Math 3435 — HW 2 Solutions

1.2.7 (a) Characteristic curves: $\frac{dy}{dx} = \frac{x}{y}$

$\Rightarrow \int y\,dy = \int x\,dx \Rightarrow x^2 - y^2 = C$

$\Rightarrow u = f(C) = f(x^2-y^2)$ (f should be differentiable)

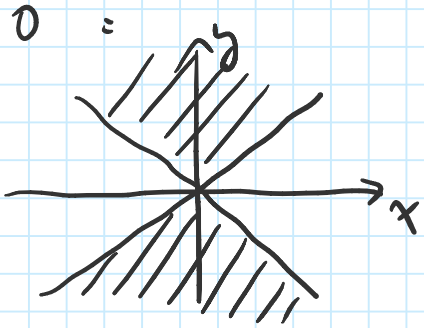
$u(0,y) = e^{-y^2} \Rightarrow f(0-y^2) = e^{-y^2}$

let $x = -y^2 \Rightarrow f(x) = e^x$ for $x \le 0$.

Conclusion: $u = f(x^2-y^2)$, f differentiable and

$f(x) = e^x$ for $x \le 0$.

(b) From part (a) the solution is uniquely determined for $x^2 - y^2 \le 0$:



1.2.10 Characteristic curves: $\frac{dy}{dx} = 1 \Leftrightarrow y = x + C$

Let $w(x) = u(x, x+C)$ $\Leftrightarrow C = y - x$

then $w'(x) = u_x + u_y$, and the original PDE becomes

$$w'(x) + w(x) = e^{x+2(x+C)} = e^{3x+2C}$$

Using integrating factor:

$$(e^x w(x))' = e^x \cdot e^{3x+2C} = e^{4x+2C}$$

$$\Rightarrow e^x w(x) = \int e^{4x+2C} = \frac{1}{4}e^{4x+2C} + f(C)$$

the const. depending on C.

$$\Rightarrow w(x) = \frac{1}{4}e^{3x+2C} + e^{-x}f(C)$$

$$\Rightarrow u(x, x+C) = \frac{1}{4}e^{3x+2C} + e^{-x}f(C)$$

$$\Rightarrow u(x,y) = \frac{1}{4}e^{3x+2(y-x)} + e^{-x}f(y-x)$$

$$= \frac{1}{4}e^{x+2y} + e^{-x}f(y-x)$$

Now $u(x,0) = 0 \Rightarrow \frac{1}{4}e^{x} + e^{-x}f(-x) = 0$

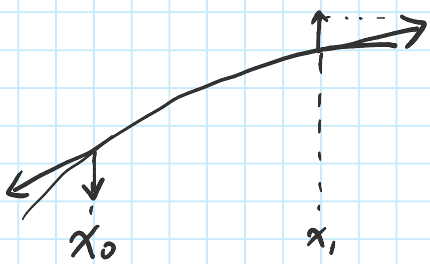
$$\Rightarrow f(-x) = -\frac{1}{4}e^{2x}$$

$$\Rightarrow f(x) = -\frac{1}{4}e^{-2x}$$

$$\Rightarrow u(x,y) = \frac{1}{4}e^{x+2y} + e^{-x}\left(-\frac{1}{4}e^{-2(y-x)}\right)$$

$$= \frac{1}{4}e^{x+2y} - \frac{1}{4}e^{x-2y}$$

1.3.1 Consider any segment of a string:



Vertical component of the forces acting on this segment:

From tension: $T\cdot u(x_1,t) - T\cdot u(x_0,t)$

From resistance: $-\int_{x_0}^{x_1} k \cdot u_t(x,t)\,dx$

proportionality constant, $k>0$

Newton's law $\Rightarrow$

$$T u(x_1,t) - T u(x_0,t) - \int_{x_0}^{x_1} k\cdot u_t(x,t)\,dx = \int_{x_0}^{x_1} \rho\, u_{tt}(x,t)\,dx$$

