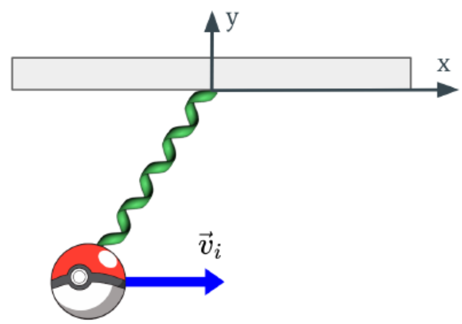
Physics 2211 GPS Week 4

Problem #1

One end of a spring is attached to the ceiling at the origin and the other is attached to a ball of mass m which is free to move around. The spring has a rest length L_0 and stiffness k_s. At time $t = 0$ the ball is located at position $\vec{r}_i\langle -2L_0, -4L_0, 0\rangle$ and has a velocity $\vec{v}_i = \langle v_0, 0, 0\rangle$ where v_0 is a positive constant. The gravitational force acting on the ball due to the Earth points in the negative y-direction.



A. Calculate the net force acting on the ball at time $t = 0$.

$$\vec{F}_{net} = \vec{F}_{grav} + \vec{F}_{spring}$$

$$\vec{F}_{grav} = \langle 0, -mg, 0\rangle$$

$$\begin{aligned}\vec{F}_{spring} &= -k_s(|L| - L_0)\hat{L}\\ &= -k_s\left(\sqrt{(-2L_0)^2 + (-4L_0)^2 + 0^2} - L_0\right)\frac{\langle -2L_0, -4L_0, 0\rangle}{\sqrt{(-2L_0)^2 + (-4L_0)^2 + 0^2}}\\ &= -(2\sqrt{5} - 1)k_sL_0\langle -\tfrac{1}{\sqrt{5}}, -\tfrac{2}{\sqrt{5}}, 0\rangle\end{aligned}$$

$$\vec{F}_{net} = \left\langle \frac{(2\sqrt{5}-1)k_sL_0}{\sqrt{5}}, \frac{2(2\sqrt{5}-1)k_sL_0}{\sqrt{5}} - mg, 0\right\rangle$$

B. Calculate the new position of the ball a short time later $t = \Delta t$.

Update the velocity:

$$\vec{v}(\Delta t) = \vec{v}(0) + \frac{\vec{F}_{net}}{m}\Delta t$$

$$\vec{a}_{net} \quad = \left\langle \tfrac{(2\sqrt{5}-1)k_s L_0}{\sqrt{5}m}, \tfrac{2(2\sqrt{5}-1)k_s L_0}{\sqrt{5}m} - g, 0 \right\rangle$$

$$\begin{aligned}\vec{v}(\Delta t) &= \langle v_0, 0, 0\rangle + \left\langle \tfrac{(2\sqrt{5}-1)k_s L_0}{\sqrt{5}m}, \tfrac{2(2\sqrt{5}-1)k_s L_0}{\sqrt{5}m} - g, 0 \right\rangle \Delta t \\ &= \left\langle v_0 + \tfrac{(2\sqrt{5}-1)k_s L_0}{\sqrt{5}m}\Delta t, \tfrac{2(2\sqrt{5}-1)k_s L_0}{\sqrt{5}m}\Delta t - g\Delta t, 0 \right\rangle\end{aligned}$$

Update the position:

$$\vec{r}(\Delta t) = \vec{r}(0) + \vec{v}_{avg}\Delta t$$

$$\vec{v}_{avg} \quad \approx \vec{v}(\Delta t)$$

$$\begin{aligned}\vec{r}(\Delta t) &= \langle -2L_0, -4L_0, 0\rangle + \left\langle v_0 + \tfrac{(2\sqrt{5}-1)k_s L_0}{\sqrt{5}m}\Delta t, \tfrac{2(2\sqrt{5}-1)k_s L_0}{\sqrt{5}m}\Delta t - g\Delta t, 0 \right\rangle \Delta t \\ &= \left\langle -2L_0 + v_0\Delta t + \tfrac{(2\sqrt{5}-1)k_s L_0}{\sqrt{5}m}\Delta t^2, -4L_0 + \left(\tfrac{2(2\sqrt{5}-1)k_s L_0}{\sqrt{5}m} - g\right)\Delta t^2, 0 \right\rangle\end{aligned}$$

C. Give a descriptive name to the position and velocity at $t = \Delta t$. For the next 2 parts, you may write your answers in terms of the position and velocity at $t = \Delta t$. You do not need to substitute in your answers from the first 2 parts.

Determine the new net force acting on the ball at time $t = \Delta t$.

Now at $t = \Delta t$

$$\vec{F}_{net} = \vec{F}_{grav} + \vec{F}_{spring}$$

$$\vec{F}_{grav} = \langle 0, -mg, 0 \rangle$$

$$\begin{aligned}\vec{F}_{spring} &= -k_s(|L| - L_0)\hat{L} \\ &= -k_s(|r(\vec{\Delta} t)| - L_0)\frac{\vec{r}(\Delta t)}{|\vec{r}(\Delta t)|}\end{aligned}$$

$$\vec{F}_{net} = \vec{F}_{grav} - k_s(|r(\vec{\Delta} t)| - L_0)\frac{\vec{r}(\Delta t)}{|\vec{r}(\Delta t)|}$$

