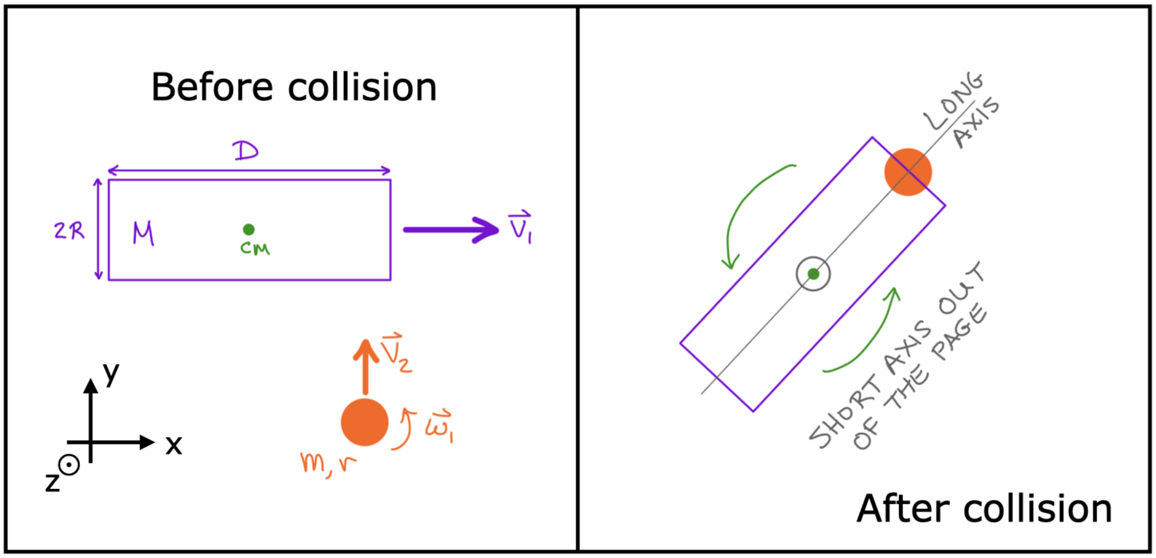
Problem #1

A spaceship with mass M can be modeled as a thick solid cylinder of length D and radius R. It travels through space with speed v_1 to the right, and it is not rotating about any axis. A small, solid, spherical asteroid (mass m, radius r) travels with speed v_2 in the $+\hat{y}$ direction, and it rotates about its own CM counterclockwise with angular speed ω_1. The asteroid and spaceship collide in such a way that the asteroid gets embedded on the front end of the spaceship. After the collision, the ship+asteroid system is rotating counterclockwise about the spaceship's short axis, with an unknown angular speed ω_2.



A. [10 pts] Determine the total angular momentum of the ship+asteroid system immediately before the collision. Use the center of mass of the ship as the reference point.

Immediately before collision,

$$\vec{L}_{ship} = 0, \quad \vec{L}_{ast} = \vec{r}_{ast} \times m\vec{v}_2 + I_{ast}\vec{\omega}_1$$

$$\vec{r}_{ast} \times m\vec{v}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ D/2 & -R-r & 0 \\ 0 & mv_2 & 0 \end{vmatrix} = \frac{mv_2 D}{2}\hat{z}$$

assumes asteroid is just touching ship

$$I_{ast}\vec{\omega}_1 = \frac{2}{5}mr^2\omega_1\hat{z}$$

$$\Rightarrow \boxed{\vec{L}_{total,i} = \left[\frac{mv_2 D}{2} + \frac{2}{5}mr^2\omega_1\right]\hat{z}}$$

B. [10 pts] Determine the final angular speed ω_2 for the ship+asteroid system after the collision. The moment of inertia of a solid cylinder about its short axis is $I_c = (1/12)MD^2 + (1/4)MR^2$, and the moment of inertia of a solid sphere about its center of mass is $I_s = (2/5)mr^2$. You don't need to simplify the final answer.

