Practice Midterm 1

Math 3435, Spring 2019, UConn

Date: 02/27/2019, Wednesday
Duration: 1h

Instructions:

- No calculator is allowed.
- Unless it is indicated in the questions that only a final answer is required, you need to show your work to get credit.
- If you have work on the back of a page, please indicate this on the front of the page.

Name: ______________________________

Husky Card ID: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total	40	

Question 1. Solve $u_x + u_y + u = e^{x+2y}$ with $u(x,0) = 0$ by the method of characteristic curves.

Done in HW2-Solutions posted on the course website.

$$\begin{cases} h(x) = \dfrac{x^2}{20} + \dfrac{4}{5} e^{\frac{x}{4}} + C_1 \\ g(x) = \dfrac{x^2}{5} - \dfrac{4}{5} e^{-x} + C_2 \end{cases}$$

also $C_1 + C_2 = 0$ by $(*)$. Thus

$$u(x,t) = h(t+4x) + g(t-x)$$

$$= \frac{(t+4x)^2}{20} + \frac{4}{5} e^{\frac{t+4x}{4}} + \frac{(t-x)^2}{5} - \frac{4}{5} e^{-t+x}$$

Question 2. Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = e^x$.

Rewrite $u_{xx} - 3u_{xt} - 4u_{tt}$

$$= (\partial_x^2 - 3\partial_x \partial_t - 4\partial_t^2)\, u$$

$$= (\partial_x - 4\partial_t)(\partial_x + \partial_t)\, u = 0$$

$$\Rightarrow \begin{cases} (\partial_x + \partial_t)\, u = v & ① \\ (\partial_x - 4\partial_t)\, v = 0 & ② \end{cases}$$

②: charateristic curves $\dfrac{dt}{dx} = -4 \quad t = -4x + C$

$$\Rightarrow \quad v = f(C) = f(t+4x)$$

①: charateristic curves $\dfrac{dt}{dx} = 1 . \quad t = x + C$

Let $w(x) = u(x, x+C)$,

then $w'(x) = u_x(x, x+C) + u_y(x, x+C)$

$$① \Rightarrow w'(x) = v = f(t+4x) = f(x + C + 4x) = f(5x+C)$$

$$\Rightarrow \quad w(x) = \int f(5x+C)\,dx = h(5x+C) + g(C)$$

(Here $h'(y) = \frac{1}{5} f(y)$)

$C = t - x \Rightarrow u(x,t) = h(t+4x) + g(t-x)$.

Now $u(x,0) = h(4x) + g(-x) = x^2 \Rightarrow 4h'(4x) - g'(-x) = 2x$ $(*)$

$$u_t(x,0) = h'(4x) + g'(-x) = e^x$$

$$\Rightarrow h'(4x) = \frac{1}{5}(2x + e^x) \Rightarrow g'(-x) = \frac{4}{5} e^x - \frac{2}{5} x \Rightarrow$$

$$h'(x) = \frac{1}{5}\left(\frac{1}{2} x + e^{\frac{x}{4}}\right), \quad g'(x) = \frac{4}{5} e^{-x} + \frac{2}{5} x \Rightarrow (\text{Top})$$

Question 3. Prove uniqueness of the solution to the wave equation $u_{tt} - c^2 u_{xx} = 0$ satisfying the initial conditions $u(x,0) = \phi(x) \in C^2$, $u_t(x,0) = \psi(x) \in C^1$.

Let u_1, u_2 be two solutions to the PDE problem. Then $u = u_1 - u_2$ satisfies

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = 0 \quad (\Rightarrow u_x(x,0) = 0) \\ u_t(x,0) = 0 \end{cases}$$

It suffices to show $u(x,t) = 0$ for all x, t.

Define $E(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t) + c^2 u_x^2(x,t)\, dx$

$E(t)$ is conserved $\Rightarrow$

$$E(t) = E(0) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,0) + c^2 u_x^2(x,0)\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} 0^2 + c^2 \cdot 0^2 \, dx$$

$$= 0 \quad \text{for all } t.$$

$\Rightarrow \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t) + c^2 u_x^2(x,t)\, dx = 0$

$\Rightarrow u_t(x,t), u_x(x,t) = 0$ for all x, t.

$\Rightarrow u(x,t) =$ a constant for all x, t

$= 0$ since $u(x,0) = 0$.

Question 4. Provide a solution to the heat equation $u_t - ku_{xx} = 0$ satisfying the initial condition $u(x,0) = \phi(x)$ where $\phi(x) = 1$ for $x > 0$ and $\phi(x) = 3$ for $x < 0$.

Solution formula :

$$u(x,t) = \frac{1}{\sqrt{\pi}}(B-A)\int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,dp + \frac{1}{2}(A+B)$$

is a solution to $u_t - k u_{xx} = 0$ satisfying the initial condition $u(x,0) = \phi(x) = \begin{cases} A, & x<0 \\ B, & x>0 \end{cases}$.

Here $A = 3$, $B = 1$

$$\Rightarrow u(x,t) = -\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,dp + 2 .$$