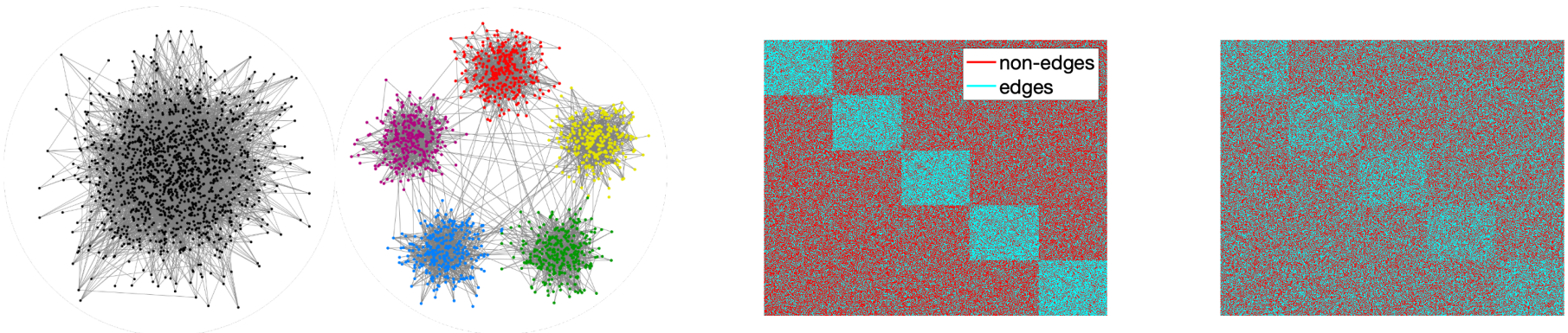

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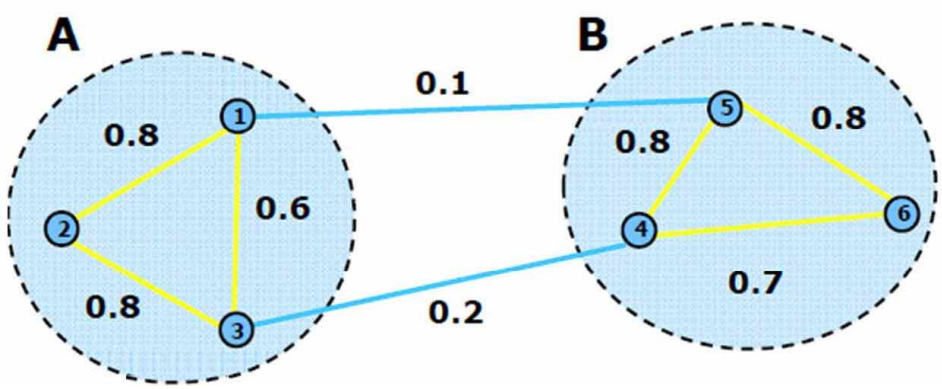
Unlike standard graph clustering settings, we do not require that the intra-cluster edge probabilities to be different from those of inter-cluster edges

- implicitly achieved by the sign or directionality of the edges

Graph Cuts

- consider a partition of the graph G into two subgraphs A and B
- the Cut(A, B) will be given by the sum of the weights of the set of edges that connect the two groups

$$\text{cut(A,B)} := \sum_{i \in A, j \in B} w_{ij} \tag{2}$$



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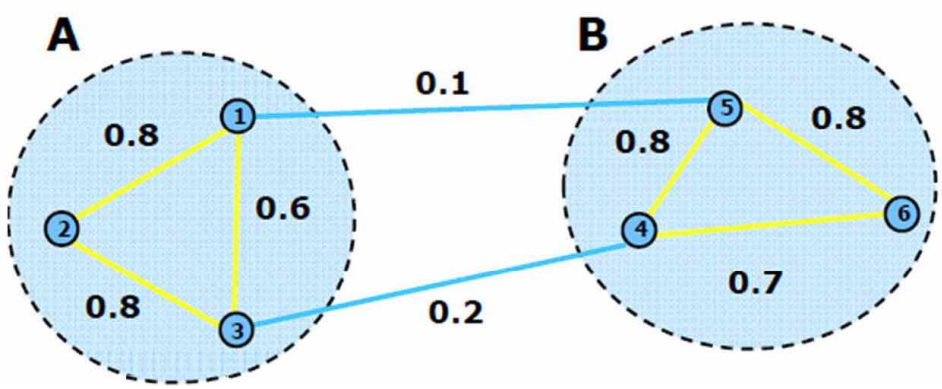

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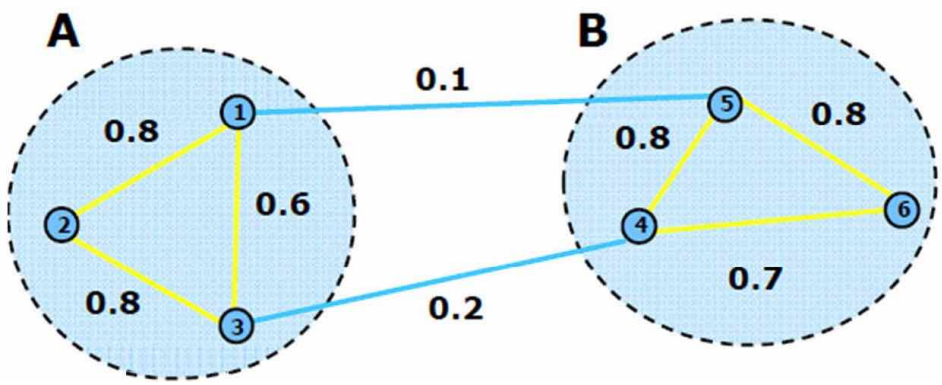

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- the notion of a **cut** is a fundamental concept in graph clustering
- aim to find a partition/split of G into A and B in order to minimize the resulting cut.

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Spectral clustering of (unsigned) graphs

- A popular approach is to perform spectral clustering:
  - **Idea:** Embed V into $\mathbb{R}^k$ and perform k means clustering.
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Laplacian $L = D - A$ is PSD; captures the cut if x is a cluster indicator vector

$$x^T L\, x = \sum_{(i,j)\in E} (x_i - x_j)^2 = \text{total edge weight leaving the cluster}$$

Clustering (unsigned) graphs in a financial context

Hamed Amini, Yudong Chen, Andreea Minca, Xin Qian

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- theoretical analysis provides exact recovery guarantees of the inlier nodes with high probability

Signed Graphs and Signed Graph Cuts

- Many applications involve graphs where edge weights can take negative values (dissimilarity) as well
  - **Social networks:** users are friends (“+" edge) or enemies (“–" edge)
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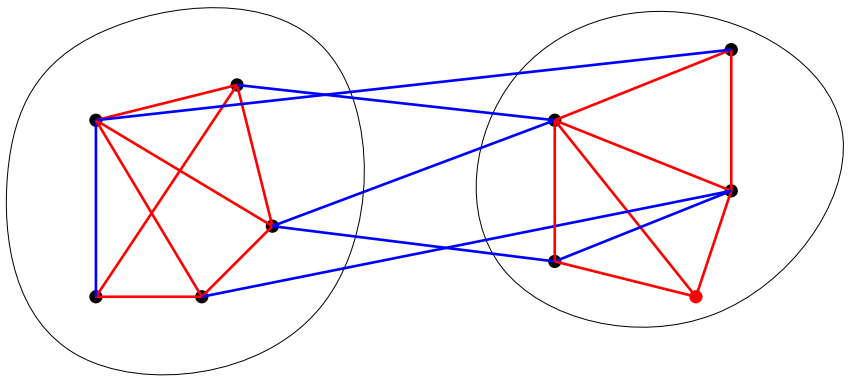

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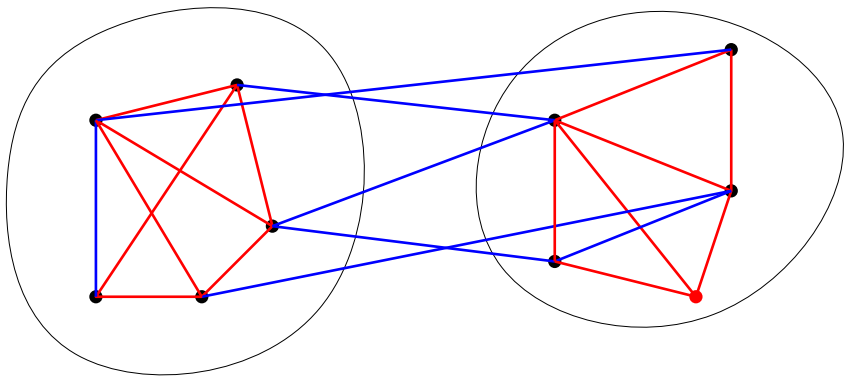

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Goal:
Maximize the sum of weights of: intra cluster positive edges plus inter cluster negative edges

Motivation from Statistical Arbitrage

Stat arb broadly refers to

- technical short-term mean-reversion strategies
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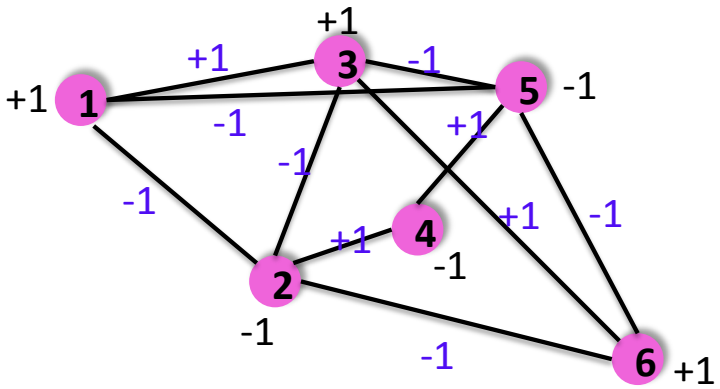
Pairs-trading - widely assumed to be the *ancestor* of statistical arbitrage:

- If stocks X and Y are in the same industry or have similar characteristics (e.g. Pepsi and Coca Cola), one expects the returns of the two stocks to track each other after controlling for beta.
- cluster of size 2

Signed clustering with $k = 2$ (group synchronization over $\mathbb{Z}_2$)



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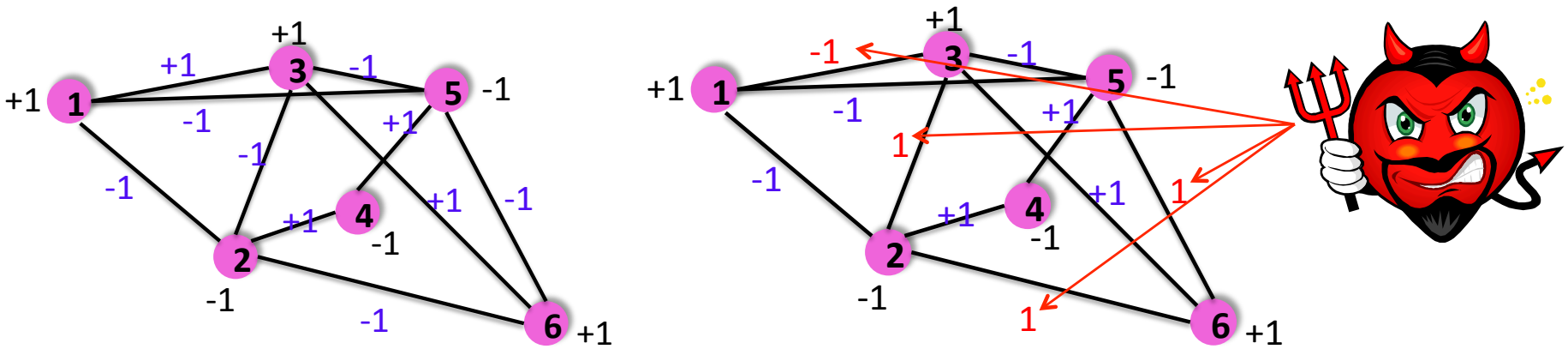

Figure: Synchronization over $\mathbb{Z}_2$ (left: clean, right: noisy)

- unknown group elements $z_1, z_2, \ldots, z_N \in \mathbb{Z}_2$ (eg. ± 1) correspond to the vertices of a measurement graph G
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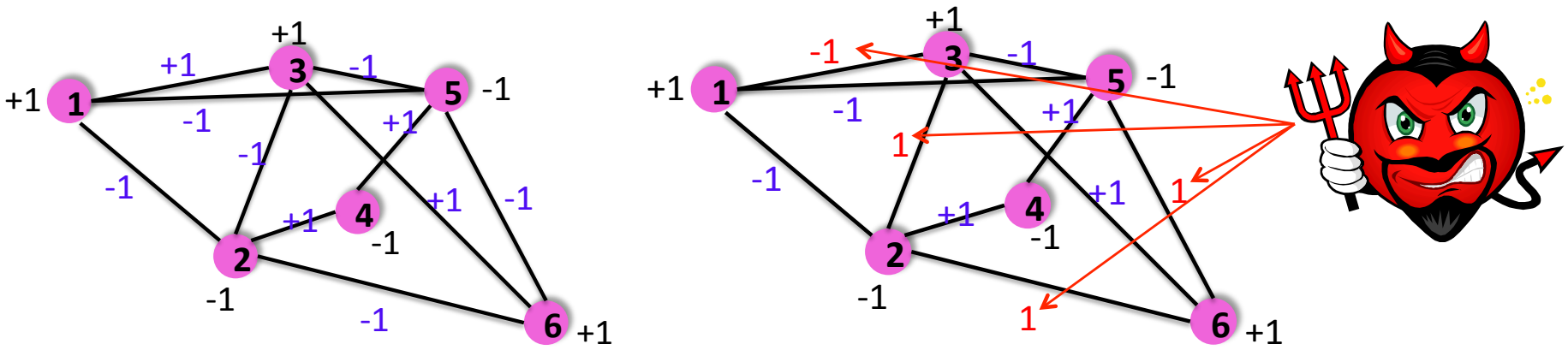

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- $z_i z_j$ encodes the measured similarity between nodes (eg., stocks) i and j
- a potential noise model for the measurement graph is

$$A_{ij} = \begin{cases} z_i z_j & (i,j) \in E \text{ and the measurement is correct,} \\ -z_i z_j & (i,j) \in E \text{ and the measurement is incorrect,} \\ 0 & (i,j) \notin E \end{cases}$$

Signed clustering with $k = 2$ (group synchronization over $\mathbb{Z}_2$)

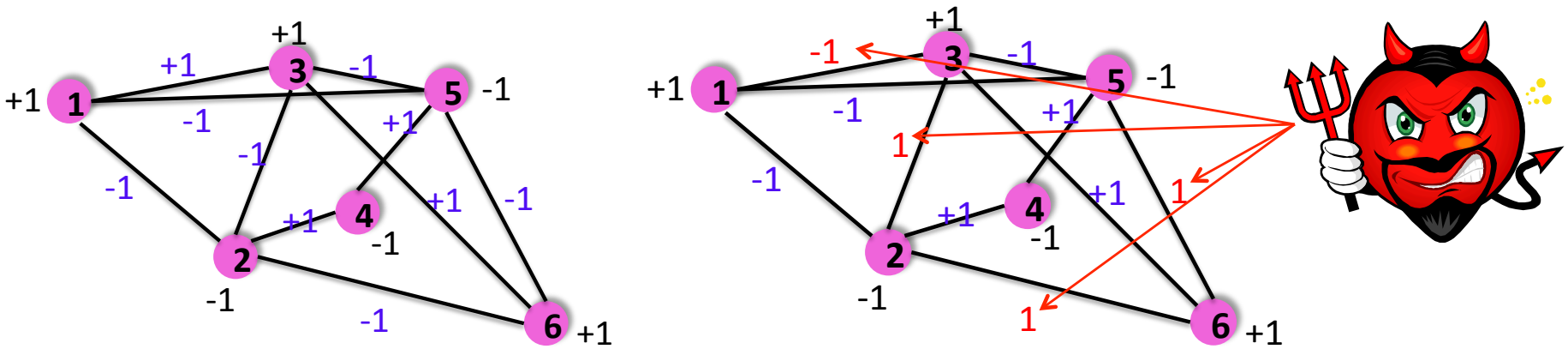

Figure: Synchronization over $\mathbb{Z}_2$ (left: clean, right: noisy)

- unknown group elements $z_1, z_2, \ldots, z_N \in \mathbb{Z}_2$ (eg. ± 1) correspond to the vertices of a measurement graph G
- $z_i z_j$ encodes the measured similarity between nodes (eg., stocks) i and j
- a potential noise model for the measurement graph is

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- original solution: $z_1, \ldots, z_n \in \pm 1^n \quad (\mathbb{Z}_2 = \{-1, +1\})$
- task: estimate approx. solution $x_1, \ldots, x_N \in \pm 1^N$ such that we satisfy as many pairwise group relations (happy edges) in $\mathbb{Z}_2$ as possible.

Signed clustering with k=2 (Synchronization over $\mathbb{Z}_2$)

- Consider maximizing the following quadratic form (intra-cluster happiness)

$$\max_{x_1,\ldots,x_N \in \mathbb{Z}_2^N} \sum_{i,j=1}^{N} x_i A_{ij} x_j = \max_{x_1,\ldots,x_N \in \mathbb{Z}_2^N} x^T A x, \tag{4}$$

whose maximum is attained at $x = z$ (noise-free data). NP-hard, but relax to

$$\max_{\sum_{i=1}^{N} |x_i|^2 = N} \sum_{i,j=1}^{N} x_i A_{ij} x_j = \max_{\|x\|^2 = N} x^T A x \tag{5}$$

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Alternatively, formulate synchronization as a least squares problem, by minimizing the following quadratic form (minimize unhappy edges)

$$\min_{x \in \mathbb{Z}_2^N} \sum_{(i,j) \in E} (x_i - A_{ij} x_j)^2 = \ldots = \min_{x \in \mathbb{Z}_2^N} x^T (\overline{D} - A) x$$

- (psd) Signed Laplacian $\overline{L} = \overline{D} - A$, diagonal matrix $\overline{D}$ with $\overline{D}_{ii} = \sum_{i=1}^{n} |A_{ij}|$.

Signed Clustering: Signed Laplacians & Balanced Ratio Cuts

- Kunegis et al. (2010) solved a signed version of the 2-way ratio-cut problem via the Signed (combinatorial) graph Laplacian $\overline{L} = \overline{D} - A$.
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Balanced Ratio Cut:

$$\min_{\{x_1,\ldots,x_k\} \in I} \left(\sum_{c=1}^{k} \frac{x_c^T (D^+ - A) x_c}{x_c^T x_c} \right). \tag{7}$$

- where x_c is the cluster indicator vector corresponding to cluster c.

Signed clustering: a generalized eigenproblem formulation

H: unsigned graph, adj. matrix W ($W_{ij} > 0$) for any cluster $C \subset V$

$$\mathrm{cut}_H(C, \overline{C}) := \sum_{i \in C, j \in \overline{C}} W_{ij}$$

the total weight of edges crossing from C to $\overline{C}$.

Volume(C): sum of degrees of nodes in C; $\mathrm{vol}_H(C) = \sum_{i \in C} \sum_{j=1}^{n} W_{ij}$

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Ideally, want C s.t. both (8) and (9) are small. "Merge" obj. (8)+(9)

$$\min_{C \subset V} \frac{\mathrm{cut}_{G^+}(C, \overline{C}) + \tau^- \mathrm{vol}_{G^-}(C)}{\mathrm{cut}_{G^-}(C, \overline{C}) + \tau^+ \mathrm{vol}_{G^+}(C)}, \tag{10}$$

$\tau^+, \tau^- > 0$ denote trade-off/regularization parameters.

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Natural extension to $k > 2$ disjoint clusters $C_1, \cdots, C_k$

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For a subset $C_i \subset V$, the normalized indicator vector

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renders (11) as the discrete optimization problem

$$\min_{C_1,\cdots,C_k} \sum_{i=1}^{k} \frac{x_{C_i}^T (L^+ + \tau^- D^-) x_{C_i}}{x_{C_i}^T (L^- + \tau^+ D^+) x_{C_i}}, \tag{13}$$

which is NP-hard.

- L^+ (resp. L^-) denotes the Laplacian of G^+ (resp. G^-), and
- D^+ (resp. D^-) denotes a diagonal matrix with the degrees of G^+ (resp. G^-).

Signed clustering: a generalized eigenproblem formulation

Drop the discreteness constraint & allow each $x_{C_i} \in \mathbb{R}^n$

- new set of vectors $z_1, \ldots, z_k \in \mathbb{R}^n$ orthonormal w.r.t. $L^- + \tau^+ D^+$
  - $z_i^T (L^- + \tau^+ D^+) z_i = 1$, and
  - $z_i (L^- + \tau^+ D^+) z_j = 0$, for $i \neq j$

  leads to the following modified version of (13)

$$\min_{z_i (L^- + D^+) z_j = \delta_{ij}} \sum_{i=1}^{k} \frac{z_i^T (L^+ + \tau^- D^-) z_i}{z_i^T (L^- + \tau^+ D^+) z_i}. \tag{14}$$

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INPUT: A signed weighted graph G ($G = G^+ \cup G^-$)

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• In experiments, we use LOBPCG – a preconditioned eigensolver for solving large positive definite generalized eigenproblems.

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  i, j lie in **same** cluster

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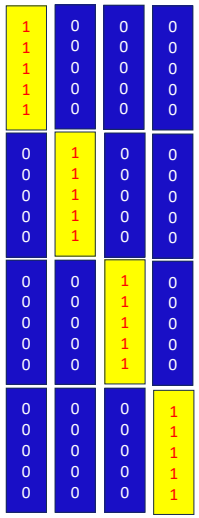
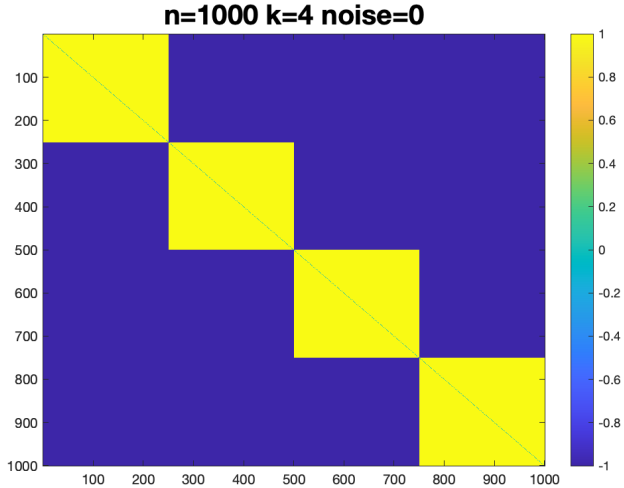
- performance guarantees (misclustering rate) for SPONGE for SSBM ($k \geq 2$)

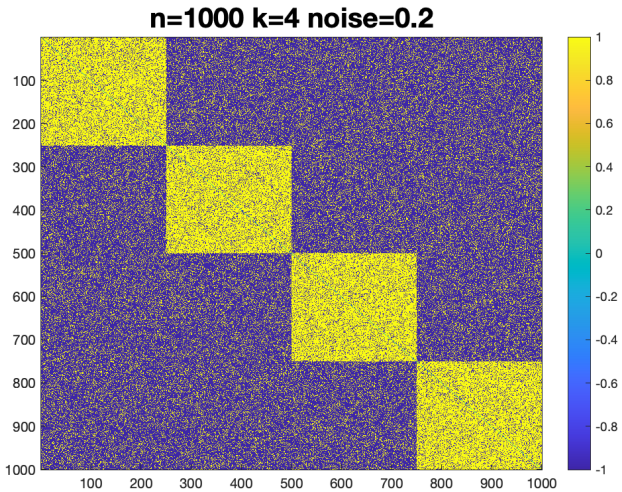
SSBM instances at varying noise levels



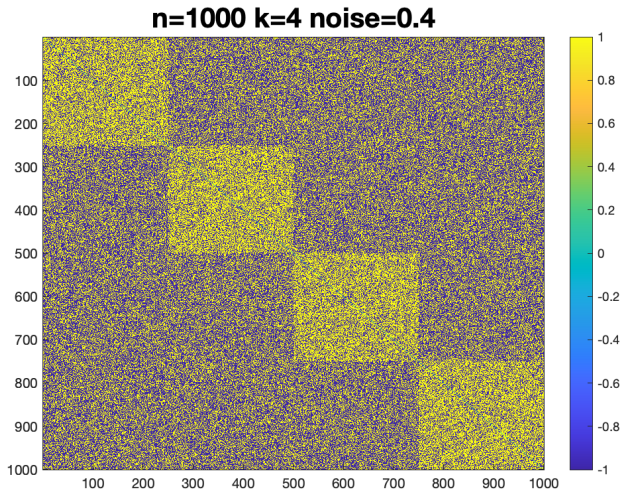
(a) Cluster membership matrix



(b) $\eta = 0$



(c) $\eta = 0.2$



(d) $\eta = 0.4$

Figure: Instances of SSBM with $n = 1000$ nodes, $k = 4$ clusters, $p = 1$ edge density.

Robustness guarantees for the SSBM

- consider the embedding given by the k smallest eigenvectors of

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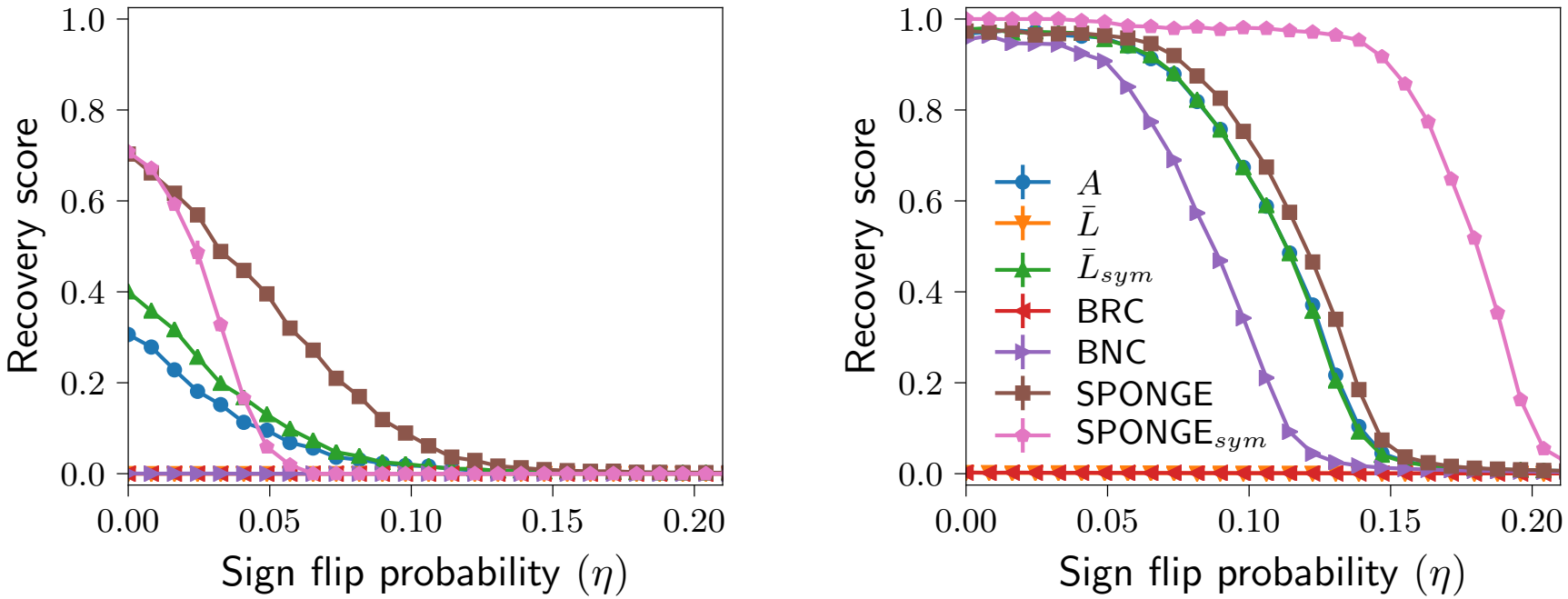
- provided the sparsity parameter $p = \Omega(\ln n/n)$; extended to $p = \Theta(1/n)$.

Numerical comparison of state-of-the-art methods

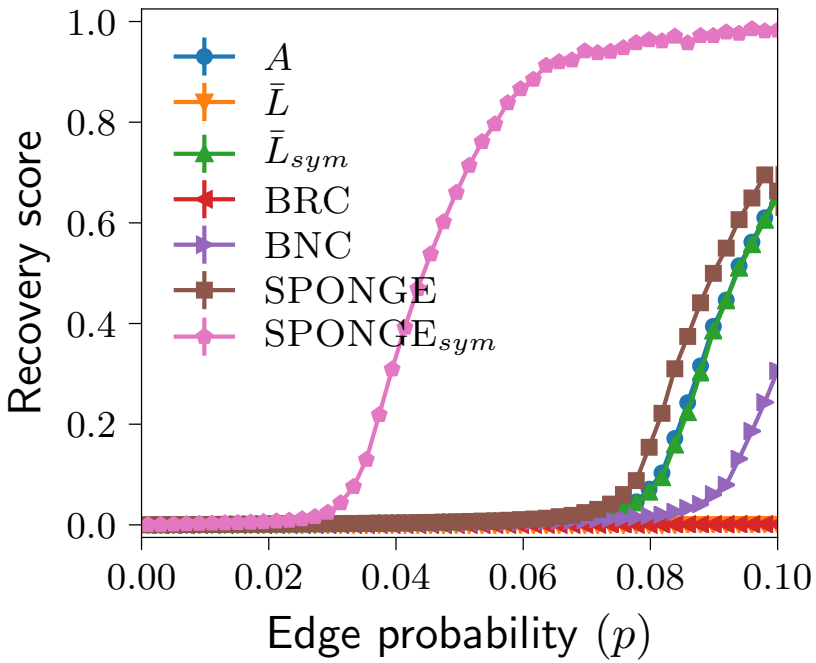
The recovery score used here is the Adjusted Rand Index (ARI) (a slightly modified version of the Rand Index).



