Math 3435 – HW3 Solutions

1. (1) $u_n(t) = \frac{1}{n} e^{n^2 t} \sin nx$

$(u_n)_t = n e^{n^2 t} \sin nx$.

$(u_n)_x = e^{n^2 t} \cos nx$, $(u_n)_{xx} = -n e^{n^2 t} \sin nt$

$\Rightarrow (u_n)_t + (u_n)_{xx} = 0$.

(2) $u_n(0,t) = u_n(\pi,t) = 0$. clear!

$u_n(x,0) = \frac{1}{n} \sin nx \to 0$, as $n \to \infty$. clear!

(3) $t > 0$, $\frac{1}{n} e^{n^2 t} \to \infty$, as $n \to \infty$.

but $\lim_{n\to\infty} \sin nx \neq 0$ (except for $x = k\pi$, $k = 0, \pm 1, \pm 2, \dots$)

Thus $\lim_{n\to\infty} \frac{1}{n} e^{n^2 t} \sin nx \neq 0$ (except for $x = k\pi$, $k = 0, \pm 1, \pm 2$)

1.4.3. Claim 1: The total heat $H(t) = \int_D u(x,t)\,dx$ is a constant w.r.t. t.

Claim 2: The steady-state temperature is a constant C over the region D.

Assuming Claim 1 & claim 2, we have

$$\underbrace{\int_D f(x)\,dx}_{\text{initial heat}} = \underbrace{\int_D C\,dx}_{\text{heat after a long time (steady-state)}}$$

$$\Rightarrow C = \frac{\int_D f(x)\,dx}{\int_D dx}$$

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Proof of claim 1:

Complete insulation $\Rightarrow$ $\frac{\partial u}{\partial n} = 0$ on ∂D

$\uparrow$ boundary of D

Then

$$\frac{d}{dt} H(t) = \int_D u_t(x,t)\, dx$$
$$= \int_D k \Delta u(x,t)\, dx$$
$$= k \int_D \vec{\nabla} \cdot (\vec{\nabla} u)\, dx$$

$$\underline{\underline{\text{divergence thm}}} \quad k \int_{\partial D} \vec{\nabla} u \cdot \vec{n} = k \int_{\partial D} \frac{\partial u}{\partial n} = 0$$

$\Rightarrow$ $H(t)$ is constant w.r.t. time.

Proof of claim 2: Let $u(x)$ be the steady-state temperature. Then $\Delta u = 0$. Let $v = u^2$.

Then $\vec{\nabla} v = 2u \cdot \vec{\nabla} u$,

$$\Rightarrow \Delta v = \vec{\nabla} \cdot (\vec{\nabla} v) = \vec{\nabla} \cdot (2u \cdot \vec{\nabla} u)$$
$$= 2 \vec{\nabla} u \cdot \vec{\nabla} u + 2u \cdot \vec{\nabla}(\vec{\nabla} u)$$
$$= 2 |\vec{\nabla} u|^2 + 2 \cdot u \cdot \cancel{\Delta u}^{\,0}$$
$$= 2 |\vec{\nabla} u|^2$$

$$\int_D \Delta v \ \underline{\underline{\text{divergence thm}}} \ \int_{\partial D} \frac{\partial v}{\partial n} = \int_{\partial D} \vec{\nabla} v \cdot \vec{n}$$

$$= \int_{\partial D} 2u \cdot (\vec{\nabla} u \cdot \vec{n}) = \int_{\partial D} 2u \cdot \cancel{\frac{\partial u}{\partial n}}^{\,0} = 0$$

$$\Rightarrow \int_D |\vec{\nabla} u|^2 = \frac{1}{2}\int \Delta u = 0$$ 3

$|\vec{\nabla} u|^2$ being a nonnegative continuous function

$$\Rightarrow |\vec{\nabla} u|^2 = 0, \Rightarrow \vec{\nabla} u = 0 \text{ over } D$$

$\Rightarrow u(x)$ is a constant over D.

