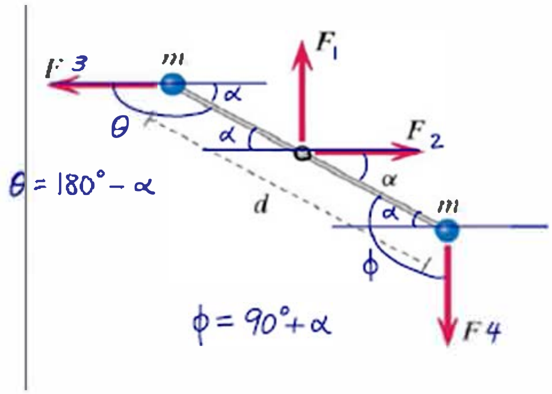
Problem #1

A barbell is mounted on a nearly frictionless axle through its center of mass. The rod has negligible mass and a length d. Each ball has a mass m. At the instant shown, there are four forces of equal magnitude F applied to the system, with the directions indicated. At this instant, the angular velocity is ω_i, counterclockwise (positive), and the bar makes an angle α (which is less than 45 degrees) with the horizontal.



(a) Calculate the magnitude of the net torque on the barbell about the center of mass.

$$\vec{\tau}_1 = \vec{r}_1 \times \vec{F}_1 = 0 \quad b/c \ \vec{r}_1 = 0$$

$$\vec{\tau}_2 = \vec{r}_2 \times \vec{F}_2 = 0 \quad b/c \ \vec{r}_2 = 0$$

$$\vec{\tau}_3 = \vec{r}_3 \times \vec{F}_3 = r_3 F_3 \sin\theta \ (\hat{z}) = \frac{d}{2} F \sin(180° - \alpha) \ (\hat{z}) = \frac{dF}{2} \sin\alpha \ (\hat{z})$$

$$\vec{\tau}_4 = \vec{r}_4 \times \vec{F}_4 = r_4 F_4 \sin\phi \ (-\hat{z}) = \frac{d}{2} F \sin(90° + \alpha) \ (-\hat{z}) = \frac{dF}{2} \cos\alpha \ (-\hat{z})$$

$$\vec{\tau}_{Net} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \vec{\tau}_4 = \frac{dF}{2} \sin\alpha \ (\hat{z}) + \frac{dF}{2} \cos\alpha \ (-\hat{z})$$

$$\Rightarrow \boxed{|\vec{\tau}_{Net}| = \frac{dF}{2} \left| \sin\alpha - \cos\alpha \right|}$$

Since $\alpha < 45°$, then $\cos\alpha > \sin\alpha$, which means $|\vec{\tau}_4| > |\vec{\tau}_3|$

$\Rightarrow$ $\boxed{\text{direction of } \vec{\tau}_{Net} \text{ is } (-\hat{z}), \text{ into the page}}$

(b) Select the statement that accurately describes the situation in the figure:

A. α is less than 45 degrees, so the net torque on the system is in the clockwise direction. This means the torque is out of the page.

B. α is less than 45 degrees, so the net torque on the system is in the counterclockwise direction. This means the torque is into the page.

C. α is less than 45 degrees, so the net torque on the system is in the counterclockwise direction. This means the torque is out of the page.

D. α is less than 45 degrees, so the net torque on the system is in the clockwise direction. This means the torque is into the page.

(c) Determine the moment of inertia, about the center of mass, for the barbell.