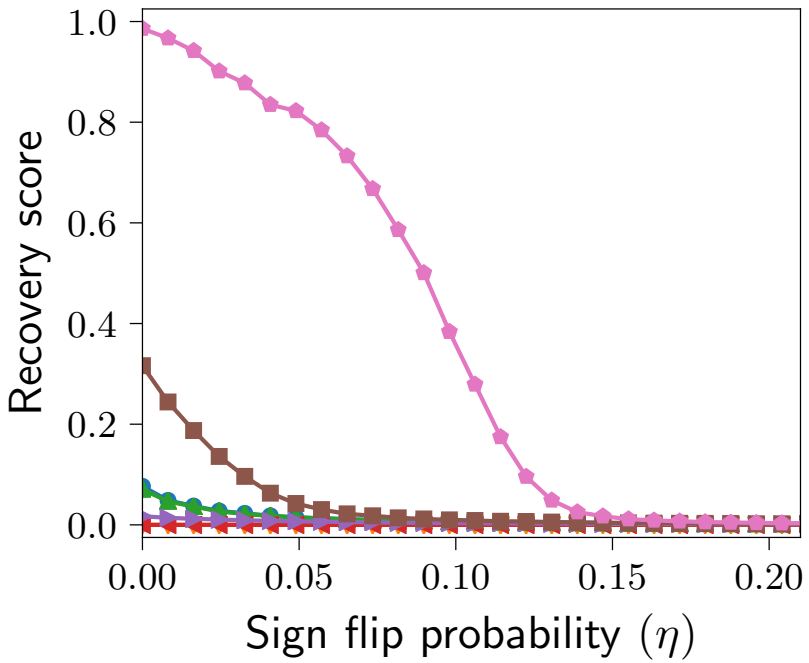
(a) $k = 5, p = 0.001$ (b) $k = 50, p = 0.1$

Figure: ARI recovery scores versus η for increasing k, with communities of equal size and $n = 10000$.

Numerical comparison of state-of-the-art methods



(a) $k = 50, \eta = 0.1$



(b) $k = 20, p = 0.01$

Figure: ARI recovery scores for $n = 10000$, as a function of:
(Left:) the edge probability p, for $\eta = 0.1$ and $k = 50$ equally-sized clusters;
(Right:) the sign flipping probability η for $k = 20, p = 0.01$.

Financial time series clustering

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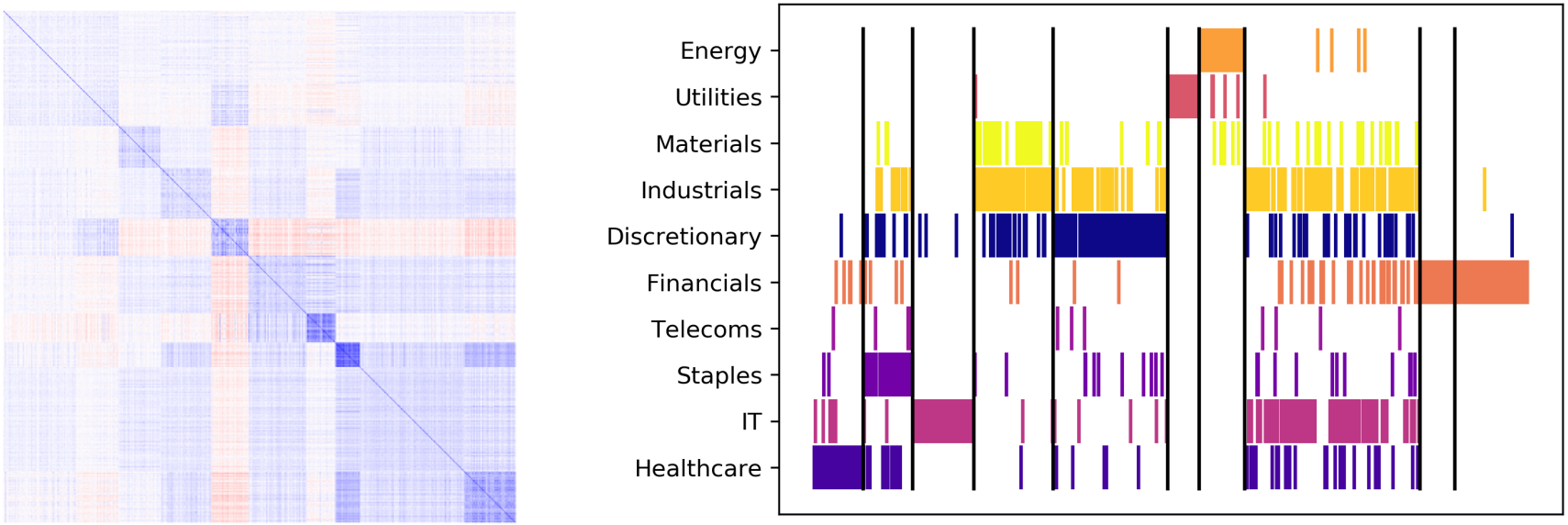

Figure: Left: the adjacency matrix A with rows/columns sorted in accordance to cluster membership. Right: Sector decomposition of the recovered clusters (based on a standard classification of the US economy into sectors).

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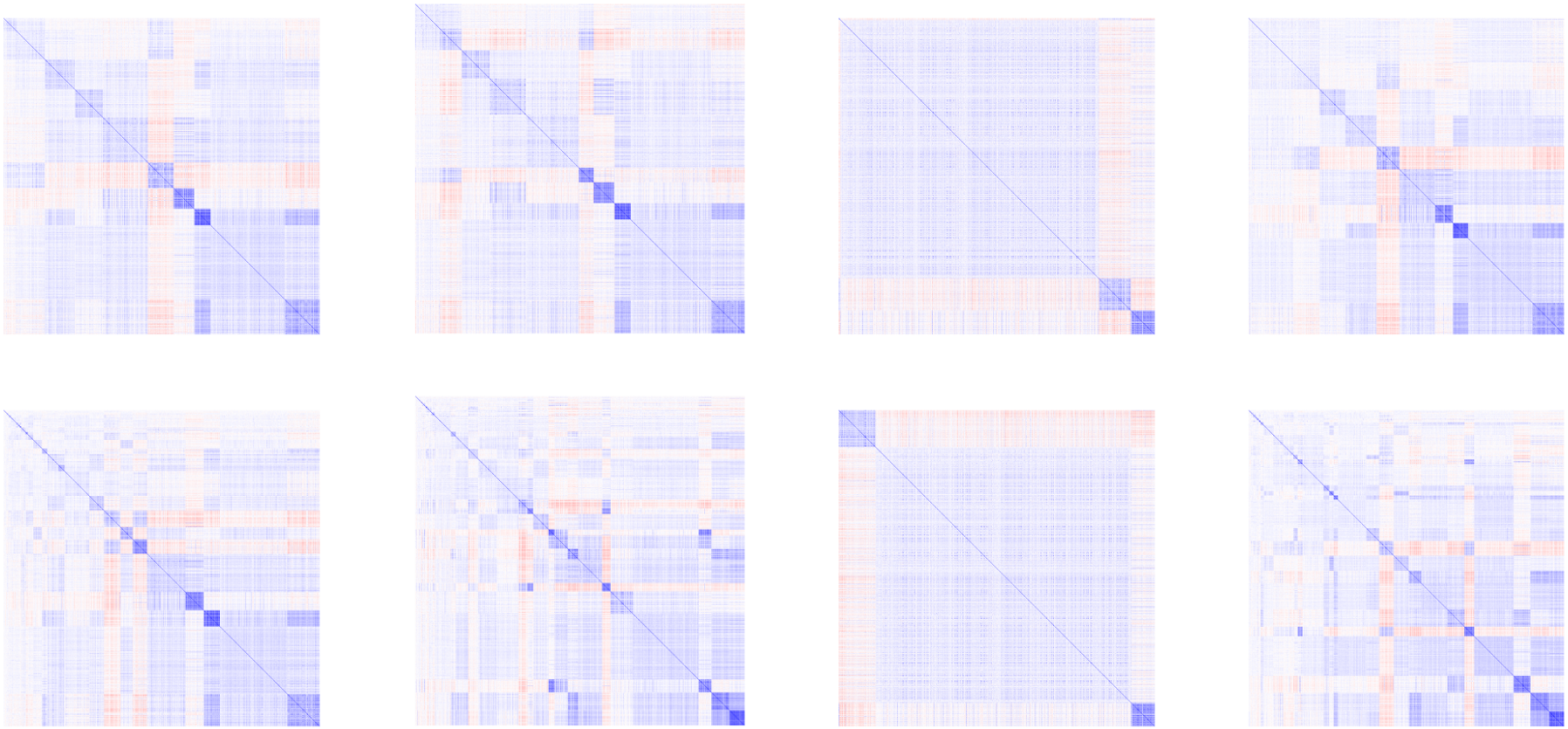

(e) SPONGE (f) SPONGE$_{sym}$ (g) BNC (h) $\bar{L}_{sym}$

Figure: Adjacency matrix of the S&P 1500 data, sorted by cluster membership; $k = 10$ (top) and $k = 30$ (bottom).

Cluster centric portfolios & Markowitz – Problem setup

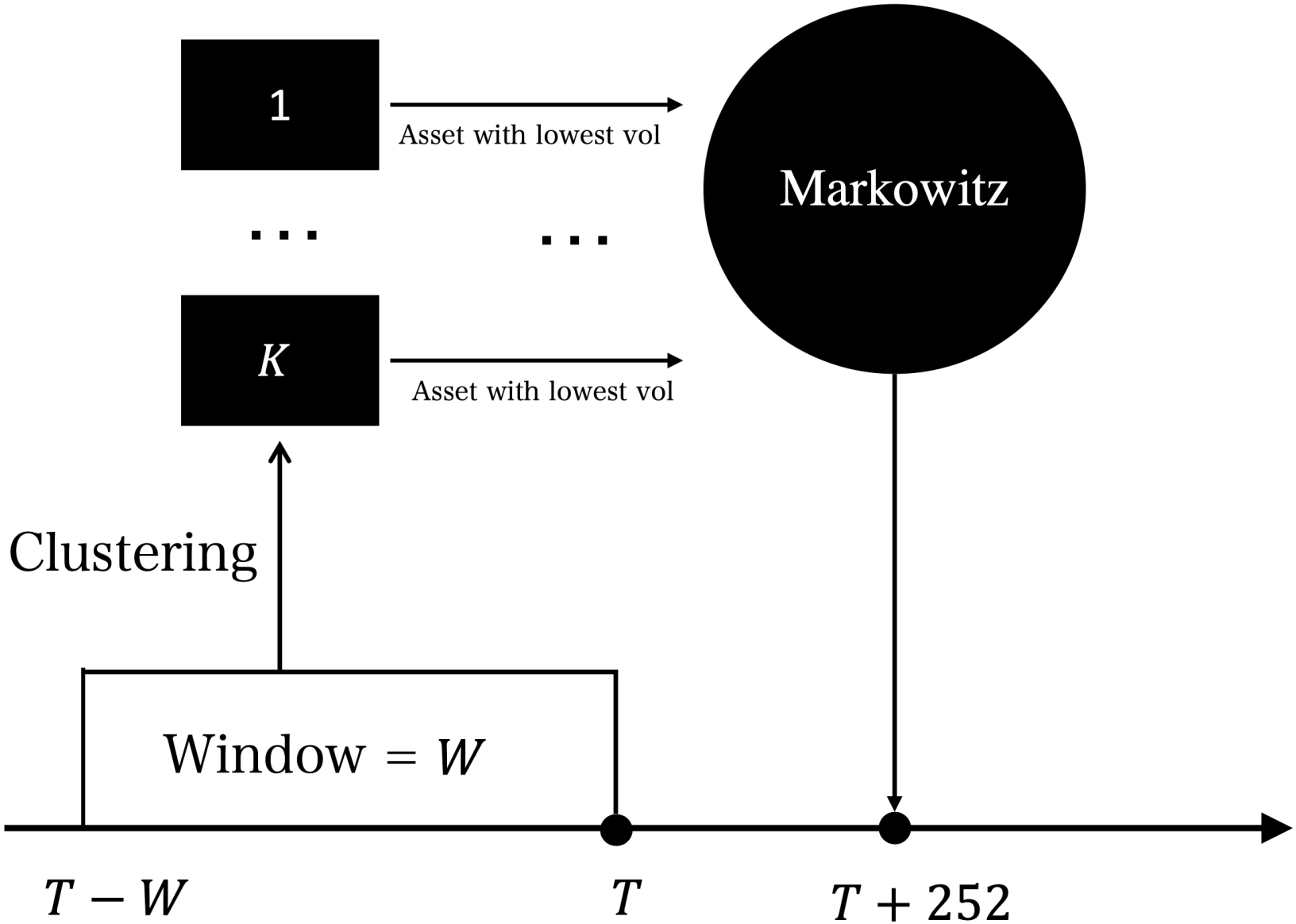

Figure: Experimental setup for the clustering and portfolio construction.

Cluster centric portfolios & Markowitz – Experimental results (I)

Table: Performance of portfolios across different clustering algorithms

Panel A: 25 clusters, 500 day lookback, 252 day holding period			
	Compound Return (%)	Return (bps/day)	Sharpe ratio
SPONGE	11.15	4.19	0.62
SPONGE$_{sym}$	8.86	3.37	0.48
Hermitian	8.76	3.33	0.52
Spectral	8.42	3.21	0.49
SPY	6.59	2.53	0.32
Panel B: 50 clusters, 500 day lookback, 252 day holding period			
	Compound Return (%)	Return (bps/day)	Sharpe ratio
SPONGE	10.62	4.00	0.58
SPONGE$_{sym}$	10.61	4.00	0.55
Hermitian	8.55	3.26	0.49
Spectral	8.06	3.08	0.44
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Cluster centric portfolios & Markowitz – Experimental results (II)

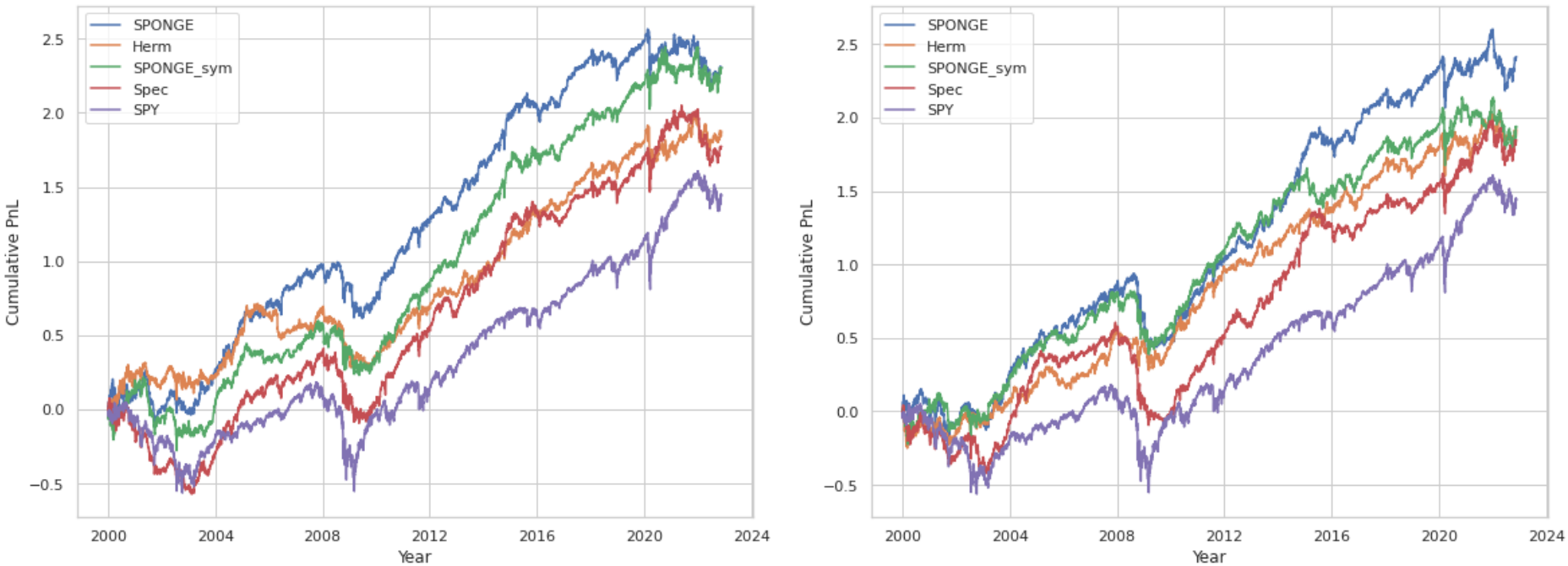

Figure: Markowitz strategy with 50 (left) and 25 (right) clusters; 252 days rebalance frequency, 500 days lookback window on the empirical correlation matrix.

Further extensions/use cases

- two-stage approach:
  - Stage I: construct an index for each cluster (with weights of assets in each cluster inversely proportional to the distance to the cluster centroid)
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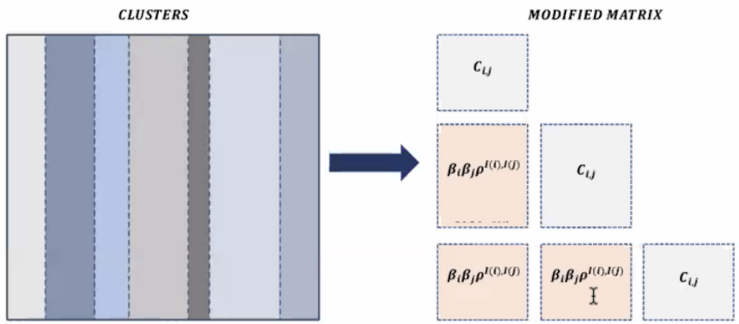

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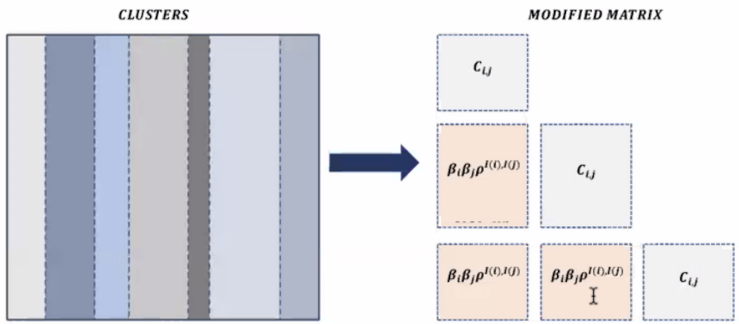

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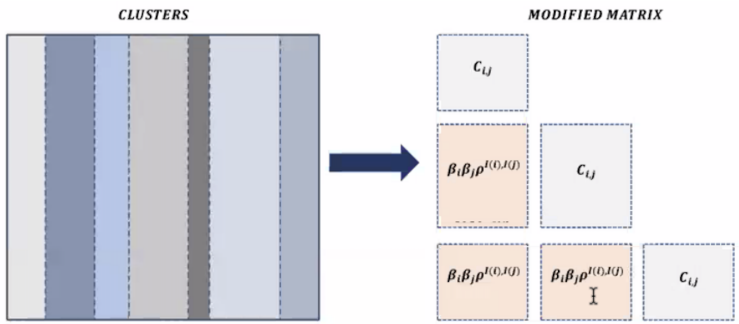

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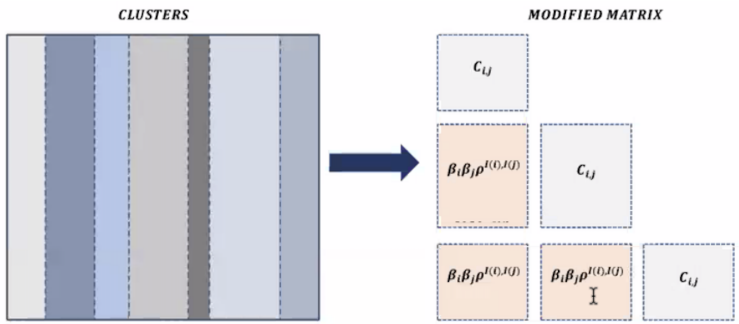

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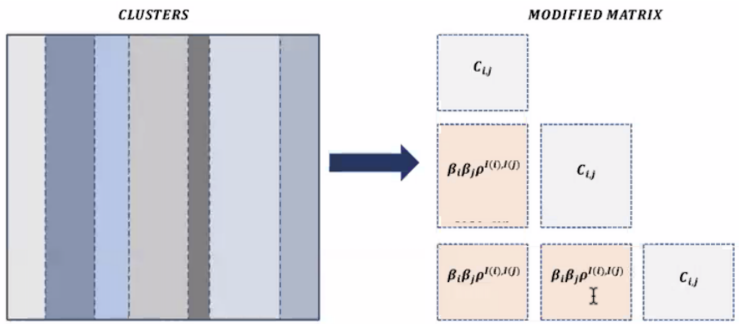

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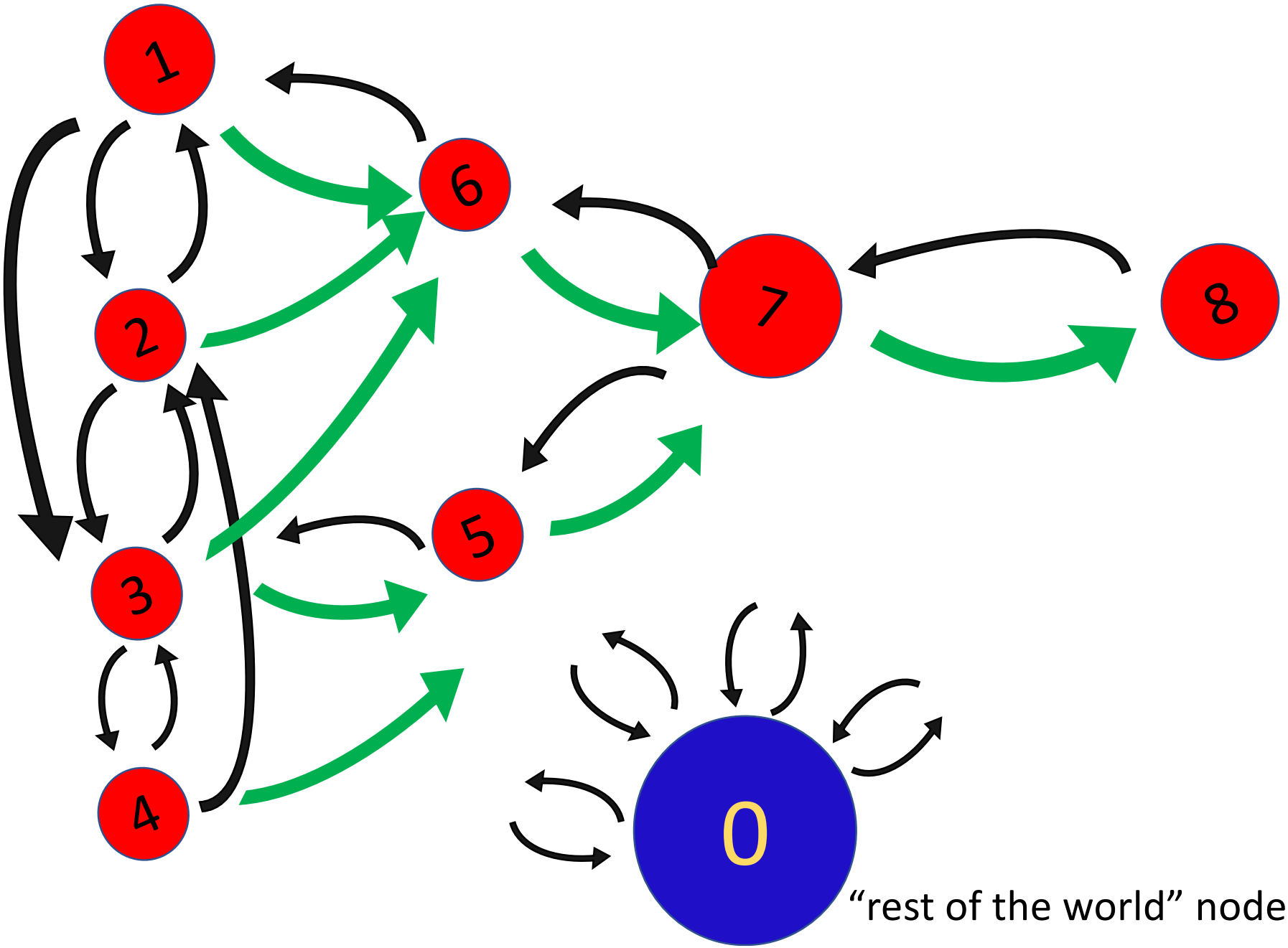

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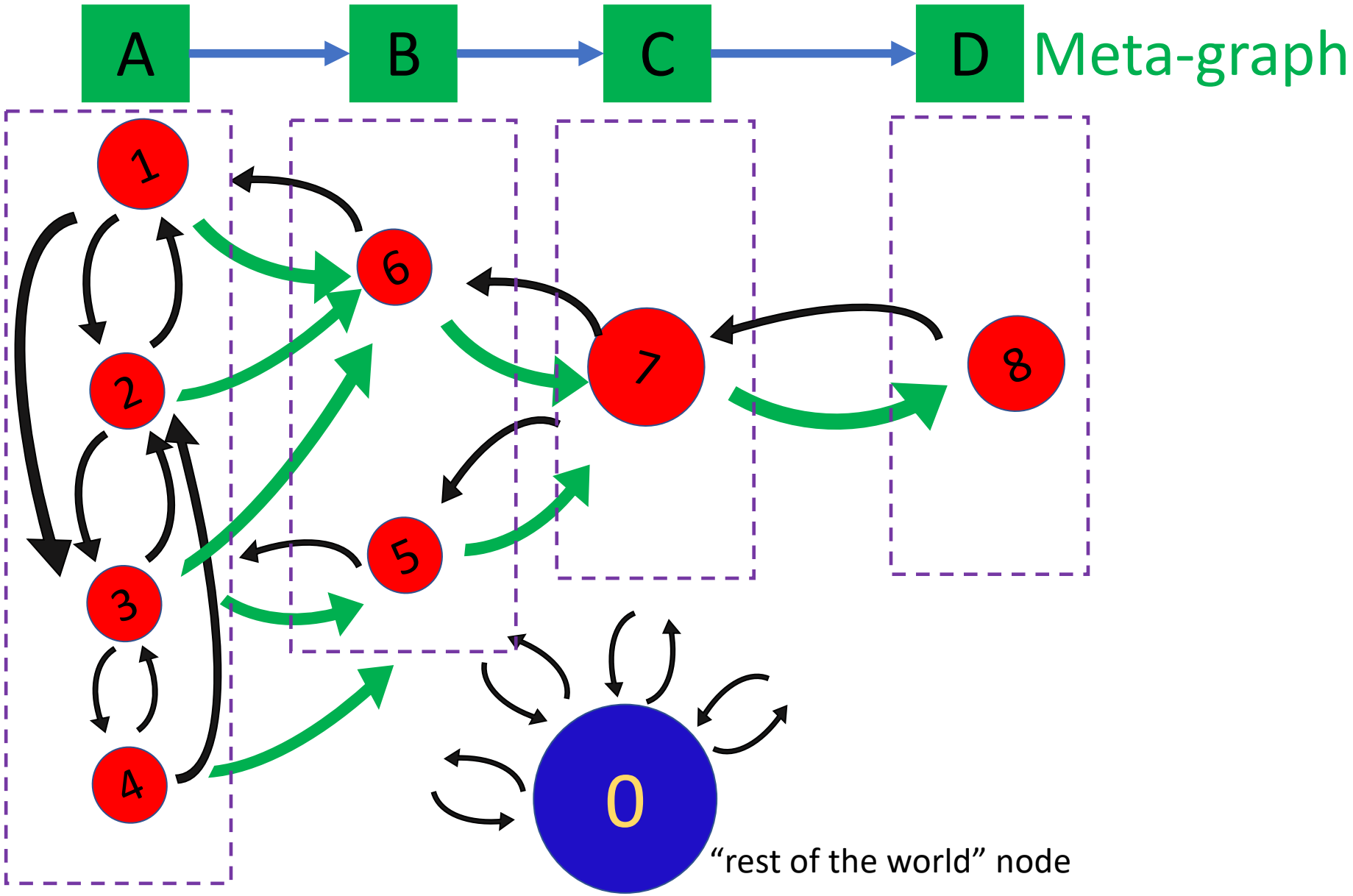
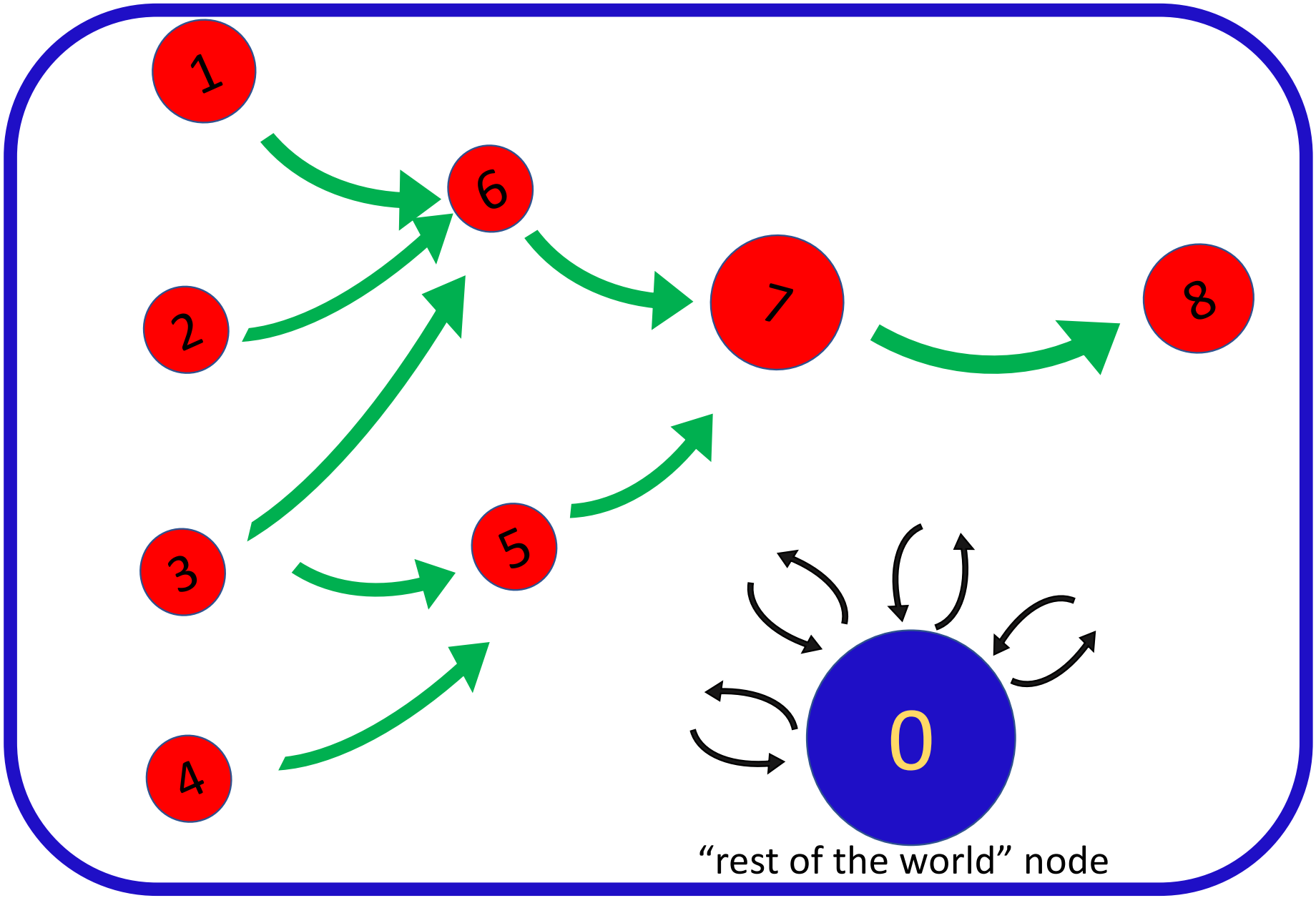
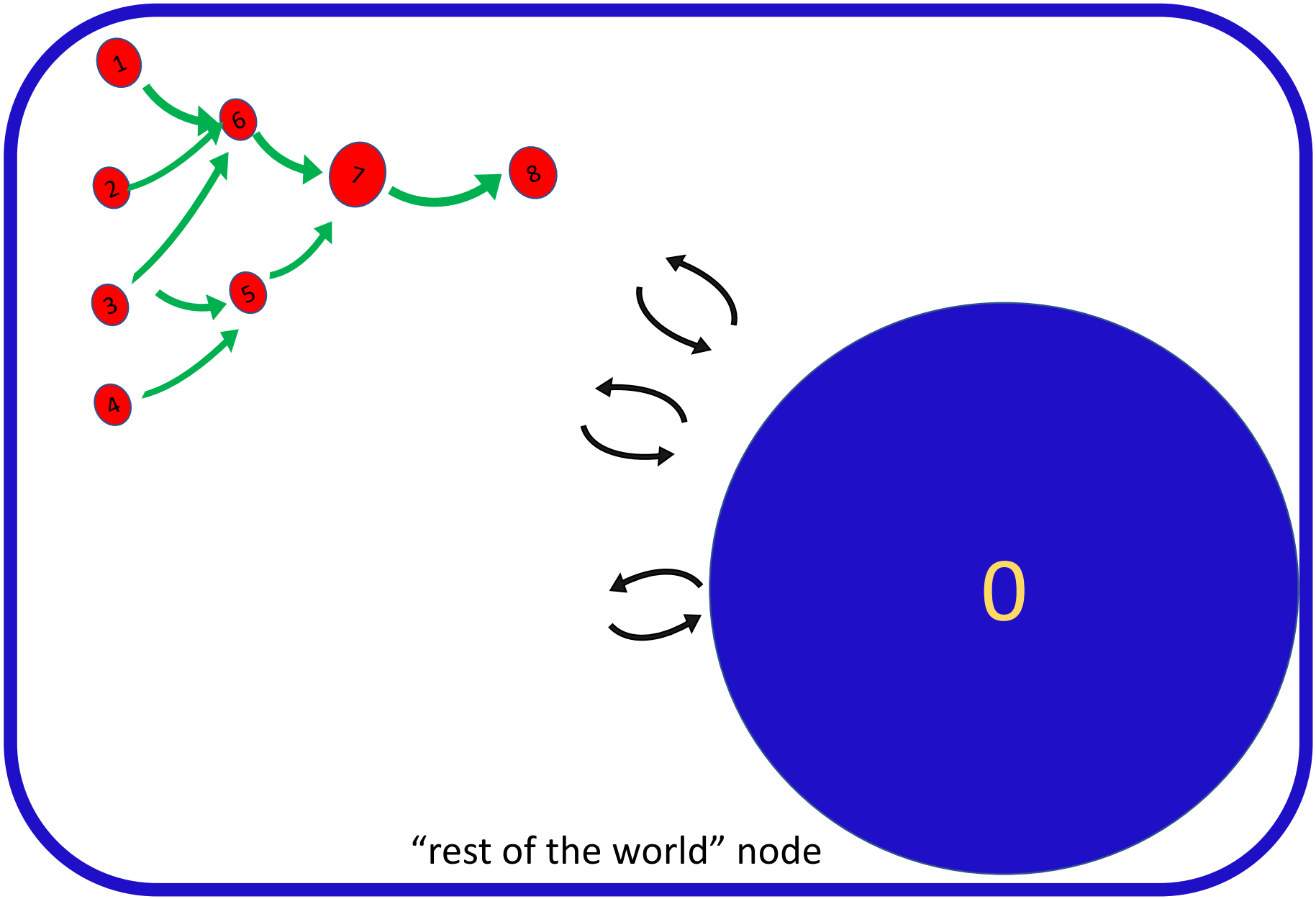


