①

2. (1) Since $\gamma = i\beta$, (*) implies

$$e^{i2\beta l} = \frac{(i\beta - a_0)(i\beta - a_1)}{(i\beta + a_0)(i\beta + a_1)}$$

Then

$$\tan\beta l = \frac{\sin\beta l}{\cos\beta l} = \frac{\frac{1}{2i}\left(e^{i\beta l} - e^{-i\beta l}\right)}{\frac{1}{2}\left(e^{i\beta l} + e^{-i\beta l}\right)}$$

$$= \frac{e^{i2\beta l} - 1}{i\left(e^{i2\beta l} + 1\right)}$$

$$= \frac{\frac{(i\beta - a_0)(i\beta - a_1)}{(i\beta + a_0)(i\beta + a_1)} - 1}{i\left(\frac{(i\beta - a_0)(i\beta - a_1)}{(i\beta + a_0)(i\beta + a_1)} + 1\right)}$$

$$= \frac{(i\beta - a_0)(i\beta - a_1) - (i\beta + a_0)(i\beta + a_1)}{i\left[(i\beta - a_0)(i\beta - a_1) + (i\beta + a_0)(i\beta + a_1)\right]}$$

$$= \frac{-i2(a_0 + a_1)\beta}{i2\left(-\beta^2 + a_0 a_1\right)}$$

$$= \frac{(a_0 + a_1)\beta}{\beta^2 - a_0 a_1}$$

(2) $\lambda = \beta^2$, $\gamma = i\beta$

$$X = Ce^{\gamma x} + De^{-\gamma x}$$
$$= Ce^{i\beta x} + De^{-i\beta x}$$

②

Using Euler's identity, we may rewrite

$$X = C' \cos\beta x + D' \sin\beta x.$$

Using $X'(0) - a_0 X(0) = 0$

$$\Rightarrow \quad -C'\beta \sin\beta 0 + D'\beta \cos\beta 0$$
$$- a_0\left(C'\cos\beta 0 + D'\sin\beta 0\right) = 0$$

$$\Rightarrow \quad D'\beta - a_0 C' = 0$$

$$\Rightarrow \quad D' = \frac{a_0}{\beta} C'$$

Thus $X = C' \cos\beta x + \frac{a_0}{\beta} C' \sin\beta x$

$$= C'\left(\cos\beta x + \frac{a_0}{\beta}\sin\beta x\right).$$

(3) $X = Ce^{\gamma x} + De^{-\gamma x}$

Since $\cosh\gamma x = \dfrac{e^{\gamma x} + e^{-\gamma x}}{2}$

$\sinh\gamma x = \dfrac{e^{\gamma x} - e^{-\gamma x}}{2}$

We may rewrite

$$X = C'\cosh\gamma x + D'\sinh\gamma x$$

Using $X'(0) - a_0 X(0) = 0$.

③

$$\Rightarrow \quad C'\gamma \sinh\gamma 0 + D'\gamma\cosh\gamma 0$$
$$- a_0\left(C'\cosh\gamma 0 + D'\sinh\gamma 0\right) = 0$$
$$\Rightarrow \quad D'\gamma - a_0 C' = 0$$
$$D' = \frac{a_0}{\gamma} C'$$
$$\Rightarrow \quad X = C'\cosh\gamma x + \frac{a_0}{\gamma} C' \sinh\gamma x$$
$$= C'\left(\cosh\gamma x + \frac{a_0}{\gamma}\sinh\gamma x\right)$$

3. (b) Orthogonality: for $n \neq m$,

$$\int_0^l \sin\frac{n\pi x}{l}\,\overline{\sin\frac{m\pi x}{l}}\,dx = \int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx$$
$$= \int_0^l \frac{1}{2}\left[\cos\frac{(n-m)\pi x}{l} - \cos\frac{(n+m)\pi x}{l}\right]dx$$
$$= \frac{1}{2}\left[\frac{l}{(n-m)\pi}\sin\frac{(n-m)\pi x}{l} - \frac{l}{(n+m)\pi}\sin\frac{(n+m)\pi x}{l}\right]\Bigg|_{x=0}^{x=l}$$
$$= 0$$

