Rotating Reel [30 pts]

A reel consists of a cylinder of radius R and mass $6M$ with 4 very small (i.e. point) masses M attached at the outer rim of the cylinder (see Figure 1). A reel can freely rotate around a fixed axis through its center. A light rope is wound around the cylinder. At the initial state the reel is motionless. Then a force of constant magnitude F is applied to the rope. At the final state the rope is unwound distance b while the reel acquires angular speed ω (see Figure 2).

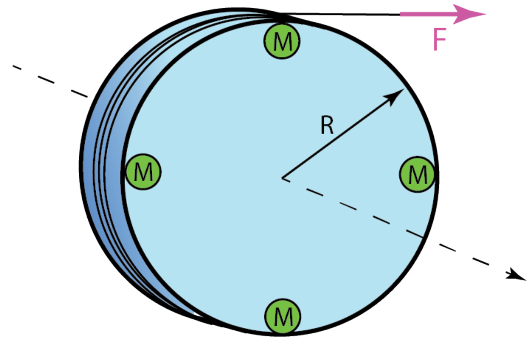


Figure 1. Initial state

Answer all questions in this problem in terms of known quantities R, M, F, b.

1. [10 pts] Determine the total moment of inertia I of the reel.

$$I_{cylinder} = \frac{1}{2}(6M)R^2 = 3MR^2$$

$$I_{point} = 4(MR^2) = 4MR^2$$

$$I = I_{cylinder} + I_{point} = 7MR^2$$

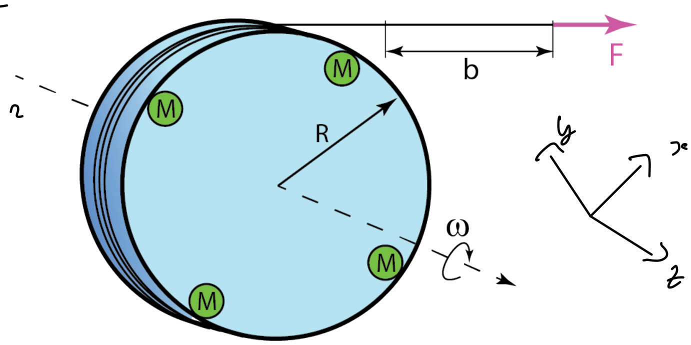


Figure 2. Final state

2. [20 pts] Determine the angular speed ω of the reel at the final state.

$$\Delta E = \cancel{\Delta K_{trans}}^{0} + \cancel{\Delta U}^{0} + \Delta K_{rot} = W$$

$$W = Fb \quad \Rightarrow \quad \Delta K_{rot} = \frac{1}{2} I w^2 - \frac{1}{2} \cancel{I w_i^2}^{0} = Fb$$

$$\Rightarrow \quad w = \sqrt{\frac{2Fb}{I}} = \sqrt{\frac{2Fb}{7MR^2}}$$

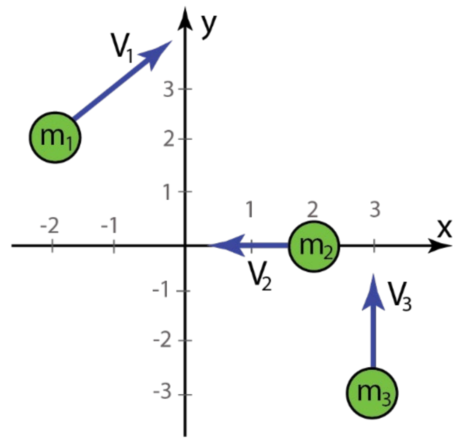
Center of Mass [30 pts]

Three small particles have masses $m_1 = 7.0\ kg,\ m_2 = 5.0\ kg$, and $m_3 = 8.0\ kg$ and are located at

$\vec{r}_1 = \langle -2.0, 2.0, 0.0\rangle m,\ \vec{r}_2 = \langle 2.0, 0.0, 0.0\rangle m,$ and $\vec{r}_3 = \langle 3.0, -3.0, 0.0\rangle m.$

Velocities of these particles are:

$\vec{v}_1 = \langle 5.0, 4.0, 0.0\rangle m/s,\ \vec{v}_2 = \langle -3.0, 0.0, 0.0\rangle m/s,$ and $\vec{v}_3 = \langle 0.0, 4.0, 0.0\rangle m/s.$



1. [8 pts] Find the position $\vec{r}_{CM}$ of the center of mass of this system.