Transactions in a financial network (ii)



Transactions in a financial network (iii)



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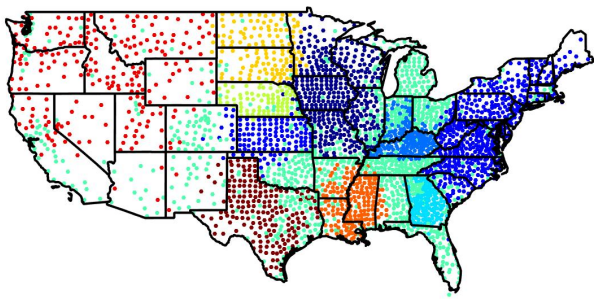

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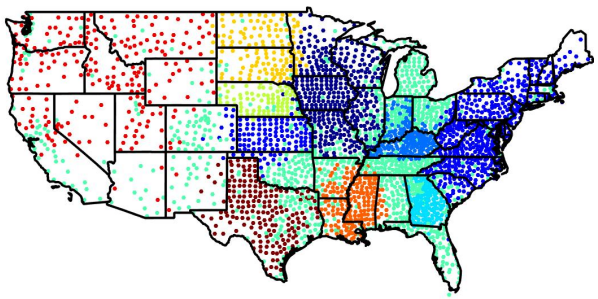

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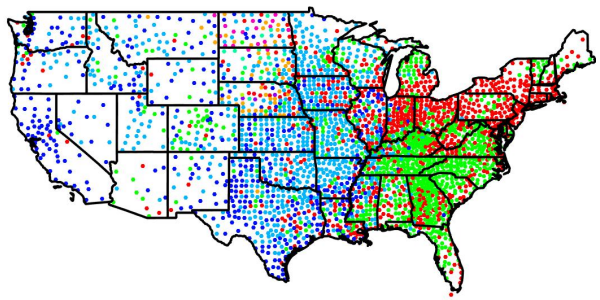

(q) HERMITIAN CLUSTERING

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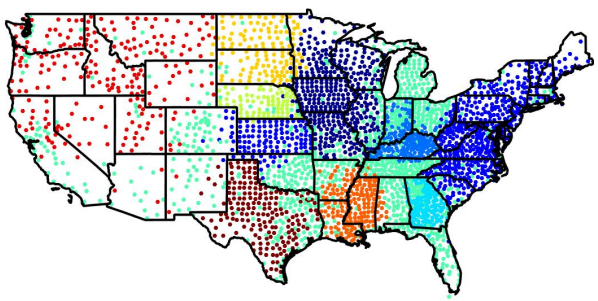

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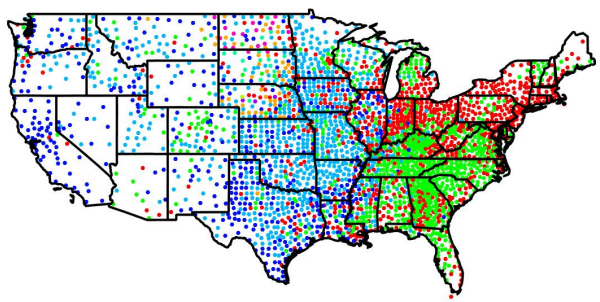

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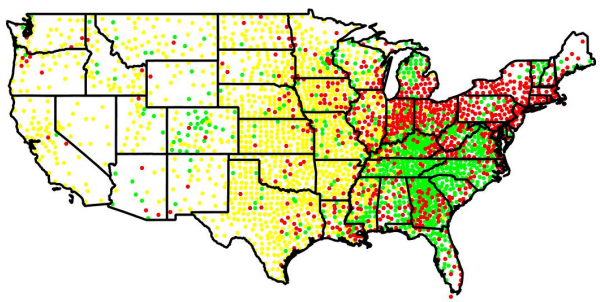

(u) HERMITIAN CLUSTERING: TOP PAIR

A new algorithm for clustering directed graphs

- complex-valued Hermitian adjacency matrix $A \in \mathbb{C}^{N \times N}$ of G

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- the Hermitian clustering algorithm uncovers the ”higher-order" structure between the clusters

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Random graphs from the DSBM with parameters:

- $k \geq 2$ represents the number of clusters/communities $C_1, C_2, \ldots, C_k$

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Example - directed stochastic block model (DSBM)

Matrix F - understood as the adjacency matrix of a weighted directed graph which represents the ***meta-graph*** describing relations between the clusters

$$F = \begin{pmatrix} 0.50 & 0.25 & 0.75 \\ 0.75 & 0.50 & 0.25 \\ 0.25 & 0.75 & 0.50 \end{pmatrix}$$

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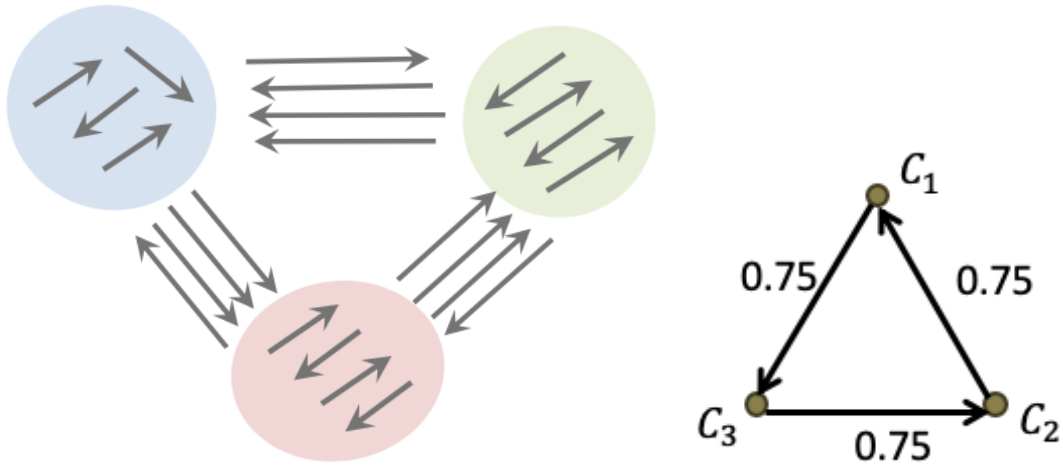

Figure: Circular flow.

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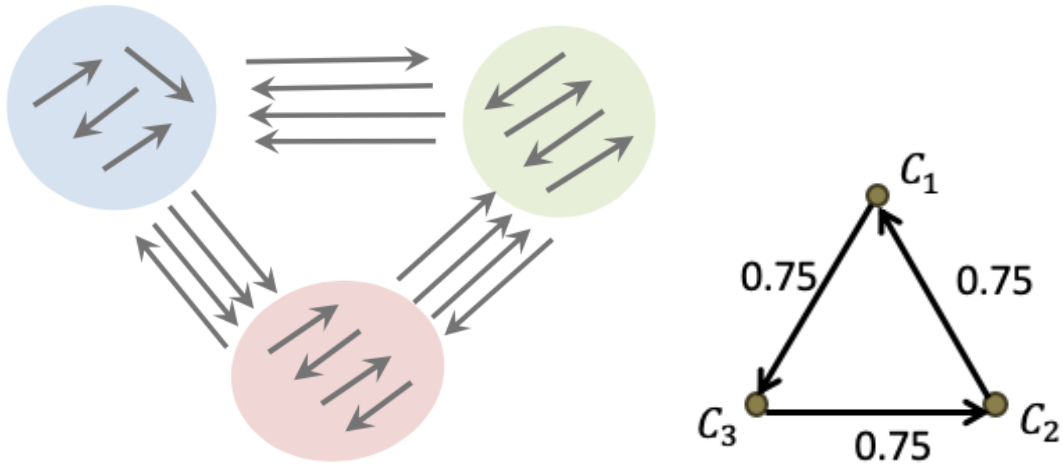


Figure: Circular flow.

- let $k = 3$, $n_1 = n_2 = n_3$, $p = q = 0.2$, and
- G consists of 3 clusters C_1, C_2 and C_3 of same size; any pair of vertices is connected by an edge with the same probability p
- directions of edges inside a cluster are chosen uniformly at random
- directions of the edges crossing different clusters are chosen non-uniformly and are defined by F.
- in expectation, all the vertices in G have the same in- and out-degrees.

Algorithm 5 Spectral clustering for a digraph

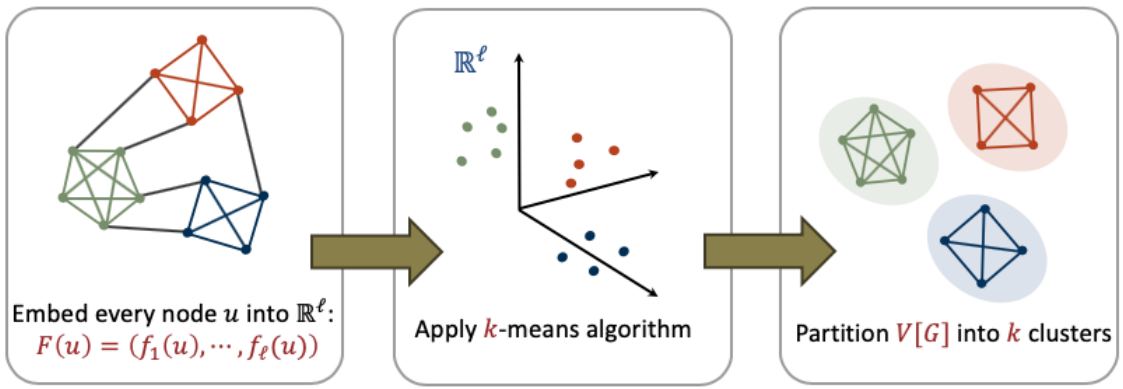
INPUT: A directed graph $G = (V, E)$, $k \geq 2$

1. Build Hermitian adjacency matrix A

$$A_{u,v} = \begin{cases} i & ; \text{ if } u \mapsto v \\ -i & ; \text{ if } v \mapsto u \\ 0 & ; \text{ if } \textit{otherwise} \end{cases} . \tag{19}$$

2. Consider the normalized Hermitian Laplacian matrix
3. Compute its top eigenvalues/eigenvectors pairs $\{(\lambda_1, f_1), \ldots, (\lambda_\ell, f_\ell)\}$
4. Apply a k-means algorithm to the resulting eigen-embedding

For general directed graphs with initial adj. matrix M set $A = (M - M^T) \cdot i$



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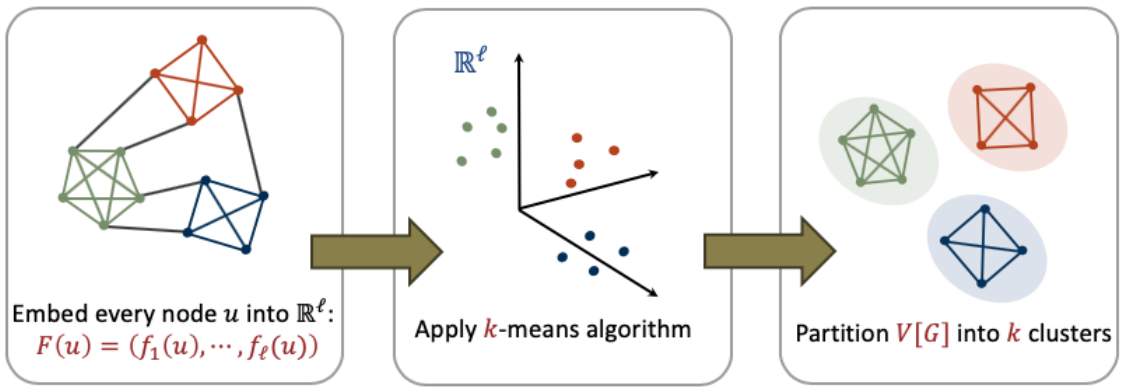

Thm. Let $G \sim \mathcal{G}(k, n, p = q, F)$. If $\tilde{\rho} \geq C\,(k/\theta)\sqrt{(1/pn)\,\log n}$ then whp the number of misclassified vertices is $O\left(k^2/(\tilde{\rho}^2\,\theta^2\,p)\,\log n\right)$.

- Previous spectral methods count the number of common parents or children or both:

$$(M^\top M)_{uv} = |\{w : w \rightsquigarrow u \text{ and } w \rightsquigarrow v\}|, \tag{20}$$

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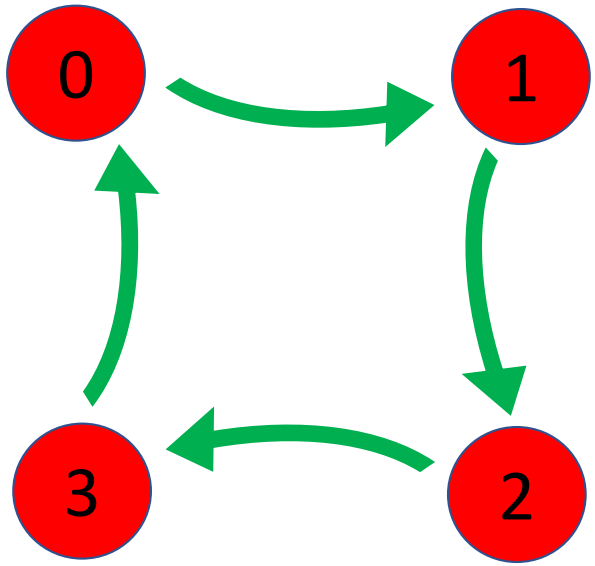
Meta-graph structures on $k = 4$ nodes



(a) Clean cycle meta-graph

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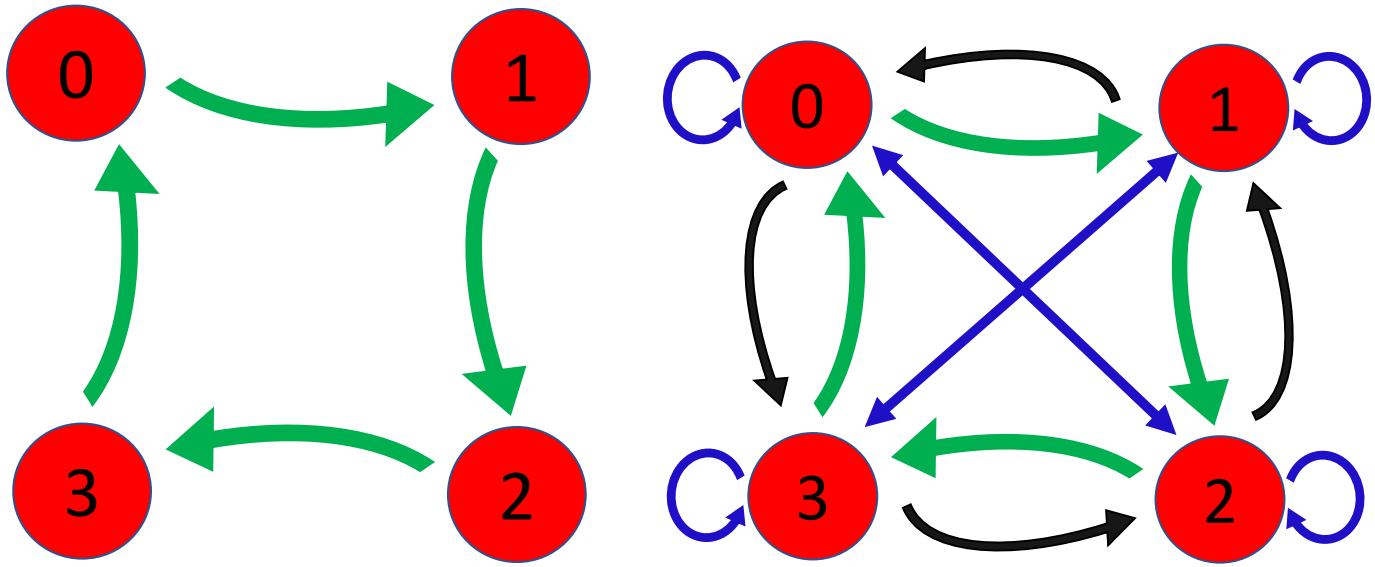

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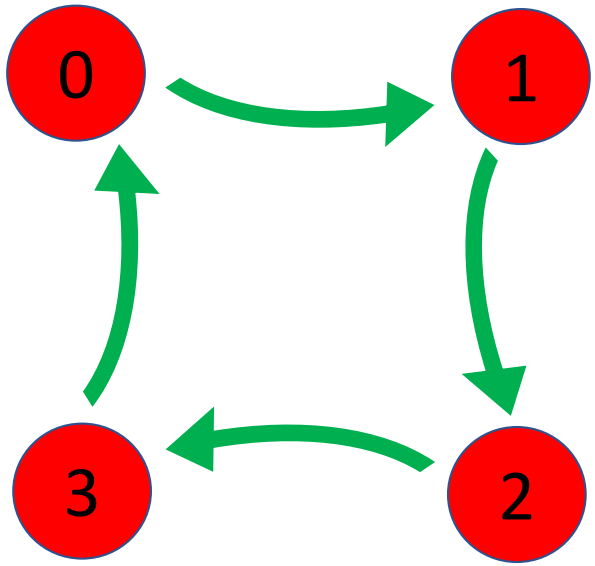

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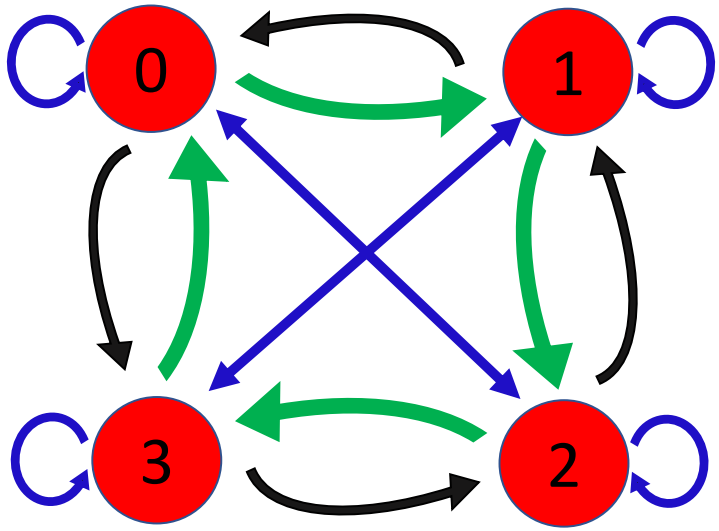

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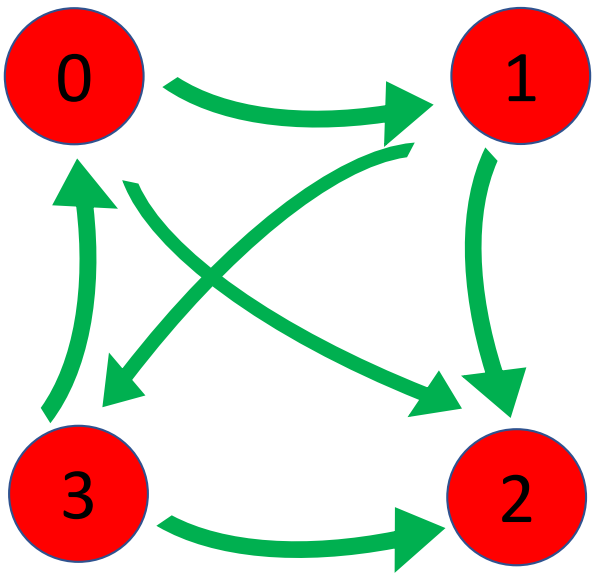

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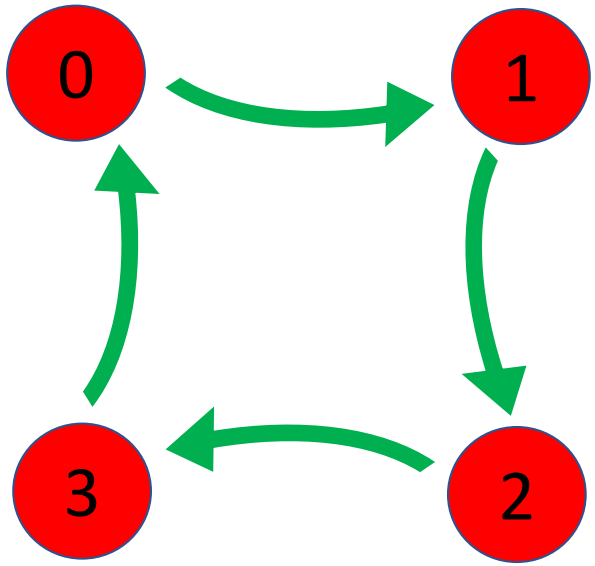

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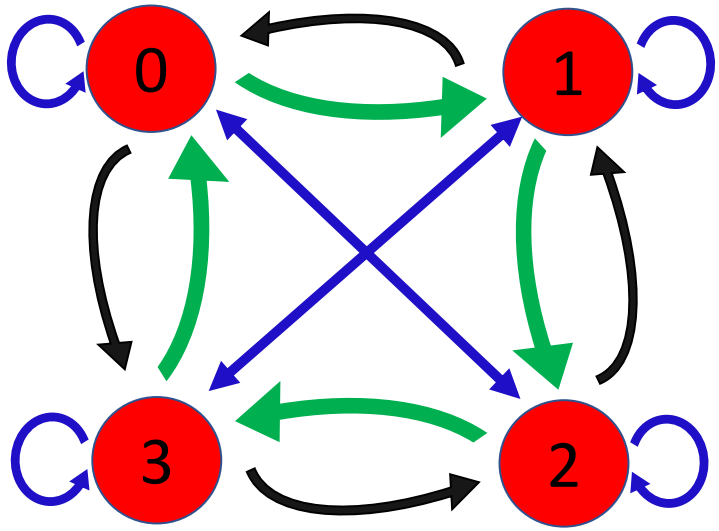

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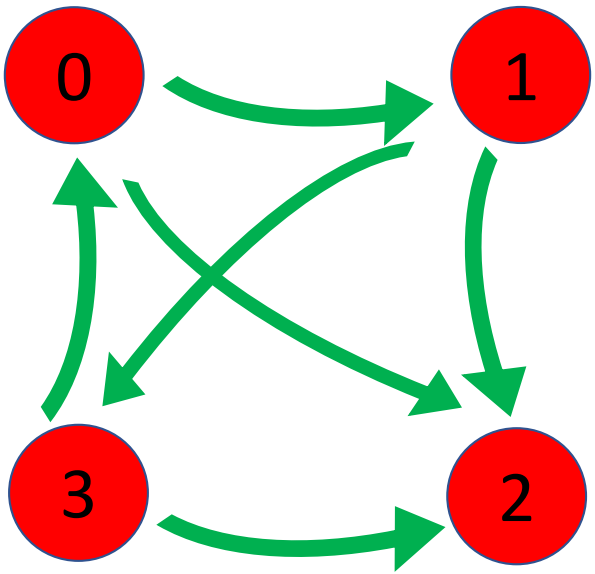

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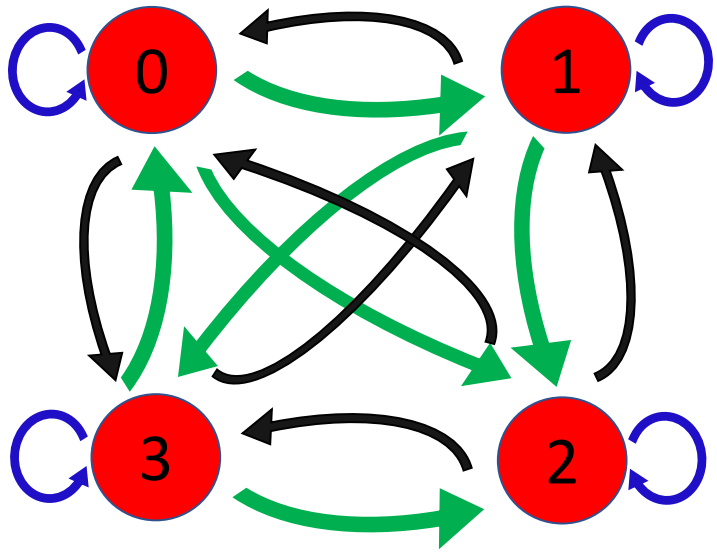

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Normalization of A_G

- D the diagonal matrix with $D_{jj} = \sum_{\ell=1}^{N} |A_{j\ell}|$

$$\text{HERM-RW}: \quad A_{\text{rw}} = D^{-1} A, \tag{23}$$

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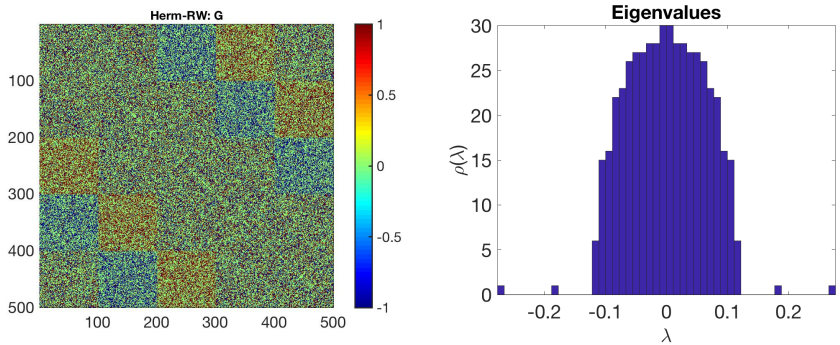

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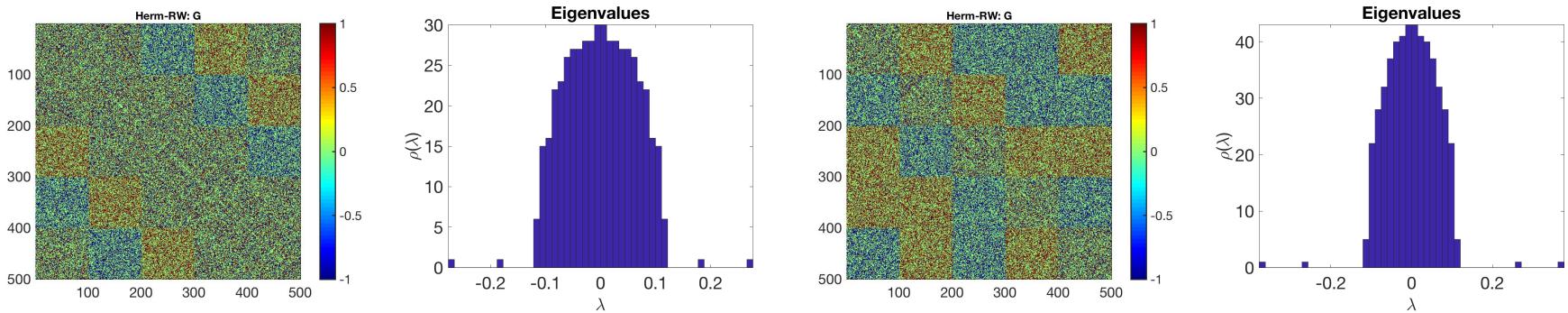
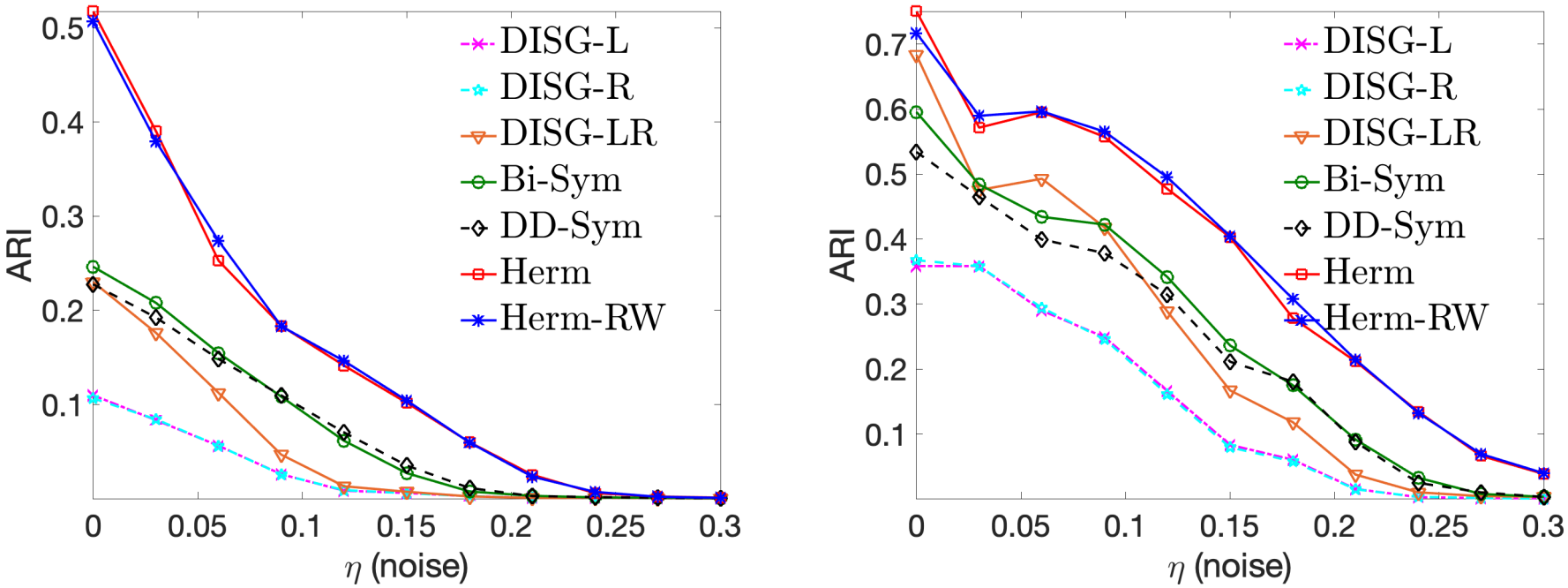


(a) Cycle meta-graph: G + Spectrum (b) Complete meta-graph: G + Spectrum

Figure: Adjacency matrices and spectra of A_{rw}.