Differentiate w.r.t. x_1, we get

$$T u_{xx}(x_1,t) - k u_t(x_1,t) = \rho u_{tt}(x_1,t)$$

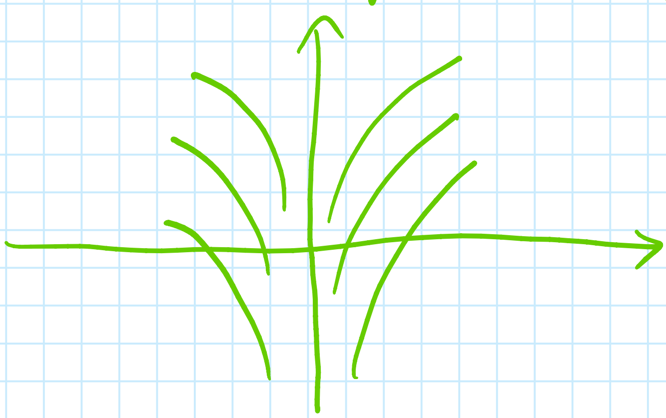
$$\Rightarrow \quad u_{tt} - c^2 u_{xx} + r u_t = 0.$$

where $c = \sqrt{\frac{T}{\rho}}$, $r = \frac{k}{\rho}$.

2. Characteristic curves: you may either write $\frac{dy}{dx} = \frac{1}{x}$ or $\frac{dx}{dy} = \frac{x}{1}$, but the latter is better in the sense that the special solution $x=0$ is included in the latter but not in the former. Solving the ODE, we get the solutions

$$y = \ln|x| + C \quad \text{or} \quad x = Ce^y$$

Still, the latter expression is better in two ways: first, it includes the special solution $x=0$ while the former does not; second, each C provides only one connected piece of characteristic curve, while ~~in the~~ former, each C provides two pieces: one for $x>0$, the other one for $x<0$.



Restricting u on each characteristic curve: if we were to use the expression $y = \ln|x| + C$, then we should consider the three cases separately: ④

① $x > 0$. Let $w(x) = u(x, \ln x + C)$

② $x < 0$, Let $w(x) = u(x, \ln(-x) + C)$

③ $x = 0$, then consider $u(0, y)$.

① $w'(x) = u_x + \frac{1}{x} u_y$, then the original PDE becomes

$$x w'(x) + x w(x) = x$$

$$\Rightarrow w' + w = 1$$

$$\Rightarrow (e^x w)' = e^x \Rightarrow e^x w = \int e^x dx$$
$$= e^x + f(C)$$

$$\Rightarrow w = 1 + e^{-x} f(C)$$

$$\Rightarrow u(x, y) = 1 + e^{-x} f(y - \ln x)$$

② Similarly, for $x < 0$, we get

$$u(x, y) = 1 + e^{-x} g(y - \ln(-x))$$

③ For $x = 0$, the PDE becomes $u_y = 0$

$$\Rightarrow u(0, y) = \text{a constant } C$$

Conclusion:

$$u(x, y) = \begin{cases} 1 + e^{-x} f(y - \ln x), & x > 0 \\ 1 + e^{-x} g(y - \ln(-x)), & x < 0 \\ C & x = 0. \end{cases}$$

Now if we were to use the expression $x = Ce^y$, then we would let

$$w(y) = u(Ce^y, y) \quad \text{then}$$

$$w'(y) = Ce^y u_x + u_y, \quad \text{then the original}$$

PDE becomes

$$w'(y) + Ce^y w(y) = Ce^y.$$

$$\Rightarrow \left(e^{Ce^y} w(y)\right)' = e^{Ce^y} Ce^y$$

$$\Rightarrow e^{Ce^y} w(y) = \int e^{Ce^y} Ce^y \, dy$$

$$= e^{Ce^y} + h(C)$$

$$\Rightarrow w(y) = 1 + e^{-Ce^y} h(C)$$

$$\Rightarrow u(x,y) = 1 + e^{-x} h(xe^{-y}).$$

Last step: Use $u(x,0) = e^x$.

