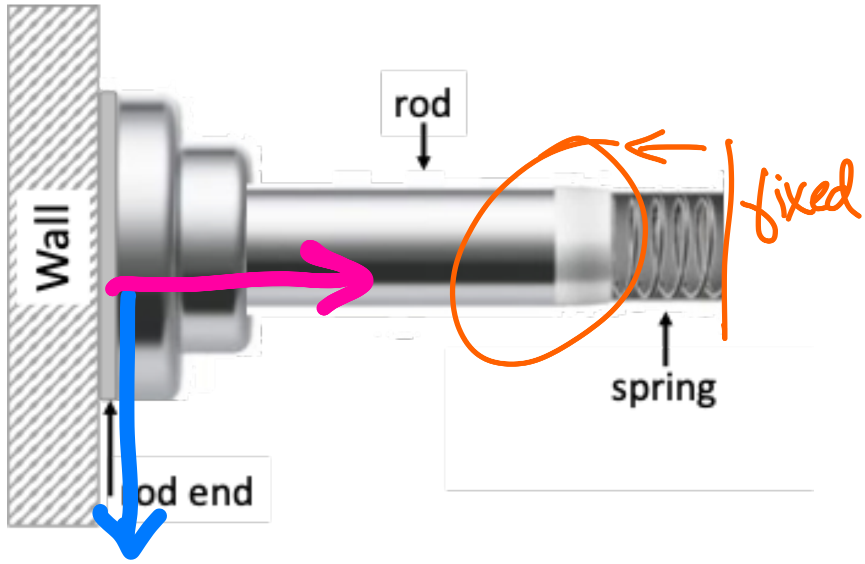
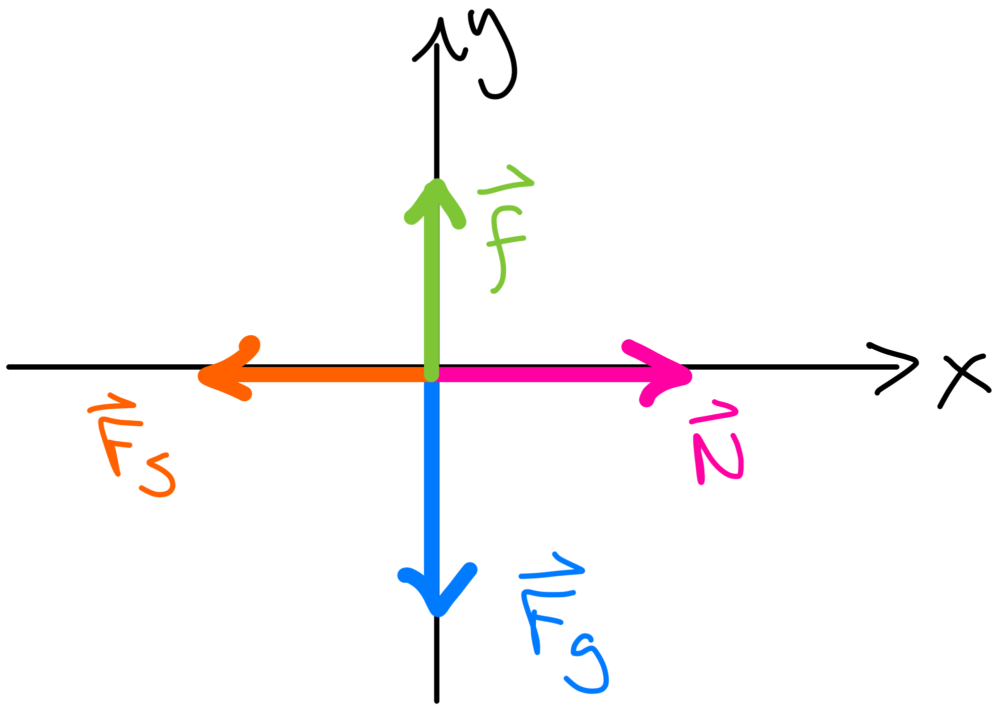
Curtain Rod [40 pts]

The working principle of a tension curtain rod is that the rod is attached to a spring pressing the rod's end against the wall (see diagram). The spring has stiffness $k = 500$ N/m and is compressed by an amount $s = 5$ cm from its relaxed length. The rod and spring are horizontal, and encased in an external tube (not shown).