Comparison with state-of-the-art



(a) Circular pattern (b) Complete meta-graph

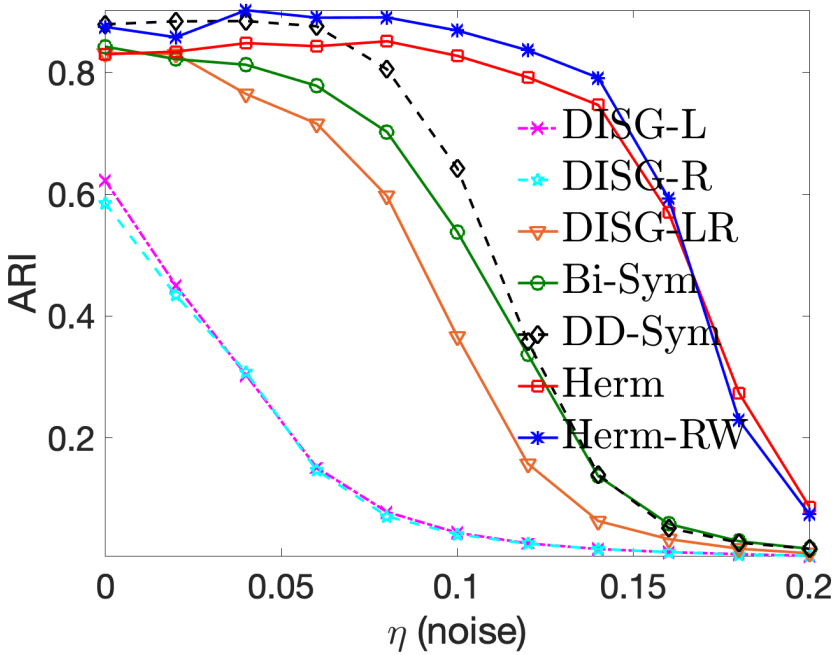
Figure: Recovery rates for the DSBM with $k = 5$, $N = 5000$, at sparsity $p = 0.5\%$. Averaged over 10 runs.

- **DISG**: K. Rohe, T. Qin, and B. Yu. *Co-clustering directed graphs to discover asymmetries and directional communities*. Proceedings of the National Academy of Sciences (2016); regularized graph Laplacian

$$L_{ij} = \frac{A_{ij}}{O^{\tau}_{ii} P^{\tau}_{jj}} = \left[(O^{\tau})^{-1/2} A (P^{\tau})^{-1/2} \right]_{ij}$$

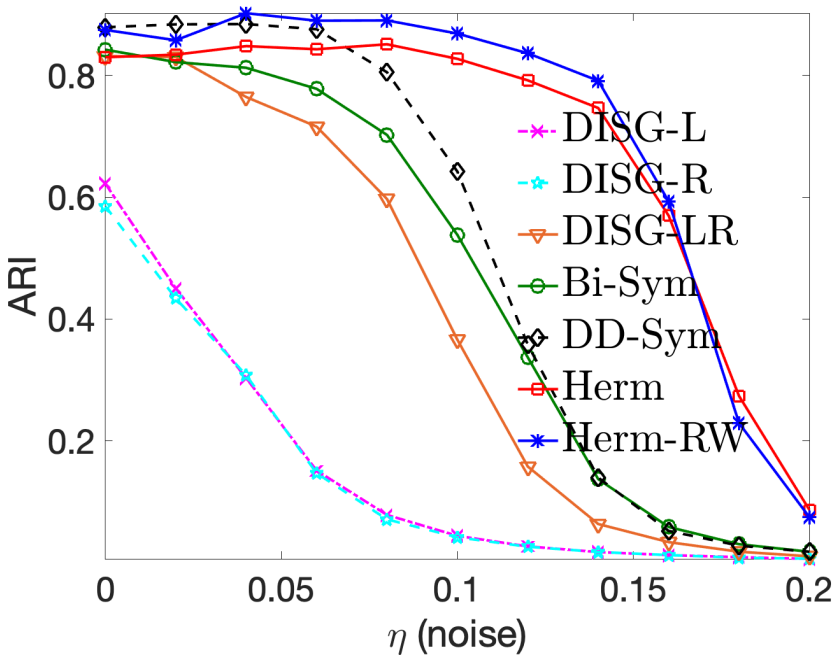
- **{Bi,DD}-Sym** V. Satuluri and S. Parthasarathy. *Symmetrizations for Clustering Directed Graphs*. In Proc. of the 14th ICEDT (2011)

Experiments - large k



(a) $p = 0.02$

Experiments - large k



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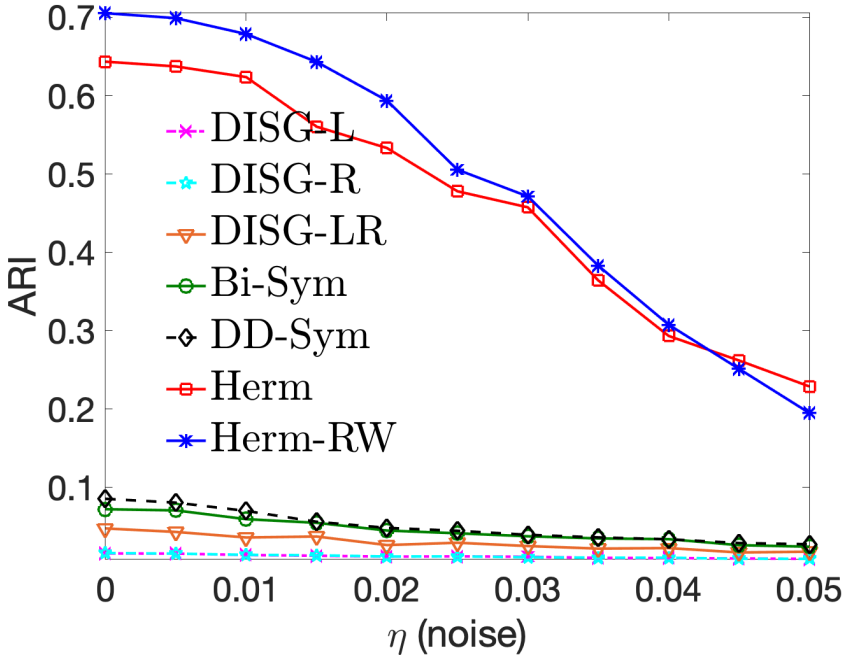
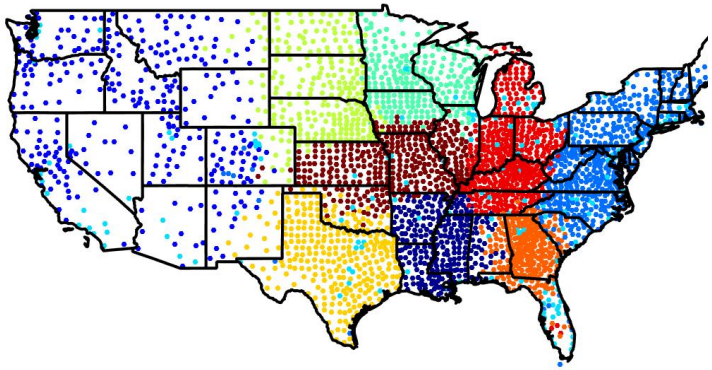


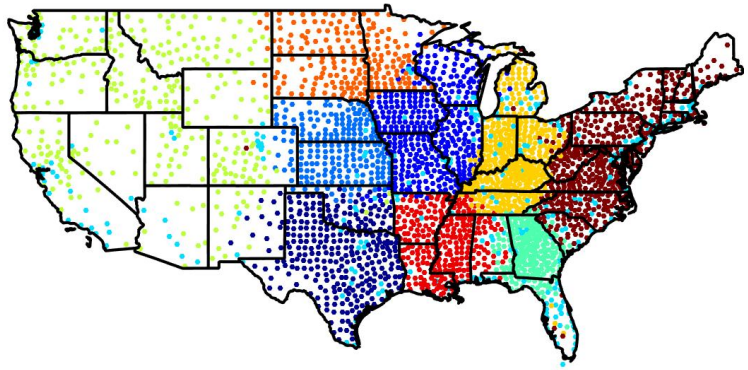
(b) $p = 0.01$

Figure: Recovery rates for the complete meta-graph in the DSBM with $k = 50$, $N = 5000$, two sparsity values p. Averaged over 10 runs.

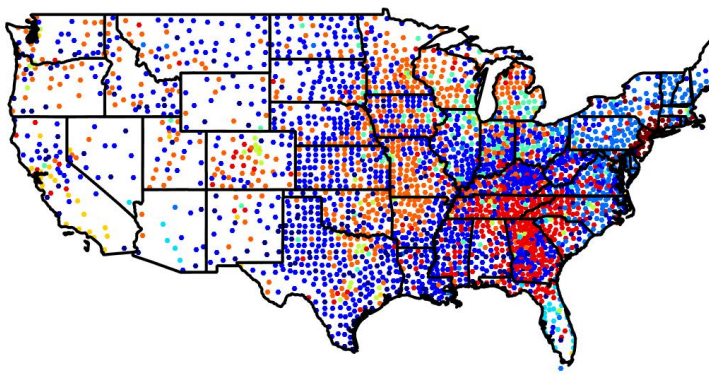
Clustering the US Migration Network



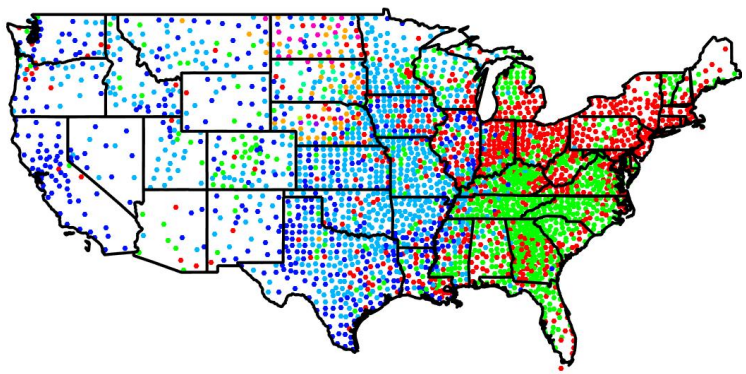
(a) DISGLR



(b) DD-SYM



(c) HERM



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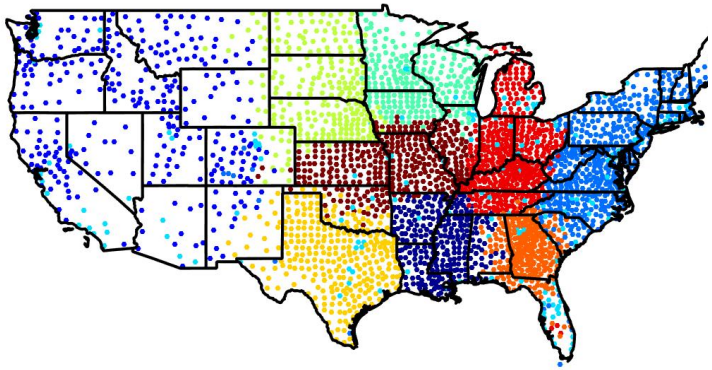

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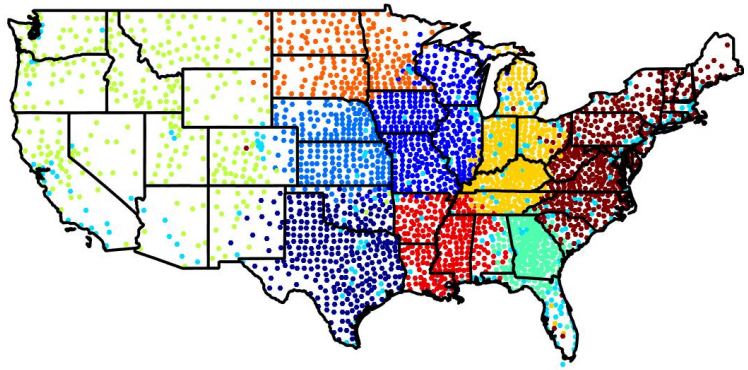

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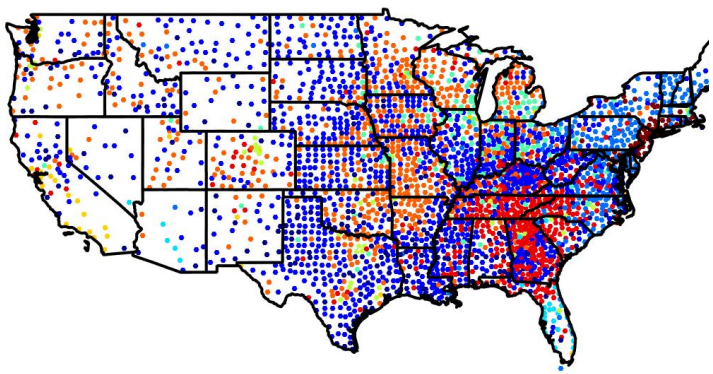

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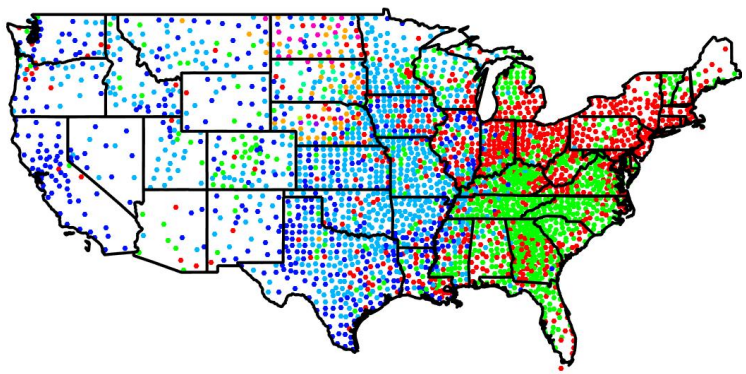

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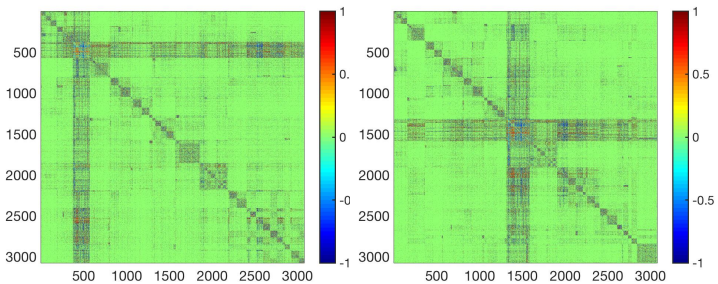

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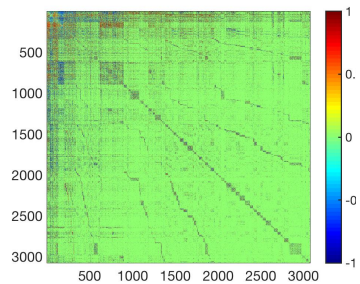

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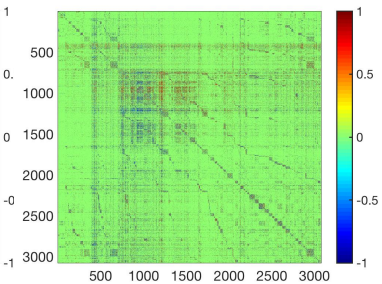

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Cut Imbalance Ratio

Given a pair of clusters (X, Y), the Cut Imbalance ratio CI is defined by

$$\mathsf{CI}(X, Y) = \frac{w(X, Y)}{w(X, Y) + w(Y, X)}, \tag{25}$$

- $w(X, Y) = \sum_{j \in X, \ell \in Y} w(j, \ell)$ denotes the total weight of all edges flowing from X to Y
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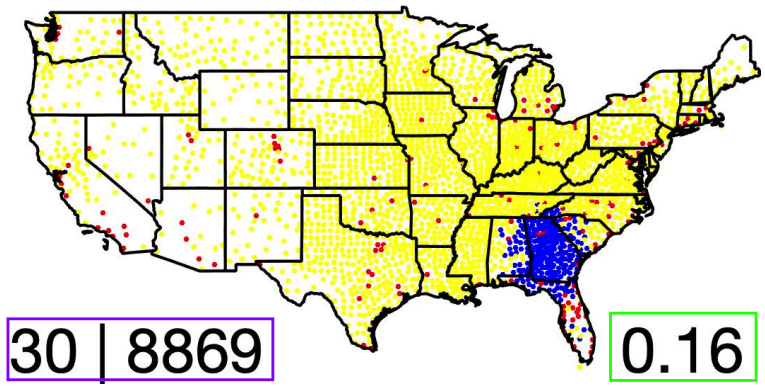
$$\mathrm{TopCI}^{\mathrm{vol}} = \sum_{t=1}^{M} \mathrm{CI}^{\mathrm{vol}}(C_{j_t}, C_{\ell_t}) \tag{28}$$

where (C_{j_t}, C_{ℓ_t}) denotes the t-th largest $\mathrm{CI}^{\mathrm{vol}}$ cut imbalance pair.

US Migration - top largest size-normalized cut imbalance pair



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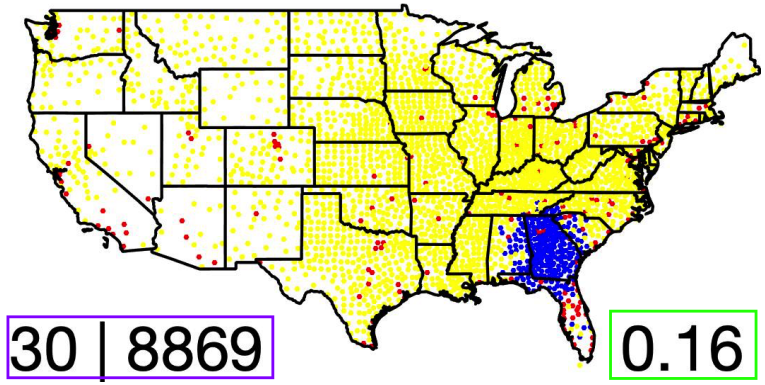

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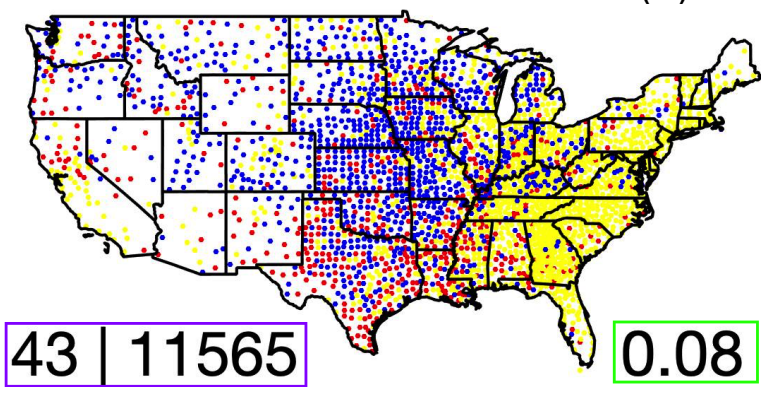

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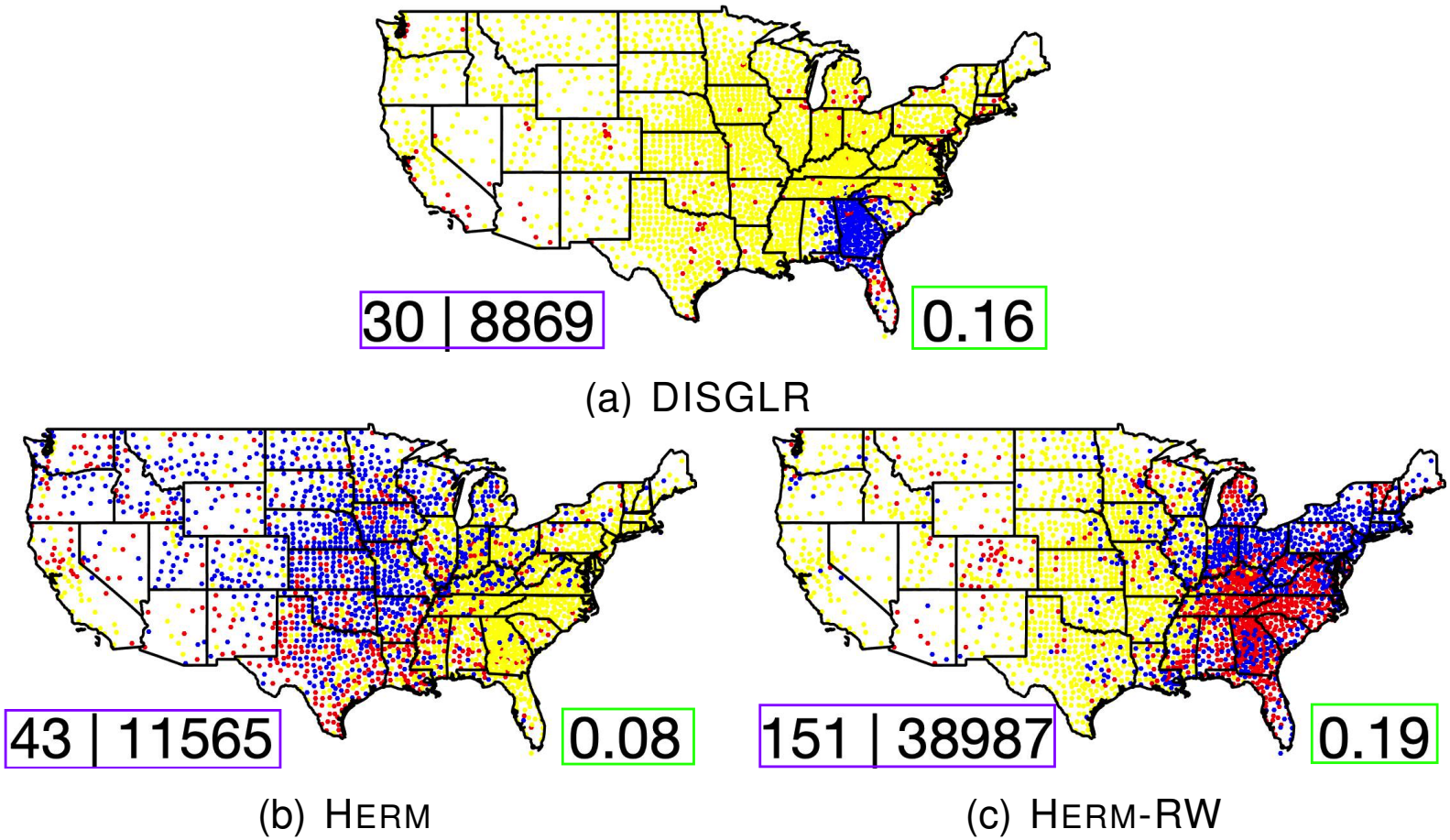

(a) DISGLR

(b) HERM

(c) HERM-RW

Figure: The top four largest size-normalized cut imbalance pairs for the US-MIGRATION-I data with $k = 10$ clusters. Red denotes the source cluster, and blue denotes the destination cluster. For each plot, the bottom left text contains the numerical values (rounded to nearest integer) of the normalized CI^{size} and CI^{vol} pairwise cut imbalance values (the higher the better), and the bottom right text contains the CI cut imbalance value in $[0, 1]$ (the farther from 0.5 the better).

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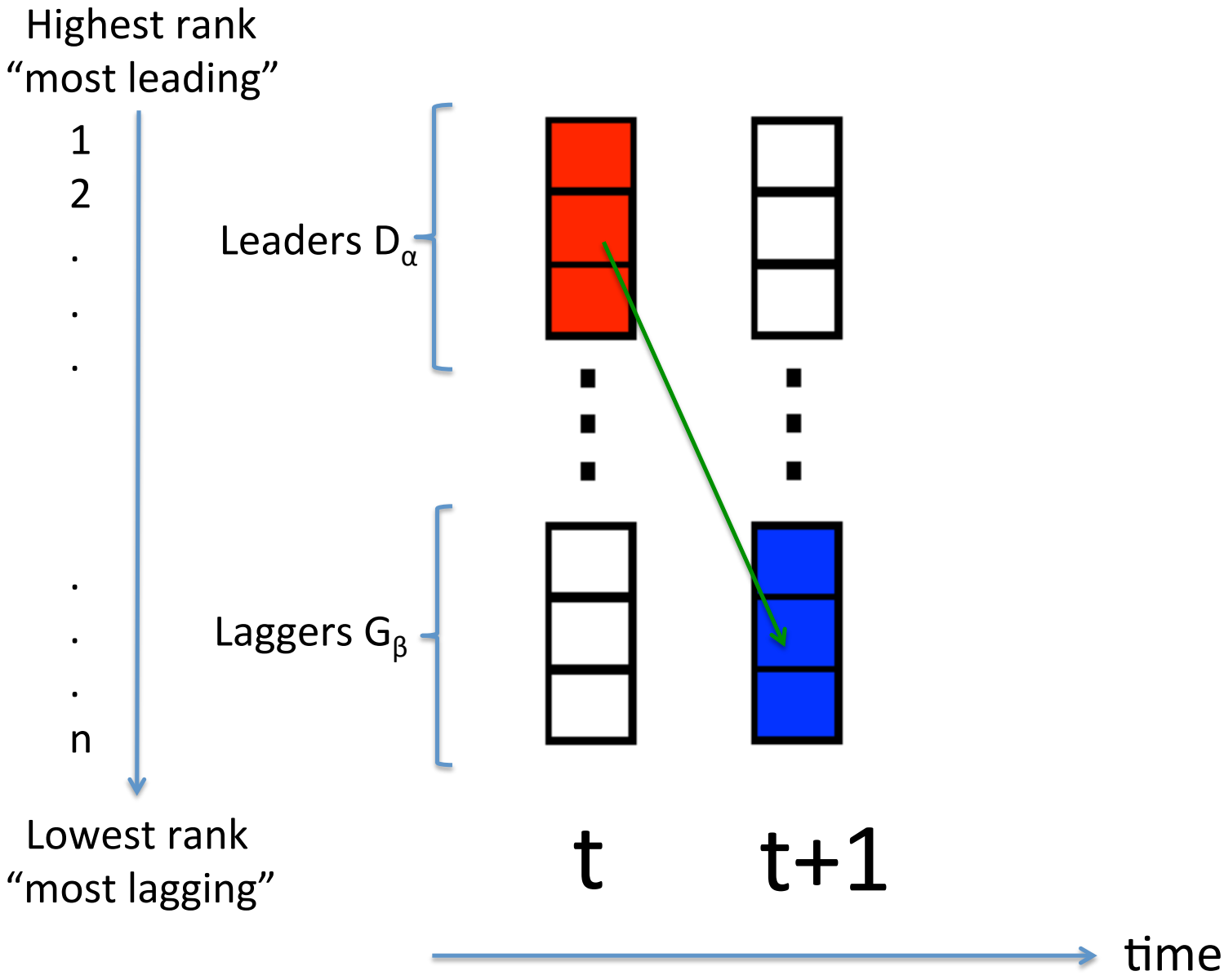
  - compute rankings, predict the laggers catching up
  - cluster the network, and infer leading and lagging clusters, and leverage for prediction task


Highest rank
“most leading”
1
2
.
.
.
Leaders D_α
.
.
.
n
Laggers G_β
Lowest rank
“most lagging”
t
t+1
time

Why lead-lag relationships?