$\vec{L}_f = I_{total}\vec{\omega}_2$. $I_{ship} = \frac{1}{12}MD^2 + \frac{1}{4}MR^2$, and $I_{sat} = \frac{2}{5}mr^2 + m\left(\frac{D}{2}\right)^2$

$\Rightarrow I_{total} = \frac{1}{12}MD^2 + \frac{1}{4}MR^2 + \frac{2}{5}mr^2 + \frac{mD^2}{4}$

Since $\vec{L}_f = \vec{L}_i$,

$$I_{total}\vec{\omega}_2 = \left[\frac{mv_2 D}{2} + \frac{2}{5}mr^2\omega_1\right]\hat{z}$$

$$\Rightarrow \boxed{\omega_2 = \frac{\frac{mv_2 D}{2} + \frac{2}{5}mr^2\omega_1}{\frac{1}{12}MD^2 + \frac{1}{4}MR^2 + \frac{2}{5}mr^2 + \frac{mD^2}{4}}}$$

Problem #2

Approximate the total angular momentum of the solar system. Make sure to list all simplifying assumptions you've made. You can find planetary information (masses, distances, speeds, etc) online.

NOTE: This is an open-ended question with no one correct answer. Let the students explore a bit. Here are some questions you might consider asking to guide the students if they need it.

1. What does it mean conceptually to calculate the angular momentum of the Solar System?

2. How would you go about actually numerically calculating/estimating the angular momentum of the Solar System?

3. As an example, give an order of magnitude estimate of the contribution of Earth to the total angular momentum of the Solar System. Take into account the Earth's rotational angular momentum and its orbital angular momentum. (You may neglect the obliquity, i.e. tilt of its rotational axis, of Earth. As an extra challenge, however, you could try including the effect of the Earth's 23.5 degree tilt.)

4. What object gives the largest contribution to the Solar System's angular momentum, and what is its angular momentum? It turns out that this object dominates the Solar System's calculated angular momentum and therefore can be used to estimate the order of magnitude of the Solar System's angular momentum. Estimate the order of magnitude of the Solar System's angular momentum.

Why do we care? What can we do with this information? Learn something about the history of the Solar System (i.e. its formation and anomalies like Uranus' rotation).

(Pick a location about which to find L, most naturally the center of the Sun)

(Assume they all orbit in circular orbits in a plane (not true, but approximately) and they all spin on an axis perpendicular to orbital plane

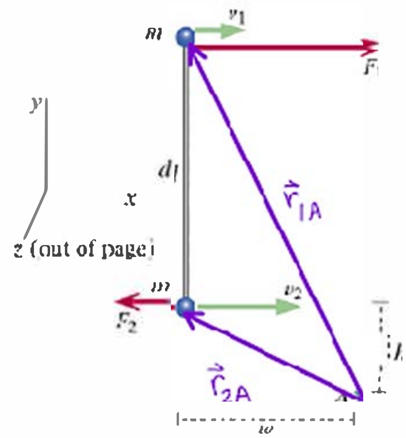
(Why is translational and rotational angular momentum additive? See

https://scripts.mit.edu/~srayyan/PERwiki/index.php?title=Module_3_--_Angular_Momentum_of_a_Rigid_Body_both_Rotating_and_Translating

for proof (for 2 bodies, then can be generalized)

Problem #3

In the figure two small objects each of mass $m = 0.235$ kg are connected by a lightweight rod of length $d = 1.20$ m. At a particular instant they have speeds $v_1 = 25$ m/s and $v_2 = 58$ m/s and are subjected to external forces $F_1 = 41$ N and $F_2 = 16$ N. A point is located distances $w = 0.80$ m and $h = 0.32$ m from the bottom object. No other external forces are acting on this system.