$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3}{m_1 + m_2 + m_3}$$

$$= \langle 1, -0.5, 0 \rangle\ m$$

2. [8 pts] Find the velocity $\vec{V}_{CM}$ of the center of mass of this system.

$$\vec{V}_{cm} = \frac{m_1 \vec{V}_1 + m_2 \vec{V}_2 + m_3 \vec{V}_3}{m_1 + m_2 + m_3}$$

$$= \langle 1, 3, 0 \rangle\ m/s$$

3. [4 pts] Find the translational kinetic energy K_{trans} of this system.

$$K_{trans} = \frac{1}{2}(m_1 + m_2 + m_3)\, V_{cm}^2$$

$$= 100 \text{ J}$$

4. [8 pts] Find the total kinetic energy K_{tot} of this system.

$$K_{tot} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2$$

$$= 230 \text{ J}$$

5. [2 pts] Find the kinetic energy of this system relative to the center of mass K_{rel}.

$$K_{tot} = K_{trans} + K_{rel} \Rightarrow$$

$$K_{rel} = K_{tot} - K_{trans} = 130 \text{ J}$$

Projectile Launch [40 pts]

A projectile (rocket) of mass m is launched from the surface of the Earth with the initial speed $V_i = \sqrt{\frac{5GM}{3R}}$ where G is the universal gravitational constant, M is the mass of the Earth, and R is its radius (see Figure 1).

1. [10 pts] Determine the total energy of the projectile in the initial state (at the launch time, Figure 1). Express your answer in terms of known quantities G, M, m, R.

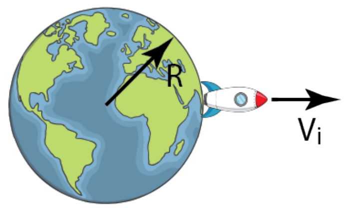


Figure 1. Initial state

$$E_i = U_i + K_i = -\frac{GMm}{R} + \frac{1}{2} m v_i^2$$

$$= -\frac{GMm}{R} + \frac{1}{2} m \left(\frac{5GM}{3R}\right)$$

$$= \frac{-6GMm}{6R} + \frac{5GMm}{6R} = -\frac{GMm}{6R}$$

2. [10 pts] At the final state the projectile is at the maximum height h relative to the Earth's surface and is momentarily at rest (see Figure 2). Express the total energy of the projectile in the final state in terms of given quantities and h.

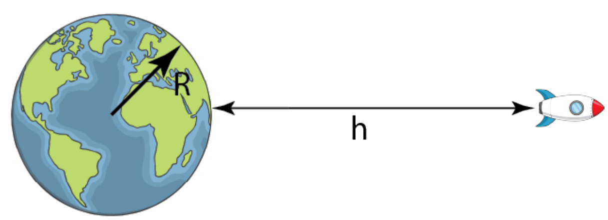


Figure 2. Final state

$$E_f = U_f + \cancel{K_f}^{\,0} = -\frac{GMm}{R+h}$$

3. [10 pts] Determine the maximum height h of the projectile relative to the Earth's surface in terms of R.

$$E_i = E_f$$

$$\Rightarrow \quad -\frac{GMm}{R+h} = -\frac{GMm}{6R} \quad \Rightarrow \quad 6R = R + h$$

$$\Rightarrow \quad h = 5R$$

4. [10 pts] Sketch the gravitational potential energy and the total energy of the projectile between initial and final states as a function *of the distance to the center of the Earth* r on the provided graph.