- Stock returns are difficult to predict due to low signal to noise ratio
- Changes in stock prices of some firms tend to follow that of other firms, and this relationship between stock prices is often referred to as *lead-lag* relationships
- Detecting lead-lag relationships is not straightforward

It would be useful if one could:

1. Detect lead-lag relationships
2. Use lead-lag relationships to predict stock returns
3. Employ these relationships to build portfolios

Outline of our lead-lag project

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5. We find that the portfolios based on lead-lag relationships are sensitive to rebalancing frequencies. Portfolios rebalanced once a day consistently outperform the bidaily, weekly, bi-weekly, tri-weekly, and monthly rebalanced portfolios

Lead-lag literature in finance

Lead-lag relationships are intensively studied in the literature of return predictability

- Lo and MacKinlay (1990): Firms with large market cap lead firms with small market cap
- Badrinath et al. (1995): Firms with high institutional ownership lead firms with low institutional ownership
- Chordia and Swaminathan (2000): High volume/turnover stocks lead low volume/turnover stocks
- Hou (2007): large cap stocks lead small cap stocks in each industry
- Brennan et al. (2015): High investment coverage stocks lead low coverage stocks
- Parsons et al. (2020): stocks who are not in the same sector but have the same headquarter show lead-lag relationships, with large cap ones leading small cap ones
- Huang et al. (2022): stocks with frequent, gradual price updates lead those with infrequent, dramatic price updates.

Related literature – source of lead-lag

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- some assets receive information faster, and others take longer to receive information and to incorporate the price change
- if the slow information diffusion hypothesis holds, it is natural to
  - extend beyond pre-determined leader-follower identities
  - use data-driven methods to better understand dynamics of information flow in financial markets.

Pairwise lead-lag detection methodology

- finance & econometrics literature: typical methods in causality relationship detection rely on the Granger causality test (Granger 1969)
  - linear regression test that assesses if the inclusion of the lagged values of one series in a linear regression model significantly improves the fit and predictive power of the model compared to the original model.
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  - we use cross-correlation as one of the methods to detect lead-lag as well.
- we also introduce a method based on signatures to handle nonlinearity in the data.
- just a means to an end in terms of detecting higher-order structure

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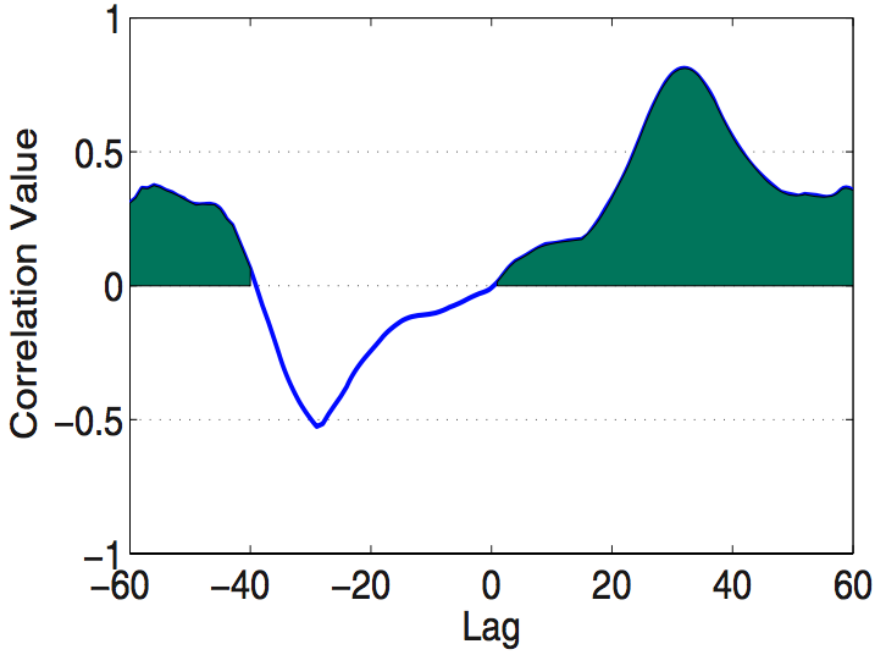
Cross-correlations and the lead-lag matrix



Wu et al, 2010

Options for building the $n \times n$ pairwise comparison matrix:

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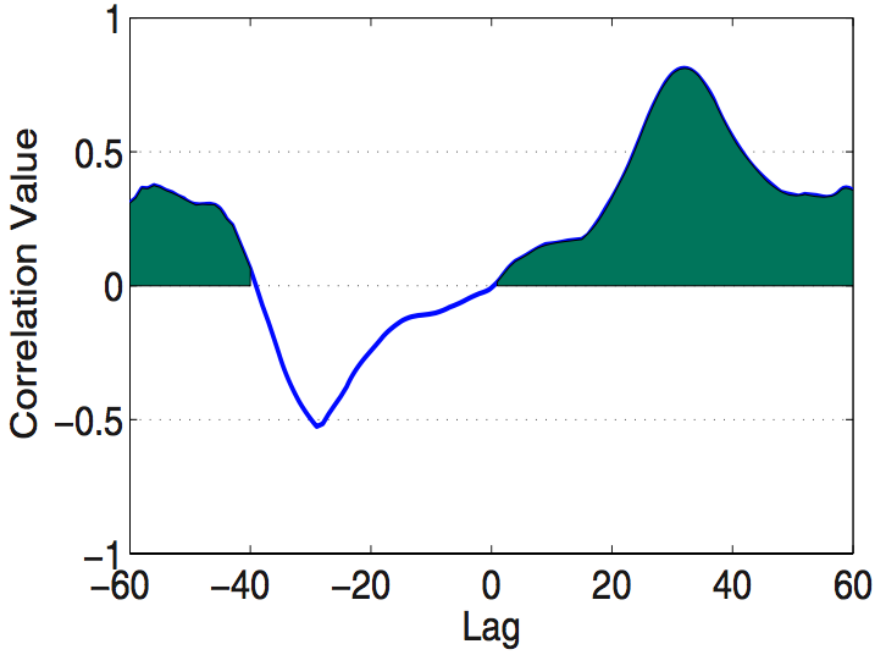

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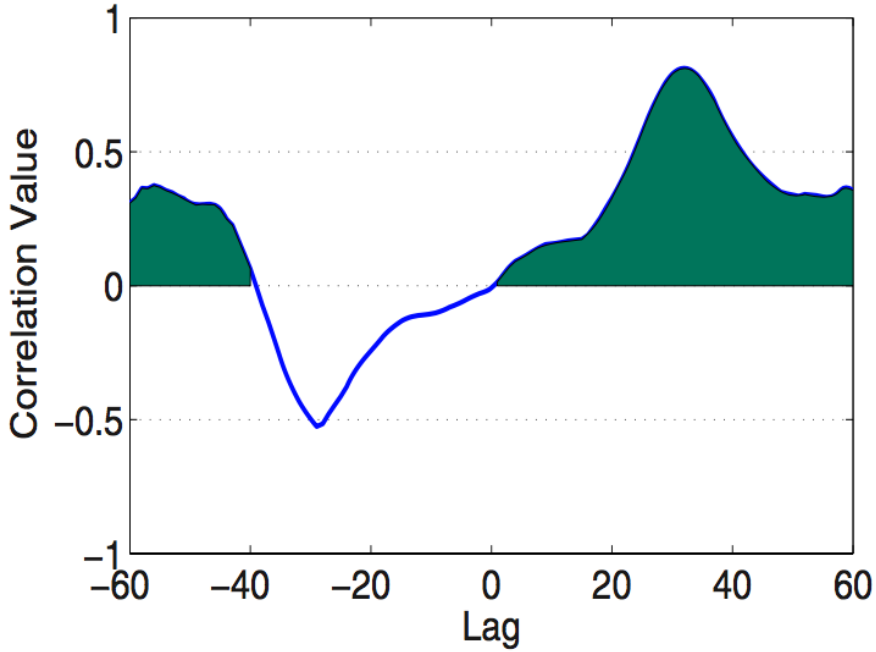

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3. C_{ij}: second-order signatures of the two time series

$$C_{ij}(t-m,t) = \iint_{t-m<u<v<t} dX_i(u)\, dX_j(v) - dX_j(u)\, dX_i(v)$$

Lead-lag detection methodology/framework

In a simple linear model, consider the regression

$$r_{i,t} = \beta_\ell r_{j,t-\ell} + \epsilon_t \tag{29}$$

i, j represent different assets from a universe of n assets, and t indexes time.

- for each pair of assets (i, j), identify one of the two as the leader and the other as the follower
- ideally, we want to design an algorithm such that by assigning a score C_{ij} to each asset pair (i, j), the score satisfies two conditions:
  1. the sign of the score determines the direction of the lead-lag relationship (i.e. the sign of ℓ)
  2. the magnitude of the score determines the strength of the relationship. (i.e. proportional to β_ℓ)
- yields an $n \times n$ matrix denoted as the lead-lag matrix C (comparison matrix), on which we can perform ranking/clustering algorithms.

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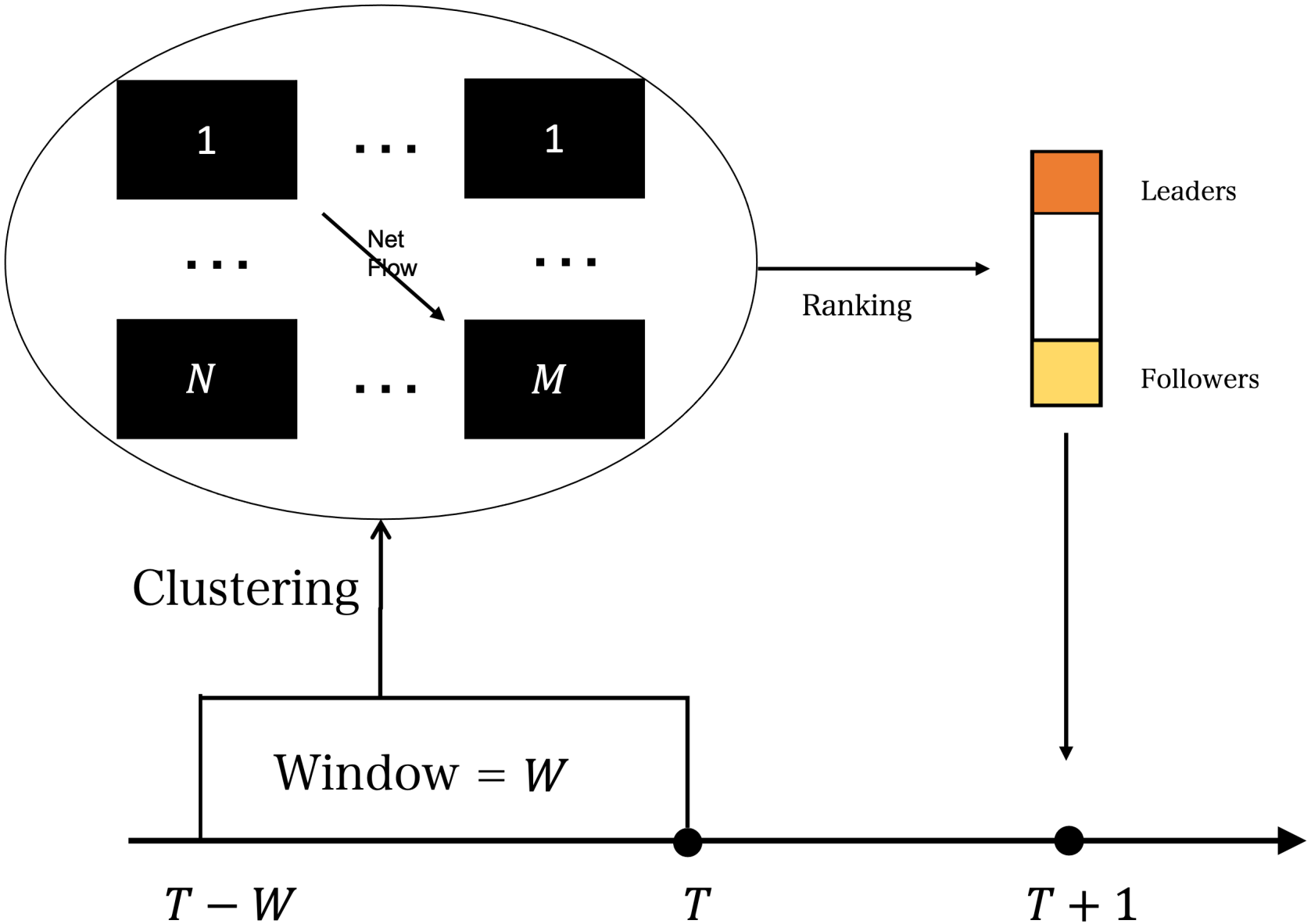
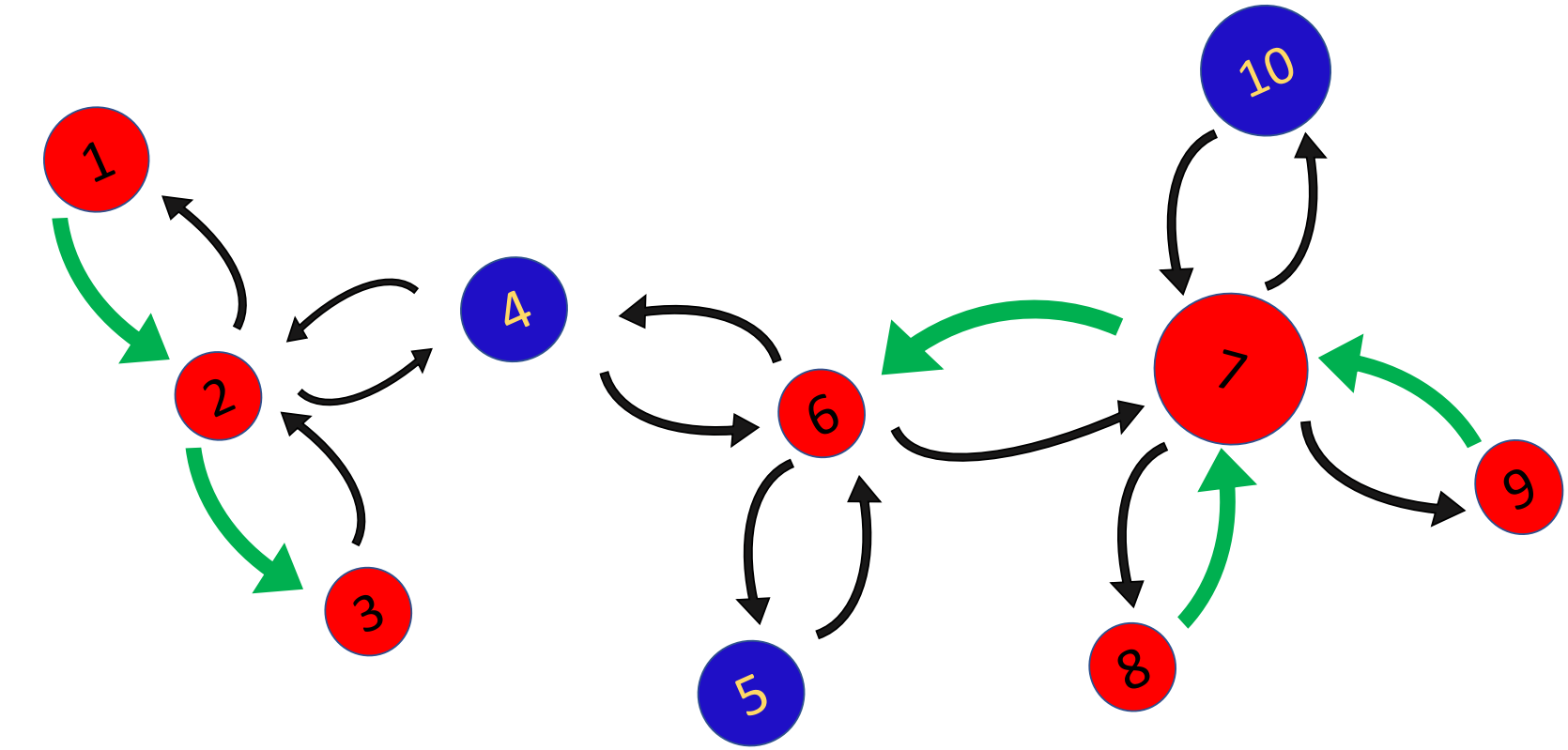
MetaClusters diagram



Uncover underlying directed meta-graph structure



- Bennett, C. and Reinert, *"Detection and clustering of lead-lag networks for multivariate time series with an application to financial markets"*, Machine Learning (2022)

Data description

- universe of 5325 NYSE equities spanning from 04-01-2000 to 31-12-2019
- Wharton's CRSP database – restricting our attention to equities trading on the same exchange to avoid spurious lead-lag effects due to non-synchronous trading
- daily closing prices from which we compute daily log-returns
- subset to symbols with at least 2.5 years' worth of non-missing data
- subset the largest 500 equities in average daily traded dollar volume
- result in a data set of 434 equities
- reduces the risk of spurious lead-lag effects due to non-synchronous trading

Cluster meta-flow matrix & cluster leadingness metric

- total *net flow* between any two clusters is given by the net of the normalized weights between all edges directed from one cluster to another

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$$F_{ij} = \frac{1}{|C_i|\,|C_j|} \sum_{u \in C_i, v \in C_j} [A_{uv} - A_{vu}],$$

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- the diagonal of F consists of zeros: $F_{ii} = 0$, $\forall\, i \in 1, \ldots, k$
- define a metric for the *leadingness* of each cluster $i \in \{1, \ldots, k\}$

$$L(i) := \frac{1}{|C_i|} \sum_{u \in C_i, v \in \{1,\ldots,n\}} [A_{uv} - A_{vu}]. \tag{30}$$

Thus, $L(i)$ averages the row-sums of the skew-symmetric matrix $A - A^T$ for nodes within the cluster C_i;

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- the row-sums of the lead-lag matrix are measure of the total tendency of the equity corresponding to the row to be a leader
- obtain a ranking of the clusters from
  - the most leading cluster (largest row-sum value) (label 0)
  - to the most lagging cluster (smallest row-sum value) (label $k - 1$)

From the stock lead-lag matrix to the cluster meta-flow matrix

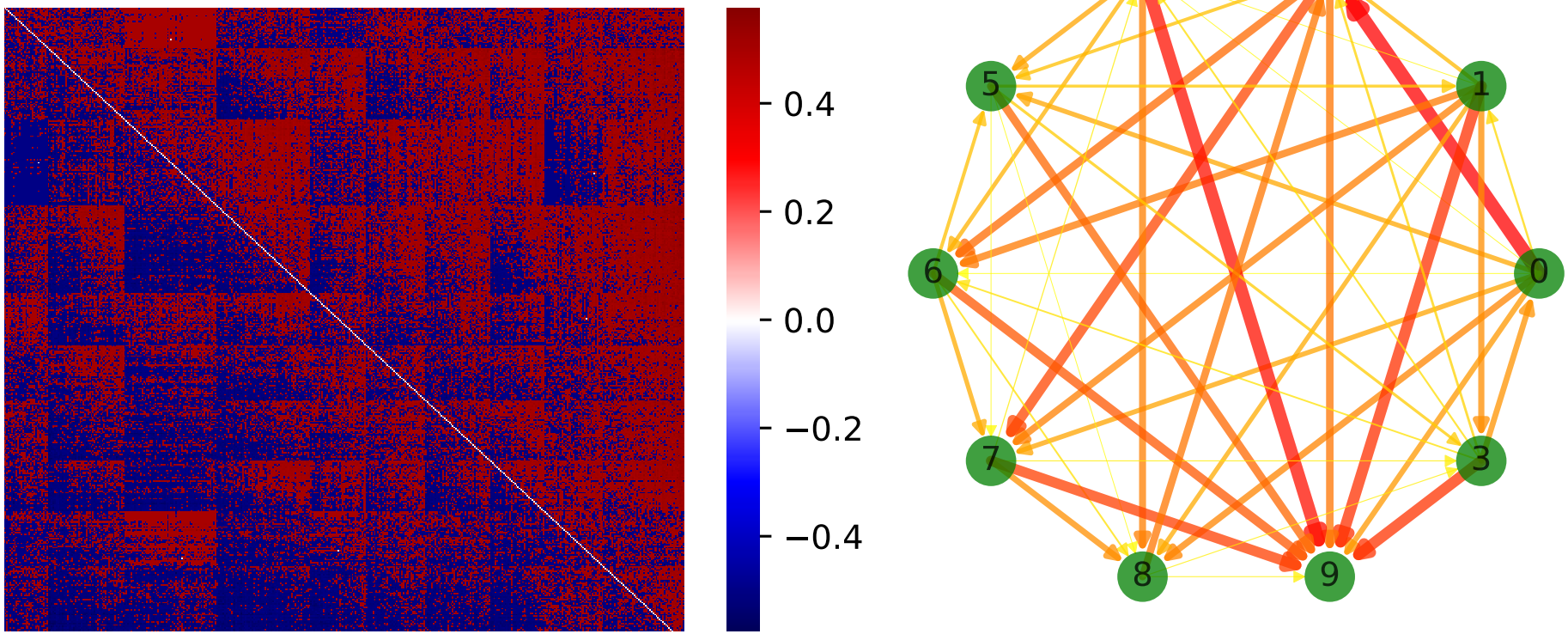

Figure: Left: Heatmap of the double-sorted lead-lag $n \times n$ matrix $A - A^T$. The rows and columns of the matrix index the $n = 434$ equities, and are categorised by cluster membership (labelled by the leadingness metric. Within each cluster, we sort the equities by their respective row-sum in $A - A^T$, a proxy for their individual leadingness. Right: Meta-flow network for Hermitian RW clusters; clusters are represented by nodes and larger edge weights are depicted by bolder colours and thicker lines. Cluster 0: most leading; Cluster 9 most lagging.

Clusters vs GICS

Retail	90
Manufacturing	67
Construction	66
Mining	58
Trans., Util. & other	54
Fin., Ins. & RE	46
Wholesale	43
Services	9
Agri., Forest. & Fish.	1

Table: Number of equities in each SIC industry sector.

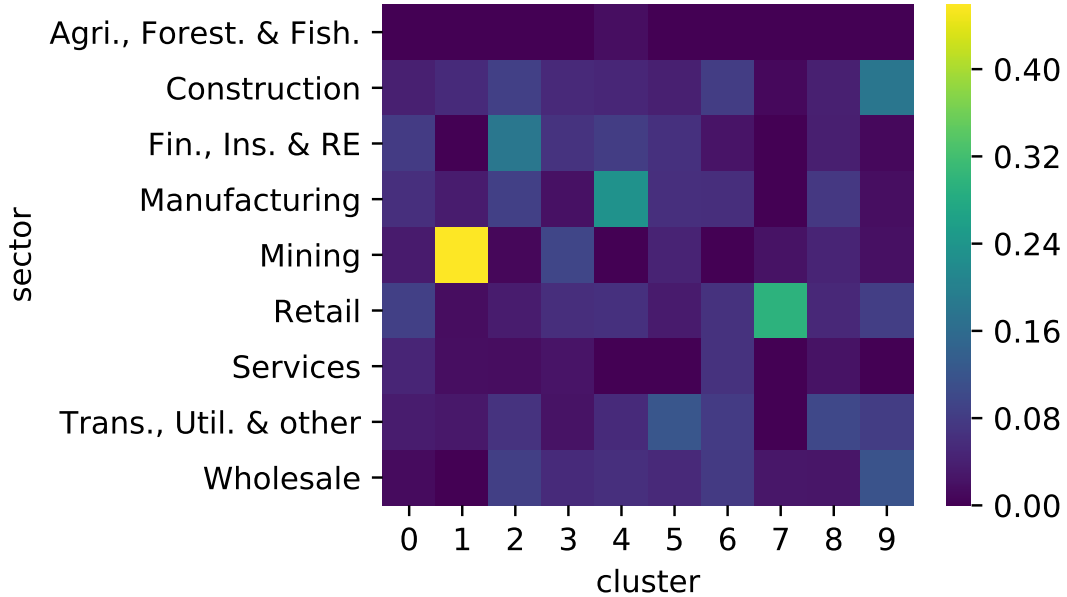

Figure: The Jaccard similarity coefficient between the Hermitian RW clusters and industry clusters. Cluster 0 is most leading; cluster 9 is most lagging.

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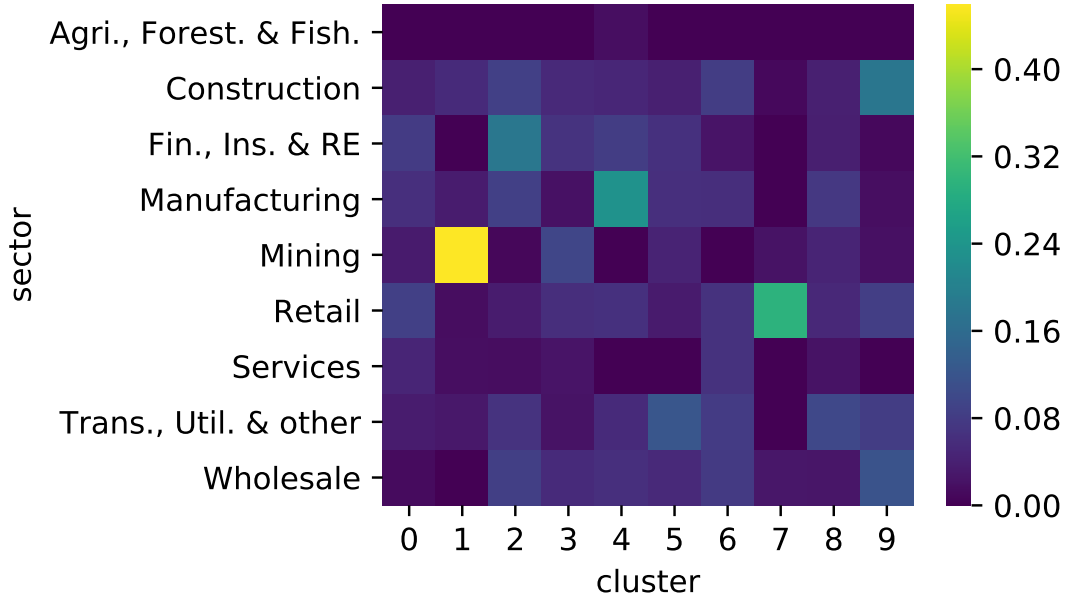

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- test for a statistically significant time dependence in daily returns using a permutation test on the spectrum of the Hermitian adj. mtx. $\tilde{A} = i(A - A^T)$
- since lead-lag cluster structure is associated with the largest eigenvalues of the $\tilde{A}$, the permutation test statistic is the largest eigenvalue of $\tilde{A}$
- reject the null hypothesis with p-value $p < 0.005$, and conclude that there is significant temporal structure in US equity markets.

Data-driven clustering with known lead-lag mechanisms

Recovered clusterings **cannot** be explained by the three previously hypothesized mechanisms in the empirical finance lead-lag literature

1. Sector membership induces clustered lead-lag effects
   - Biely and Thurner [2008] fin find associations between sector membership and lead-lag structure on the high-frequency scale of returns
2. Equities with higher trading volume are hypothesized to lead lower volume equities
   - disparities in trading volume across equities can lead to non-synchronous trading lead-lag effects
   - clustering structure may be induced by ordering equities based on quantiles of average trading volume
3. Larger capitalization equities are hypothesized to lead lower capitalization equities; this market cap mechanism can produce lead-lag effects partly via non-trading effects and partly via other channels
   - large stocks may lead small stocks via volatility spillovers.
   - clustering structure may be induced by ordering equities based on quantiles of market capitalization.

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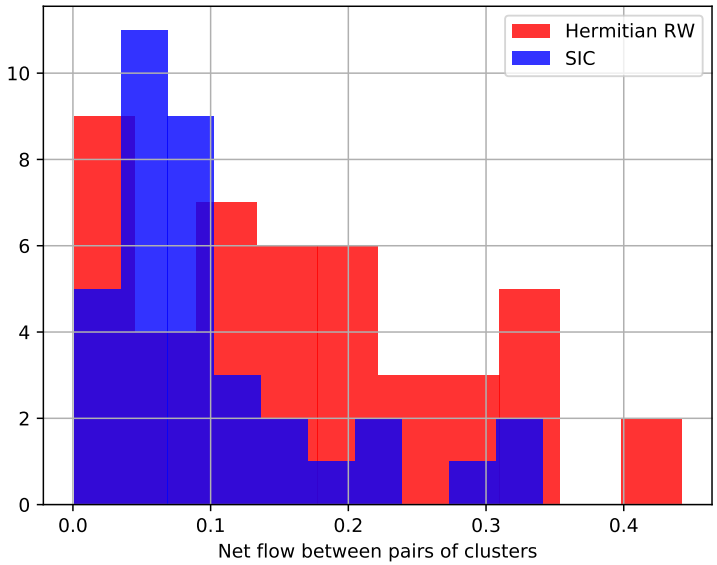
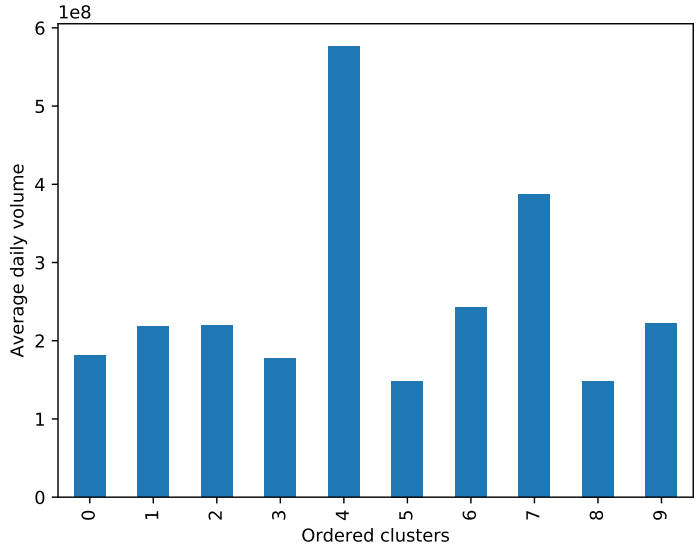
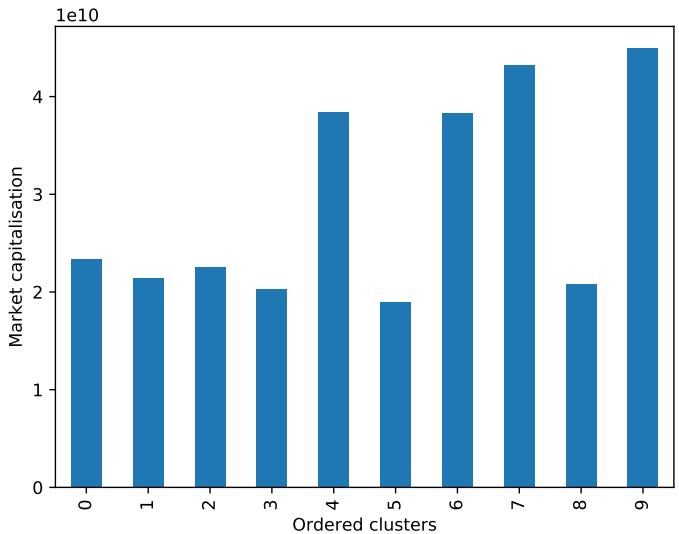


(a) Hist. of Herm-RW and SIC clustering meta-flow edge weights.



