2.(1) As $x \geq ct$, $t > 0$,

$x - ct$, $x + ct \geq 0$

$\Rightarrow \phi_{odd}(x-ct) = \phi(x-ct)$, $\phi_{odd}(x+ct) = \phi(x+ct)$

$\psi_{odd}(y) = \psi(y)$, for all $x-ct \leq y \leq x+ct$.

Thus

$$\frac{1}{2}\left[\phi_{odd}(x-ct) + \phi_{odd}(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)\,dy$$
$$= \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy,$$

for $x \geq ct$, $t > 0$.

As $0 < x < ct$,

$x - ct < 0$, $x + ct > 0$

$\Rightarrow \phi_{odd}(x-ct) = -\phi(ct-x)$, $\phi_{odd}(x+ct) = \phi(x+ct)$

$$\psi_{odd}(y) = \begin{cases} -\psi(-y), & x-ct \leq y < 0 \\ \psi(y), & 0 \leq y \leq x+ct \end{cases}$$

Thus

$$\frac{1}{2}\left[\phi_{odd}(x-ct) + \phi_{odd}(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)\,dy$$
$$= \frac{1}{2}\left[-\phi(ct-x) + \phi(x+ct)\right] + \frac{1}{2c}\left(\int_{x-ct}^{0} -\psi(-y)\,dy + \int_{0}^{x+ct} \psi(y)\,dy\right)$$
$$= \frac{1}{2}\left[\phi(x+ct) + \phi(-x+ct)\right] + \frac{1}{2c}\left(\int_{-x+ct}^{0} \psi(y)\,dy + \int_{0}^{x+ct} \psi(y)\,dy\right)$$
$$= \frac{1}{2}\left[\phi(x+ct) + \phi(-x+ct)\right] + \frac{1}{2c}\int_{-x+ct}^{x+ct} \psi(y)\,dy.$$

(2) Duhamel's principle:

$$u = \mathcal{S}'(t)\phi + \mathcal{S}(t)\psi + \int_0^t \mathcal{S}(t-s)f(\cdot, s)\,ds$$

is a solution to

$$\begin{cases} u_{tt} - c^2 u_{xx} = f \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \\ u(0,t) = 0 \end{cases}$$

From part (1), we know that

$$\mathcal{S}(t)\psi = \begin{cases} \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy, & x \geq ct \\ \frac{1}{2c}\int_{-x+ct}^{x+ct} \psi(y)\,dy, & 0 < x < ct \end{cases}$$

Thus

$$\mathcal{S}(t-s)f(\cdot,s) = \begin{cases} \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy, & x \geq c(t-s) \\ \frac{1}{2c}\int_{-x+c(t-s)}^{x+c(t-s)} f(y,s)\,dy, & 0 < x < c(t-s) \end{cases}$$

If $x \geq ct$, then $x \geq c(t-s)$ for any $0 \leq s \leq t$.

Thus $\mathcal{S}(t-s)f(\cdot,s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy$ and

$$\int_0^t \mathcal{S}(t-s)f(\cdot,s)\,ds = \int_0^t \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$$

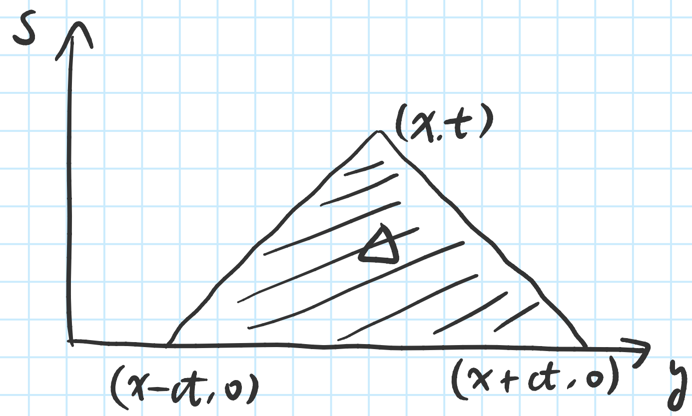
$$= \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$$

$$= \frac{1}{2c}\iint_\Delta f$$

where $\Delta = \{(y,s) : 0 \leq s \leq t, \ x - c(t-s) \leq y \leq x + c(t-s)\}$

~~is~~ the triangle region with vertices (x,t), $(x-ct,0)$, $(x+ct,0)$.

③



If $0 < x < ct$, then

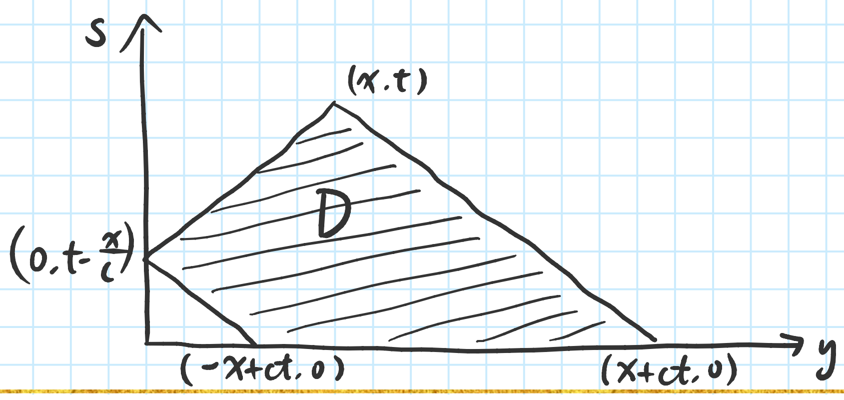
$$\mathcal{S}(t-s)f(\cdot,s) = \begin{cases} \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy, & x \geq c(t-s) \iff s \geq t-\frac{x}{c} \\ \frac{1}{2c}\int_{-x+c(t-s)}^{x+c(t-s)} f(y,s)\,dy, & 0 < x < c(t-s) \iff s < t-\frac{x}{c} \end{cases}$$