Fourier series: $\phi \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$$b_n = \frac{\left(\phi, \sin\frac{n\pi x}{l}\right)}{\left(\sin\frac{n\pi x}{l}, \sin\frac{n\pi x}{l}\right)} = \frac{\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx}{\int_0^l \sin^2\frac{n\pi x}{l}\,dx}$$

Note that $\int_0^l \sin^2\frac{n\pi x}{l}\,dx = \int_0^l \frac{1-\cos\frac{2n\pi x}{l}}{2}\,dx$ (4)

$$= \frac{l}{2}$$

Thus $b_n = \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx$

(2) Orthogonality: for $n\neq m$

$$\int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx = \int_0^l \frac{1}{2}\left[\cos\frac{(n+m)\pi x}{l} + \cos\frac{(n-m)\pi x}{l}\right]dx$$

$$= \frac{1}{2}\left[\frac{l}{(n+m)\pi}\sin\frac{(n+m)\pi x}{l} + \frac{l}{(n-m)\pi}\sin\frac{(n-m)\pi x}{l}\right]\Bigg|_{x=0}^{x=l}$$

$$= 0$$

Fourier series: $\phi \sim \sum_{n=0}^{\infty} A_n\cos\frac{n\pi x}{l}$

$$A_n = \frac{\left(\phi, \cos\frac{n\pi x}{l}\right)}{\left(\cos\frac{n\pi x}{l}, \cos\frac{n\pi x}{l}\right)} = \frac{\int_0^l \phi(x)\cos\frac{n\pi x}{l}\,dx}{\int_0^l \cos^2\frac{n\pi x}{l}\,dx}$$

$$= \begin{cases} \frac{1}{l}\int_0^l \phi(x)\,dx, & \text{if } n=0 \\ \frac{2}{l}\int_0^l \phi(x)\cos\frac{n\pi x}{l}\,dx, & \text{if } n=1,2,\dots \end{cases}$$

Thus we may rewrite the Fourier series as

$$\phi \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$$

where

$$a_n = \frac{2}{l}\int_0^l \phi(x)\cos\frac{n\pi x}{l}\,dx, \quad n=0,1,2,\dots$$

(3) Orthogonality: the set is

$$\left\{\cos\frac{n\pi x}{l},\ n=0,1,\dots;\ \sin\frac{n\pi x}{l},\ n=1,2,\dots\right\}$$

$$\int_{-l}^{l} \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx = \cdots = 0, \quad \text{for } n\neq m.$$

5

$$\int_{-l}^{l} \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\, dx = \cdots = 0. \qquad n \neq m.$$

$$\int_{-l}^{l} \cos\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\, dx$$ (it may happen that $n=m$ for this case!)

$$= \int_{-l}^{l} \frac{1}{2}\left[\sin\frac{(m+n)\pi x}{l} + \sin\frac{(m-n)\pi x}{l}\right] dx$$

$$= \cdots = 0.$$

Fourier series: $\phi \sim A_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$

$$A_0 = \frac{(\phi, 1)}{(1,1)} = \frac{\int_{-l}^{l} \phi(x)\, dx}{\int_{-l}^{l} 1\, dx} = \frac{1}{2l}\int_{-l}^{l} \phi(x)\, dx$$

$$a_n = \frac{(\phi, \cos\frac{n\pi x}{l})}{(\cos\frac{n\pi x}{l}, \cos\frac{n\pi x}{l})} = \frac{\int_{-l}^{l} \phi(x) \cos\frac{n\pi x}{l}\, dx}{\underbrace{\int_{-l}^{l} \cos^2\frac{n\pi x}{l}\, dx}_{=l\ (n\neq 0)}}$$

$$= \frac{1}{l}\int_{-l}^{l} \phi(x)\cos\frac{n\pi x}{l}\, dx$$

$$b_n = \frac{(\phi, \sin\frac{n\pi x}{l})}{(\sin\frac{n\pi x}{l}, \sin\frac{n\pi x}{l})} = \frac{\int_{-l}^{l} \phi(x) \sin\frac{n\pi x}{l}\, dx}{\underbrace{\int_{-l}^{l} \sin^2\frac{n\pi x}{l}\, dx}_{=l\ (n\neq 0)}}$$

$$= \frac{1}{l}\int_{-l}^{l} \phi(x)\sin\frac{n\pi x}{l}\, dx.$$

Note that we may let

$$a_0 = \frac{1}{l}\int_{-l}^{l} \phi(x)\, dx$$

then we may rewrite the Fourier series into

$$\phi \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$$

(4) Orthogonality: for $n \neq m$.