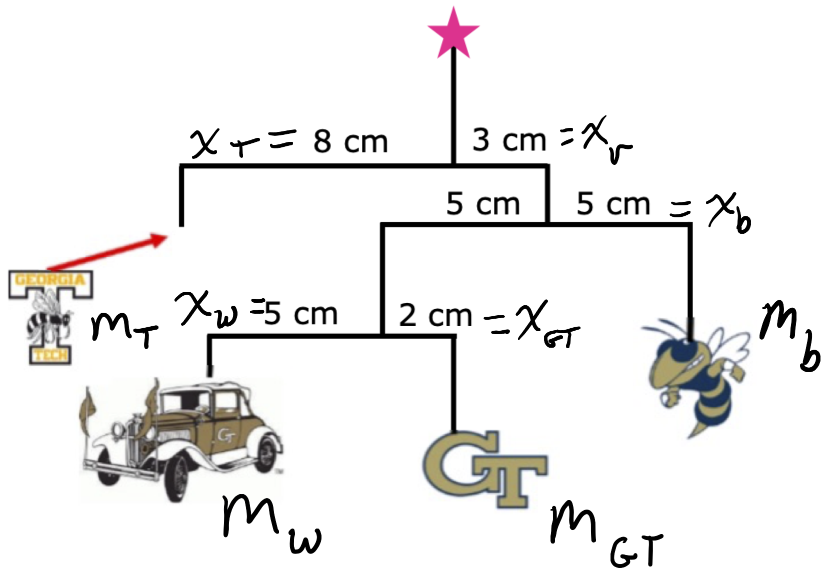
$$I_{CM} = m_1 r_1^2 + m_2 r_2^2 = m\left(\frac{d}{2}\right)^2 + m\left(\frac{d}{2}\right)^2 = \frac{md^2}{4} + \frac{md^2}{4} =$$

$$= \frac{\cancel{2}md^2}{\cancel{4}_{2}} = \boxed{\frac{1}{2}md^2}$$

Problem #2

1. You found a GT mobile in a store but it's missing a piece (a "T", of course). You buy it anyway and make a T to add to the mobile. You measure the lengths of all the (horizontal) arms of the mobile (measure-ments in the figure) and you find that Buzz has a mass of $m_b = 300$ g. What should be the mass of the T (m_T), so that when you at-tach it the mobile stays balanced (unmoving)?

Hints: (1) a balanced mobile experiences zero net gravitational torque; (2) notice that the Wreck and GT are attached to an arm that is the same length as the arm holding up Buzz; (3) remember to use standard SI units in your final answer.



Let's balance each level of the mobile.

$m_w x_w = m_{GT} x_{GT}$. Also,

$m_b x_b = (m_{GT} + m_w) x_b \Rightarrow m_b = m_{GT} + m_w$.

Therefore, the topmost right arm of the mobile has a combined mass of $2m_b$. Finally, balancing the top arms,

$m_T x_T = (2m_b) x_r$

$$\Rightarrow m_T = \frac{2m_b x_r}{x_T} = \frac{2(.3\,kg)(0.03\,m)}{(0.08\,m)} = \boxed{0.225\ kg}$$

Alternatively,

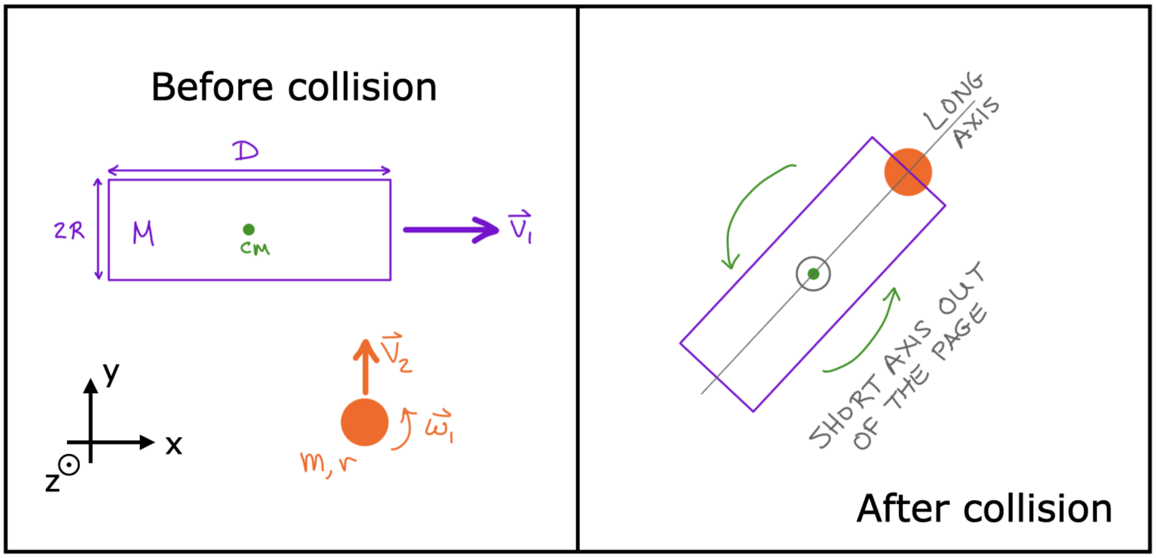
$$|\vec{\tau}_T| = |\vec{\tau}_r| \Rightarrow x_T F_T = x_r F_r$$