If we were to use the general solution in the last page:

① $x > 0$, $u(x,y) = 1 + e^{-x} f(y - \ln x)$

$$\Rightarrow u(x,0) = 1 + e^{-x} f(-\ln x) = e^x$$

$$\Rightarrow f(-\ln x) = e^x (e^x - 1)$$

$$\Rightarrow f(y) = e^{e^{-y}} (e^{e^{-y}} - 1)$$

$$\Rightarrow u(x,y) = 1 + e^{-x} e^{e^{-(y-\ln x)}} \left(e^{e^{-(y-\ln x)}} - 1\right)$$

$$= 1 + e^{xe^{-y} - x} (e^{xe^{-y}} - 1)$$

② $x<0$ $\quad u(x,y) = 1+e^{-x} g(y-\ln(-x))$

$\Rightarrow u(x,0) = 1+e^{-x} g(-\ln(-x)) = e^{x}$

$\rightarrow g(-\ln(-x)) = e^{x}(e^{x}-1)$

$\Rightarrow g(y) = e^{-e^{-y}}(e^{-e^{-y}}-1)$

$\rightarrow u(x,y) = 1+e^{-x} e^{-e^{-(y-\ln(-x))}}\left(e^{-e^{-(y-\ln(-x))}}-1\right)$

$= 1+e^{xe^{-y}-x}\left(e^{xe^{-y}}-1\right)$

③ $x=0$. $\quad u(0,y) = C = e^{0} = 1$

Combining ① ② ③,

$\Rightarrow u(x,y) = 1+e^{xe^{-y}-1}\left(e^{xe^{-y}}-1\right)$.

But if we use the other expression of general solutions: $u(x,y) = 1+e^{-x}h(xe^{-y})$, then it's much simpler:

$\Rightarrow u(x,0) = 1+e^{-x}h(x) = e^{x}$

$\rightarrow h(x) = e^{x}(e^{x}-1)$

$\Rightarrow u(x,y) = 1+e^{-x} e^{xe^{-y}}\left(e^{xe^{-y}}-1\right)$

$= 1+e^{xe^{-y}-x}\left(e^{xe^{-y}}-1\right)$

1.3.9 $\vec{F} = (x^2+y^2+z^2)\cdot\langle x, y, z\rangle$.

$= \langle (x^2+y^2+z^2)x, (x^2+y^2+z^2)y, (x^2+y^2+z^2)z \rangle$

$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}\left((x^2+y^2+z^2)\cdot x\right) + \frac{\partial}{\partial y}\left((x^2+y^2+z^2)y\right) + \frac{\partial}{\partial z}\left((x^2+y^2+z^2)z\right)$

7

$$= 2x\cdot x + x^2+y^2+z^2 + 2y\cdot y + x^2+y^2+z^2 + 2z\cdot z + x^2+y^2+z^2$$
$$= 5(x^2+y^2+z^2)$$

$$\Rightarrow \iiint_D \vec{\nabla}\cdot\vec{F}\,dx\,dy\,dz = \iiint_D 5(x^2+y^2+z^2)\,dx\,dy\,dz$$

By spherical coordinates,

$$\iiint_D 5(x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 5\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^{\pi}\sin\phi\,d\phi \int_0^a 5\rho^4\,d\rho = 4\pi a^5$$

Now $\vec{n} = \frac{1}{a}\langle x, y, z\rangle$, then

$$\vec{F}\cdot\vec{n} = a^2\langle x, y, z\rangle\cdot\frac{1}{a}\langle x, y, z\rangle = a\cdot(x^2+y^2+z^2)$$
$$= a^3$$

(on the sphere of radius a, $x^2+y^2+z^2=a^2$)

Then
$$\iint_{\partial D}\vec{F}\cdot\vec{n}\,dS = \iint_{\partial D} a^3\,dS$$
$$= a^3\,\text{Area}(\text{Sphere of radius } a)$$
$$= a^3\cdot 4\pi a^2 = 4\pi a^5$$

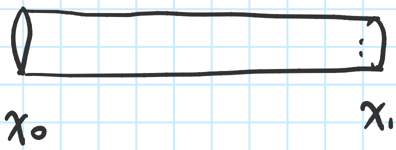
$\Rightarrow$ LHS = RHS, checked !

