Derivation of kinematics formulas

$$x_f = x_i + v_{ix} \Delta t$$

$$y_f = y_i + v_{iy} \Delta t - \frac{1}{2} g (\Delta t)^2$$

Without explicit calculus

Newton's 2nd: $\vec{v}_f = \vec{v}_i + \frac{\vec{F}_{net}}{m} \Delta t$

Gravity near Earth: $\vec{F}_{net} = \langle 0, -mg, 0 \rangle$

Position update: $\vec{r}_f = \vec{r}_i + \vec{v}_{AVG} \Delta t$

For a constant force, $\vec{v}_{AVG}$ = arithmetic average

$$\vec{v}_{AVG} = \frac{\vec{v}_i + \vec{v}_f}{2}$$

$$2 \vec{v}_{AVG} = \vec{v}_i + \vec{v}_f$$

$$2 \vec{v}_{AVG} - \vec{v}_i = \vec{v}_f$$

Now we equate this $\vec{v}_f$ to Newton's 2nd 's $\vec{v}_f$

$$2\vec{v}_{Avg} - \vec{v}_i = \vec{v}_i + \frac{\vec{F}_{net}}{m}\Delta t$$

$$2\vec{v}_{Avg} = \vec{v}_i + \vec{v}_i + \frac{\vec{F}_{net}}{m}\Delta t$$

$$2\vec{v}_{Avg} = 2\vec{v}_i + \frac{\vec{F}_{net}}{m}\Delta t$$

$$\vec{v}_{Avg} = \vec{v}_i + \frac{1}{2}\frac{\vec{F}_{net}}{m}\Delta t$$

Now we use this $\vec{v}_{Avg}$ in the position update:

$$\vec{r}_f = \vec{r}_i + \vec{v}_{Avg}\Delta t$$

$$\vec{r}_f = \vec{r}_i + \left(\vec{v}_i + \frac{1}{2}\frac{\vec{F}_{net}}{m}\Delta t\right)\Delta t$$

$$\boxed{\vec{r}_f = \vec{r}_i + \vec{v}_i\Delta t + \frac{1}{2}\frac{\vec{F}_{net}}{m}(\Delta t)^2}$$

With $\vec{F}_{net} = \langle 0, -mg, 0\rangle$,

$$\boxed{x_f = x_i + v_{ix}\Delta t}$$

$$y_f = y_i + v_{iy}\Delta t + \frac{1}{2}\left(\frac{-\cancel{m}g}{\cancel{m}}\right)(\Delta t)^2$$

$$\Rightarrow \boxed{y_f = y_i + v_{iy}\Delta t - \frac{1}{2}g(\Delta t)^2}$$

Derivation using calculus explicitly

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$

$\vec{F}_{net}$ = constant

$$\vec{F}_{net}\, dt = d\vec{p}$$

first integration

$$\int \vec{F}_{net}\, dt = \int d\vec{p}$$

$$\vec{F}_{net} \int_{t_0}^{t_f} dt = \int_{\vec{p}_i}^{\vec{p}_f} d\vec{p}$$

Note: we'll let $t_0 = 0$ be the initial time to simplify the derivation. Later on, you can replace t with $t_f - t_i$ and it's the same thing

$$\vec{F}_{net}\,(t)\Big|_0^{t_f} = \vec{p}\,\Big|_{\vec{p}_i}^{\vec{p}_f}$$

$$\vec{F}_{net}\, t_f = \vec{p}_f - \vec{p}_i$$

$$\frac{\vec{F}_{net}}{m} t_f = \frac{\vec{p}_f}{m} - \frac{\vec{p}_i}{m}$$

$$\frac{\vec{F}_{net}}{m} t_f = \vec{v}_f - \vec{v}_i$$

Let $t = t_f$ and $\vec{v} = \vec{v}_f$, then:

$$\frac{\vec{F}_{net}}{m} t = \vec{v} - \vec{v}_i$$

$\vec{v}_i$ is constant (initial conditions)

$$\frac{\vec{F}_{net}}{m} t + \vec{v}_i = \vec{v}$$

$$\vec{V} = \vec{v}_i + \frac{\vec{F}_{net}}{m} t$$

Remember that $\vec{V} = \frac{d\vec{r}}{dt}$

$$\frac{d\vec{r}}{dt} = \vec{v}_i + \frac{\vec{F}_{net}}{m} t$$

$$d\vec{r} = \left(\vec{v}_i + \frac{\vec{F}_{net}}{m} t\right) dt$$

Second integration

$$\int_{\vec{r}_i}^{\vec{r}_f} d\vec{r} = \int_{t_0=0}^{t_f} \vec{v}_i \, dt + \frac{\vec{F}_{net}}{m} \int_{t_0=0}^{t_f} t \, dt$$

$$\vec{r} \Big|_{\vec{r}_i}^{\vec{r}_f} = \vec{v}_i t \Big|_0^{t_f = t} + \frac{\vec{F}_{net}}{m} \left(\frac{1}{2} t^2\right) \Big|_0^{t_f = t}$$

$$\vec{r}_f - \vec{r}_i = \vec{v}_i t + \frac{1}{2} \frac{\vec{F}_{net}}{m} t^2$$

Remember that you can substitute $t = t_f - t_i$ to make it an interval, meaning:

$$t \longrightarrow \Delta t$$

$$\vec{r}_f - \vec{r}_i = \vec{v}_i t + \frac{1}{2}\frac{\vec{F}_{net}}{m} t^2$$

$$\vec{r}_f = \vec{r}_i + \vec{v}_i t + \frac{1}{2}\frac{\vec{F}_{net}}{m} t^2$$

With $\vec{F}_{net} = \langle 0, -mg, 0 \rangle$, then:

$$\boxed{x_f = x_i + v_{ix} t}$$

$$y_f = y_i + v_{iy} t + \frac{1}{2}\left(\frac{-\cancel{m}g}{\cancel{m}}\right) t^2 =$$

$$\Rightarrow \boxed{y_f = y_i + v_{iy} t - \frac{1}{2} g t^2}$$

And thus we arrive at the same kinematics equations that we did in the other derivation.