Week 12

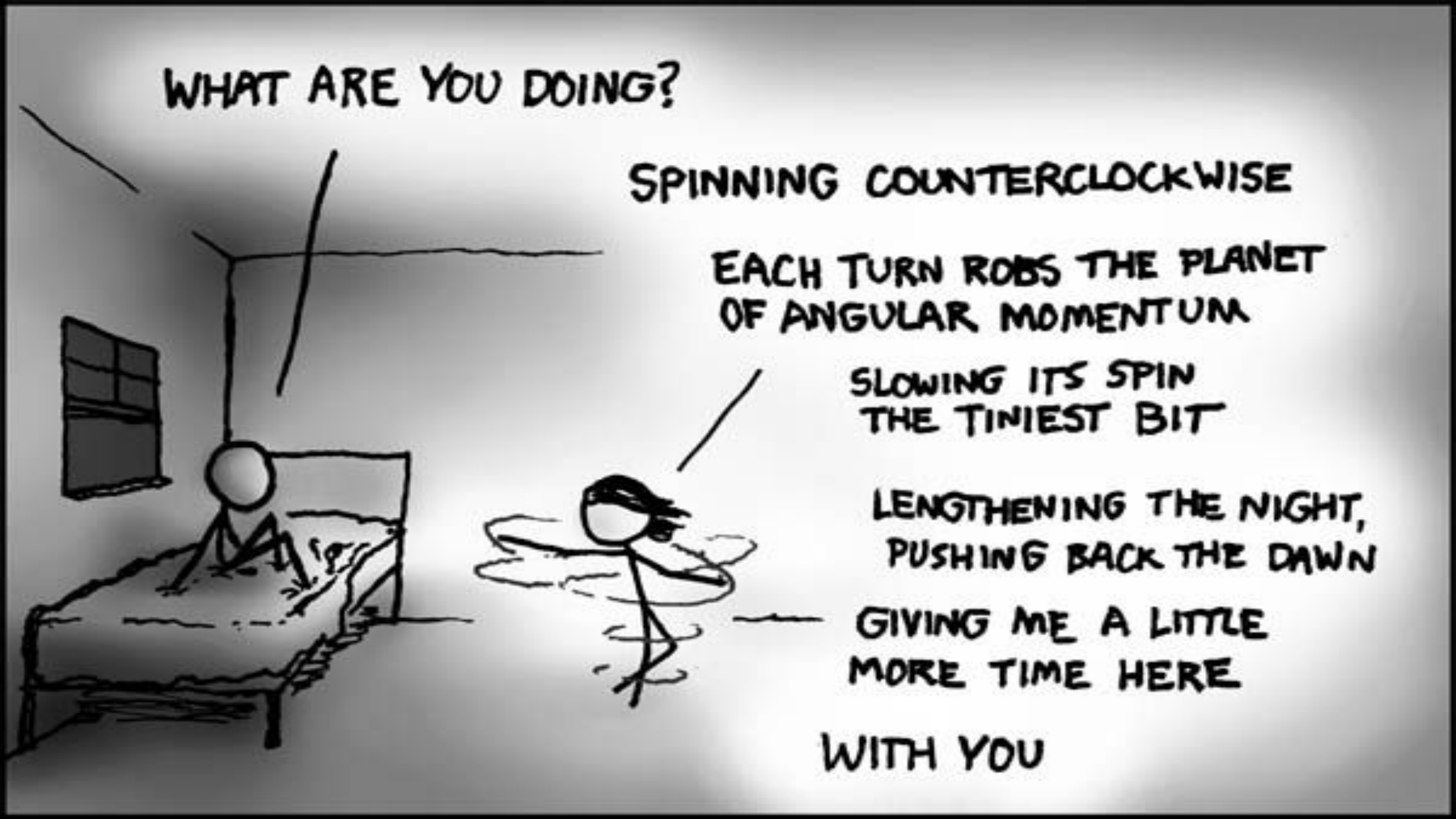
Angular Momentum

Topics for this week

1. Angular Momentum
2. Cross products

By the end of the week

1. Be able to quantify rotations
2. Compute the vector product for any two vectors
3. Be able to laugh at old Simpson's jokes and web comics


