Calculus BC Equations

Limits

If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$, or $\frac{\infty}{\infty}$
then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ $\quad \lim_{x \to 0} \frac{\sin x}{x} = 1$

Applications of Differentiation

$V = \frac{4}{3}\pi r^3 \quad V = \frac{1}{3}h\pi r^2$

Integration

$$\int u\,dv = uv - \int v\,du \qquad \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Applications of Integration

$\pi \int_a^b (R_2)^2 - (R_1)^2 dx$

Infinite Series

Monotonic Sequence A sequence $\{a_n\}$ that is nondecreasing (i.e. $\{1, 1, 2, 3\}$) where

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$$

or if terms are nonincreasing like

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$$

Bounded Monotonic Sequence A bounded monotonic sequence converges. A sequence is bounded if it bounded above by M and below by N such that $N < a_n < M, \forall n \geq 0$.

Infinite Series Infinite series are defined as

$$S = \sum_{n=1}^{\infty} a_n$$

where $S_n$denotes the n^{th} partial sum

Convergence: For an infinite series $S = \sum a_n$, where $S_n$denotes the n^{th} partial sum, if the sequence $\{S_n\}$ converges to S then the series $S = \sum a_n$ converges. The limit S is called the sum of the series.

Integral Test: For an infinite series $S = \sum f(x)$ if the improper integral $\int f(x) = L$ converges then the series converges and if the improper integral $\int f(x)$ does not exist or is infinity, it diverges. It does not give any information about the actual sum of the series.

P series:

$$N = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

$P = 1$ diverges $P > 1$ converges $P < 1$ diverges $0 > P > 1$ diverges

Taylor Polynomials If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n$$

is defined as the **nth** degree **taylor polynomial**.

Taylor Series If f is infinitely differentiable, then f is represented exactly by the series, centered at $x = c$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Parametric Equations

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

Distance Formula: $\quad \Delta s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$x(t) = x_0 + v_{x0} + \frac{1}{2}at^2$ $\quad$ $y(t) = y_0 + v_{y0} + \frac{1}{2}at^2$

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Projectile Motion

Maximum Height: $\quad H = \frac{v_o^2 \sin\theta^2}{2g}$

Horizontal Range: $\quad R = \frac{v_o^2 \sin 2\theta}{g}$

Flight Time: $\quad t = \frac{2v_{y0} sin\theta}{g}$

Example

Eliminating the Parameter:
Finding a rectangular equation that represents the graph of a set of parametric equations is called *eliminating the parameter*.

1. Parametric Equations $\quad x = t^2 - 4 \quad y = \frac{t}{2}$
2. Solve for t in one equation.
   $x = (2y)^2 - 4$
3. Rectangular Equation
   $x = 4y^2 - 4$

Polar Equations

$$\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

$x = r\cos\theta$ $\quad$ $x^2 + y^2 = r^2$

$y = r\cos\theta$ $\quad$ $\tan\theta = \frac{y}{x}$

$r = a\cos 2\theta$ $\quad$ $r = a\cos 3\theta$ $\quad$ $r = a\cos 4\theta$ $\quad$ $r = a\cos 6\theta$

Vectors

Angle Between Two Vectors
If θ is the angle between two nonzero vectors $\vec{u}$ and $\vec{v}$, then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

Alternatively,

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

This form can be used to calculate the dot product without knowing the component form of the vectors.

Vector-Valued Functions

A function of the form:

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

Can also be written as:

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$$

Projectile Motion equations (without air resistance) can be written as Vector Valued Functions:

$$\vec{s}(t) = \langle x_o + v_{xo}t, y_o + v_{yo}t - \frac{1}{2}gt^2\rangle$$

$$\vec{v}(t) = \langle v_{xo}, v_{yo} - gt\rangle$$

$$\vec{a}(t) = \langle 0, -g\rangle$$

Differential Equations

- Separable Differentiable Equations
  1. Separate the variables into standard form:

     $$F(y)\,dy = G(x)\,dx$$

- First Order Differentiable
  1. Rearrange equation into standard form:
     $y' + py = q$
  2. Integrating factor:
     $u(x) = e^{\int pdx}$
  3. Multiply both sides:
     $uy' + upy = uq$
     $(uy)' = uq$
  4. Integrate:
     $uy = \int (uq)dx$

- Euler's Method Formula

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-\cdots+(-1)^n(x-1)^n+\cdots$$

$$1-x+x^2-x^3+x^4-x^5+\cdots+(-1)^n x^n+\cdots$$

$$(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots+\frac{(-1)^{n-1}(x-1)^n}{n}+\cdots$$

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots+\frac{x^n}{n!}+\cdots$$

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots$$

$$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$$

$$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}+\cdots$$

$$x+\frac{x^3}{2\cdot 3}+\frac{1\cdot 3x^5}{2\cdot 4\cdot 5}+\frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7}+\cdots+\frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)}+\cdots$$

$$1+kx+\frac{k(k-1)x^2}{2!}+\frac{k(k-1)(k-2)x^3}{3!}+\frac{k(k-1)(k-2)(k-3)x^4}{4!}+\cdots$$

Limits

- Limits at infinity
  - Three possibilities for horizontal asymptotes
- Removeable vs. non-removeable discontinuity
  - One-sided limits
- L'Hôpital's Rule
  - Conditions for use

Differentiation

- Definition of derivative at a point
- Derivatives of polynomials, trig, and exponential functions
- Differentiation rules
- Equation of a tangent line to a curve
- Interpreting the signs of the first and second derivative
  - Find the min or max of a function
- Sketching the first and second derivative from a graph
- Implicit differentiation

Applications of Differentiation

- Optimization
  - Distance, area, volume
- Newton's Method
- Related Rates
  - Distance, area, volume, depth, ladder, and shadows

Integration

- Reimann Sums
- Difference between area and definite integral
- Integration by substitution
- Integration by parts
- Partial fractions
  - Distinct linear, repeated linear, quadratic and repeated quadratic factors
- Improper integrals

Applications of Integration

- Volumes of revolution
- Work done by a variable force
- Average value of a function
- Arc length of a curve
- Area between two curves

Infinite Series

- P-series
- Geometric Series
- Convergence or divergence
  - Integral test, ratio test, comparison test and root test
- Taylor polynomial approximation with desired accuracy
- Taylor and Maclaurin series for elementary functions
  - Radius of convergence
  - Interval of convergence

Parametric Equations

- Converting to/from rectangular functions
- Difference between $\frac{dy}{dx}$, $\frac{dy}{dt}$ and $\frac{dx}{dt}$
- Second derivative of a parametric equation
- Arc length
- Projectile Motion: Range, hangtime, and max height

Polar Equations

- Converting to/from rectangular functions
- Area and arc length of polar functions

Vectors

- Dot product of two vectors
- Differentiation and integration of vector-valued functions (Initial value problems)
- Tangential acceleration and centripetal acceleration

Differential Equations

- Logistic differential equations and population growth
- Standard form of first order linear differential equations
- Solve by integrating factor