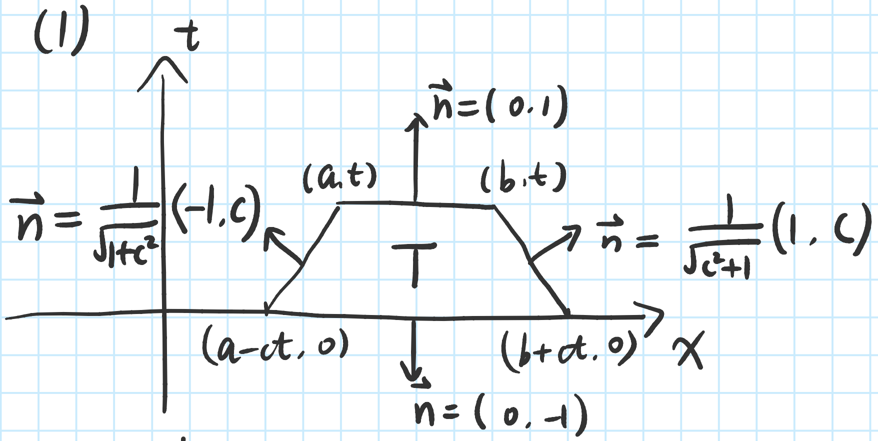
2. (1)



$(1, c)$

$(c, -1)$

①

Divergence thm: $\displaystyle\int_{\partial T}\vec{F}\cdot\vec{n}\,dl=\iint_T \cancel{\operatorname{div}\vec{F}}^{\,0}\cdot dx\,dy=0$

$$\left(\begin{aligned}\operatorname{div}\vec{F}&=\partial_x\left(c^2u_xu_t\right)+\partial_t\left(-\tfrac{1}{2}\left(u_t^2+c^2u_x^2\right)\right)\\&=c^2u_{xx}u_t+\cancel{c^2u_xu_{xt}}-u_tu_{tt}-\cancel{c^2u_xu_{xt}}\\&=-u_t\left(\cancel{u_{tt}-c^2u_{xx}}^{\,0}\right)\\&=0\end{aligned}\right)$$

$\partial T=\text{top}\sqcup\text{bottom}\sqcup\text{left}\sqcup\text{right}$

$$\int_{top}\vec{F}\cdot\vec{n}\,dl=-\frac{1}{2}\int_a^b\left[u_t^2(x,t)+c^2u_x^2(x,t)\right]dx$$

$$\int_{bottom}\vec{F}\cdot\vec{n}\,dl=\frac{1}{2}\int_{a-ct}^{b+ct}\left[u_t^2(x,0)+c^2u_x^2(x,0)\right]dx$$

$$=\frac{1}{2}\int_{a-ct}^{b+ct}\left[\psi^2+c^2(\phi')^2\right]dx$$

$$\int_{left}\vec{F}\cdot\vec{n}\,dl=\frac{1}{\sqrt{1+c^2}}\int_{left}\left[-c^2u_xu_t-\frac{1}{2}c\left(u_t^2+c^2u_x^2\right)\right]dl$$

$$=-\frac{c}{2\sqrt{1+c^2}}\int_{left}\left(u_t+cu_x\right)^2dl\le 0$$

$$\int_{right}\vec{F}\cdot\vec{n}\,dl=\frac{1}{\sqrt{1+c^2}}\int_{right}\left[c^2u_xu_t-\frac{1}{2}c\left(u_t^2+c^2u_x^2\right)\right]dl$$

$$=-\frac{c}{2\sqrt{1+c^2}}\int_{right}\left(u_t-cu_x\right)^2dl\le 0$$

$$0=\int_{\partial T} \vec{F}\cdot\vec{n}\, dl = \int_{top} + \int_{bottom} + \underbrace{\int_{left} + \int_{right}}_{\leq 0}$$ ②

$$\Rightarrow \int_{top} + \int_{bottom} \geq 0$$

$$\Rightarrow \frac{1}{2}\int_a^b \left[u_t^2(x,t) + c^2 u_x^2(x,t)\right] dx$$

$$\leq \frac{1}{2}\int_{a-ct}^{b+ct} \left[\psi^2 + c^2(\phi')^2\right] dx$$

(2) From (1), we get

$$\frac{1}{2}\int_a^b \left[u_t^2(x,t) + c^2 u_x^2(x,t)\right] dx \leq E(0) < \infty$$

for any a, b. Let $a \to -\infty$, $b \to \infty$, we get $E(t) \leq E(0)$. In particular $E(t)$ is finite.

(3) Completely analogous to (1).

(4) From (3), let $a \to -\infty$, $b \to \infty$, we get $E(0) \leq E(t)$. Combined with (2), we get

$$E(t) = E(0).$$

3. It's clear that $u = u_1 - u_2$ satisfies the PDE system

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \phi_1 - \phi_2 \\ u_t(x,0) = \psi_1 - \psi_2 \end{cases}$$

By d'Alembert's formula,

$$(u_1 - u_2)(x,t) = \frac{1}{2}\left[(\phi_1-\phi_2)(x-ct) + (\phi_1-\phi_2)(x+ct)\right] +$$

③

$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} (\psi_1 - \psi_2)(y)\, dy$$

$$\Rightarrow \left|(u_1 - u_2)(x,t)\right| \le \frac{1}{2}\left(\|\phi_1 - \phi_2\| + \|\phi_1 - \phi_2\|\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \underbrace{\|\psi_1 - \psi_2\|}_{\text{constant now}}\, dy$$

$$= \|\phi_1 - \phi_2\| + \|\psi_1 - \psi_2\| \cdot \frac{1}{2c}\int_{x-ct}^{x+ct} dy$$

$$= \|\phi_1 - \phi_2\| + \|\psi_1 - \psi_2\| \cdot t$$

$$\le \|\phi_1 - \phi_2\| + T \cdot \|\psi_1 - \psi_2\|,$$

for any $-\infty < x < \infty$, $0 \le t \le T$.

$$\Rightarrow \|u_1 - u_2\|_T \le \|\phi_1 - \phi_2\| + T\|\psi_1 - \psi_2\|.$$

(Here "max" is better replaced by "supremum", but we will not bother to discuss this mathematical detail.)