WHAT ARE YOU DOING?
SPINNING COUNTERCLOCKWISE
EACH TURN ROBS THE PLANET OF ANGULAR MOMENTUM
SLOWING ITS SPIN THE TINIEST BIT
LENGTHENING THE NIGHT, PUSHING BACK THE DAWN
GIVING ME A LITTLE MORE TIME HERE
WITH YOU

Angular Momentum

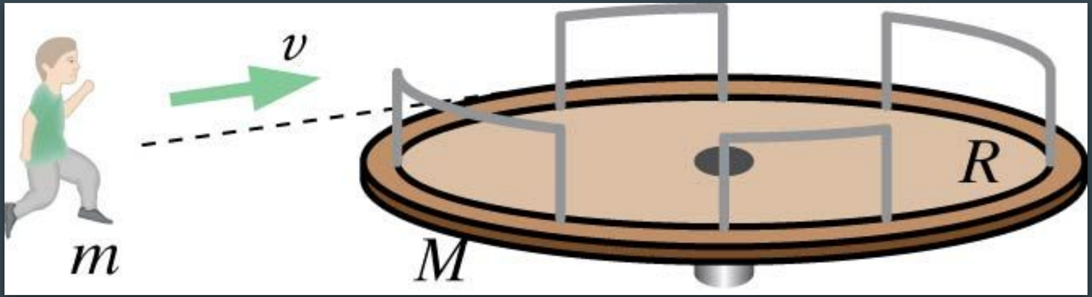
- The angular momentum of an object is a measure of its rotational motion
    - A conserved vector quantity
    - Defined relative to a point “A”

$$\vec{L}_A = \vec{r}_A \times \vec{p}$$

- Angular momentum encompasses
    - Translations of an object around a reference point
    - Rotations of an object about its own center of mass
        - Just like we saw with energy!

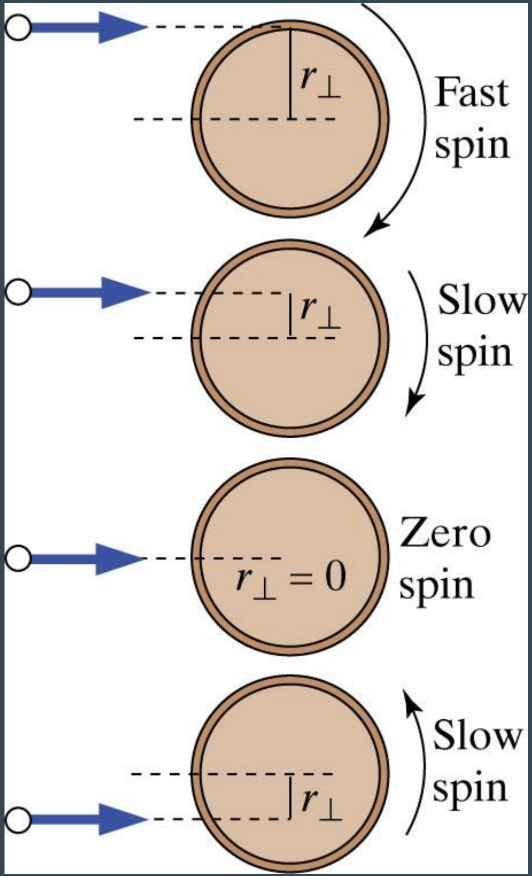
Example: The merry-go-round

- Consider the example of a child running at a constant speed who then jumps onto a merry-go-round
  - The momentum principle tells us about the impulse provided by the merry-go-round axle
  - The energy principle gives us the change in internal energy of the system when the child and merry-go-round stick together
- How can we get information about the rotation rate of the child and merry-go-round after the collision?