1. [10 pts] Draw a free body diagram of all the forces acting on the rod end. Consider that the end carries the total weight of the rod, and the rod is held in place so it doesn't slip down.



2. [10 pts] What is the maximum mass m_0 that the rod can have without slipping down on the wall? The coefficient of static friction of the rod end on the wall is $\mu_{s1} = 0.6$.

$$\vec{F}_{netx} = 0$$
$$\vec{F}_s + \vec{N} = 0$$
$$N - F_s = 0$$
$$N = F_s$$
$$N = k|s|$$

$$\vec{F}_{nety} = 0$$
$$\vec{f} + \vec{F}_g = 0$$
$$f - mg = 0$$
$$f = mg$$

We also know that $f = \mu N = \mu k|s|$

$$\Rightarrow \mu_{s1} k|s| = m_0 g$$

$$m_0 = \frac{\mu_{s1} k|s|}{g} = \frac{(0.6)(500)(0.05)}{9.8} =$$

$$= \boxed{1.53 \text{ kg}}$$

Reminder:

$$\vec{F}_s = -k(L - L_0)\hat{i}$$
$$|\vec{F}_s| = k|L - L_0|$$
$$L - L_0 = s$$
$$|\vec{F}_s| = k|s|$$

3. [10 pts] After months in place, the rod end wears off and the rod starts to slide down. The rod's mass is m_0 (calculated in Part 2), and the new coefficient of static friction is $\mu_{s2} = 0.4$. If the rod falls at constant speed, what is the value of the coefficient of kinetic friction μ_k of the rod end on the wall?

Falls @ constant speed = dynamic equilibrium

$$\Rightarrow \vec{F}_{net} = 0$$

$F_{net\,x} = 0$

$N - F_s = 0$

$N = k|s|$

$F_{net\,y} = 0$

$f - mg = 0$

$f = m_0 g$

$f = \mu N$

$$\Rightarrow \mu_k N = \mu_k k|s|$$

$$\Rightarrow \mu_k k|s| = m_0 g$$

$$\mu_k = \frac{m_0 g}{k|s|} = \frac{(1.53)(9.8)}{(500)(0.05)} = 0.59$$

$$\sim \boxed{0.6}$$

4. [10 pts] To fix the slippage problem, the owner decides to insert a wooden block between the rod end and the wall to compress the spring further. What should be the minimum width w of the block to prevent sliding? The block is glued to the rod and its coefficient of friction on the wall is $\mu_{s3} = 0.5$.

$$F_{net\,x} = 0$$
$$N - F_s = 0$$
$$N = k(w + |s|)$$

$$F_{net\,y} = 0$$
$$f - mg = 0$$
$$f = m_0 g$$

Wood block has width w so spring's compression will be $w + |s|$

$$f = \underbrace{\mu}_{\mu_{s3}} N$$

$$\mu_{s3} k (w + |s|) = m_0 g$$

$$w + |s| = \frac{m_0 g}{\mu_{s3} k}$$

$$w = \frac{m_0 g}{\mu_{s3} k} - |s| =$$

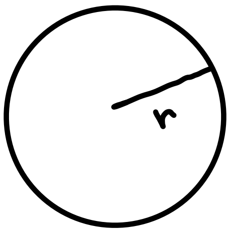
$$= \frac{(1.53)(9.8)}{(0.5)(500)} - 0.05 =$$

$$\boxed{= 0.009976 \text{ m} \sim 9.976 \text{ mm} \sim 1 \text{ cm}}$$

Supermassive Black Hole [40 pts]

A star of mass $m_s = 2 \times 10^{31}$ kg moves in uniform circular motion in its orbit around a supermassive black hole. The radius of the orbit is $r = 1.5 \times 10^{14}$ m, and it takes the star 16 Earth-years to complete one full orbit around the black hole.