(b) Average daily dollar volume by Hermitian RW cluster.



(c) Average market capitalisation by Hermitian RW cluster.

Time variation in the recovered clusterings

Recompute the clustering year-by-year using only data from the retrospective year to do so.

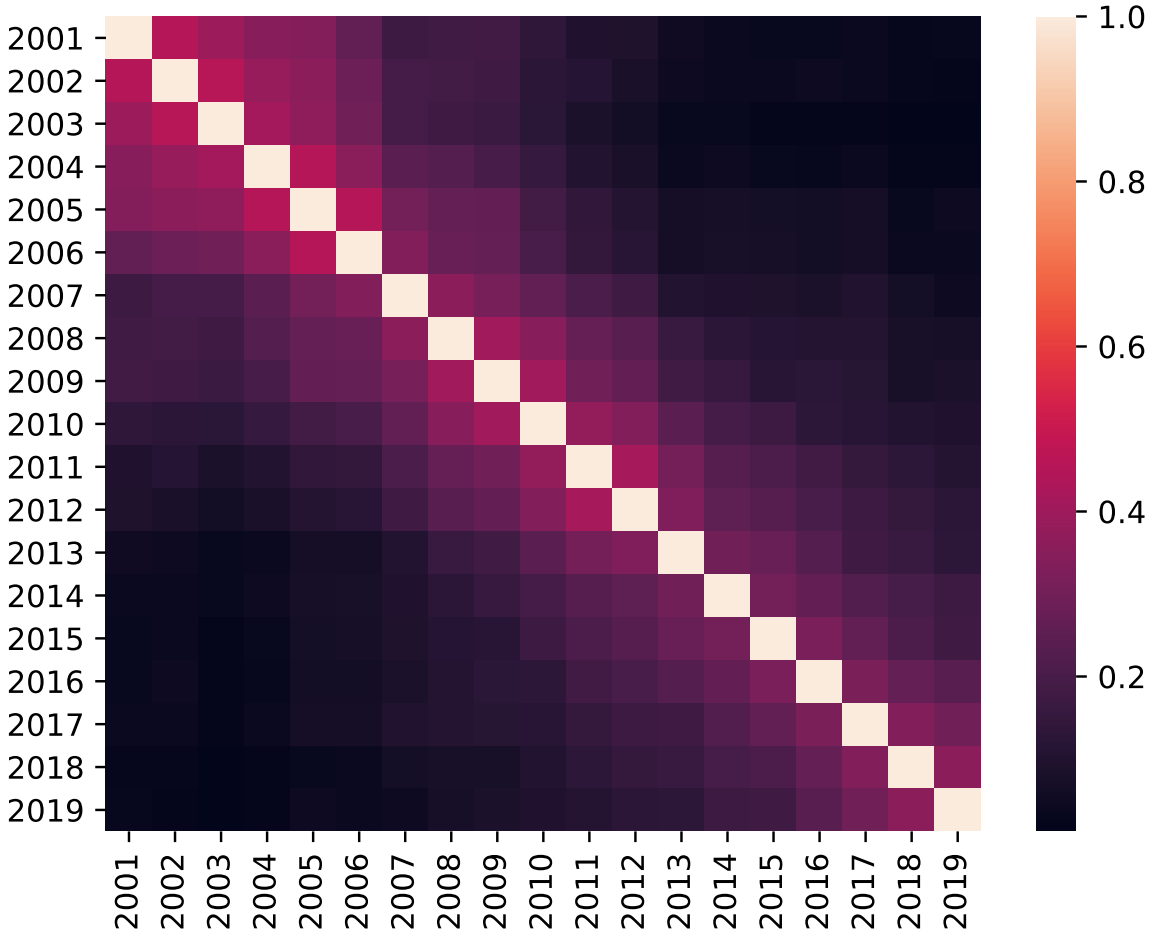


Figure: Adjusted Rand index between clusters computed on yearly snapshots of data

The relatively low ARI values between pairs of clusters indicates some –albeit low– persistence in year-to-year lead-lag structure.

MetaCluster lead-lag portfolios: strategy & cumulative P&L

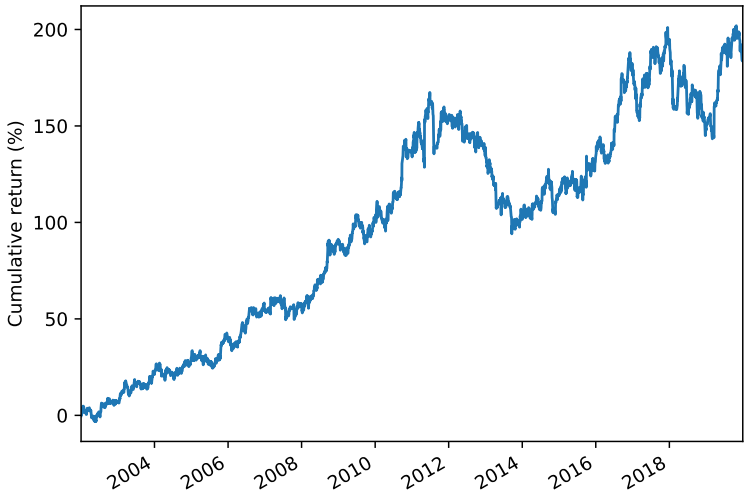


Figure: Cumulative return for the financial forecasting signal.

- MetaCluster: trading signal forecasts lagging cluster returns using smoothed leading cluster returns (update clusters every 2 months); consider only top 10% strongest lead-lag relationships between clusters

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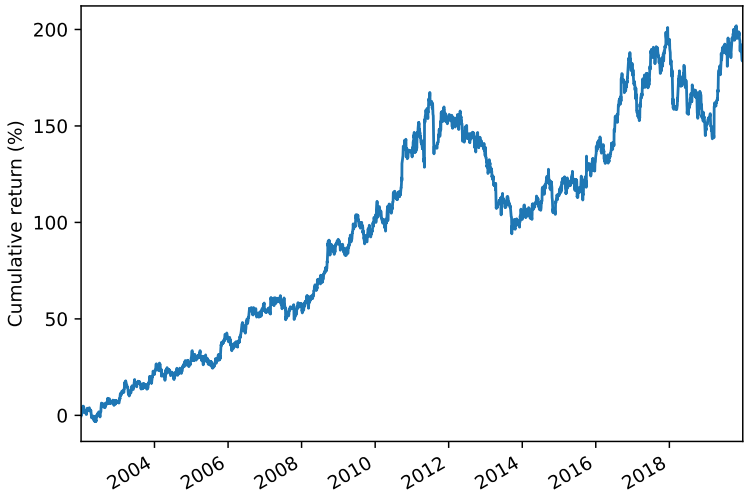

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- MetaCluster: trading signal forecasts lagging cluster returns using smoothed leading cluster returns (update clusters every 2 months); consider only top 10% strongest lead-lag relationships between clusters
- annualized Sharpe Ratio of 0.62 (the S&P500 market return has SR=0.4 on the same period
- low correlation (4%) with the market return
- mean pnl return of 2.4 bpts/day
- decay in the performance of the signal after 2012;
- can be compared with the reduction in clustering persistence post 2012
- in line with Curme et al. (2015) - the information efficiency of the market appears to increase in 2012 relative to earlier years

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GlobalRank Lead-lag Portfolios

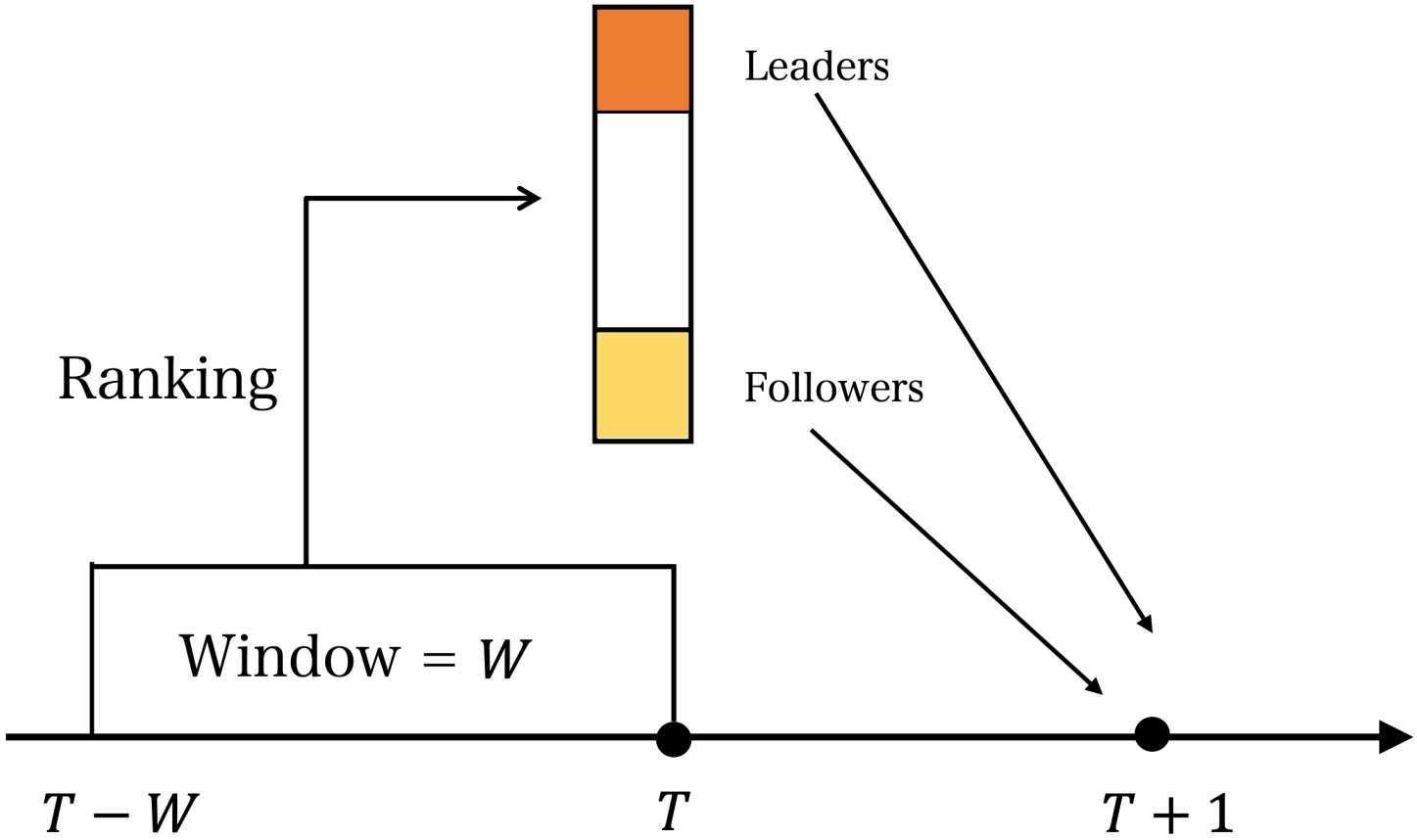

Figure: Illustration of the global ranking pipeline.

- use past 1-day CLCL return of the leaders as indicator of the future return of followers

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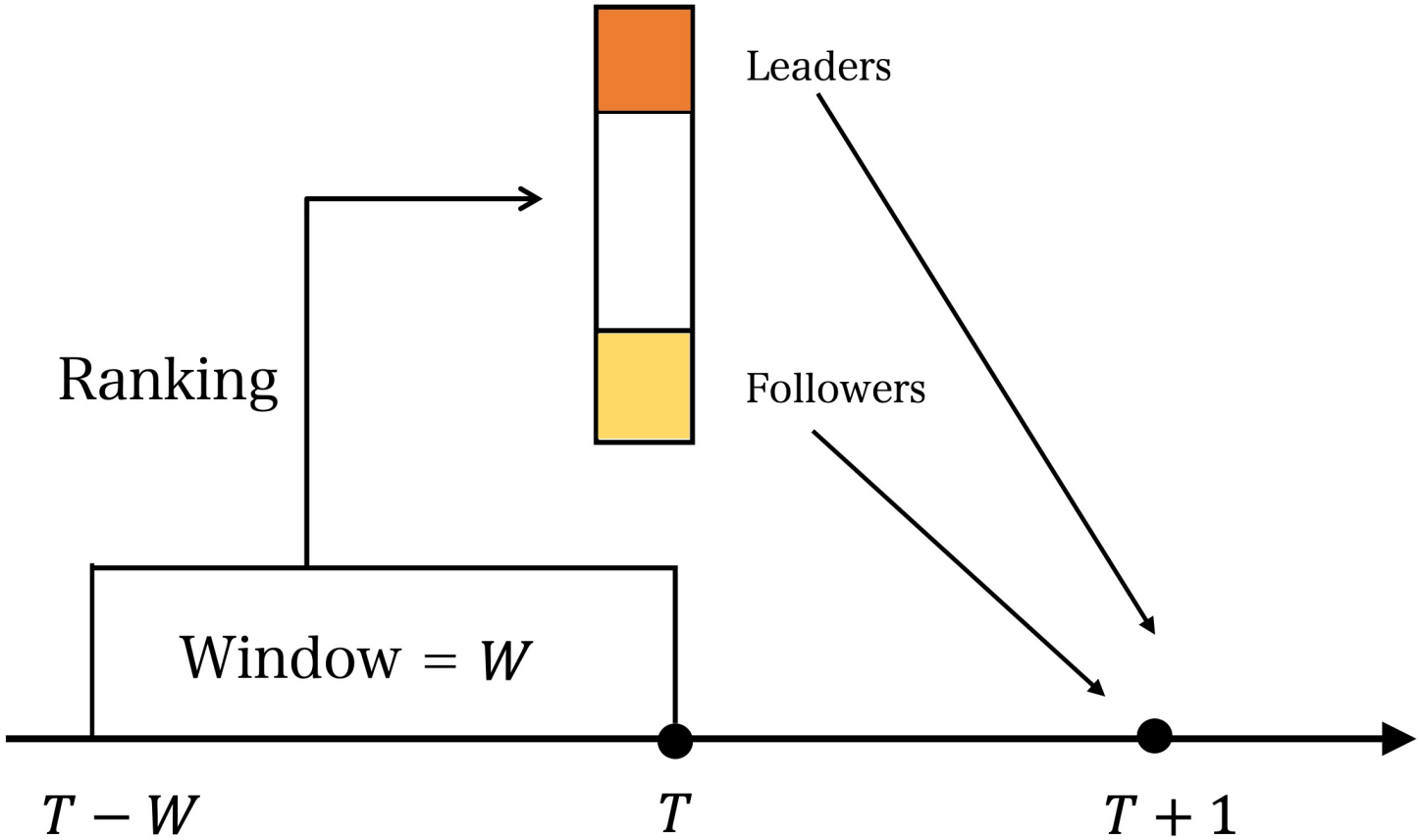

Figure: Illustration of the global ranking pipeline.

- use past 1-day CLCL return of the leaders as indicator of the future return of followers
- to keep the portfolio dollar neutral, trade SPY on the opposite direction of the signal

Fitting previous works into our framework

- interpret the ”heuristic" models as to have derived their scoring mechanism from financial intuition, and to have ranked the assets by the column average of the pairwise lead-lag matrix.

- Lo and MacKinlay (1990):
  - lead-lag score = difference between market caps of asset *i* and asset *j*
  - ranking = column average (i.e. net difference in market cap compared to all other assets)

- Chordia and Swaminathan (2000):
  - lead-lag score = difference between turnover ratios of asset *i* and asset *j*
  - ranking = column average (i.e. net difference in turnover ratio compared to all other assets)

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Extraction of partial rankings

Goal: extract partial rankings - detect clusters within which the lead-lag relationships are more consistent w.r.t an underlying ordering.

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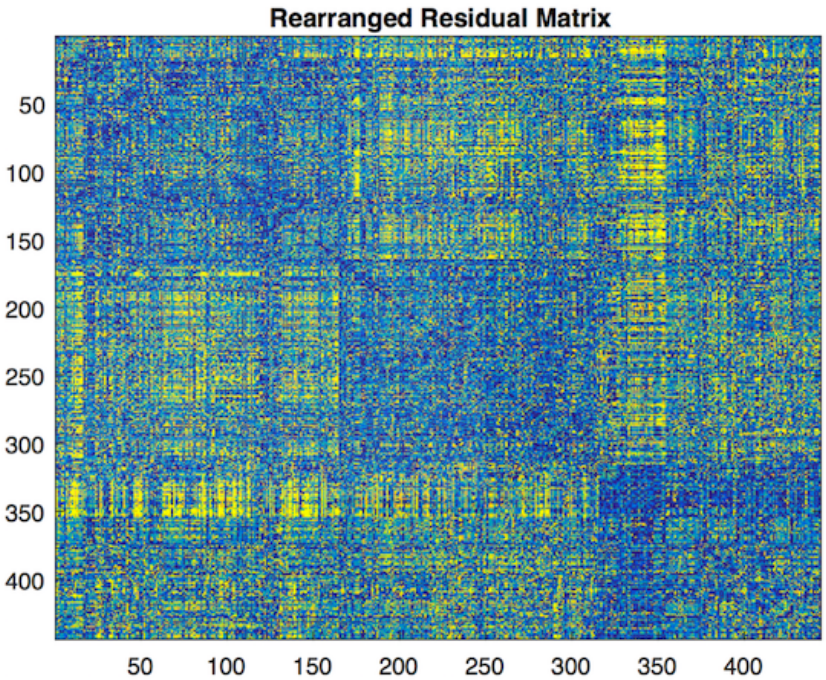
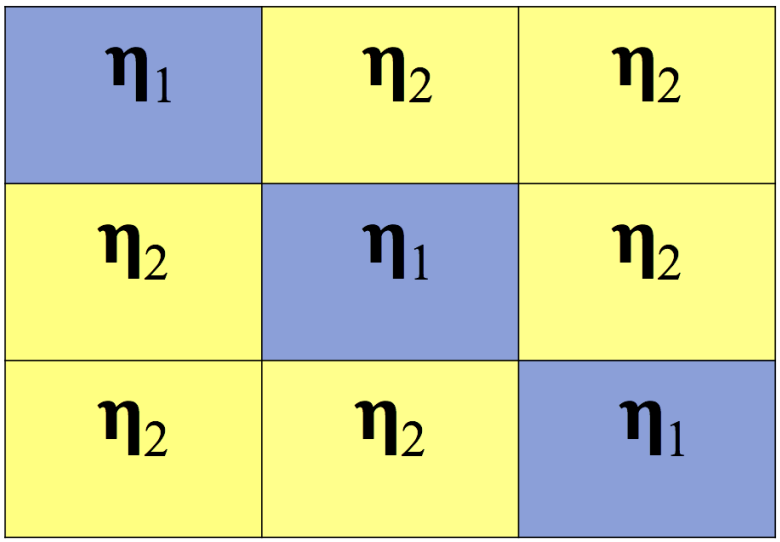
Modularity clustering of the residual matrix $R = |C - \hat{C}|$; S&P 500





Noise levels $\eta_1 < \eta_2$

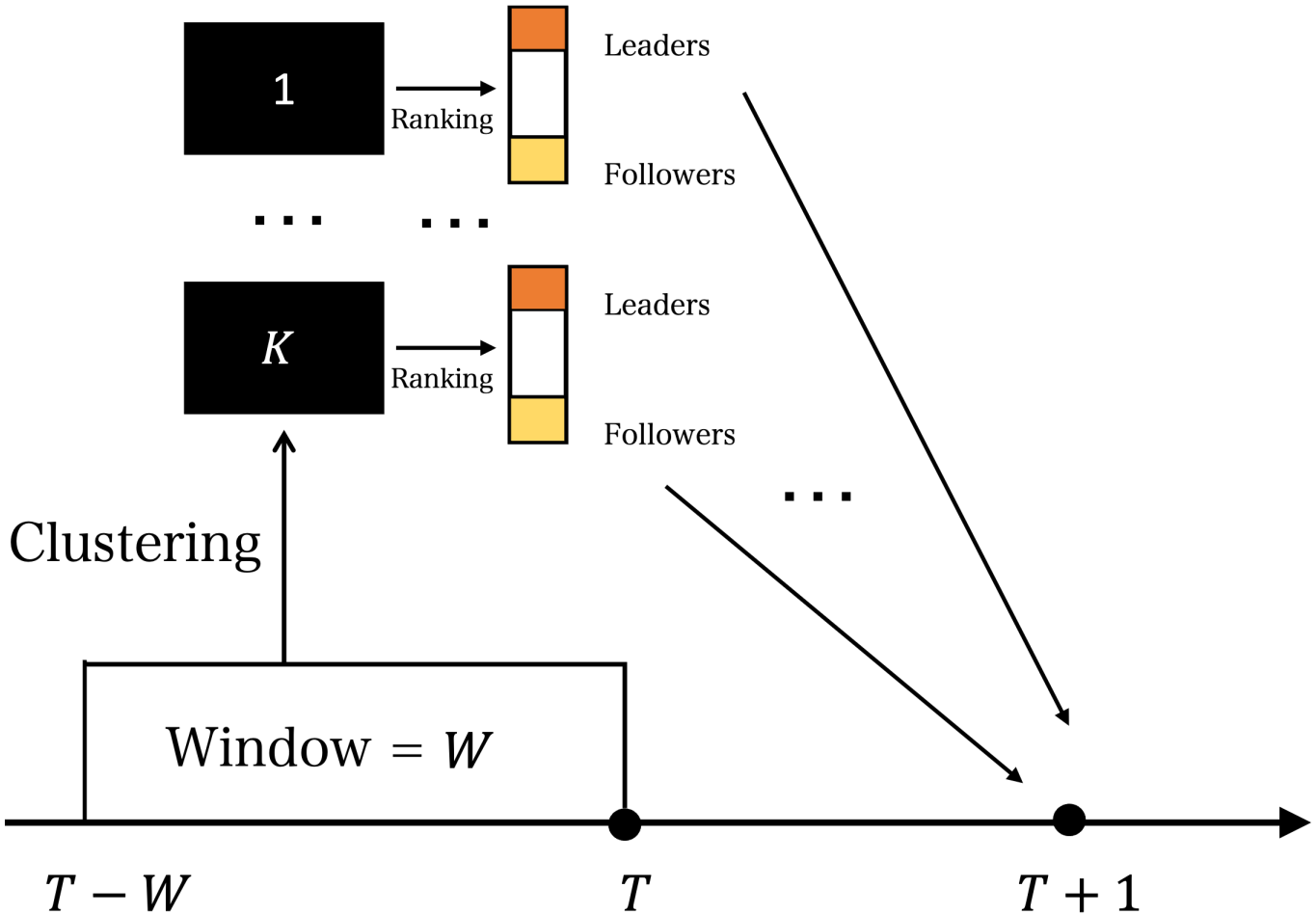
- Stochastic block model for ranking

$$C_{ij} = r_i - r_j + \text{noise}$$

heterogeneous noise (easier to solve (*rank*) within each cluster)

Cluster-Ranking Lead-lag Portfolios

- Hou (2007) and others have studied intra-industry lead-lag relationships



Clustering algorithms on comparison matrices:

- Spectral clustering
- Hermitian clustering

Data set description

- prices and share information data comes from the Center of Research in Security Prices (CRSP) daily prices database.
- sample period from July 1963 to December 2022
- to be consistent with standard literature practices, we include NYSE, Amex, NASDAQ stocks
- for realistic trading simulation results, at each trading day, we only include stocks in the top 25% by market capitalization (defined as stock end-of-day price $\times$ shares outstanding)
- industry classification information is created by linking each firm to a single, non-overlapping Fama-French 12 industry by its SIC code.
  - industries: nondurables (1), durables (2), manufacturing (3), energy (4), chemicals (5), business equipment(6), telecommunications (7), utilities (8), shops (9), healthcare (10), finance (11), and other(12).
- altogether a universe of about 550 assets on each day, on average across time

Empirical results: GlobalRank vs ClusterRank

Table: Performance of various lead-lag portfolios

Panel A: GlobalRank Lead-Lag Portfolios					
	Compound Return (%)	Return (bps/day)	Volatility (%)	Sharpe Ratio	Max Drawdown (%)
Max Cross-Cor	19.69	7.14	0.47	2.37	16.90
Avg Cross-Cor	27.97	9.79	0.70	2.21	28.64
Signature	24.87	8.82	0.59	2.38	24.67
Market Cap	6.15	2.37	0.50	0.75	63.40
Turnover	7.60	2.91	0.34	1.37	13.62
Panel B: ClusterRank Lead-lag Portfolios					
	Compound Return (%)	Return (bps/day)	Volatility (%)	Sharpe Ratio	Max Drawdown (%)
Max Cross-Cor	14.52	5.39	0.38	2.23	18.21
Avg Cross-Cor	19.01	6.91	0.49	2.23	23.49
Signature	17.50	6.40	0.42	2.43	12.28
Industry	15.75	5.81	0.43	2.18	42.29

Param. Specs in GlobalRank & ClusterRank Lead-lag Portfolios I

Table: Performances of Different Lead-lag Portfolios - Alternative Hyperparameters

Panel A: GlobalRank Lead-Lag Portfolios - 40% Leaders and Followers

	Compound Return (%)	Return (bps/day)	Volatility (%)	Sharpe Ratio	Max Drawdown (%)
Max Cross-Cor	18.36	6.70	0.44	2.43	18.9
Avg Cross-Cor	21.37	7.69	0.48	2.54	18.2
Signature	24.26	8.62	0.51	2.70	17.8
Market Cap	0.08	0.03	0.46	0.01	91.75
Turnover	8.90	3.38	0.35	1.52	13.93

Panel B: ClusterRank Lead-lag Portfolios (Hermitian Clustering) - 20% Leader and Lagger each

	Compound Return (%)	Return (bps/day)	Volatility (%)	Sharpe Ratio	Max Drawdown (%)
Max Cross-Cor	15.73	5.80	0.38	2.38	12.46
Avg Cross-Cor	21.74	7.81	1.18	1.05	38.20
Signature	24.31	8.64	0.39	3.43	9.19

Panel C: ClusterRank Lead-lag Portfolios (Spectral Clustering) - 40% Leader and Lagger each

	Compound Return (%)	Return (bps/day)	Volatility (%)	Sharpe Ratio	Max Drawdown (%)
Max Cross-Cor	15.06	5.57	0.35	2.42	11.75
Avg Cross-Cor	17.23	6.31	0.40	2.52	20.39
Signature	19.65	7.12	0.38	2.99	11.51
Industry	6.64	2.55	0.38	1.06	26.9

Param. Specs in GlobalRank & ClusterRank Lead-lag Portfolios II

Table: Performances of Different Lead-lag Portfolios - Alternative Hyperparameters

Panel D: ClusterRank Lead-lag Portfolios (Hermitian Clustering) - 40% Leader and Lagger					
	Compound Return (%)	Return (bps/day)	Volatility (%)	Sharpe Ratio	Max Drawdown (%)
Max Cross-Cor	16.08	5.92	0.37	2.51	14.77
Avg Cross-Cor	20.98	7.56	1.14	1.05	39.49
Signature	22.70	8.12	0.37	3.45	10.07
Panel E: GlobalRank Lead-lag Portfolios - 20% Leader and Lagger, $W = 30$ Day Look-back Window					
	Compound Return (%)	Return (bps/day)	Volatility (%)	Sharpe Ratio	Max Drawdown (%)
Max Cross-Cor	29.60	10.29	0.74	2.21	27.48
Avg Cross-Cor	22.72	8.13	0.51	2.19	16.91
Signature	23.64	8.42	0.59	2.61	16.72

Param. Specs in GlobalRank & ClusterRank Lead-lag Portfolios II

Table: Performances of Different Lead-lag Portfolios - Alternative Hyperparameters