$$\int_{-l}^{l} e^{i\frac{n\pi x}{l}}\, \overline{e^{i\frac{m\pi x}{l}}}\, dx = \int_{-l}^{l} e^{i\frac{n\pi x}{l}} e^{-i\frac{m\pi x}{l}}\, dx$$

$$= \int_{-l}^{l} e^{i\frac{(n-m)\pi x}{l}}\, dx = \frac{l}{i(n-m)\pi} e^{i\frac{(n-m)\pi x}{l}}\Big|_{-l}^{l}$$

$$= \frac{l}{i(n-m)\pi}\left[e^{i(n-m)\pi} - e^{-i(n-m)\pi}\right]$$

$$= 0 \qquad \left(e^{ik\pi} = e^{-ik\pi} \text{ for any integer } k\right)$$

Fourier series: $\phi \sim \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{l}}$

$$C_n = \frac{\left(\phi, e^{i\frac{n\pi x}{l}}\right)}{\left(e^{i\frac{n\pi x}{l}}, e^{i\frac{n\pi x}{l}}\right)} = \frac{\int_{-l}^{l} \phi(x)\, \overline{e^{i\frac{n\pi x}{l}}}\, dx}{\int_{-l}^{l} e^{i\frac{n\pi x}{l}} \cdot \overline{e^{i\frac{n\pi x}{l}}}\, dx}$$

$$= \frac{\int_{-l}^{l} \phi(x) e^{-i\frac{n\pi x}{l}}\, dx}{\int_{-l}^{l} e^{i\frac{n\pi x}{l}} e^{-i\frac{n\pi x}{l}}\, dx} = \frac{1}{2l}\int_{-l}^{l} \phi(x) e^{-i\frac{n\pi x}{l}}\, dx$$

(5) Orthogonality: for $n \neq m$. ⑦

$$\left(\cos\beta_n x + \frac{a_0}{\beta_n}\sin\beta_n x,\ \cos\beta_m x + \frac{a_0}{\beta_m}\sin\beta_m x\right)$$

$$= \int_0^l \left(\cos\beta_n x + \frac{a_0}{\beta_n}\sin\beta_n x\right)\left(\cos\beta_m x + \frac{a_0}{\beta_m}\sin\beta_m x\right) dx$$

$$= \int_0^l \Big(\cos\beta_n x \cos\beta_m x + \frac{a_0}{\beta_n}\sin\beta_n x \cos\beta_m x$$

$$+ \frac{a_0}{\beta_m}\cos\beta_n x \sin\beta_m x + \frac{a_0^2}{\beta_n\beta_m}\sin\beta_n x \sin\beta_m x\Big) dx$$

$$= \frac{1}{2}\int_0^l \Big\{\left[\cos(\beta_n+\beta_m)x + \cos(\beta_n-\beta_m)x\right]$$

$$+ \frac{a_0}{\beta_n}\left[\sin(\beta_n+\beta_m)x + \sin(\beta_n-\beta_m)x\right]$$

$$+ \frac{a_0}{\beta_m}\left[\sin(\beta_m+\beta_n)x + \sin(\beta_m-\beta_n)x\right]$$

$$+ \frac{a_0^2}{\beta_n\beta_m}\left[\cos(\beta_n-\beta_m)x - \cos(\beta_n+\beta_m)x\right]\Big\} dx$$

$$= \frac{1}{2}\Big\{ \frac{\sin(\beta_n+\beta_m)l}{\beta_n+\beta_m} + \frac{\sin(\beta_n-\beta_m)l}{\beta_n-\beta_m} +$$

$$+\frac{a_0}{\beta_n}\left[-\frac{\cos(\beta_n+\beta_m)l}{\beta_n+\beta_m}-\frac{\cos(\beta_n-\beta_m)l}{\beta_n-\beta_m}\right]$$

$$+\frac{a_0}{\beta_m}\left[-\frac{\cos(\beta_m+\beta_n)l}{\beta_m+\beta_n}-\frac{\cos(\beta_m-\beta_n)l}{\beta_m-\beta_n}\right]$$