4. $C: x^2+y^2=1$, $x=\cos\theta$, $y=\sin\theta$, $0\le\theta<2\pi$

$$f = x^2 - y = \cos^2\theta - \sin\theta,$$
$$ds = \sqrt{x'(\theta)^2 + y'(\theta)^2}\,d\theta = d\theta$$

$$\Rightarrow \int_C f\, ds = \int_0^{2\pi} (\cos^2\theta - \sin\theta)\, d\theta$$
$$= \pi.$$

1.3.5.



Mass of the substance $M(t) = \int_{x_0}^{x_1} u(x,t)\, dx$

Mass change rate from diffusion : $k u_x(x_1,t) - k u_x(x_0,t)$

Mass change rate from transport : $V u(x_0,t) - V u(x_1,t)$

$$\Rightarrow \frac{dM(t)}{dt} = \underbrace{\frac{d}{dt}\int_{x_0}^{x_1} u(x,t)\, dx}_{\int_{x_0}^{x_1} u_t(x,t)\, dx} = k u_x(x_1,t) - k u_x(x_0,t) + V u(x_0,t) - V u(x_1,t)$$

Differentiate w.r.t. x_1, $\Rightarrow$

$$u_t = k u_{xx} - V u_x.$$

1.3.6 $u(x,y,z,t) = u(\overset{=r}{\sqrt{x^2+y^2}}, t)$

$$u_x = u_r \cdot \frac{\partial r}{\partial x},$$
$$u_{xx} = (u_r)_x \cdot \frac{\partial r}{\partial x} + u_r \frac{\partial^2 r}{\partial x^2}$$
$$= u_{rr} \cdot \left(\frac{\partial r}{\partial x}\right)^2 + u_r \frac{\partial^2 r}{\partial x^2}$$
$$u_{yy} = u_{rr} \cdot \left(\frac{\partial r}{\partial y}\right)^2 + u_r \frac{\partial^2 r}{\partial y^2}$$
$$u_{zz} = 0$$

$$\begin{cases} \frac{\partial r}{\partial x} = \frac{x}{r} \\ \frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{1}{r} - \frac{x^2}{r^3} \\ \frac{\partial r}{\partial y} = \frac{y}{r} \\ \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} - \frac{y^2}{r^3} \end{cases}$$

$$\Rightarrow U_{xx} + U_{yy} + U_{zz}$$

$$= U_{rr}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + U_r\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right]$$

$$= U_{rr}\underbrace{\left[\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2\right]}_{=1} + U_r\left(\frac{2}{r} - \underbrace{\frac{x^2+y^2}{r^3}}_{=\frac{1}{r}}\right)$$

$$= U_{rr} + U_r/r$$

thus the heat eq. becomes

$$U_t = k(U_{rr} + 2U_r/r).$$

1.3.7. (Almost identical to 1.3.6)

$$u(x,y,z,t) = u(\underbrace{\sqrt{x^2+y^2+z^2}}_{r}, t)$$

$U_{xx} = U_{rr}\left(\frac{\partial r}{\partial x}\right)^2 + U_r \frac{\partial^2 r}{\partial x^2}$ and similarly for U_{yy}, U_{zz}.

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r},$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{r} - \frac{x^2}{r^3}, \quad \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} - \frac{y^2}{r^3}, \quad \frac{\partial^2 r}{\partial z^2} = \frac{1}{r} - \frac{z^2}{r^3}$$

$$\Rightarrow U_{xx} + U_{yy} + U_{zz}$$

$$= U_{rr}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2\right] + U_r\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2}\right]$$

$$= U_{rr}\underbrace{\left[\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2\right]}_{=1} + U_r\left(\frac{3}{r} - \underbrace{\frac{x^2+y^2+z^2}{r^3}}_{=\frac{1}{r}}\right)$$

$$= U_{rr} + 2U_r/r$$

$$\Rightarrow U_t = k(U_{rr} + 2U_r/r).$$