# Math 3435 - HW 1



**1.1.2** (a)(e) are linear the others are nonlinear

I will demonstrate (b) and (e).

(b) Check $L(u+v) = Lu + Lv$

$L(u+v) = (u+v)_x + (u+v)(u+v)_y$

$= u_x + v_x + (u+v)(u_y + v_y)$

(Used the linearity of taking partial derivatives)

$= u_x + v_x + u u_y + v u_y + u v_y + v v_y$

$Lu + Lv = u_x + u u_y + v_x + v v_y$

$\Rightarrow L(u+v) \neq Lu + Lv$, thus nonlinear.

(e) check $L(u+v) = Lu + Lv$

$L(u+v) = \sqrt{1+x^2}\cos y\,(u+v)_x + (u+v)_{yxy} - \arctan(x/y)(u+v)$

$= \sqrt{1+x^2}\cos y\,(u_x + v_x) + u_{yxy} + v_{yxy} - \arctan(x/y)(u+v)$

$Lu + Lv = \sqrt{1+x^2}\cos y\, u_x + u_{yxy} - \arctan(x/y)\, u +$
$\sqrt{1+x^2}\cos y\, v_x + v_{yxy} - \arctan(x/y)\, v$

$\Rightarrow L(u+v) = Lu + Lv$ ✓

check $L(cu) = cLu$ $\quad c$ constant

$L(cu) = \sqrt{1+x^2}\cos y\,(cu)_x + (cu)_{yxy} - \arctan(x/y)(cu)$

$= \sqrt{1+x^2}\cos y \cdot c\, u_x + c \cdot u_{yxy} - \arctan(x/y) \cdot c \cdot u$

$cLu = c(\sqrt{1+x^2}\cos y\, u_x + u_{yxy} - \arctan(x/y)\, u)$

$\Rightarrow L(cu) = cLu$ ✓ $\Rightarrow$ Linear!

1.1.3 (a) 2nd order; either nonlinear or linear inhomogeneous

(b) 2nd order; linear homogeneous

(c) 3rd order; nonlinear

(d) 2nd order; either nonlinear or linear inhomogeneous

(e) 2nd order; linear homogeneous

(f) 1st order; nonlinear

(g) 1st order; linear homogeneous

(h) 4th order; nonlinear

I will demonstrate (d).

Two points of view: <1> $Lu = u_{tt} - u_{xx} + x^2 = 0$.

Then $Lu$ is nonlinear: check $L(u+v) = Lu + Lv$:

$$L(u+v) = (u+v)_{tt} - (u+v)_{xx} + x^2$$
$$= u_{tt} + v_{tt} - (u_{xx} + v_{xx}) + x^2$$
$$Lu + Lv = u_{tt} - u_{xx} + x^2 + v_{tt} - v_{xx} + x^2$$
$$L(u+v) \neq Lu + Lv \Rightarrow \text{nonlinear!}$$

<2> Or we may rewrite the equation into

$$Lu = u_{tt} - u_{xx} = -x^2$$

Then $Lu = u_{tt} - u_{xx}$ is linear (check omitted) and the equation becomes linear inhomogeneous.

1.1.4. Let $u_1, u_2$ be two solutions of $Lu = g$. Thus $Lu_1 = g$, $Lu_2 = g$. Let $u = u_1 - u_2$. Then

$$Lu = L(u_1 - u_2) \underset{\text{by } L(u+v) = Lu + Lv}{=} Lu_1 + L(-u_2) \underset{\text{by } L(cu) = cLu}{=} Lu_1 - Lu_2 = g - g = 0.$$

Thus $u$ is the solution to $Lu=0$. ③

1.1.10 $u''' - 3u'' + 4u = 0$ is a linear homogeneous ODE with constant coefficients. There are systematic methods of solving this type of ODE:

⇝ associated equation $X^3 - 3X^2 + 4 = 0$

$\Rightarrow (X+1)(X-2)^2 = 0$

$X = -1$ is a root with multiplicity 1

⇝ associated solution is $u(x) = e^{-x}$

$X = 2$ is a root with multiplicity 2

⇝ associated solutions are $u(x) = e^{2x}, xe^{2x}$

General solutions are $C_1 e^{-x} + C_2 e^{2x} + C_3 x e^{2x}$.

where $C_1, C_2, C_3$ are arbitrary constants.

1.1.12. $u(x,y) = \sin nx \sinh ny$

$\Rightarrow u_x = n \cos nx \sinh ny$, $u_y = n \sin nx \cosh ny$

$u_{xx} = -n^2 \sin nx \sinh ny$, $u_{yy} = n^2 \sin nx \sinh ny$

$\Rightarrow u_{xx} + u_{yy} = 0$.