(a) What is the velocity of the center of mass?

$$\vec{P}_{total} = M_{total}\vec{V}_{cm} \Longrightarrow \vec{V}_{cm} = \frac{\vec{P}_{total}}{M_{total}} = \frac{\vec{p}_1+\vec{p}_2}{m_1+m_2} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} =$$

$$= \frac{(0.235)(25)\,\hat{x} + (0.235)(58)\,\hat{x}}{0.235+0.235} = \frac{\cancel{0.235}(25+58)\,\hat{x}}{2(\cancel{0.235})} =$$

$$= \boxed{41.5 \text{ m/s } \hat{x}}$$

(b) What is the total angular momentum of the system relative to point A?

$$\vec{L}_A = \vec{L}_{1A} + \vec{L}_{2A} = (\vec{r}_{1A}\times\vec{p}_1) + (\vec{r}_{2A}\times\vec{p}_2) = (h+d)(m_1v_1)(-\hat{z}) + (h)(m_2v_2)(-\hat{z}) =$$

$$= (-\hat{z})(m)\left[(h+d)v_1 + hv_2\right] = (-\hat{z})(0.235)\left[(0.32+1.20)(25) + (0.32)(58)\right] =$$

$$= (-\hat{z})(0.235)(38+18.56) = \boxed{13.29 \text{ kg m}^2/\text{s } (-\hat{z})}$$

(c) What is the rotational angular momentum of the system?

$$\vec{L}_{total} = \vec{L}_{trans} + \vec{L}_{rot} \Rightarrow \vec{L}_{rot} = \vec{L}_{total} - \vec{L}_{trans}$$

$$\checkmark\ \vec{L}_{trans} = \vec{r}_{cm} \times \vec{P}_{total} = (h + \tfrac{d}{2})(\hat{y}) \times (M_{total}\vec{V}_{cm}) =$$

$$= (0.32 + \tfrac{1.2}{2})(0.235 + 0.235)\,\hat{y} \times (41.5)\,\hat{x} = 17.9\ \text{kgm}^2/\text{s}\ (-\hat{z})$$

$$\Rightarrow \vec{L}_{rot} = 13.29\,(-\hat{z}) - 17.9\,(-\hat{z}) = (-13.29 - -17.9)\,\hat{z} =$$

$$= (-13.29 + 17.9)\,\hat{z} = \boxed{4.61\ \text{kgm}^2/\text{s}\ (\hat{z})}$$

(d) After a short time interval $\Delta t = 0.035$ s, determine the total (linear) momentum of the system?

$$\vec{P}_f = \vec{P}_i + \vec{F}_{net}\Delta t = (2)(0.235)(41.5)\,\hat{x} + (41 - 16)(\hat{x})(0.035) =$$

$$= \boxed{20.38\ \text{kgm/s}\ \hat{x}}$$

(e) Calculate the new rotational angular momentum of the system?

$$\vec{\tau}_{net} = \vec{\tau}_1 + \vec{\tau}_2 = (\vec{r}_1 \times \vec{F}_1) + (\vec{r}_2 \times \vec{F}_2) = \tfrac{d}{2}F_1\,(-\hat{z}) + \tfrac{d}{2}F_2\,(-\hat{z}) =$$

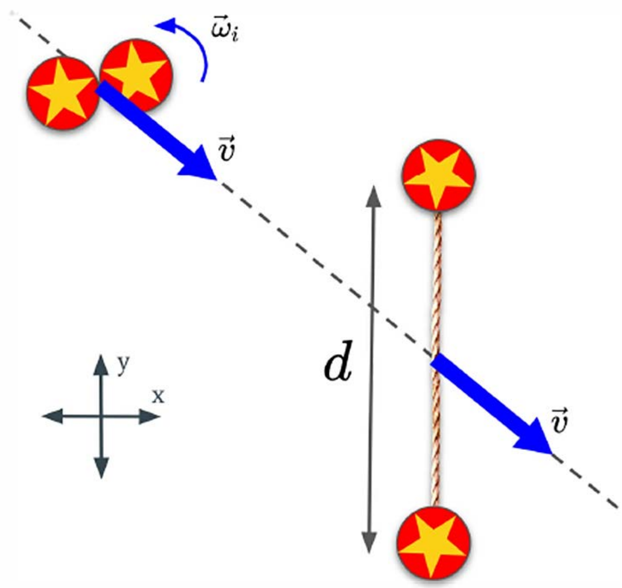
$$= (-\hat{z})\,\tfrac{d}{2}(F_1 + F_2) = (-\hat{z})\left(\tfrac{1.2}{2}\right)(41 + 16) = 34.2\ \text{Nm}\ (-\hat{z})$$