1. [20 pts] What is the mass M_{bh} of the supermassive black hole? Hint: remember Lab 3.



orbit: $d = 2\pi r$

period: $T = 16$ yrs

orbital speed: $v = \frac{d}{T} = \frac{2\pi r}{T}$

uniform circular motion

$\Rightarrow v = \text{const}$

$\Rightarrow \vec{F}_{net,\parallel} = 0$

Since $\vec{F}_{net,\parallel} = 0$, then $\underline{\vec{F}_{net}} = \underline{\vec{F}_{net\perp}}$

$$\frac{GM_{bh}\cancel{m_s}}{r^{\cancel{2}}} = \frac{\cancel{m_s}v^2}{\cancel{r}} \qquad v^2 = \frac{4\pi^2 r^2}{T^2}$$

$$\frac{GM_{bh}}{r} = \frac{4\pi^2 r^2}{T^2}$$

$$\boxed{M_{bh} = \frac{4\pi^2 r^3}{GT^2}} =$$

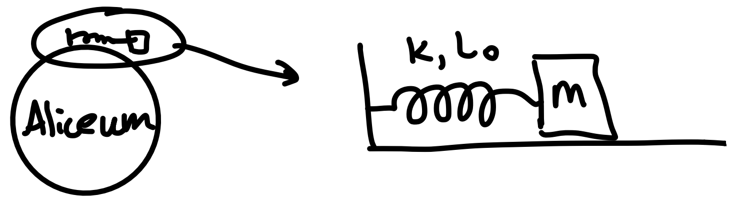
$$= \frac{(4\pi^2)(1.5e14)^3}{(6.7e\text{-}11)(504576000)^2} = \boxed{7.8 \times 10^{36}\ \text{kg}}$$

Period: $\frac{16\ \cancel{yr}}{} \left| \frac{365\ \cancel{d}}{1\ \cancel{yr}} \right| \frac{24\ \cancel{hr}}{1\ \cancel{d}} \left| \frac{60\ \cancel{min}}{1\ \cancel{hr}} \right| \frac{60\ sec}{1\ \cancel{min}} = 504576000$ sec

2. [20 pts] The star that orbits the black hole has several planets orbiting it (like the solar system). One of those planets, which we'll call Aliceum, has a mass of $M_a = 6 \times 10^{23}$ kg and a radius of $R_a = 4 \times 10^6$ m.

The inhabitants of Aliceum launch their satellites using an extra-large horizontal spring of relaxed length $L_0 = 50$ m and stiffness $k = 1.2 \times 10^6$ N/m. The spring is compressed to a length L and whatever is being launched is placed at rest in front of the spring. When the spring is released it pushes out the package, launching it into space once the package is no longer in contact with the spring.

If a satellite of mass $m = 50$ kg needs to be launched with speed $v_0 = 4500$ m/s, what should be the compressed length L of the spring when getting it ready for launch? For this calculation you can ignore the star, the black hole, and all other planets and moons and atmospheric drag.



initial: compressed to L

$s_i = L - L_0$

$v_i = 0$

final: $s_f = 0$, $v_f = v_0$

System: satellite + spring + planet

Surr: Nothing

$$\Delta E_{sys} = \cancel{W_{surr}}^{\,0}$$

$$\Delta K + \Delta U_s = 0$$

$$\frac{1}{2} m (v_f^2 - \cancel{v_i^2}) + \frac{1}{2} k (\cancel{s_f^2} - s_i^2) = 0$$

$$m v_0^2 - k s_i^2 = 0$$

$$k s_i^2 = m v_0^2$$

$$s_i^2 = \frac{m v_0^2}{k} \Rightarrow s_i = \sqrt{\frac{m v_0^2}{k}} = \sqrt{\frac{(50)(4500)^2}{1.2e6}} = 29\,\text{m}$$

The compression of the spring is 29m $\Rightarrow -s$

$$-s = L - L_0 \Rightarrow L = L_0 - s = 50 - 29$$

$$\Rightarrow \boxed{L = 21\,\text{m}}$$