	Compound Return (%)	Return (bps/day)	Volatility (%)	Sharpe Ratio	Max Drawdown (%)
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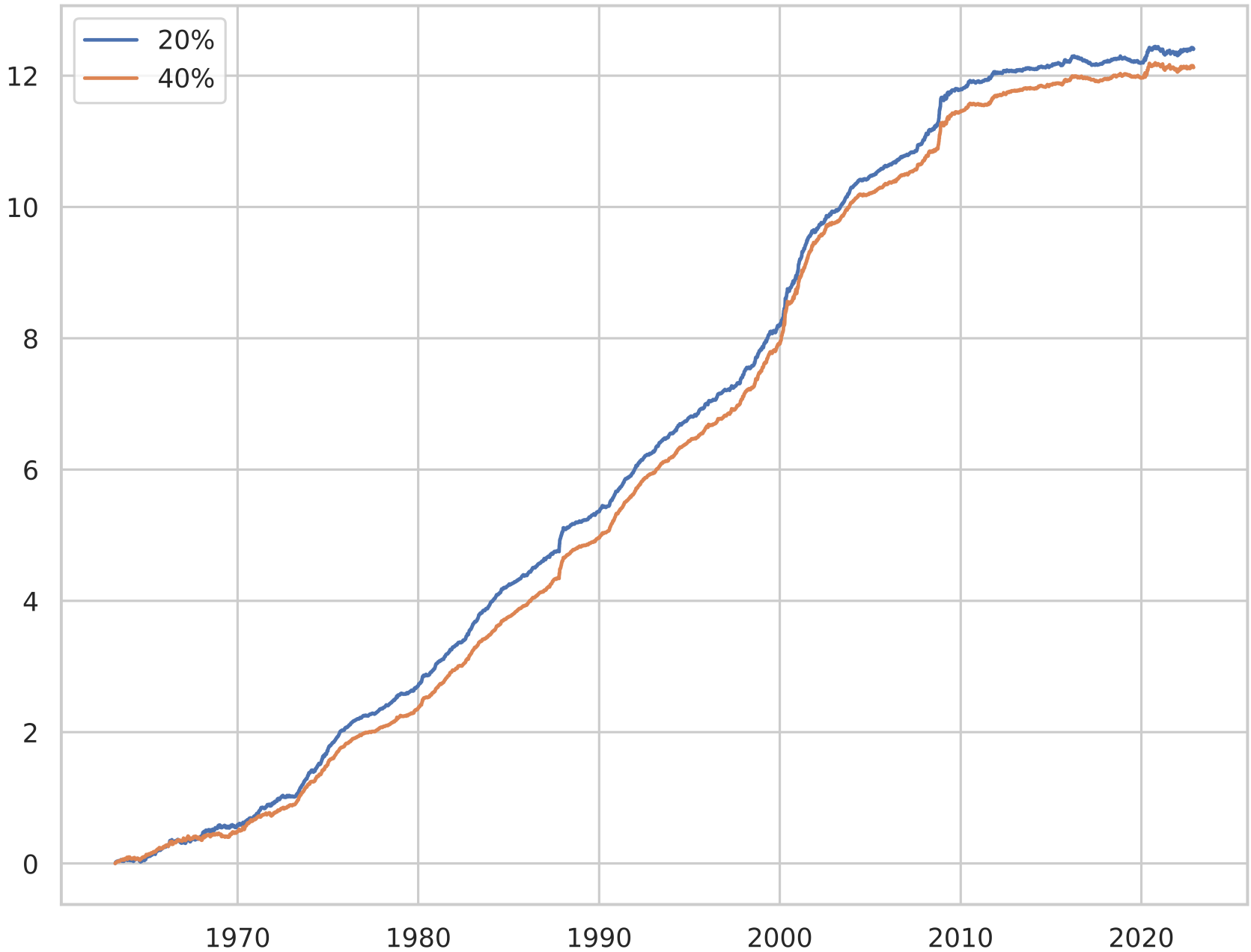
Results are Robust to Hyper Parameters

Cumsum PnL - Hermitian Clustering 1970-2022



Turnovers analysis

- for a portfolio that trades $w_1^t, ..., w_n^t$ dollar notional for stocks $s_1, ..., s_n$ at time t, the turnover of the portfolio at time t is defined as

$$\text{Turnover}^t = \frac{\sum_{i=1}^{n} |w_i^t - w_i^{t-1}|}{\sum_{i=1}^{n} |w_i^{t-1}|},$$

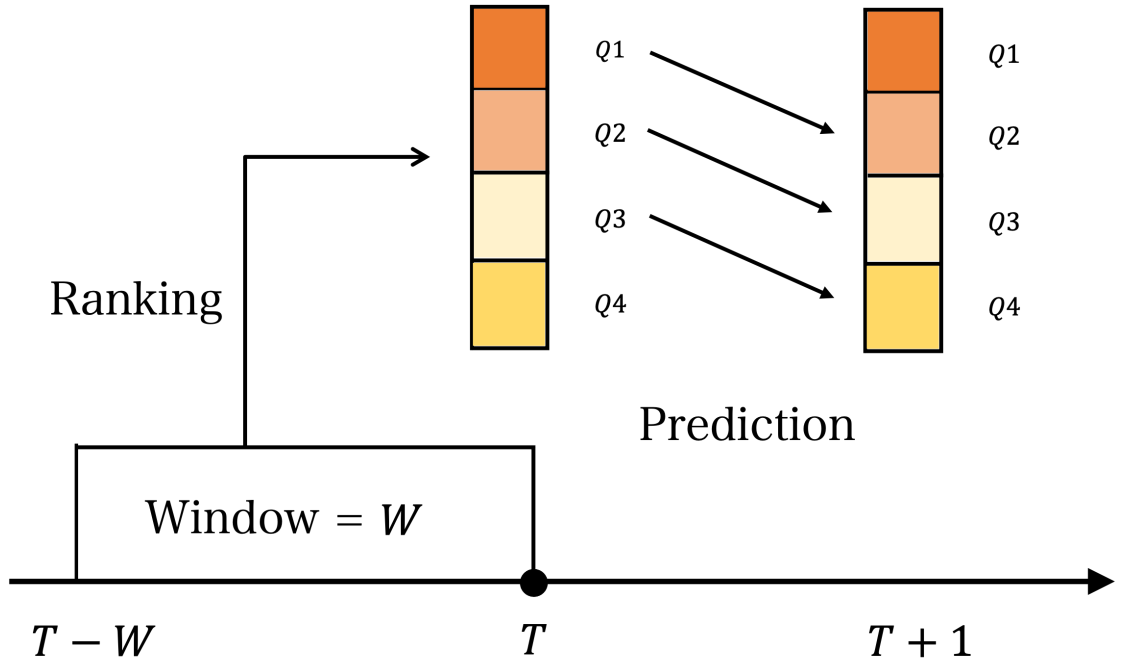
where $|w_i^t - w_i^{t-1}|$ is the change in dollar value held by the portfolio on asset s_i over the re-balance period.

Table: Turnover ratio of various portfolios

	Avg turnover ratio (%)	Avg change in portfolio composition(%)	Proportion of sign flips(%)
Max Cross-Cor	102.5	22.6	44.3
Avg Cross-Cor	104.5	26.7	44.1
Signature	111.3	23.6	50.4
Market Cap	95.4	1.21	44.2
Turnover	94.4	1.22	43.3

Waterfall GlobalRank lead-lag relationships

- what about the middle 60% of stocks not included in the analysis?

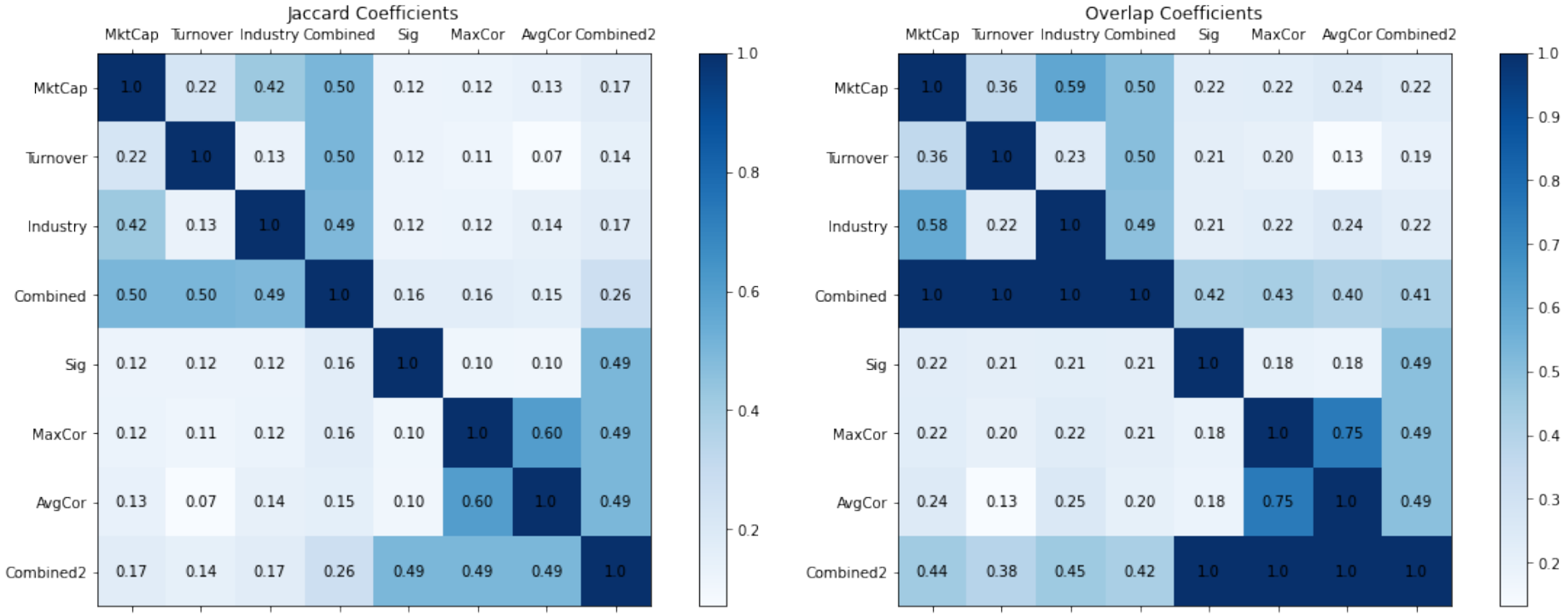


- intermediate lead-lag relationships exist
- leads to larger traded universe

Table: Waterfall Lead-lag Relationships

	Compound Return (%)	Return (bps/day)	Volatility (%)	Sharpe Ratio	Max Drawdown (%)
Max Cross-Cor	16.1	5.91	0.40	2.37	15.8
Avg Cross-Cor	14.8	5.48	0.35	2.52	14.1
Signature	20.4	7.36	0.40	2.89	14.7

Similarity across portfolio types



(a) Jaccard coefficients

(b) Overlap coefficients

Figure: (a) Diagram for the Jaccard coefficients among different lead-lag portfolios; a higher Jaccard coefficient represents higher similarity. (b) Diagram for the overlap coefficients among different lead-lag portfolios; a higher overlap coefficient represents higher similarity. Covered period: 2019-2022.

Clusters composition

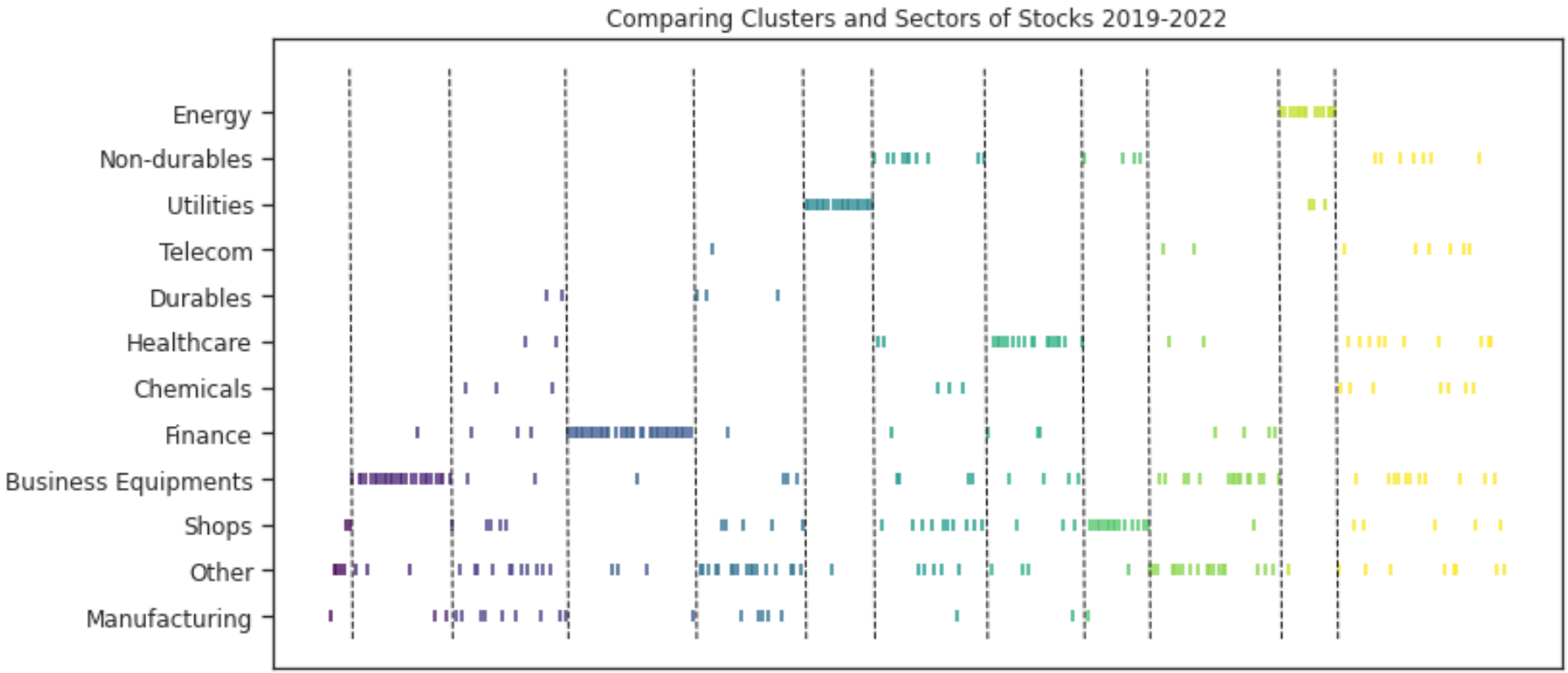
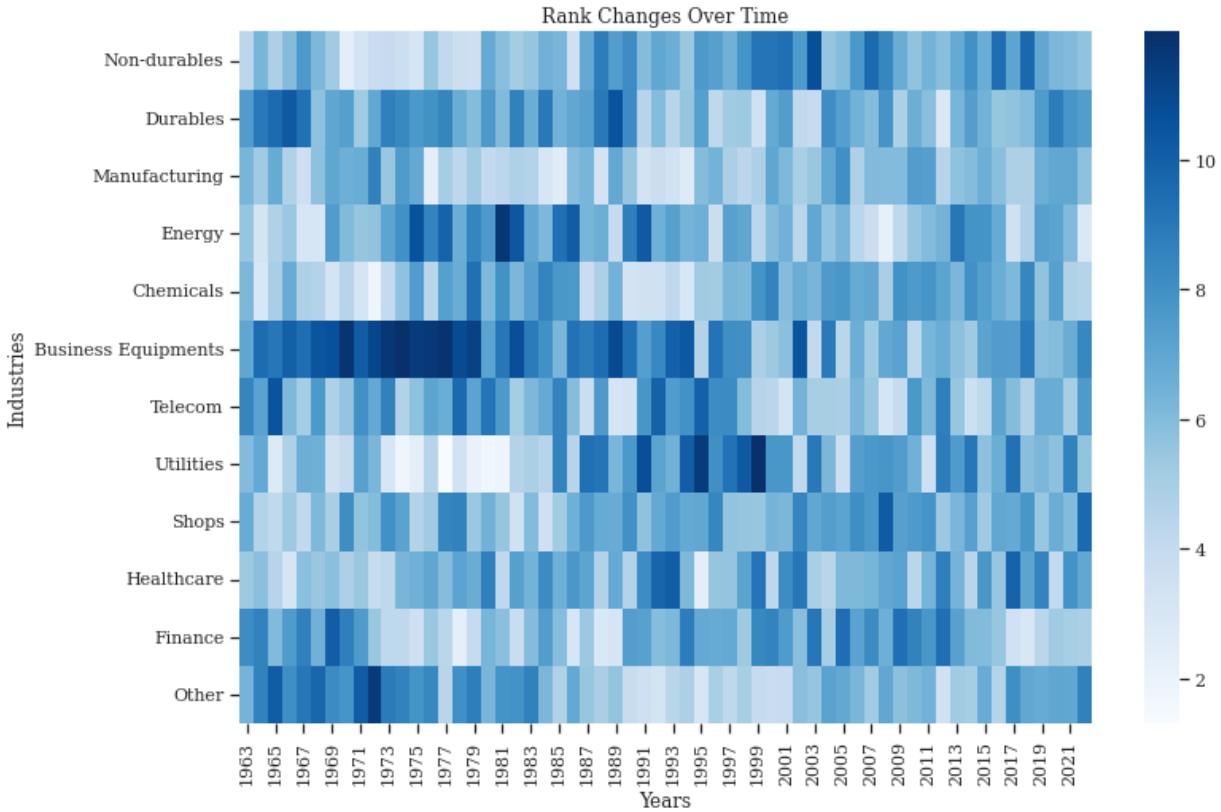


Figure: A comparison between the clusters using the Signature matrices from 1 January 2019 to 31 December 2022. The area between black vertical dashes represents each cluster formed with Hermitian clustering. There are 376 stocks that are traded every day in this time period.

Evolution across time of sector ranks



Manufacturing	0.476
Chemicals	0.485
Energy	0.494
Telecom	0.495
Healthcare	0.496
Utilities	0.498
Non-durables	0.499
Shops	0.501
Finance	0.503
Other	0.504
Durables	0.511
Business Equipments	0.554

Figure: Left: rank evolution of sector averages. Right: average rank across time (sorted from most leading to most lagging sectors).

Revisiting the slow information diffusion hypothesis

- if the hypothesis holds, lead-lag relationship should slowly disappear as we reduce the time frequency by which we detect the relationship/ rebalance the portfolio.

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- explore a multiscale approach, and consider daily, bi-daily, weekly, bi-weekly, tri-weekly, and monthly portfolios.

Revisiting the slow information diffusion hypothesis

- if the hypothesis holds, lead-lag relationship should slowly disappear as we reduce the time frequency by which we detect the relationship/ rebalance the portfolio.
- explore a multiscale approach, and consider daily, bi-daily, weekly, bi-weekly, tri-weekly, and monthly portfolios.
- eg. for monthly, consider the previous-month cumulative return of the leaders, and use that to predict return of the followers over the next month

Lead-lag relationships at multiple frequencies

Table: Performance of lead-lag portfolios at various frequencies (return is per period).

Panel A: Signature Portfolio for various frequencies

	Compound Return(%)	Return (bps/per)	Volatility(%)	Sharpe Ratio	Max Drawdown(%)
Daily	24.87	8.82	0.59	2.37	24.67
Bi-Daily	11.02	8.30	0.76	1.23	24.17
Weekly	12.33	23.09	1.34	1.22	26.65
Bi-Weekly	8.37	31.94	1.69	0.95	19.59
Tri-Weekly	7.30	42.9	2.01	0.87	24.56
Monthly	7.81	65.8	2.56	0.86	22.50

Panel B: MaxCor Portfolio for various frequencies

	Compound Return(%)	Return (bps/per)	Volatility(%)	Sharpe Ratio	Max Drawdown(%)
Daily	19.69	7.14	0.48	2.36	16.90
Bi-Daily	13.36	9.96	0.66	1.70	32.64
Weekly	13.32	24.84	1.09	1.62	28.48
Bi-Weekly	9.51	36.12	1.47	1.24	23.16
Tri-Weekly	8.86	50.64	1.80	1.15	16.95
Monthly	6.91	61.20	2.35	0.86	21.5

Panel C: AvgCor Portfolio for various frequencies

	Compound Return(%)	Return (bps/per)	Volatility(%)	Sharpe Ratio	Max Drawdown(%)
Daily	27.97	9.79	0.70	2.21	28.64
Bi-Daily	16.46	12.10	0.96	1.41	52.67
Weekly	16.41	30.19	1.56	1.38	37.26
Bi-Weekly	12.63	47.31	1.99	1.19	33.79
Tri-Weekly	11.65	65.80	2.39	1.13	24.91
Monthly	9.76	85.36	3.02	0.94	19.52

A multi-frequency perspective

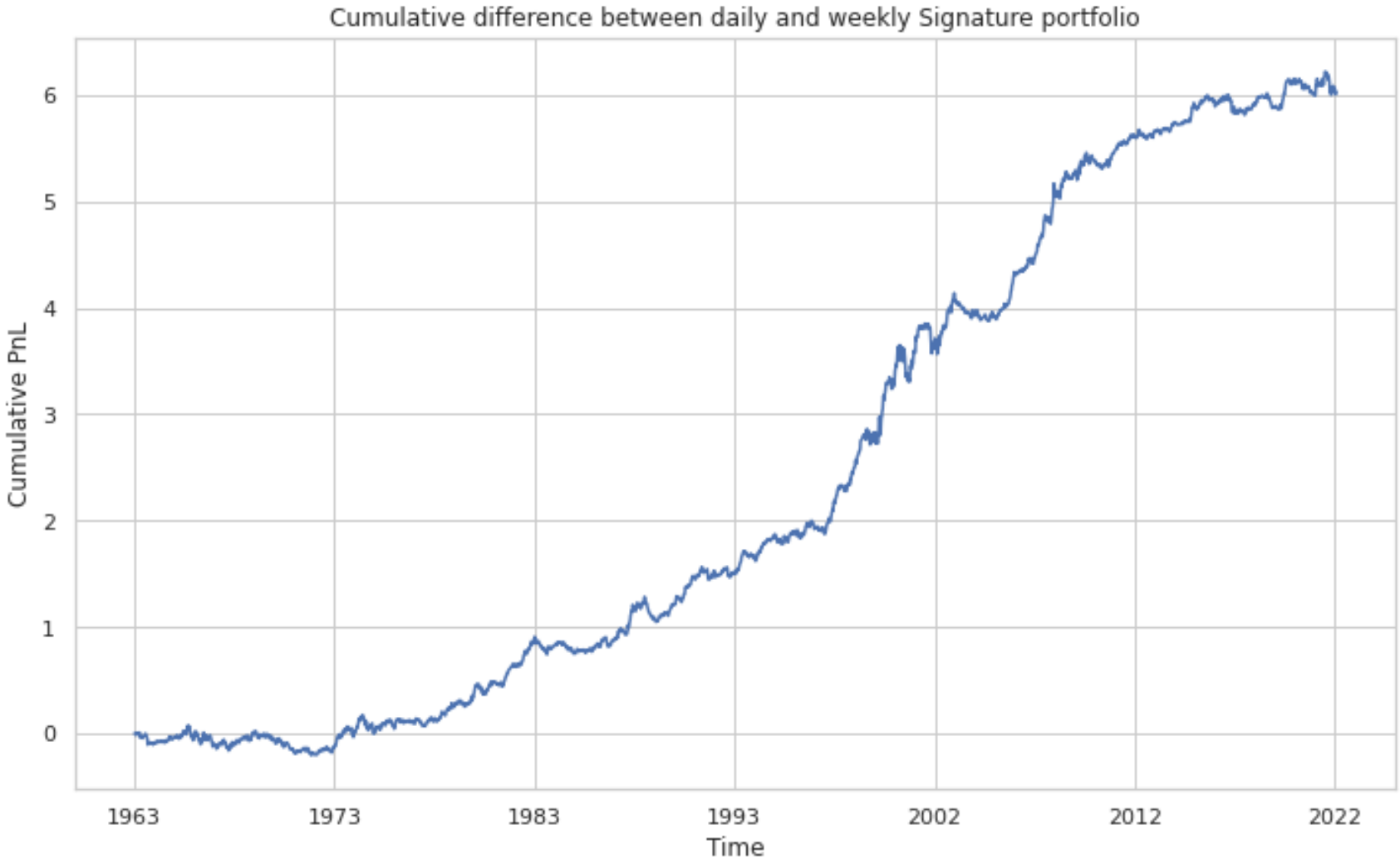

Figure: Cumulative sum of the difference between the daily lead-lag Signature portfolio and the weekly lead-lag Signature portfolio

Historical progression of lead-lag relationships

Table: Dates when daily lead-lag portfolios first out-perform lower frequency lead-lag portfolios by a certain margin.

Panel A: First occurrence of a cumulative excess return of 50% btw daily and lower-frequency portfolios

	Raw	Bi-Daily	Weekly	Bi-Weekly	Tri-Weekly	Monthly
Signature	1968.07	1973.04	1981.09	1980.09	1978.10	1971.02
Max Cross-Corr	1968.03	1974.07	1996.02	1982.07	1975.08	1970.08
Avg Cross-Corr	1968.02	1974.01	1995.06	1985.02	1982.08	1973.12

Panel B: First occurrence of a cumulative excess return of 100% btw daily and lower-frequency portfolios

	Raw	Bi-Daily	Weekly	Bi-Weekly	Tri-Weekly	Monthly
Signature	1973.04	1974.09	1987.10	1981.11	1981.03	1974.12
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Potential hypotheses:

- historically, the market got faster and faster - information that used to take weeks or months to arrive at the followers now takes much less time, causing the daily lead-lag portfolio to ultimately outperform the lower frequency ones.

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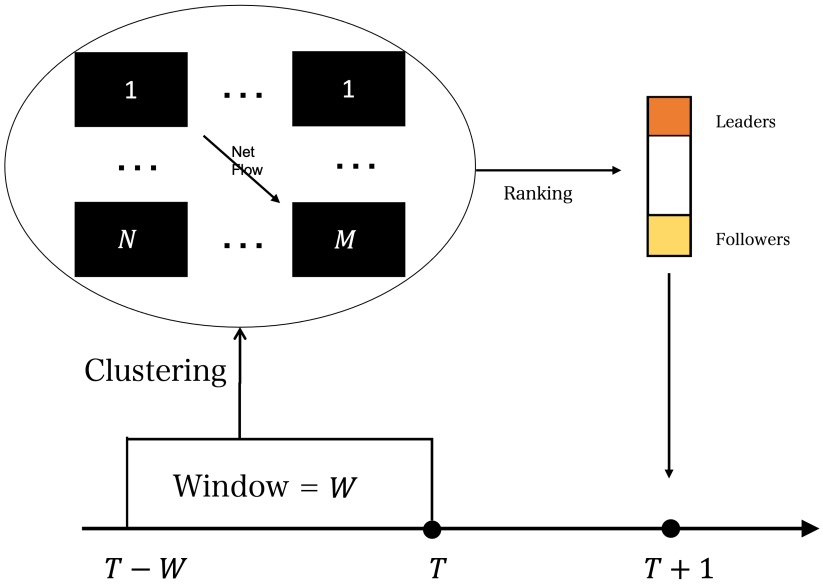
  - how do we measure the speed of the market?
  - it is difficult to establish causal relationship in finance!
- early days, only few had the capacity to trade daily or even intraday; but if you already can trade bi-daily, there's no reason to not go daily; caused the bi-daily portfolio to be outperformed by the daily one right away.

Data-driven lead-lag detection pipeline

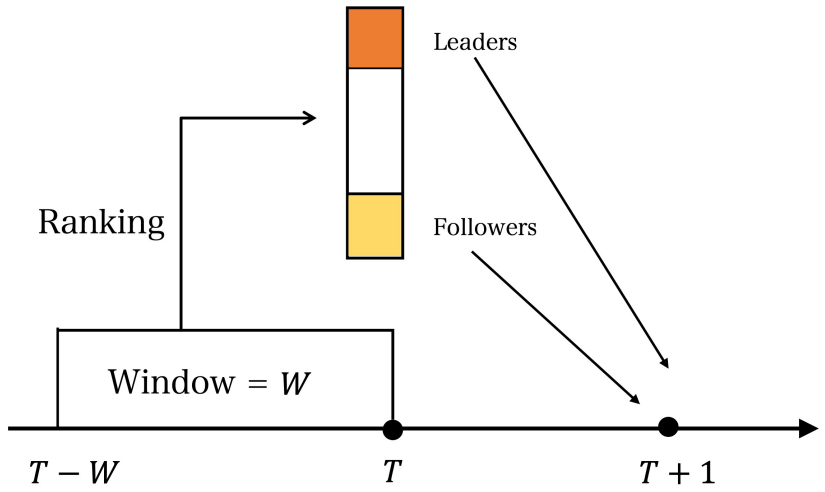
Outline:

1. measure pairwise lead-lag relationships
2. extrapolate pairwise relationships to higher-order relationships
   - (A) MetaCluster: build cluster-driven portfolios that lead-lag each other, or
   - (B) GlobalRank: extract ranking-based ”global" leaders and laggers/followers, or
   - (C) ClusterRank: combine clustering & ranking for lead-lag extraction
3. construct a portfolio to test/evaluate/exploit the uncovered lead-lag relationship.