$$\vec{L}_{rot\,f} = \vec{L}_{rot\,i} + \vec{\tau}_{net}\Delta t = 4.61\,(\hat{z}) + (34.2)(0.035)(-\hat{z}) = \boxed{3.4\ \text{kgm}^2/\text{s}\ (\hat{z})}$$

Problem #4

A yoyo can be approximated as a solid disk of mass m and radius R. Two identical such yoyos have their strings tied together and are wound so that the two yoyos are touching each other. These stuck together yoyos are ejected into deep space far from any other objects. Shortly after being ejected, the center of mass of the yoyos has an initial velocity $\vec{v}$ as indicated in the diagram. At this instant, the stuck together yoyos are rotating about the center of mass counterclockwise with an angular speed $|\vec{\omega}_i|$. As the yoyos fly through space the strings unwind so that at some later time all of the string has unwound from each yoyo. At this time, the velocity of the center of mass is $\vec{v}$ and the distance between the centers of the yoyos is d.

Determine the unknown angular velocity (magnitude and direction) of the center of mass for the tied together yoyos in the final state. You can neglect the mass of the string and you can assume that the yoyos are tied to the string so that the string is not slipping on the axle of the yoyo.



Key insight: the angular frequency that the yoyo rotates about its own enter of mass is the same as the angular frequency that the center of mass of the yoyo rotates about the center of mass of the system.
The net external torque is zero, so the angular momentum is conserved,

$$\Delta\vec{L} = 0$$

$$\begin{aligned}\vec{L}_i &= \vec{L}_{cm} + \vec{L}_{rot,i}\\ &= \vec{L}_{cm} + 2(\vec{L}_{yo,cm} + \vec{L}_{yo,rot}),\end{aligned}$$

where $\vec{L}_{cm}$ is the angular momentum of the system's center of mass about some point (it doesn't matter which), $\vec{L}_{yo,cm}$ is the angular momentum of the center of mass of the yoyo about the system's center of mass, and $\vec{L}_{yo,rot}$ is the angular momentum of the yoyo about its own center of mass.

$$\begin{aligned}\vec{L}_{yo,cm} &= \vec{r}_{yo,cm} \times \vec{p}_{yo,cm}\\ &= Rm\vec{\omega}_i R = mR^2\vec{\omega}_i\end{aligned}$$

$$\begin{aligned}\vec{L}_{yo,rot} &= I\vec{\omega}\\ &= \frac{1}{2}mR^2\vec{\omega}_i\end{aligned}$$

$$\vec{L}_i = \vec{L}_{cm} + 3mR^2\vec{\omega}_i$$

By the same logic for the final time,

$$\begin{aligned}\vec{L}_{yo,cm} &= \vec{r}_{yo,cm} \times \vec{p}_{yo,cm} \\ &= \frac{d}{2}m\vec{\omega}_f\frac{d}{2} = \frac{md^2}{4}\vec{\omega}_f\end{aligned}$$

$$\begin{aligned}\vec{L}_{yo,rot} &= I\vec{\omega} \\ &= \frac{1}{2}mR^2\vec{\omega}_f\end{aligned}$$

$$\vec{L}_f = \vec{L}_{cm} + \left(\frac{md^2}{2} + mR^2\right)\vec{\omega}_f$$

Equating the initial and final, the angular momentum of the system's center of mass cancels and therefore

$$\vec{\omega}_f = \frac{3R^2}{\frac{d^2}{2} + R^2}\vec{\omega}_i$$