List of Eigenvalue Problems and Solutions

(With Dirichlet, Neumann, or Periodic boundary conditions)

① $\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \\ X(L) = 0 \end{cases}$

Solution: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \dots$

$X_n = \sin \frac{n\pi x}{L}$.

① variant $\begin{cases} X'' + \lambda X = 0 \\ X(\alpha) = 0 \\ X(\beta) = 0 \end{cases}$ $(\alpha < \beta)$

Solutions: $\lambda_n = \left(\frac{n\pi}{\beta - \alpha}\right)^2$, $n = 1, 2, \dots$

$X_n = \sin \frac{n\pi(x-\alpha)}{\beta - \alpha}$

② $\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0 \\ X'(L) = 0 \end{cases}$

Solution: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, \dots$

$X_n = \cos \frac{n\pi x}{L}$

② variant $\begin{cases} X'' + \lambda X = 0 \\ X'(\alpha) = 0 \\ X'(\beta) = 0 \end{cases}$ $(\alpha < \beta)$

Solutions: $\lambda_n = \left(\frac{n\pi}{\beta - \alpha}\right)^2$, $n = 0, 1, \dots$

$X_n = \cos \frac{n\pi(x-\alpha)}{\beta - \alpha}$

③
$$\begin{cases} X''+\lambda X = 0 \\ X(-L) = X(L) \\ X'(-L) = X'(L) \end{cases}$$

Solution: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, \cdots$

$X_0 = 1$.

$X_n = A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}$, $n \geq 1$.

③ variant
$$\begin{cases} X''+\lambda X = 0 \\ X(\alpha) = X(\beta) \\ X'(\alpha) = X'(\beta) \end{cases} \qquad (\alpha < \beta)$$

Solution: $\lambda_n = \left(\frac{2n\pi}{\beta-\alpha}\right)^2$, $n = 0, 1, \cdots$

$X_0 = 1$.

$X_n = A_n \cos\frac{2n\pi x}{\beta-\alpha} + B_n \sin\frac{2n\pi x}{\beta-\alpha}$, $n \geq 1$.

Example
$$\begin{cases} \Theta''+\lambda\Theta = 0 \\ \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases} \qquad \Theta = \Theta(\theta)$$

Solution: $\lambda_n = \left(\frac{2n\pi}{2\pi-0}\right)^2 = n^2$, $n = 0, 1, \cdots$

$\Theta_0 = 1$.

$\Theta_n = A_n \cos\frac{2n\pi\theta}{2\pi-0} + B_n \sin\frac{2n\pi\theta}{2\pi-0}$

$= A_n \cos n\theta + B_n \sin n\theta$, $n \geq 1$.

④
$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \\ X'(L) = 0 \end{cases}$$

Solution: $\lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2$, $n = 0, 1, \cdots$

$X_n = \sin\frac{(n+\frac{1}{2})\pi x}{L}$.

④ variant $\begin{cases} X'' + \lambda X = 0 \\ X(\alpha) = 0 \\ X'(\beta) = 0 \end{cases}$ $(\alpha < \beta)$

Solution: $\lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{\beta-\alpha}\right)^2$, $n = 0, 1, \cdots$

$X_n = \sin\frac{(n+\frac{1}{2})\pi(x-\alpha)}{\beta-\alpha}$.

⑤ $\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0 \\ X(L) = 0 \end{cases}$

Solution: $\lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2$, $n = 0, 1, \cdots$

$X_n = \cos\frac{(n+\frac{1}{2})\pi x}{L}$.

⑤ variant $\begin{cases} X'' + \lambda X = 0 \\ X'(\alpha) = 0 \\ X(\beta) = 0 \end{cases}$ $(\alpha < \beta)$

Solution: $\lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2$, $n = 0, 1, \cdots$

$X_n = \cos\frac{(n+\frac{1}{2})\pi(x-\alpha)}{\beta-\alpha}$.