$$\Rightarrow x_T m_T \cancel{g} = x_r (m_w + m_{GT} + m_b) \cancel{g}. \quad m_b = m_w + m_{GT}$$

$$\Rightarrow m_T = \frac{2m_b x_r}{x_T} = 0.225\ kg.$$

Problem # 3

A spaceship with mass M can be modeled as a thick solid cylinder of length D and radius R. It travels through space with speed v_1 to the right, and it is not rotating about any axis. A small, solid, spherical asteroid (mass m, radius r) travels with speed v_2 in the $+\hat{y}$ direction, and it rotates about its own CM counterclockwise with angular speed ω_1. The asteroid and spaceship collide in such a way that the asteroid gets embedded on the front end of the spaceship. After the collision, the ship+asteroid system is rotating counterclockwise about the spaceship's short axis, with an unknown angular speed ω_2.



A. [10 pts] Determine the total angular momentum of the ship+asteroid system immediately before the collision. Use the center of mass of the ship as the reference point.

Immediately before collision,

$$\vec{L}_{ship} = 0, \quad \vec{L}_{ast} = \vec{r}_{ast} \times m\vec{v}_2 + I_{ast}\vec{\omega}_1$$

$$\vec{r}_{ast} \times m\vec{v}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ D/2 & -R-r & 0 \\ 0 & mv_2 & 0 \end{vmatrix} = \frac{mv_2 D}{2}\hat{z}$$

$$I_{ast}\vec{\omega}_1 = \frac{2}{5}mr^2\omega_1\hat{z}$$

$$\Rightarrow \boxed{\vec{L}_{total,i} = \left[\frac{mv_2 D}{2} + \frac{2}{5}mr^2\omega_1\right]\hat{z}}$$

B. [10 pts] Determine the final angular speed ω_2 for the ship+asteroid system after the collision. The moment of inertia of a solid cylinder about its short axis is $I_c = (1/12)MD^2 + (1/4)MR^2$, and the moment of inertia of a solid sphere about its center of mass is $I_s = (2/5)mr^2$. You don't need to simplify the final answer.

$\vec{L}_f = I_{total}\vec{\omega}_2$. $I_{ship} = \frac{1}{12}MD^2 + \frac{1}{4}MR^2$, and $I_{sat} = \frac{2}{5}mr^2 + m\left(\frac{D}{2}\right)^2$

$$\Rightarrow I_{total} = \frac{1}{12}MD^2 + \frac{1}{4}MR^2 + \frac{2}{5}mr^2 + \frac{mD^2}{4}$$