(steady-state)

1.4.4. $\cancel{u_t}^{\,0} - k\,u_{xx} = f \qquad k = 1.$

$$\Rightarrow u_{xx} = -f$$

$0 < x < \frac{l}{2} \qquad f = 0 \Rightarrow u_{xx} = 0 \Rightarrow$

$$u = ax + b$$

$\frac{l}{2} < x < l \qquad f = H. \Rightarrow u_{xx} = -H. \Rightarrow$

$$u = -\frac{1}{2}Hx^2 + cx + d$$

Conditions: $u(0) = u(l) = 0$

$$\lim_{x\to\frac{l}{2}-} u(x) = \lim_{x\to\frac{l}{2}+} u(x)$$

$$\lim_{x\to\frac{l}{2}-} u'(x) = \lim_{x\to\frac{l}{2}+} u'(x)$$

$$-H\left(\frac{l}{2}\right)^2 + c\frac{l}{2}$$

$$=$$

$$\Rightarrow \begin{cases} b = 0 \\ -\frac{1}{2}Hl^2 + cl + d = 0 \\ a\cdot\frac{l}{2} + b = -\frac{1}{2}H\left(\frac{l}{2}\right)^2 + c\cdot\frac{l}{2} + d \\ a = -H\cdot\frac{l}{2} + c \end{cases}$$

$$\Rightarrow \begin{cases} a = \frac{Hl}{8} \\ b = 0 \\ c = \frac{5Hl}{8} \\ d = -\frac{Hl^2}{8} \end{cases}$$

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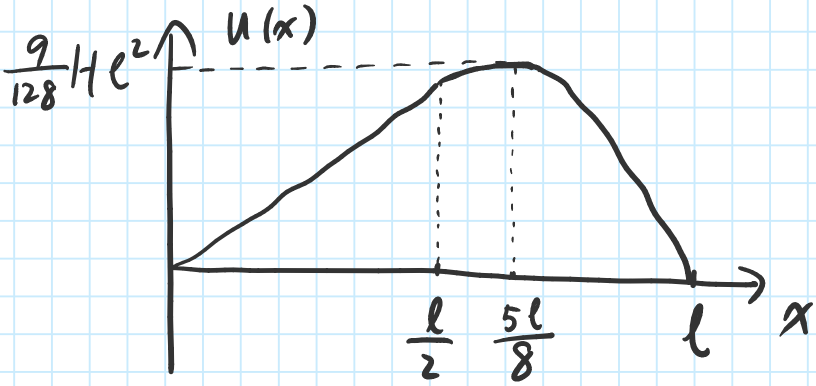
$$u(x) = \begin{cases} \dfrac{Hl}{8}x, & 0 < x < \dfrac{l}{2} \\ -\dfrac{H}{2}x^2 + \dfrac{5Hl}{8}x - \dfrac{Hl^2}{8}, & \dfrac{l}{2} < x < l. \end{cases}$$

(b) Graph of $u(x)$



$$\left(-\frac{H}{2}x^2 + \frac{5Hl}{8}x - \frac{Hl^2}{8} = -\frac{H}{2}\left(x - \frac{5l}{8}\right)^2 + \frac{9}{128}Hl^2\right)$$

The point $x = \frac{5l}{8}$ is the hottest, with temperature $\frac{9}{128}Hl^2$.

1.5.1 $\quad u'' + u = 0$

$\iff \quad u = C_1 \cos x + C_2 \sin x$

$u(0) = 0 \quad \Rightarrow \quad C_1 = 0 \quad \Rightarrow \quad u = C_2 \sin x$

$u(L) = 0 \quad \Rightarrow \quad C_2 \sin L = 0$

i) If $\sin L \neq 0$ then $C_2 = 0$, $\Rightarrow u = 0$

$\Rightarrow$ the solution is unique.

ii) If $\sin L = 0$, then C_2 can be any value.