$$+\frac{a_0^2}{\beta_n\beta_m}\left[\frac{\sin(\beta_n-\beta_m)l}{\beta_n-\beta_m}-\frac{\sin(\beta_n+\beta_m)l}{\beta_n+\beta_m}\right]\Bigg\}$$

$$=\frac{1}{2}\Bigg\{\left(1-\frac{a_0^2}{\beta_n\beta_m}\right)\frac{\sin(\beta_n+\beta_m)l}{\beta_n+\beta_m}$$

$$+\left(1+\frac{a_0^2}{\beta_n\beta_m}\right)\frac{\sin(\beta_n-\beta_m)l}{\beta_n-\beta_m}$$

$$-\left(\frac{a_0}{\beta_n}+\frac{a_0}{\beta_m}\right)\frac{\cos(\beta_n+\beta_m)l}{\beta_n+\beta_m}$$

$$-\left(\frac{a_0}{\beta_n}-\frac{a_0}{\beta_m}\right)\frac{\cos(\beta_n-\beta_m)l}{\beta_n-\beta_m}\Bigg\}$$

$$=\frac{1}{2}\Bigg\{\left(1-\frac{a_0^2}{\beta_n\beta_m}\right)\frac{\sin\beta_n l\cos\beta_m l+\sin\beta_m l\cos\beta_n l}{\beta_n+\beta_m}$$

$$+\left(1+\frac{a_0^2}{\beta_n\beta_m}\right)\frac{\sin\beta_n l\cos\beta_m l-\sin\beta_m l\cos\beta_n l}{\beta_n-\beta_m}$$

$$-\frac{a_0}{\beta_n\beta_m}\left(\cos\beta_n l\cos\beta_m l-\sin\beta_n l\sin\beta_m l\right)$$

$$+\frac{a_0}{\beta_n\beta_m}\left(\cos\beta_n l\cos\beta_m l+\sin\beta_n l\sin\beta_m l\right)\Bigg\}$$

$$= \frac{\cos\beta_n l \cos\beta_m l}{2} \left\{ \left(1 - \frac{a_0^2}{\beta_n\beta_m}\right) \frac{\tan\beta_n l + \tan\beta_m l}{\beta_n + \beta_m} \right.$$

$$+ \left(1 + \frac{a_0^2}{\beta_n\beta_m}\right) \frac{\tan\beta_n l - \tan\beta_m l}{\beta_n - \beta_m}$$

$$- \frac{a_0}{\beta_n\beta_m}\left(\cancel{1} - \tan\beta_n l \tan\beta_m l\right)$$

$$\left. + \frac{a_0}{\beta_n\beta_m}\left(\cancel{1} + \tan\beta_n l \tan\beta_m l\right) \right\}$$

$$= \frac{\cos\beta_n l \cos\beta_m l}{2} \left\{ \left(1 - \frac{a_0^2}{\beta_n\beta_m}\right) \frac{\frac{(a_0+a_1)\beta_n}{\beta_n^2 - a_0 a_1} + \frac{(a_0+a_1)\beta_m}{\beta_m^2 - a_0 a_1}}{\beta_n + \beta_m} \right.$$

$$+ \left(1 + \frac{a_0^2}{\beta_n\beta_m}\right) \frac{\frac{(a_0+a_1)\beta_n}{\beta_n^2 - a_0 a_1} - \frac{(a_0+a_1)\beta_m}{\beta_m^2 - a_0 a_1}}{\beta_n - \beta_m}$$

$$\left. + \frac{2a_0}{\beta_n\beta_m} \frac{(a_0+a_1)\beta_n}{\beta_n^2 - a_0 a_1} \frac{(a_0+a_1)\beta_m}{\beta_m^2 - a_0 a_1} \right\}$$

$$= \frac{\cos\beta_n l \cos\beta_m l \,(a_0+a_1)}{2(\beta_n^2 - a_0 a_1)(\beta_m^2 - a_0 a_1)} \left\{ \left(1 - \frac{a_0^2}{\beta_n\beta_m}\right)(\beta_n\beta_m - a_0 a_1) \right.$$

$$+ \left(1 + \frac{a_0^2}{\beta_n\beta_m}\right)(-\beta_n\beta_m - a_0 a_1)$$

$$\left. + 2a_0(a_0+a_1) \right\}$$

$$= 0$$

Checked !