Thus

$$\int_0^t \mathcal{S}(t-s)f(\cdot,s)\,ds = \int_0^{t-\frac{x}{c}} \frac{1}{2c}\int_{-x+c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds + \int_{t-\frac{x}{c}}^{t} \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$$

$$= \frac{1}{2c}\iint_D f$$

where $D = \left\{(y,s):\ 0 \leq s \leq t-\frac{x}{c},\ -x+c(t-s) \leq y \leq x+c(t-s)\right\}$
$\cup \left\{(y,s):\ t-\frac{x}{c} \leq s \leq t,\ x-c(t-s) \leq y \leq x+c(t-s)\right\}$



④

Note that $\int_0^t \mathcal{S}(t-s)f(\cdot,s)\,ds$ is a sol. to

$$(*)\begin{cases} u_{tt} - c^2 u_{xx} = f \\ u(x,0) = 0, \quad u_t(x,0) = 0 \\ u(0,t) = 0 \end{cases}$$

by the Duhamel's principle, we have proved that

$$u = \int_0^t \mathcal{S}(t-s)f(\cdot,s)\,ds = \frac{1}{2c}\iint_{D(x,t)} f$$

is a sol. to $(*)$, where $D(x,t)$ is as defined in the question.

3. We will first find ϕ_{ext} on $[-1,0]$ using the oddness of $\phi_{ext}' - \phi_{ext}$ at $x=0$ and the continuity of ϕ_{ext} at $x=0$; then we will find ϕ_{ext} on $[1,3]$ using the oddness of $\phi_{ext}' + 2\phi_{ext}$ at $x=1$ and the continuity of ϕ_{ext} at $x=1$ and $x=2$.

For $x \in [-1,0]$, we have

$$\begin{cases} \phi_{ext}'(x) - \phi_{ext}(x) = -[\phi'(-x) - \phi(-x)] \\ \qquad\qquad\qquad\qquad = -[1+x]. \\ \phi_{ext}(0) = \phi(0) = 0 \end{cases}$$

This is a first order linear ODE. Solving it, we get

$$\phi_{ext} = 2 + x - 2e^x, \qquad x \in [-1,0]$$

For $x \in [1,2]$, we have

5

$$\begin{cases} \phi_{ext}'(x) + 2\phi_{ext}(x) = -\left(\phi'(2-x) + 2\phi(2-x)\right) \\ \qquad = -\left(1 + 2(2-x)\right) = 2x-5 \\ \phi_{ext}(1) = \phi(1) = 1. \end{cases}$$

Solving this to get

$$\phi_{ext}(x) = x - 3 + 3e^{2-2x}, \quad x \in [1,2].$$

For $x \in [2,3]$, we have $(2-x \in [-1,0])$

$$\begin{cases} \phi_{ext}'(x) + 2\phi_{ext}(x) = -\left(\phi_{ext}'(2-x) + 2\phi_{ext}(2-x)\right) \\ \qquad = -\left(1 - 2e^{2-x} + 2(4 - x - 2e^{2-x})\right) \\ \qquad = -9 + 2x + 6e^{2-x} \\ \phi_{ext}(2) = \left(x - 3 + 3e^{2-2x}\right)\Big|_{x=2} \\ \qquad = -1 + 3e^{-2} \end{cases}$$

Solving this. to get

$$\boxed{\phi_{ext}(x) = x - 5 + 6e^{2-x} + 3e^{2-2x} - 4e^{4-2x}, \quad x \in [2,3]}$$

(2)

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi_{ext}(y)\,dy$$ is a solution

to

$$\begin{cases} u_t - k u_{xx} = 0, & \\ u(x,0) = \phi_{ext}(x). & -\infty < x < \infty. \end{cases}$$

Note that $w(x,t) = u_x(x,t) - u(x,t)$ is also a solution to the heat equation $w_t - k w_{xx} = 0$:

$$(u_x - u)_t - k(u_x - u)_{xx}$$

$$= (u_t - k u_{xx})_x - (u_t - k u_{xx}) = 0.$$

Thus $w = u_x - u$ solves

$$\begin{cases} w_t - k w_{xx} = 0 \\ w(x,0) = u_x(x,0) - u(x,0) = \phi_{ext}'(x) - \phi_{ext}(x). \end{cases}$$

Since $w(x,0) = \phi_{ext}'(x) - \phi_{ext}(x)$ is odd at $x=0$, this implies that $w(0,t) = 0$ for all $t > 0$. Thus $u_x(0,t) - u(0,t) = 0$ for all $t > 0$.

Similarly, $v = u_x + 2u$ solves

$$\begin{cases} v_t - k v_{xx} = 0 \\ v(x,0) = \phi_{ext}'(x) + 2\phi_{ext}(x). \end{cases}$$

Since $v(x,0) = \phi_{ext}'(x) + 2\phi_{ext}(x)$ is odd at $x = 1$, this implies that $v(1,t) = 0$ for all $t > 0$.

Thus $u_x(1,t) + 2u(1,t) = 0$ for all $t>0$. 7

We have shown that

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi_{ext}(y)\,dy$$

solves

$$\begin{cases} u_t - k u_{xx} = 0 \\ u(x,0) = \phi_{ext}(x) = \phi(x), \quad 0<x<1, \\ u_x(0,t) - u(0,t) = 0 \\ u_x(0,t) + 2u(0,t) = 0 \end{cases}$$