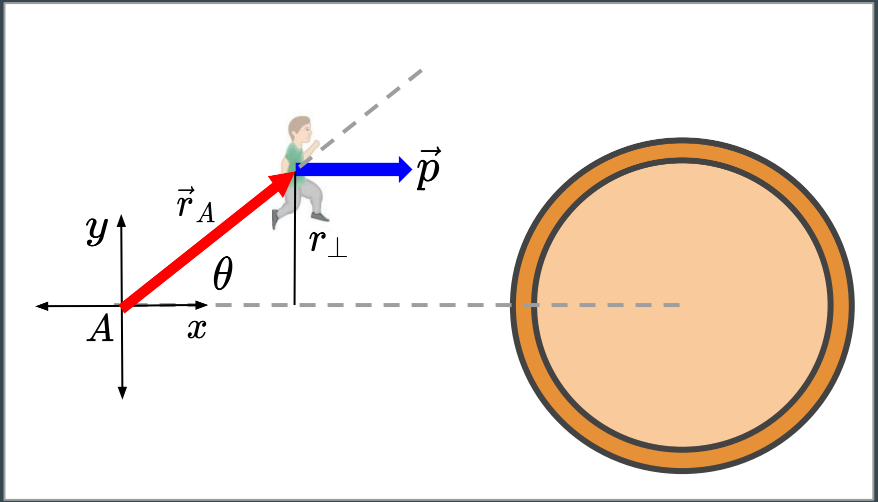
Example: The merry-go-round

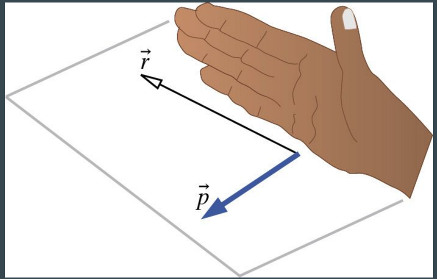
- What effect does the location of the collision between the child and the merry-go-round have on the spin rate?
  - The spin rate should be proportional to the child's initial momentum
    - velocity and mass!
  - The spin rate should be proportional to the distance from the axis of rotation
- Where the child lands will determine the direction of spin
  - Above the axis gives clockwise rotations
  - Below the axis gives counterclockwise rotations

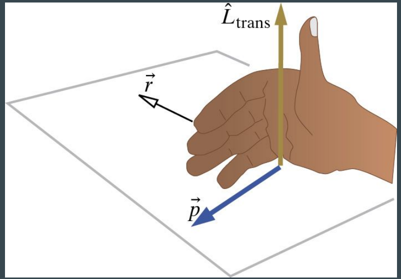


Example: The merry-go-round

- To get the direction of translational angular momentum for the child we need a specific reference point
  - Choose a point “A” to translational angular momentum
    - Rotations around this point “A”
- The right-hand rule
  - With the tails of “r” and “p” together, point your fingers along “r”
  - Rotate your palm towards “p”
  - Your thumb points in the direction of the translational angular momentum







Calculating the cross product

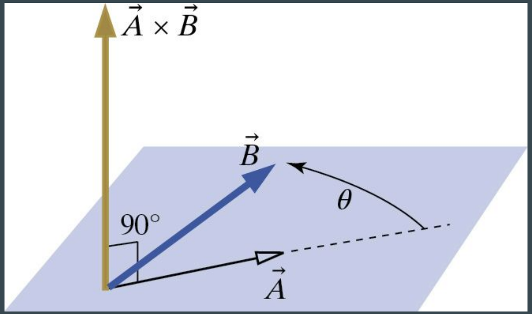
- The “x” in the definition of angular momentum is called the cross product or vector product (it's a vector quantity)
- Direction of the product is found by the right-hand rule
  - We use a right-handed coordinate system

$$\hat{x} \times \hat{y} = \hat{z} \qquad \hat{y} \times \hat{z} = \hat{x} \qquad \hat{z} \times \hat{x} = \hat{y}$$