1.2.1. Characteristic curves: $\frac{dx}{dt} = \frac{3}{2}$

$\Rightarrow x = \frac{3}{2}t + C \Leftrightarrow C = x - \frac{3}{2}t$

$\Rightarrow u = f(C) = f(x - \frac{3}{2}t)$

$u(x,0) = f(x) = \sin x$.

$\Rightarrow u = \sin(x - \frac{3}{2}t)$ is the solution.

1.2.2. Let $V = u_y$ then $3u_y + u_{xy} = 0$ becomes

$$\begin{cases} 3V + V_x = 0 & ① \\ u_y = V & ② \end{cases}$$

① is an ODE w.r.t. the variable $x$.

Solutions to $3y + y' = 0$ are $y = Ce^{-3x}$

$\Rightarrow$ solutions to $3V + V_x = 0$ are $V = \underbrace{C(y)}\, e^{-3x}$.

(constant w.r.t. $x$ depending on $y$)

Then ② $\Rightarrow$

$$u_y = C(y)\, e^{-3x}$$

$$u = \int u_y \, dy = \int C(y)\, e^{-3x} \, dy$$

$$= e^{-3x} \int C(y)\, dy + \underbrace{B(x)}$$

(constant w.r.t. $y$ depending on $x$)

Let $A(y) = \int C(y)\, dy$,

then $u = e^{-3x} A(y) + B(x)$.

check: $u_y = e^{-3x} A'(y)$, $u_x = -3e^{-3x} A(y) + B'(x)$

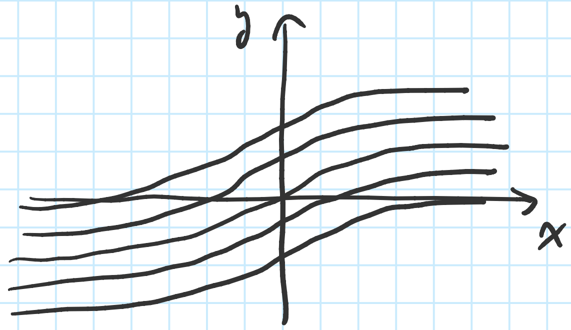
$u_{xy} = -3e^{-3x} A'(y)$. $\Rightarrow 3u_y + u_{xy} = 0$.

Conclusion: solutions are $u = e^{-3x} A(y) + B(x)$,

where $A(y)$ and $B(x)$ are differentiable.

1.2.3. Characteristic curves $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

$$\int dy = \int \frac{dx}{1+x^2} \Rightarrow y = \arctan x + C$$



$$\Rightarrow u = f(C) = f(y - \arctan x)$$
$(f$ differentiable$)$

2. ① $u_{xxx} = 0$ is an ODE w.r.t the $x$ variable.

Solutions to $y'''(x) = 0$ are $y = C_1 + C_2 x + C_3 x^2$

$\Rightarrow$ Solutions to $u_{xxx} = 0$ are $u = C_1(y) + C_2(y)x + C_3(y)x^2$.

By checking, $u = C_1(y) + C_2(y)x + C_3(y)x^2$ indeed gives the solutions.

② $u_{xxy} = 0 \Leftrightarrow (u_{xx})_y = 0 \Leftrightarrow u_{xx} = f(x)$

$$\Leftrightarrow u_x = \underbrace{\int f(x)\,dx}_{= F(x)} + \underbrace{g(y)}_{\substack{\text{constant w.r.t. } x \\ \text{depending on } y}}$$

$$\Leftrightarrow u = \int \big(F(x) + g(y)\big)\,dx + h(y)$$

$$= \underbrace{\int F(x)\,dx}_{= k(x)} + g(y)x + h(y)$$

By checking, $u = k(x) + g(y)x + h(y)$ indeed are solutions. $(k, g, h$ differentiable$)$

3. $u_{xyz} = 0 \Leftrightarrow u_{xy} = \underbrace{f(x,y)}_{\substack{\text{constant w.r.t. } z \\ \text{depending on } x,y}}$

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$$\Leftrightarrow \quad u_x = \underbrace{\int f(x,y)\,dy}_{= F(x,y)} + \underbrace{g(x,z)}_{\substack{\text{constant w.r.t. } y \\ \text{depending on } x,\,z}}$$

$$\Leftrightarrow \quad u = \underbrace{\int F(x,y)\,dx}_{= k(x,y)} + \underbrace{\int g(x,z)\,dx}_{= G(x,z)} + \underbrace{h(y,z)}_{\substack{\text{constant w.r.t. } x \\ \text{depending on } y,\,z}}$$

By checking. $u = k(x,y) + G(x,z) + h(y,z)$ indeed are solutions. ( $k, G, h$ differentiable)