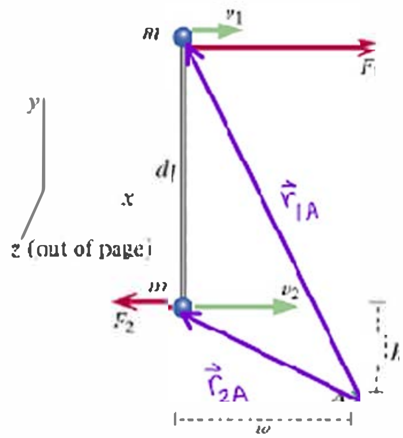
Since $\vec{L}_f = \vec{L}_i$,

$$I_{total}\vec{\omega}_2 = \left[\frac{mv_2D}{2} + \frac{2}{5}mr^2\omega_1\right]\hat{z}$$

$$\Rightarrow \boxed{\omega_2 = \frac{\frac{mv_2D}{2} + \frac{2}{5}mr^2\omega_1}{\frac{1}{12}MD^2 + \frac{1}{4}MR^2 + \frac{2}{5}mr^2 + \frac{mD^2}{4}}}$$

Problem # 4

In the figure two small objects each of mass $m = 0.235$ kg are connected by a lightweight rod of length $d = 1.20$ m. At a particular instant they have speeds $v_1 = 25$ m/s and $v_2 = 58$ m/s and are subjected to external forces $F_1 = 41$ N and $F_2 = 16$ N. A point is located distances $w = 0.80$ m and $h = 0.32$ m from the bottom object. No other external forces are acting on this system.



(a) What is the velocity of the center of mass?

$$\vec{P}_{total} = M_{total}\vec{V}_{cm} \Longrightarrow \vec{V}_{cm} = \frac{\vec{P}_{total}}{M_{total}} = \frac{\vec{p}_1+\vec{p}_2}{m_1+m_2} = \frac{m_1\vec{v}_1+m_2\vec{v}_2}{m_1+m_2} =$$

$$= \frac{(0.235)(25)\,\hat{x} + (0.235)(58)\,\hat{x}}{0.235+0.235} = \frac{\cancel{0.235}(25+58)\,\hat{x}}{2(\cancel{0.235})} =$$

$$= \boxed{41.5\ \text{m/s}\ \hat{x}}$$

(b) What is the total angular momentum of the system relative to point A?

$$\vec{L}_A = \vec{L}_{1A} + \vec{L}_{2A} = (\vec{r}_{1A}\times\vec{p}_1) + (\vec{r}_{2A}\times\vec{p}_2) = (h+d)(m_1v_1)(-\hat{z}) + (h)(m_2v_2)(-\hat{z}) =$$

$$= (-\hat{z})(m)\left[(h+d)v_1 + hv_2\right] = (-\hat{z})(0.235)\left[(0.32+1.20)(25)+(0.32)(58)\right] =$$

$$= (-\hat{z})(0.235)(38+18.56) = \boxed{13.29\ \text{kg m}^2\text{/s}\ (-\hat{z})}$$

(c) What is the rotational angular momentum of the system?

$$\vec{L}_{total} = \vec{L}_{trans} + \vec{L}_{rot} \Rightarrow \vec{L}_{rot} = \vec{L}_{total} - \vec{L}_{trans}$$

$$\checkmark\ \vec{L}_{trans} = \vec{r}_{cm} \times \vec{p}_{total} = \left(h+\tfrac{d}{2}\right)(\hat{y}) \times \left(M_{total}\vec{v}_{cm}\right) =$$

$$= \left(0.32 + \tfrac{1.2}{2}\right)(0.235 + 0.235)\,\hat{y} \times (41.5)\,\hat{x} = 17.9\ \text{kgm}^2/\text{s}\ (-\hat{z})$$

$$\Rightarrow \vec{L}_{rot} = 13.29\,(-\hat{z}) - 17.9\,(-\hat{z}) = (-13.29 - -17.9)\,\hat{z} =$$

$$= (-13.29 + 17.9)\,\hat{z} = \boxed{4.61\ \text{kgm}^2/\text{s}\ (\hat{z})}$$

(d) After a short time interval $\Delta t = 0.035$ s, determine the total (linear) momentum of the system?

$$\vec{P}_f = \vec{P}_i + \vec{F}_{net}\,\Delta t = (2)(0.235)(41.5)\,\hat{x} + (41-16)(\hat{x})(0.035) =$$

$$= \boxed{20.38\ \text{kgm/s}\ \hat{x}}$$

(e) Calculate the new rotational angular momentum of the system?

