TP 2 – AOS1

Priors

1 Introduction

The purpose of the practical session is to consider various sampling distributions $p_X(\cdot|\theta)$, and to study the effect of various posterior distributions $\pi_\theta(\cdot|\eta)$. This study will be formal (computation of the posterior distribution $\pi_\theta(\cdot|X,\alpha)$ and of the posterior predictive distribution $p_X(\cdot|\eta)$) as well as graphical (representation of the prior, sampling, and aforementioned distributions).

2 Conjugate priors

2.1 Discrete sampling distributions

Binomial sampling distribution, beta prior

1 Recall the expression of the beta distribution. What is its definition domain ? On which parameters does it depend ? What are the expectation, the mode, and the variance ? Write a script which plots the prior distribution given a set of chosen hyper-parameter values. Which particular distribution can be retrieved as a special case of the beta distribution ?

2 Consider now a random variable X following a binomial sampling distribution $\mathcal{B}(n,\theta)$, with n known and $\theta \sim \text{beta}(\alpha,\beta)$. Recall the expression of the posterior distribution of θ given x, α and β, and plot $\pi_\theta(\cdot)$, $L(\cdot|x)$ and $\pi_\theta(\cdot|x;\alpha,\beta)$.

3 Let us now assume that we have previously observed $X = x$ positive outcomes out of n outcomes. What is the predictive distribution for the number X_0 of positive outcomes out of n_0 new experiments, given $X = x$ out of n, α and β ?

Poisson sampling distribution, gamma prior

4 Recall the expression of the gamma distribution, its definition domain, the parameters on which it depends. Recall its expectation, mode, and variance. Write a script which plots the prior distribution given a set of chosen hyper-parameter values.

5 Consider a random variable $X \sim \mathcal{P}(\theta)$, with $\theta \sim \text{gamma}(\alpha,\beta)$. What is the posterior distribution of θ given an iid sample $x_1,\ldots,x_n$, α and β ? Plot $\pi_\theta(\cdot)$, $L(\cdot|x_1,\ldots,x_n)$ and $\pi_\theta(\cdot|x_1,\ldots,x_n;\alpha,\beta)$, for various values of θ, α, β and n.

6 Assume that an iid sample $x_1,\ldots,x_n$ of realizations of $X \sim \mathcal{P}(\theta)$ has been observed. Show that the predictive distribution of a new outcome x_0 given the sample, α and β is a negative binomial (or Pólya) distribution. At some point, you may want to make a change of integration variable, by replacing t with $z = (\beta+n+1)t$.

2.2 Continuous sampling distributions

Exponential sampling distribution, gamma prior

7 Consider a random variable $X \sim \mathcal{E}(\theta)$, and $\theta \sim \text{gamma}(\alpha, \beta)$. What is the posterior distribution of θ given an iid sample $x_1, \dots, x_n$, α and β ?

8 Compute the predictive distribution for a new random variable $X_0 \sim \mathcal{E}(\theta)$, given $x_1, \dots, x_n$, α and β. We recall that $\Gamma(u+1) = u\,\Gamma(u)$.

Normal sampling distribution, normal-gamma prior We now consider a Gaussian random variable $X \sim \mathcal{N}(\mu, \lambda^{-1})$, where the Gaussian distribution is parameterized using the expectation μ and the *precision* $\lambda = 1/(\sigma^2)$.
Classically, a normal-gamma prior is used for parameters μ and λ:

$$\pi_\lambda(\ell|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \ell^{\alpha-1} \exp(-\beta\ell), \qquad \text{for } \ell > 0;$$

$$\pi_{\mu|\lambda}(u|\nu, \lambda, \eta) = (2\,\pi)^{-1/2} \lambda^{1/2} \exp\left(-\frac{\eta\lambda}{2}(t-\nu)^2\right), \qquad \text{for } t \in \mathbb{R}.$$

The parameter η is called the *shrinkage* parameter of the normal prior.

9 Compute the pdf of the normal-gamma prior, i.e. the joint prior pdf of (μ, λ). Display the contour plot of the normal-gamma prior for various values of α, β, ν and η.

10 Assume that we have observed an iid sample $x_1, \dots, x_n$ of realizations of a random variable $X \sim \mathcal{N}(\mu, \lambda^{-1})$. Recall the expression for the likelihood function $L(\mu, \lambda)$. Display the contour plot of the likelihood function for a given sample $x_1, \dots, x_n$.

11 Show that the posterior distribution for (μ, σ^2) given the sample, λ, α and β is the product of a gamma distribution and a normal distribution. You may drop the computation of the denominator (normalization constant). Display the prior, likelihood, and posterior contours, for various values of n.