$\Rightarrow u = C_2 \sin L$, C_2 any constant.

$\Rightarrow$ the solution is not unique.

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1.5.4. (a) We can add any constant.

(b) $\int_D f = \int_D \Delta u = \int_D \vec{\nabla}\cdot(\vec{\nabla} u)$

$$\underline{\underline{\text{divergence thm}}} \int_{\partial D} \vec{\nabla} u \cdot \vec{n} = \int_{\partial D} \frac{\partial u}{\partial n} = 0$$

(c) For part (b): For a completely insulated region to have a steady-state temperature, it is required that the total heat source is 0.

3. Proof of Theorem 1 for the hyperbolic case:

Case 1: $a_{11} = a_{22} = 0$. then the PDE becomes

$$2a_{12} u_{xy} + \cdots = 0 \qquad (*)$$

Let $\begin{cases} x = z + w \\ y = z - w \end{cases}$ then

$$\begin{cases} \partial_z = \partial_x + \partial_y \\ \partial_w = \partial_x - \partial_y \end{cases} \Rightarrow$$

$$\partial_z^2 - \partial_w^2 = (\partial_x + \partial_y)^2 - (\partial_x - \partial_y)^2$$

$$= 4\,\partial_x \partial_y$$

$$\Rightarrow \quad u_{xy} = \frac{1}{4}(u_{zz} - u_{ww})$$

$\Rightarrow$ $(*)$ is reduced to

$$u_{zz} - u_{ww} + \cdots = 0$$

(after dividing both sides by $\frac{1}{2} a_{12}$)

Case 2: At least one of a_{11}, a_{22} is non zero. Without loss of generality, we assume $a_{11} \neq 0$ (the case when $a_{22} \neq 0$ can be proved similarly)

$$(*) \Rightarrow \left(a_{11} \partial_x^2 + 2a_{12} \partial_x \partial_y + a_{22} \partial_y^2\right) u + \cdots = 0$$

$$\Rightarrow \left(\partial_x^2 + 2\frac{a_{12}}{a_{11}} \partial_x \partial_y + \frac{a_{22}}{a_{11}} \partial_y^2\right) u + \cdots = 0$$

$$\Rightarrow \underline{\left[\left(\partial_x + \frac{a_{12}}{a_{11}} \partial_y\right)^2 - \frac{a_{12}^2 - a_{11}a_{22}}{a_{11}^2} \partial_y^2\right] u + \cdots = 0}$$

Let $\begin{cases} x = z \\ y = \dfrac{a_{12}}{a_{11}} z + \sqrt{\dfrac{a_{12}^2 - a_{11}a_{22}}{a_{11}^2}}\, w \end{cases}$ (☆)

$$\Rightarrow \begin{cases} \partial_z = \partial_x + \dfrac{a_{12}}{a_{11}} \partial_y \\ \partial_w = \sqrt{\dfrac{a_{12}^2 - a_{11}a_{22}}{a_{11}^2}}\, \partial_y \end{cases}$$

$$(☆) \Rightarrow \left(\partial_z^2 - \partial_w^2\right) u + \cdots = 0.$$

that is $u_{zz} - u_{ww} + \cdots = 0$.

Proof of Theorem 2 for the parabolic case:

Follow the above procedure, we arrive at (☆).

But then $a_{12}^2 - a_{11}a_{22} = 0$, thus (☆) becomes

$$\left(\partial_x + \frac{a_{12}}{a_{11}} \partial_y\right)^2 u + \cdots = 0.$$

Let
$$\begin{cases} x = z \\ y = \frac{a_{12}}{a_{11}} z + w \end{cases}$$

this term is chosen arbitrarily just to make sure the above linear transformation is a bijection between (x, y) and (z, w)

then $\partial_z = \partial_x + \frac{a_{12}}{a_{11}} \partial_y$

then we arrive at the PDE

$$\partial_z^2 u + \cdots = 0.$$

that is $u_{zz} + \cdots = 0.$