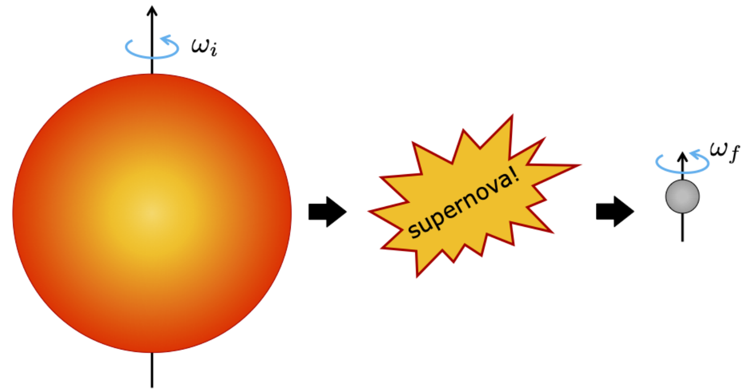
$$\vec{\tau}_{net} = \vec{\tau}_1 + \vec{\tau}_2 = (\vec{r}_1 \times \vec{F}_1) + (\vec{r}_2 \times \vec{F}_2) = \tfrac{d}{2}F_1\,(-\hat{z}) + \tfrac{d}{2}F_2\,(-\hat{z}) =$$

$$= (-\hat{z})\,\tfrac{d}{2}(F_1+F_2) = (-\hat{z})\left(\tfrac{1.2}{2}\right)(41+16) = 34.2\ \text{Nm}\ (-\hat{z})$$

$$\vec{L}_{rot\,f} = \vec{L}_{rot\,i} + \vec{\tau}_{net}\,\Delta t = 4.61\,(\hat{z}) + (34.2)(0.035)(-\hat{z}) = \boxed{3.4\ \text{kgm}^2/\text{s}\ (\hat{z})}$$

Problem #5

The red supergiant Betelgeuse has a mass $M = 2.2 \times 10^{31}$ kg and a radius of $R = 6.17 \times 10^{11}$ m, which means this star is bigger than the orbit of Mars around the Sun. At some point in the future, Betelgeuse will end its life in a supernova explosion. In the process, 90% of Betelgeuse's mass will be expelled radially outwards into space, leaving behind a neutron star remnant with $m = 0.1M$ and a radius $r = 10$ km (which would fit quite easily inside the ATL Perimeter).



1. Betelgeuse rotates with an angular speed of $\omega_i = 5.7 \times 10^{-9}$ rads/sec (it takes a whopping 35 Earth years for it to do one single spin!). Determine the angular speed ω_f of the neutron star remnant after Betelgeuse goes supernova. You can assume that (1) Betelgeuse is an isolated system, and (2) Betelgeuse can be modeled as a solid sphere both before and after going supernova.

$$\vec{L}_f = \vec{L}_i \Rightarrow I_i \omega_i = I_f \omega_f$$

$$\Rightarrow \frac{2}{5} M R^2 \omega_i = \frac{2}{5} m r^2 \omega_f . \quad m = 0.1 M$$

$$\Rightarrow \omega_f = \frac{\cancel{M} R^2}{0.1 \cancel{M} r^2} \omega_i = \frac{(6.17e11\,m)^2}{0.1(1e4\,m)^2}\left(5.7e{-9}\ \frac{rad}{sec}\right)$$

$$= \boxed{2.17e8\ \frac{rad}{sec}}$$