D. Calculate the new position of the ball a short time later $t = 2\Delta t$.

Update the velocity using the force from part C:

$$\vec{v}(2\Delta t) = \vec{v}(\Delta t) + \frac{\vec{F}_{net}}{m}(2\Delta t - \Delta t)$$

$$\vec{F}_{net} = \vec{F}_{grav} - k_s(|r(\vec{\Delta} t)| - L_0)\frac{\vec{r}(\Delta t)}{|\vec{r}(\Delta t)|}$$

$$\vec{v}(2\Delta t) = \vec{v}(\Delta t) + \left(\vec{g} - \frac{k_s}{m}(|r(\vec{\Delta} t)| - L_0)\frac{\vec{r}(\Delta t)}{|\vec{r}(\Delta t)|}\right)\Delta t$$

Update the position:

$$\vec{r}(2\Delta t) = \vec{r}(\Delta t) + \vec{v}_{avg} \times (2\Delta t - \Delta t)$$

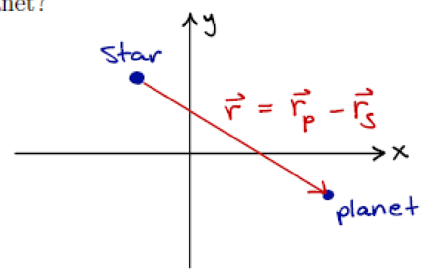
$$\vec{v}_{avg} \approx \vec{v}(2\Delta t)$$

$$\vec{r}(2\Delta t) = \vec{r}(\Delta t) + \vec{v}(\Delta t)\Delta t + \left(\vec{g} - \frac{k_s}{m}(|r(\vec{\Delta} t)| - L_0)\frac{\vec{r}(\Delta t)}{|\vec{r}(\Delta t)|}\right)\Delta t^2$$

Problem #2

A planet of mass 4×10^{24} kg is at location $< 5e11, -2e11, 0 >$ m. A star of mass 5×10^{30} kg is at location $< -2e11, 3e11, 0 >$ m. It will be useful to draw a diagram of the situation.

(a) What is the relative position vector $\vec{r}$ pointing from the $\overset{\text{initial}}{\text{star}}$ to the $\overset{\text{final}}{\text{planet}}$?



$$\vec{r} = \vec{r}_p - \vec{r}_s = \langle 5e11, -2e11, 0\rangle - \langle -2e11, 3e11, 0\rangle =$$

$$= \langle 5e11 - -2e11, -2e11 - 3e11, 0\rangle =$$

$$= \boxed{\langle 7e11, -5e11, 0\rangle \text{ m}}$$

(b) What is the (vector) force exerted on the planet by the star?

$$\checkmark \; |\vec{r}| = \sqrt{(7e11)^2 + (-5e11)^2} = \sqrt{7.4e23} = 8.6e11 \text{ m}$$

$$\checkmark \; \hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle 7e11, -5e11, 0\rangle}{8.6e11} = \langle 0.814, -0.581, 0\rangle$$

$$\checkmark \; |\vec{F}| = \frac{G m_p m_s}{|\vec{r}|^2} = \frac{(6.7e-11)(4e24)(5e30)}{7.4e23} = 1.81e21 \text{ N}$$

$$\Rightarrow \vec{F}_{\substack{\text{on planet}\\ \text{by star}}} = -|\vec{F}|\hat{r} = -(1.81e21)\langle 0.814, -0.581, 0\rangle =$$

$$= \boxed{\langle -1.47e21, 1.05e21, 0\rangle \text{ N}}$$

(c) What is the (vector) force exerted on the star by the planet?

$$\vec{F}_{\substack{\text{on star}\\ \text{by planet}}} = -\vec{F}_{\substack{\text{on planet}\\ \text{by star}}} = \boxed{\langle 1.47e21, -1.05e21, 0\rangle \text{ N}}$$

(b/c Newton's 3rd law)

(d) The velocity of the planet at this instant is $< 0.5 \times 10^4, 1.5 \times 10^4, 0 >$ m/s. Determine the velocity of the planet 2×10^7 seconds later.

$$\vec{v}_f = \vec{v}_i + (\vec{F}_{net}/m_p)\Delta t =$$

$$= \langle 0.5e4, 1.5e4, 0\rangle + \frac{2e7}{4e24}\langle -1.47e21, 1.05e21, 0\rangle =$$

$$= \langle 0.5e4, 1.5e4, 0\rangle + \langle -7350, 5250, 0\rangle =$$

$$= \boxed{\langle -2350, 20250, 0\rangle \text{ m/s}}$$

(e) Calculate the new position of the planet 2×10^7 seconds later.

Approximate $\vec{v}_{Avg} \approx \vec{v}_f$ b/c non-constant force

$$\vec{r}_{p,f} = \vec{r}_{p,i} + \vec{v}_f \Delta t = \langle 5e11, -2e11, 0\rangle + (2e7)\langle -2350, 20250, 0\rangle =$$

$$= \langle 5e11, -2e11, 0\rangle + \langle -4.7e10, 4.05e11, 0\rangle =$$

$$= \boxed{\langle 4.53e11, 2.05e11, 0\rangle \text{ m}}$$

(f) The planet has a new position and a new velocity. Starting from these new conditions, briefly explain how you would determine the new vector gravitational force exerted by the star on the planet and update the position of the planet. What approximations or simplifying assumptions did you make?

Assume star doesn't move. Calculate new $\vec{r} = \vec{r}_p - \vec{r}_s$ ($\vec{r}_p$ changed but $\vec{r}_s$ stayed the same), then use $\vec{F} = \left(\frac{G m_p m_s}{|\vec{r}|^2}\right)(-\hat{r})$ to find new vector gravitational force. Use the new $\vec{F}$ to find the new $\vec{v}$, and then use new $\vec{v}$ to find new $\vec{r}_p$