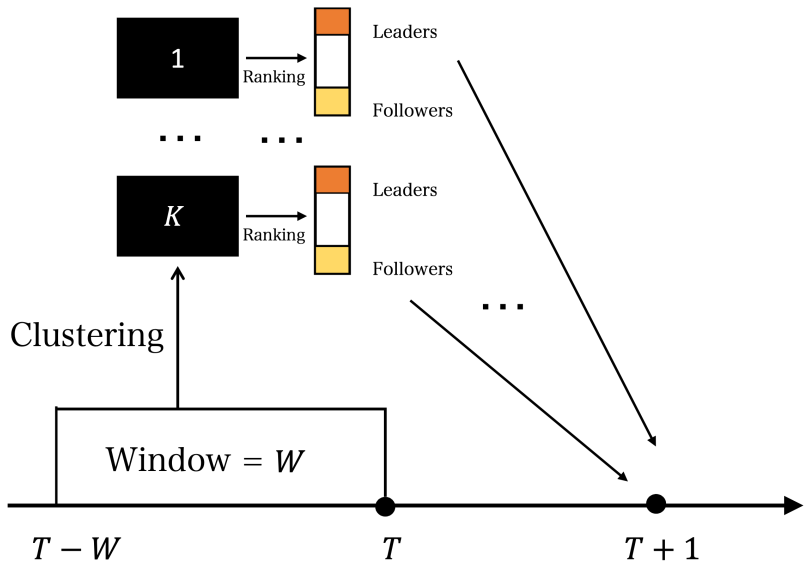
Lead-lag detection frameworks



(a) MetaCluster Lead-lag



(b) GlobalRank Lead-lag



(c) ClusterRank Lead-lag

Multi-reference alignment & multifactor models with lags

- one unknown signal x; samples are shifted noisy versions $\{x_1, \ldots, x_n\}$

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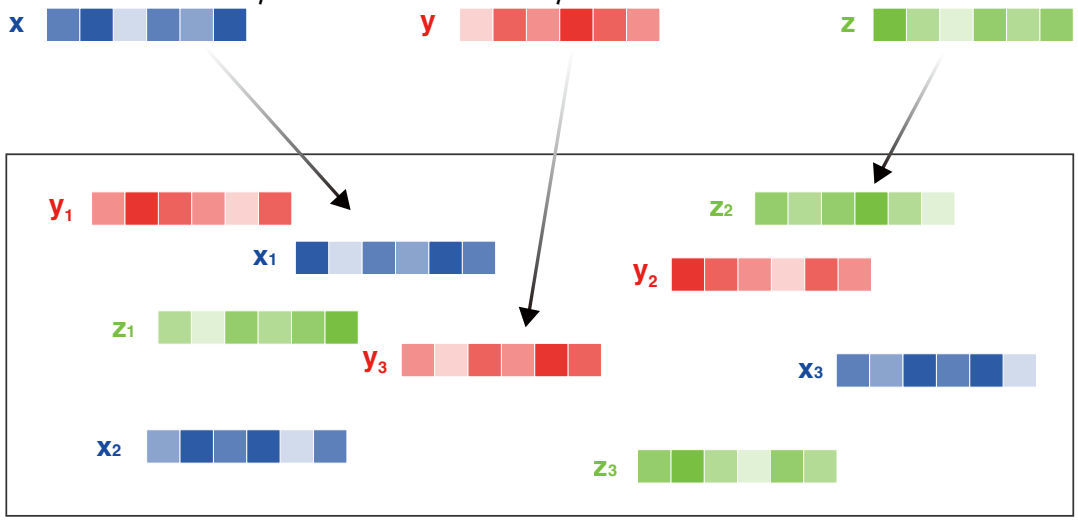

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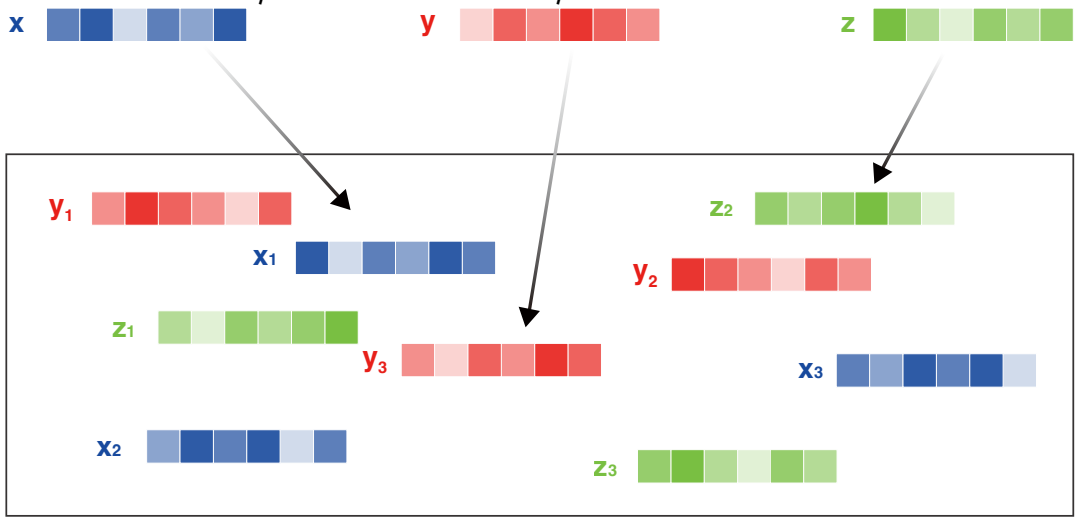

Multifactor models:

- $r_{n\times 1}$: cross-sectional returns of instruments
- $L_{n\times k}$: loadings matrix, with $L_{i,j}$ is the exposure of instrument i to factor j
- $f_{k\times 1}$: the vector of returns of the k factors

$$r = Lf + \epsilon$$

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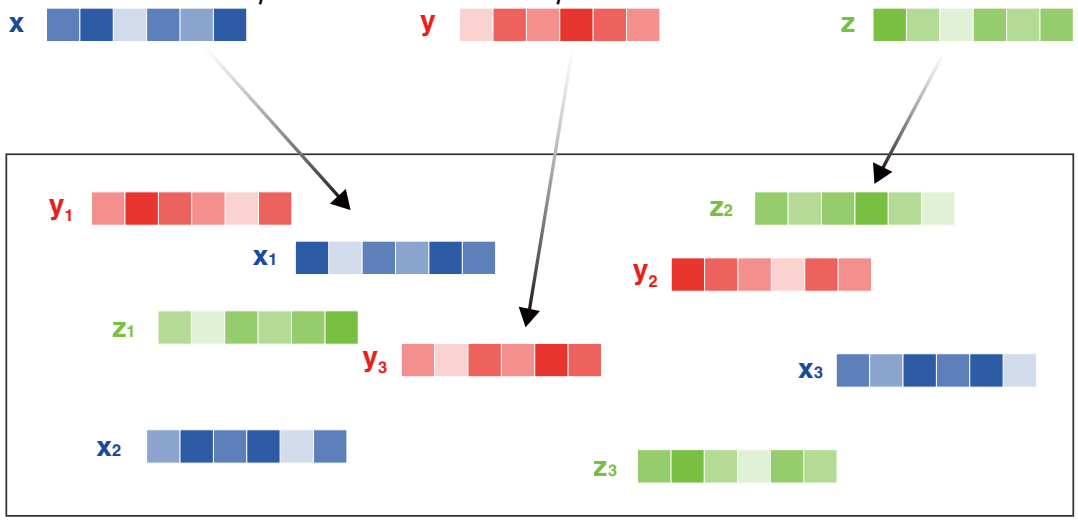

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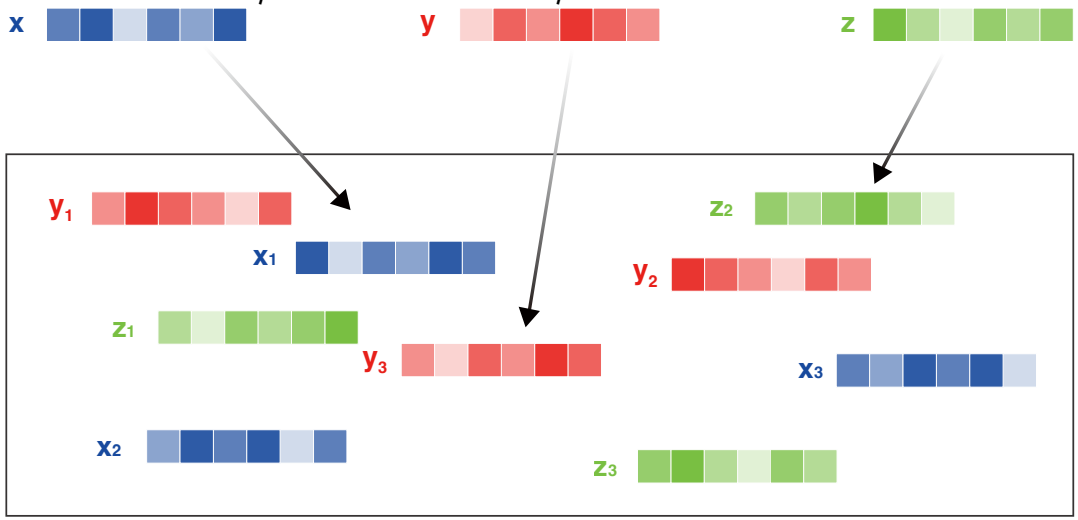

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Multifactor models with lags:

- $G_{n\times k}$: $G_{i,j}$ lag of instrument i to factor j
- $r_i = L_{i,:} \cdot h(G_{i,:};\ f^t, f^{t-1}, \ldots) + \epsilon_i$

Single membership model with $k = 3$ factors/signals

$$L = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right] \qquad G = \left[\begin{array}{ccc} 3 & 0 & 0 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ m & 0 & 0 \\ 2 & 0 & 0 \\ \hline 0 & 5 & 0 \\ 0 & m & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 3 & 0 \\ \hline 0 & 0 & 3 \\ 0 & 0 & 2 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 4 \\ 0 & 0 & m \end{array}\right]$$

Left: factor membership matrix L (assuming all β's are equal to 1). Right: lag matrix, capturing the (nonzero) lag of each stock to its corresponding factor.

Single membership model with $k = 3$ factors/signals

$$L = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right] \qquad G = \left[\begin{array}{ccc} 3 & 0 & 0 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ m & 0 & 0 \\ 2 & 0 & 0 \\ \hline 0 & 5 & 0 \\ 0 & m & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 3 & 0 \\ \hline 0 & 0 & 3 \\ 0 & 0 & 2 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 4 \\ 0 & 0 & m \end{array}\right]$$

Left: factor membership matrix L (assuming all β's are equal to 1). Right: lag matrix, capturing the (nonzero) lag of each stock to its corresponding factor.

Ongoing work: employ clustering to propose an algorithm that recovers the matrix G, and leverage this for prediction tasks.

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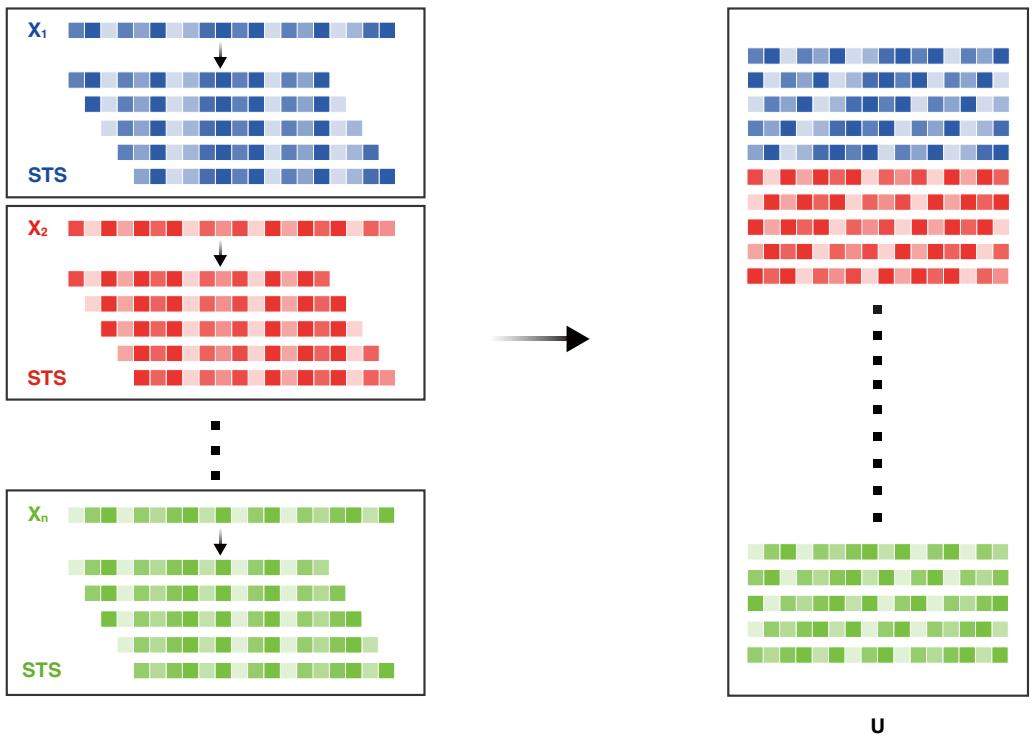


Figure: Left: STS are extracted from each time series by a sliding window. Right: An enlarged universe matrix U

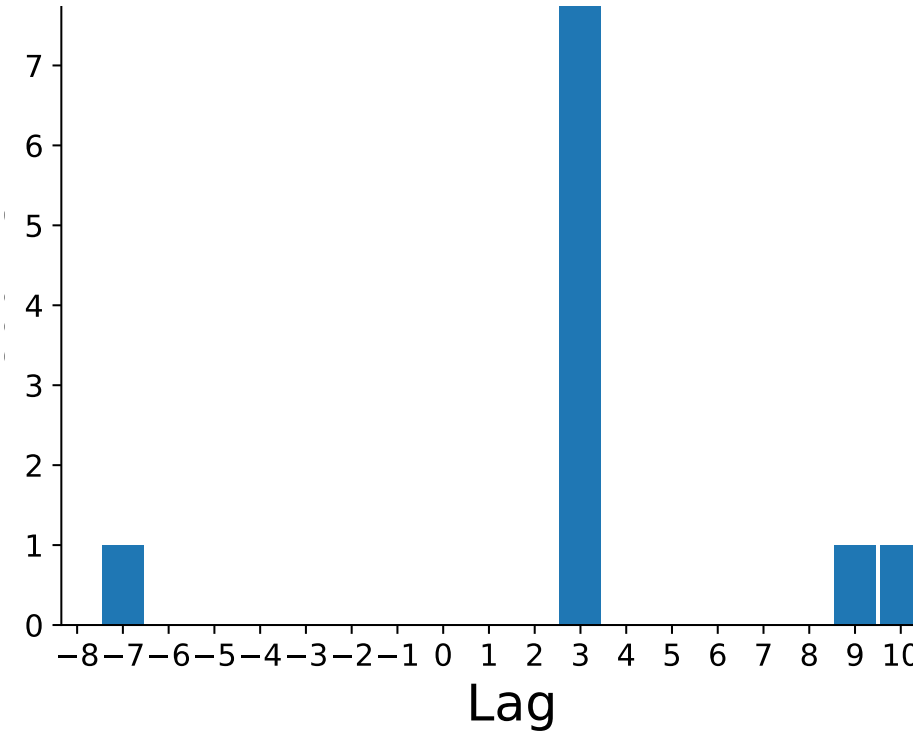
Zhang, C., Shestopaloff, Zohren, *Robust Detection of Lead-Lag Relationships in Lagged Multi-Factor Models* (2023) https://arxiv.org/abs/2305.06704

Methodology

Example: Y_i^t denotes the substring time series (STS) corresponding to stock i starting at day t.

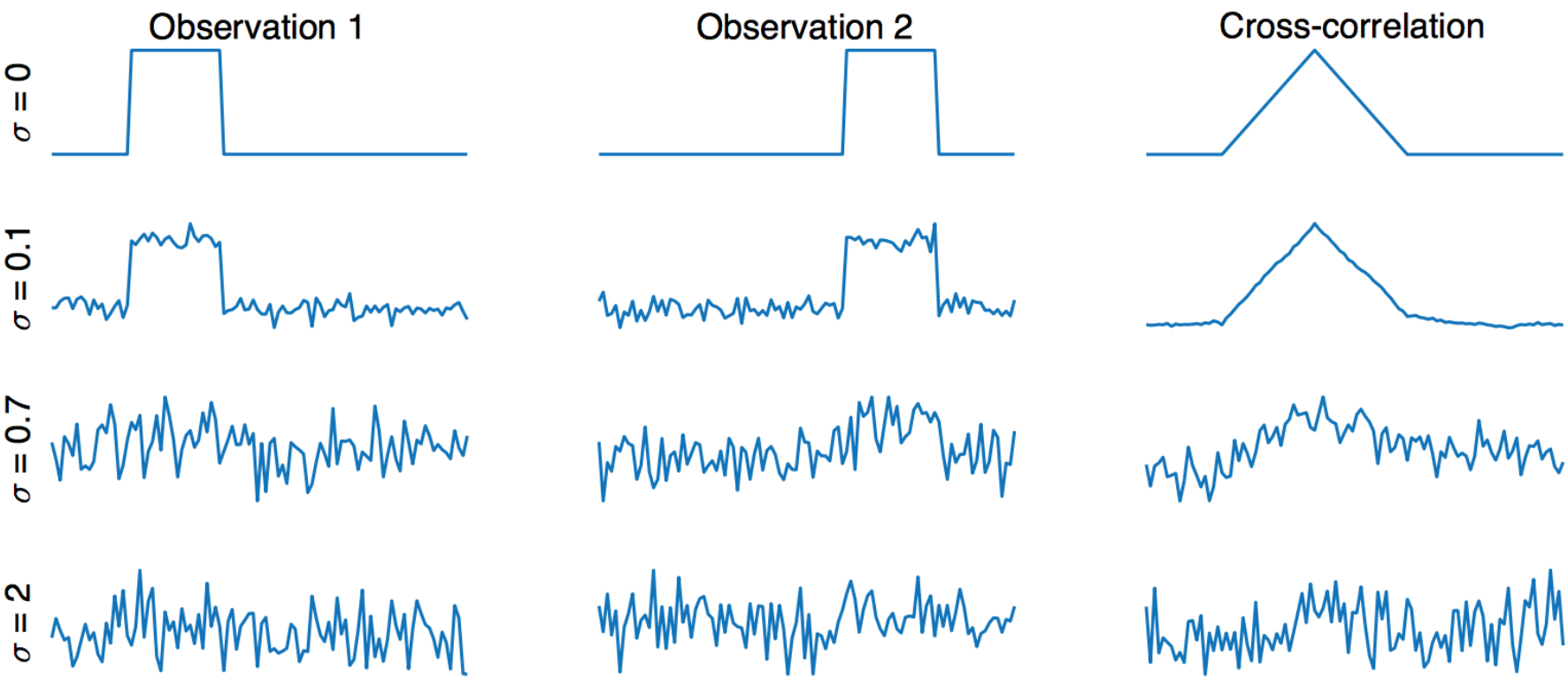
Cluster	Subsequence	Lag
ϕ_1	(Y_1^9, Y_2^2)	-7
ϕ_2	(Y_1^7, Y_2^{10})	3
ϕ_3	(Y_1^1, Y_2^4)	3
ϕ_4	(Y_1^2, Y_2^5)	3
ϕ_5	(Y_1^0, Y_2^3)	3
ϕ_6	(Y_1^6, Y_2^9)	3
ϕ_7	(Y_1^3, Y_2^6)	3
ϕ_8	(Y_1^4, Y_2^7)	3
ϕ_9	(Y_1^5, Y_2^8)	3
ϕ_{10}	$(Y_1^10, Y_2^0), (Y_1^10, Y_2^1)$	-10, -9
ϕ_{11}	Y_1^8	NaN

Example of calculating the relative lags of STS (substring time series) in each cluster from two time series.



Histogram of the relative lags of STS from two time series.

Multi-reference alignment



From: T. Bendory, N. Boumal, C. Ma, Z. Zhao, A. Singer, ”Bispectrum Inversion with Application to Multireference Alignment" (2016)

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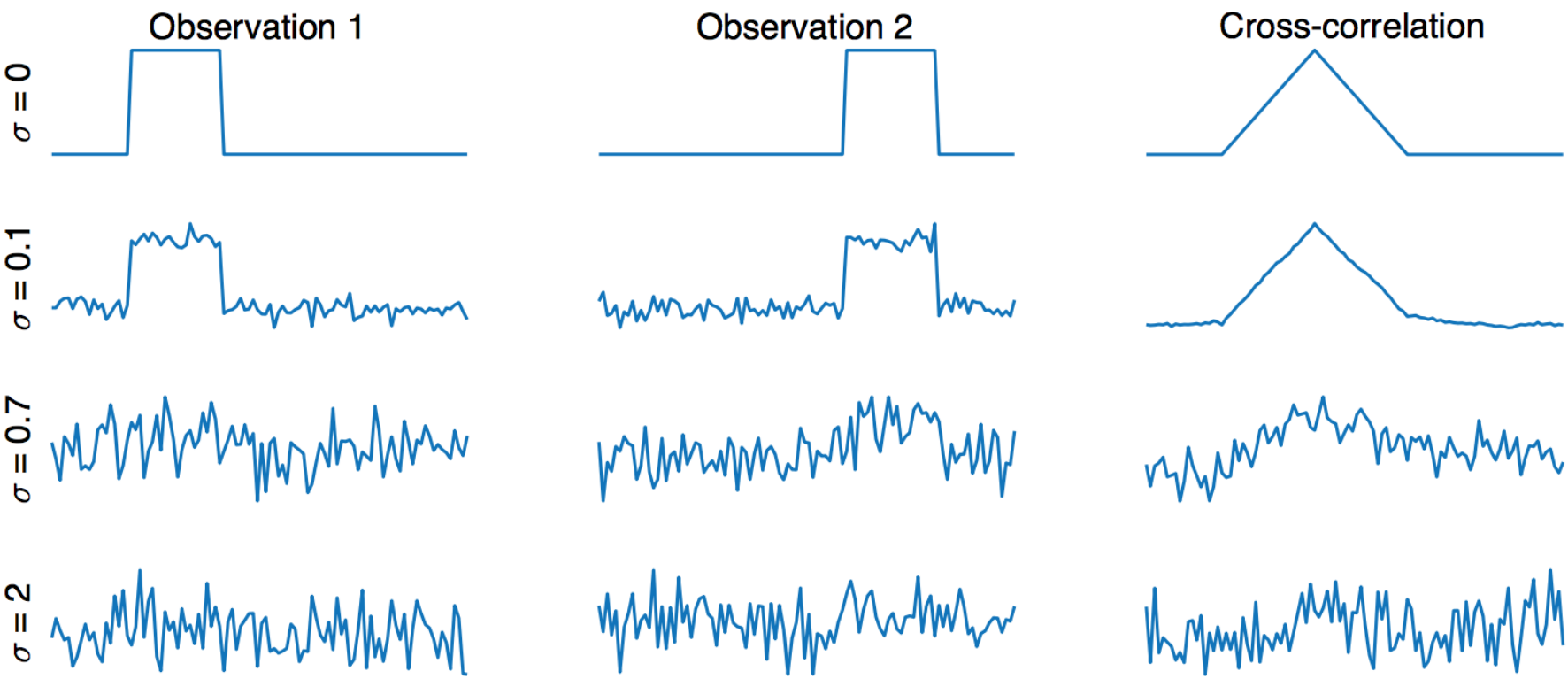

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Ongoing work: detect lags based on invariant feature representations and SDP and non-convex optimization.

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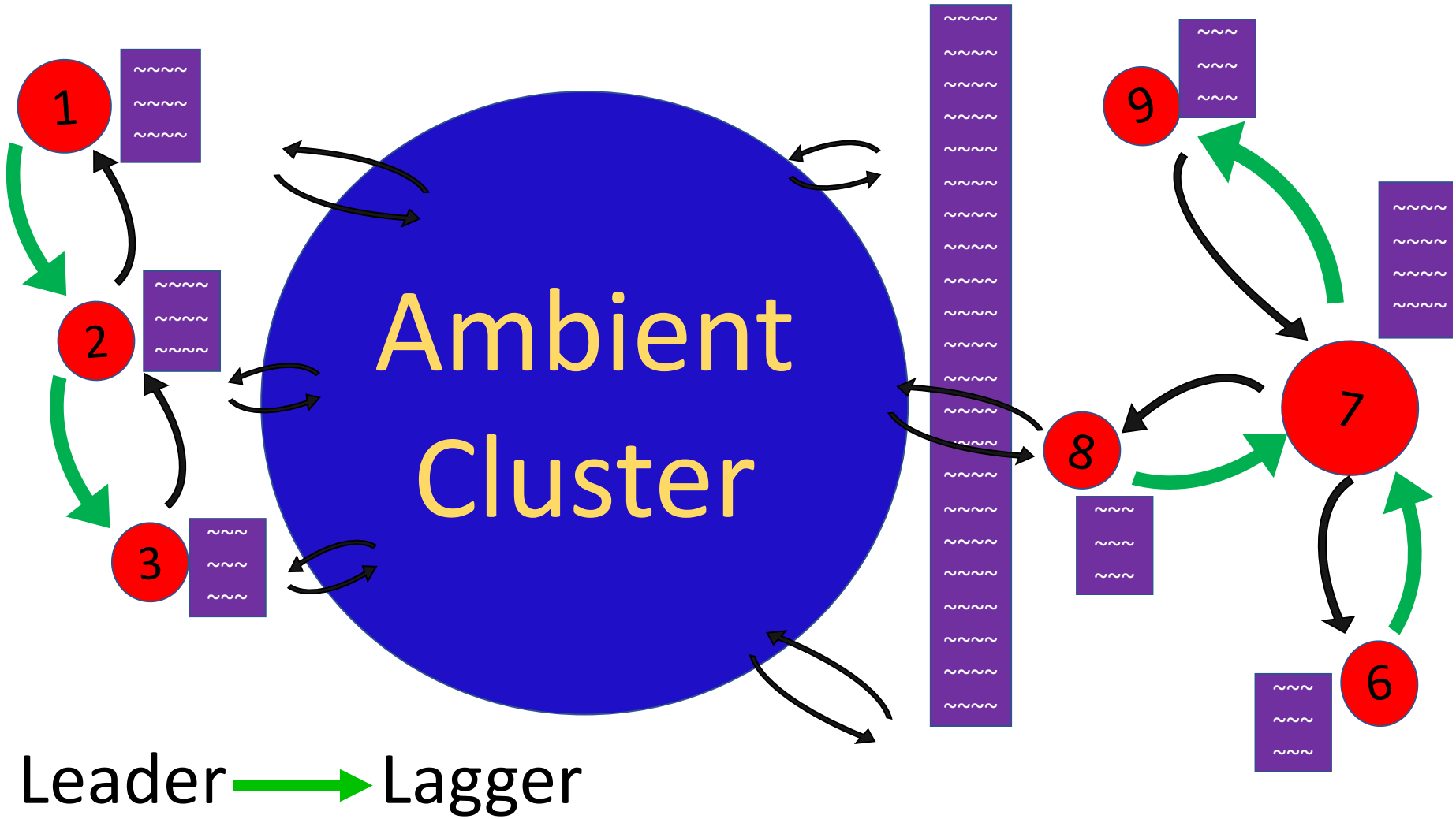
- graph neural network (GNN) for signed/directed/time series networks

Planted structure in a much larger ambient graph + Handling node level covariates



Motivates the use Graph Neural Networks for digraph clustering (ongoing work).

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$$A^+_{\gamma^+} := A^+ + \frac{\gamma^+}{n}\mathbf{1}\mathbf{1}^\top; \qquad A^-_{\gamma^-} := A^- + \frac{\gamma^-}{n}\mathbf{1}\mathbf{1}^\top$$

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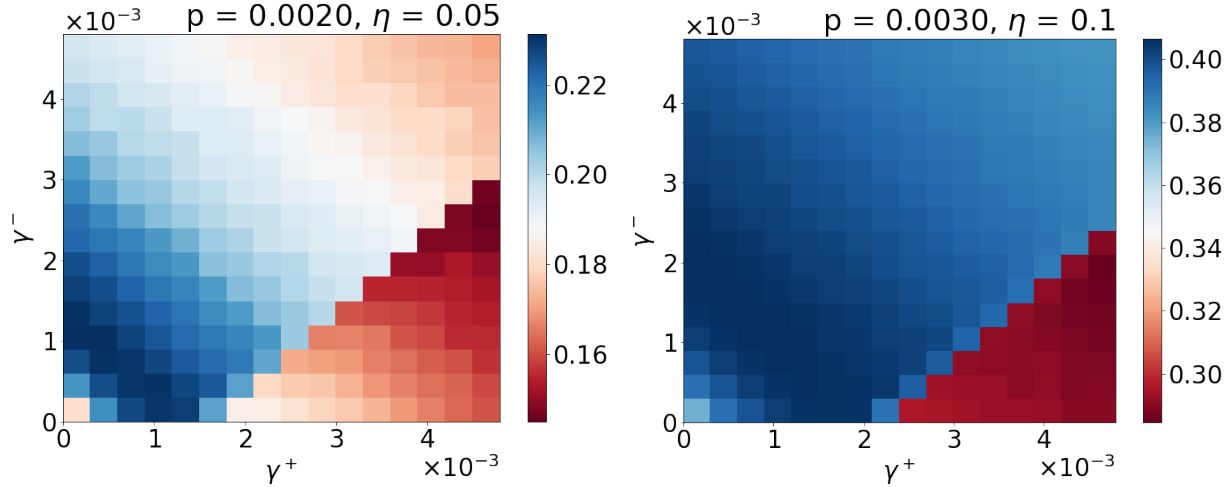
- $L^\pm_{sym,\gamma^\pm}$ the normalized Laplacians corresponding to $A^\pm_{\gamma^\pm}$
- finally, denote L_γ the regularized symmetric Signed Laplacian

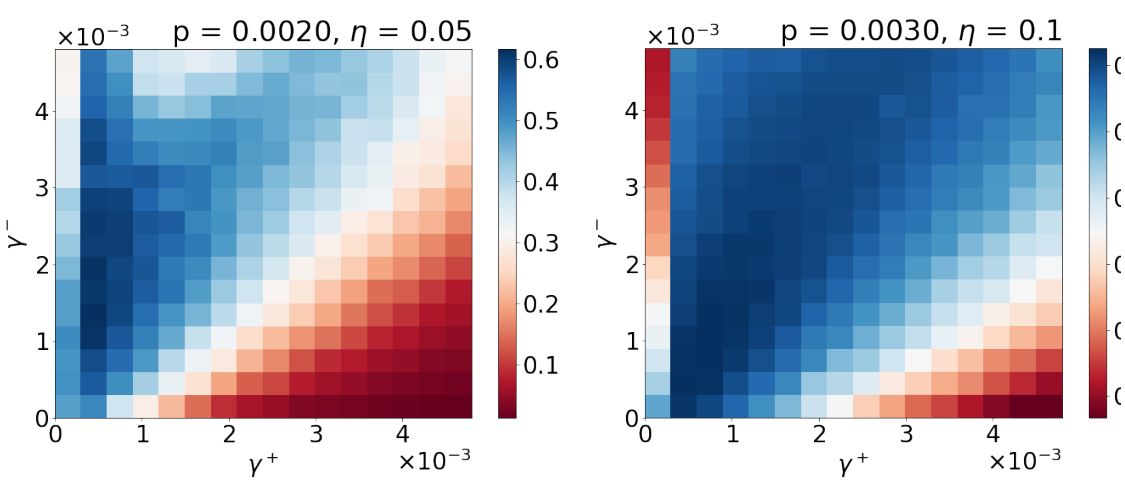
$$L_\gamma = I - (\overline{D}_\gamma)^{-1/2} A_\gamma (\overline{D}_\gamma)^{-1/2}, \tag{31}$$

with $A_\gamma = A^+_{\gamma^+} - A^-_{\gamma^-}$ and $\overline{D}_\gamma = \overline{D} + (\gamma^+ + \gamma^-)I$.

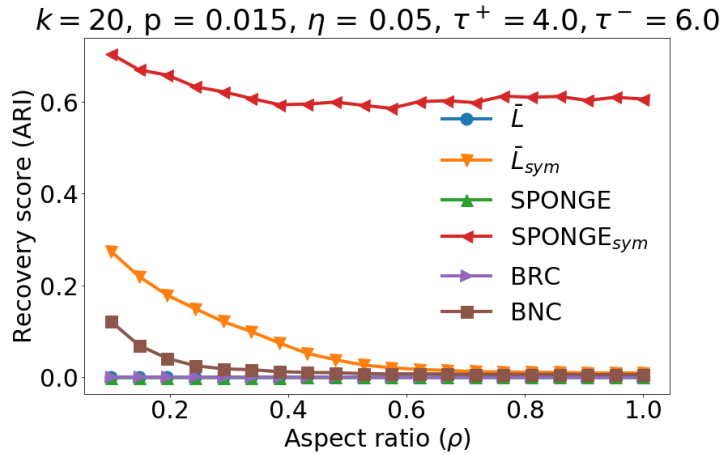
Regularization in the sparse SSBM



(a) Regularized Signed Laplacian L_γ



(b) SPONGE$_{sym}$



(c) Comparison

Figure: Heatmaps of the ARI obtained with the two sparse algorithms, L_γ and SPONGE$_{sym}$, with varying regularization parameters (γ^+, γ^-), for a SSBM in two **sparse** regimes, with $n = 5000$ and $k = 5$ clusters. Plot (c) $k = 20$.