- Magnitude of the product
  - The product of one vector and the part of the second vector that is perpendicular
    - Where the θ is the angle between the two vectors

$$\vec{A} \times \vec{B} = \vec{C}$$

$$|\vec{C}| = |\vec{A}||\vec{B}|\sin\theta$$

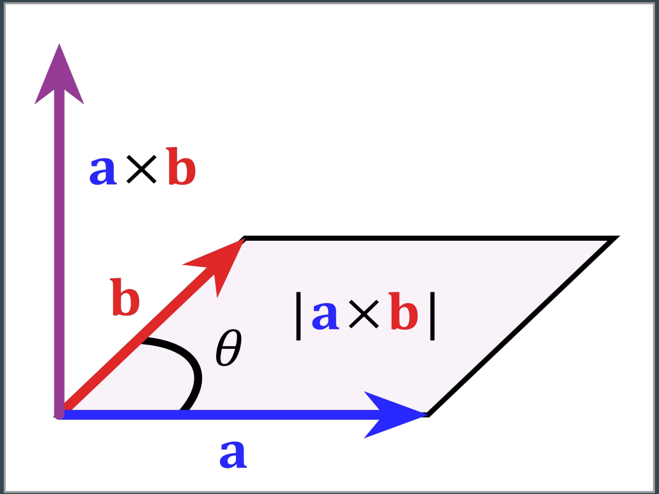


Calculating the cross product cont.

- Graphically the cross product computes a vector that is perpendicular to the plane containing the two crossed vectors
  - Proportional to the area of the parallelogram
  - Not commutative
- We can expand this product using distributivity

$$
\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1b_1(\mathbf{i} \times \mathbf{i}) + a_1b_2(\mathbf{i} \times \mathbf{j}) + a_1b_3(\mathbf{i} \times \mathbf{k}) + \\
&\quad a_2b_1(\mathbf{j} \times \mathbf{i}) + a_2b_2(\mathbf{j} \times \mathbf{j}) + a_2b_3(\mathbf{j} \times \mathbf{k}) + \\
&\quad a_3b_1(\mathbf{k} \times \mathbf{i}) + a_3b_2(\mathbf{k} \times \mathbf{j}) + a_3b_3(\mathbf{k} \times \mathbf{k})
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{a} \times \mathbf{b} = &\; -a_1b_1\mathbf{0} + a_1b_2\mathbf{k} - a_1b_3\mathbf{j} \\
&\; -a_2b_1\mathbf{k} - a_2b_2\mathbf{0} + a_2b_3\mathbf{i} \\
&\; +a_3b_1\mathbf{j} - a_3b_2\mathbf{i} - a_3b_3\mathbf{0} \\
= &\; (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}
\end{aligned}
$$



The total angular momentum

- How does the angular momentum principle generalize to multiparticle systems?
  - The same way we thought of total kinetic energy
- The total L is the superposition of the individual L

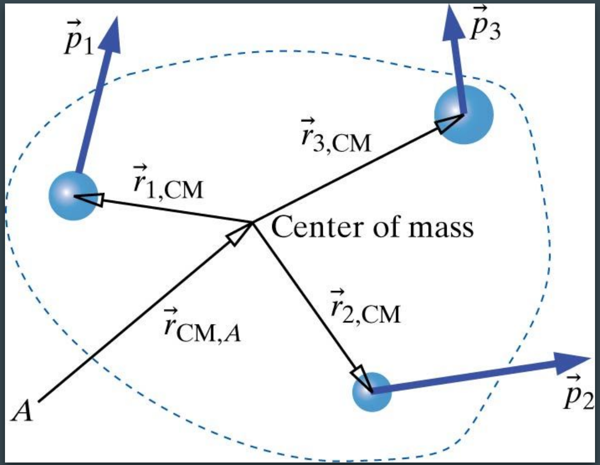
$$\vec{L}_{A,total} = \vec{L}_{A,1} + \vec{L}_{A,2} + \vec{L}_{A,3}$$

  - Define the position of each particle with respect to the center of mass

$$\vec{L}_{A,1} = (\vec{r}_{A,cm} + \vec{r}_{cm,1}) \times \vec{p}_1$$

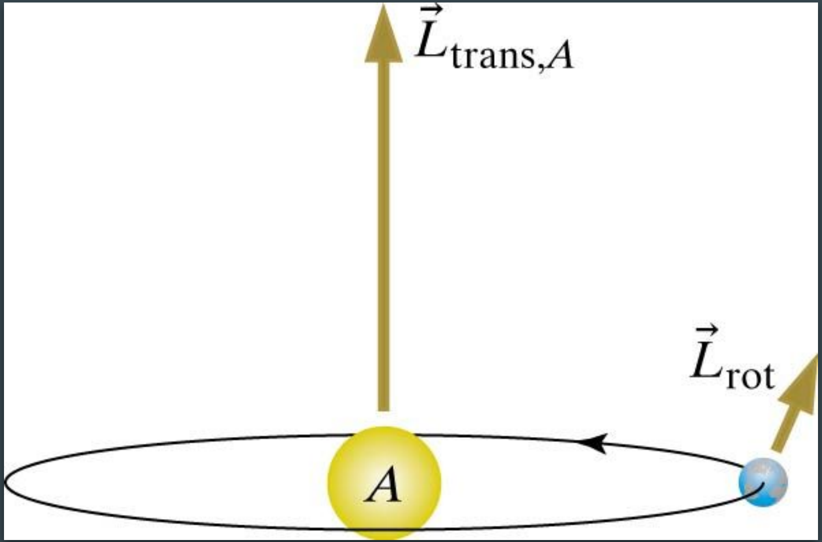
$$\vec{L}_{A,total} = (\vec{r}_{A,cm} \times \vec{p}_{total}) + \sum_{i=1}^{3} \vec{r}_{i,cm} \times \vec{p}_i$$

$$\vec{L}_{A,total} = \vec{L}_{A,trans} + \vec{L}_{cm,rot}$$



Decomposing Angular Momentum

- The translational angular momentum is associated with a rotation of the center of mass about some point A
    - Differs for different choices of the location for the point A
- The rotational angular momentum is associated with a rotation about the center of mass
    - Independent of the location of the point A and the motion of the CM
    - For solid body rotations about a single axis



$$\vec{L}_{rot} = I\vec{\omega} \quad \Rightarrow \quad K_{rot} = \frac{L_{rot}^2}{2I}$$

Changes in angular momentum

- What is the time rate of change of angular momentum for a point particle?
  - Take the derivative

$$\frac{d}{dt}(\vec{r}_A \times \vec{p}) = \left(\frac{d\vec{r}_A}{dt} \times \vec{p}\right) + \left(\vec{r}_A \times \frac{d\vec{p}}{dt}\right)$$

  - The first term is zero because those vectors are parallel
  - The second term can be simplified by substitution of the momentum principle
- Tau stands for the torque or “twist” on the system from the surroundings

$$\frac{d\vec{L}_A}{dt} = \vec{r}_A \times \vec{F}_{net} = \vec{\tau}_{A,net}$$

Torque

- We can conceptualize torque by imagining that we are using a wrench to tighten a bolt.
  - Righty Tighty: Push up on the wrench and the torque on the wrench is LF and the bolt spins into the page
  - Push to the right and the bolt has zero spin
  - Righty Tighty: Push up on the wrench close to the bolt and the torque on the wrench is LF/4 and the bolt spins into the page more slowly than before
  - Righty Tighty: Push up and out on the wrench and the torque on the wrench is LFsinθ and the bolt spins into the page more slowly than before
- What happens if we push down on the wrench?
  - Visualize the direction of the torque and the bolt with the thumb of your right hand!