2. As an approximation to the delta function. $\{\eta_t(x), t>0\}$ satisfies ①

① $\eta_t(x) \geq 0$, $-\infty < x < \infty$.

② $\int_{-\infty}^{\infty} \eta_t(x)\, dx = 1$

③ For any $\varepsilon > 0$, $\lim_{t \downarrow 0} \int_{|x|<\varepsilon} \eta_t(x)\, dx = 1$

(or equivalently $\lim_{t \downarrow 0} \int_{|x| \geq \varepsilon} \eta_t(x)\, dx = 0$)

(1)
Now by ② and a change of variable,

$$\int_{-\infty}^{\infty} \eta_t(x-y)\, dy = 1 \qquad (*)$$

Thus $u(x,t) - \phi(x) = \int_{-\infty}^{\infty} \eta_t(x-y)\phi(y)\, dy -$

$$\phi(x). \int_{-\infty}^{\infty} \eta_t(x-y)\, dy$$

$$= \int_{-\infty}^{\infty} \eta_t(x-y)\left[\phi(y) - \phi(x)\right] dy$$

(2) $\left| \int_{|y-x| \leq \delta} \eta_t(x-y)\left[\phi(y) - \phi(x)\right] dy \right|$

$\leq \int_{|y-x| \leq \delta} \left| \eta_t(x-y)\left[\phi(y) - \phi(x)\right] \right| dy$

$= \int_{|y-x| \leq \delta} \eta_t(x-y) \left| \phi(y) - \phi(x) \right| dy$ (Using property ①)

$\leq \frac{\varepsilon}{2} \int_{|y-x| \leq \delta} \eta_t(x-y)\, dy$

$\leq \frac{\varepsilon}{2} \int_{-\infty}^{\infty} \eta_t(x-y)\, dy = \frac{\varepsilon}{2}$ (Using $(*)$)

(3) $\left|\int_{|y-x|>\delta} \eta_t(x-y)[\phi(y)-\phi(x)]\,dy\right|$ ②

$$\leq \int_{|y-x|>\delta} \left|\eta_t(x-y)\cdot[\phi(y)-\phi(x)]\right| dy$$

$$= \int_{|y-x|>\delta} \eta_t(x-y)\left|\phi(y)-\phi(x)\right| dy \quad \text{(Using property ①)}$$

$$\leq 2A \int_{|y-x|>\delta} \eta_t(x-y)\,dy$$

$$= 2A\cdot \int_{|z|>\delta} \eta_t(z)\,dz \quad \text{(by a change of variable } z = y-x\text{)}$$

Now by Property ③ of $\{\eta_t(x),\ t>0\}$.

$$\lim_{t\downarrow 0} \int_{|z|>\delta} \eta_t(z)\,dz = 0$$

$$\Rightarrow \int_{|z|>\delta} \eta_t(z)\,dz \leq \frac{\varepsilon}{4A} \quad \text{for } t \text{ small enough.}$$

$$\Rightarrow \left|\int_{|y-x|>\delta} \eta_t(x-y)[\phi(y)-\phi(x)]\,dy\right| \leq 2A\cdot\frac{\varepsilon}{4A} = \frac{\varepsilon}{2}$$

for t small enough.

Thus

$$|u(x,t)-\phi(x)| = \left|\int_{-\infty}^{\infty} \eta_t(x-y)[\phi(y)-\phi(x)]\,dy\right|$$

$$\leq \left|\int_{|y-x|\leq\delta} \eta_t(x-y)[\phi(y)-\phi(x)]\,dy\right| + \left|\int_{|y-x|>\delta} \eta_t(x-y)[\phi(y)-\phi(x)]\,dy\right|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon \quad . \quad \text{for } t \text{ small enough.}$$

③

3. (1) $\int_{y<x} \eta_t(x-y)\,dy = \int_{z>0} \eta_t(z)\,dz$

$\int_{y>x} \eta_t(x-y)\,dy = \int_{z<0} \eta_t(z)\,dz$

Since $\int_{-\infty}^{\infty} \eta_t(z)\,dz = 1$ and $\eta_t(z)$ is even,

we have $\int_{z>0} \eta_t(z)\,dz = \int_{z<0} \eta_t(z)\,dz = \frac{1}{2}$.

Thus $\int_{y<x} \eta_t(x-y)\,dy = \int_{y>x} \eta_t(x-y)\,dy = \frac{1}{2}$.

(2) $u(x,t) - \frac{1}{2}[\phi(x-) + \phi(x+)]$

$$= \int_{-\infty}^{\infty} \eta_t(x-y)\phi(y)\,dy - \phi(x-)\cdot\int_{y<x} \eta_t(x-y)\,dy - \phi(x+)\cdot\int_{y>x} \eta_t(x-y)\,dy$$

(Using (1))

Using $\int_{-\infty}^{\infty} \eta_t(x-y)\phi(y)\,dy = \int_{y<x} \eta_t(x-y)\phi(y)\,dy + \int_{y>x} \eta_t(x-y)\phi(y)\,dy$.

we have

$u(x,t) - \frac{1}{2}[\phi(x-) + \phi(x+)]$

$$= \int_{y<x} \eta_t(x-y)[\phi(y) - \phi(x-)]\,dy + \int_{y>x} \eta_t(x-y)[\phi(y) - \phi(x+)]\,dy.$$

(3) To show $\lim_{t\downarrow 0} \int_{y<x} \eta_t(x-y)[\phi(y) - \phi(x-)]\,dy = 0$.

we need to show for any fixed $\varepsilon > 0$,

$$\left| \int_{y<x} \eta_t(x-y)\left[\phi(y) - \phi(x-)\right] dy \right| \leq \varepsilon.$$

Now $\phi(x-) = \lim_{y \uparrow x} \phi(y)$, thus there exists $\delta > 0$. such that whenever $x-\delta < y < x$,

$$|\phi(y) - \phi(x-)| \leq \frac{\varepsilon}{2}.$$

Then $\left| \int_{x-\delta<y<x} \eta_t(x-y)\left[\phi(y) - \phi(x-)\right] dy \right|$

$$\leq \int_{x-\delta<y<x} \eta_t(x-y)\,|\phi(y) - \phi(x-)|\, dy$$

$$\leq \frac{\varepsilon}{2} \int_{x-\delta<y<x} \eta_t(x-y)\, dy$$

$$\leq \frac{\varepsilon}{2} \int_{-\infty}^{\infty} \eta_t(x-y)\, dy = \frac{\varepsilon}{2} \cdot 1 = \frac{\varepsilon}{2}.$$

Now ϕ being a bounded function, there is some $A > 0$ such that $|\phi(x)| \leq A$ for all x.

Then $|\phi(y) - \phi(x-)| \leq |\phi(y)| + |\phi(x-)| \leq 2A$.

Then $\left| \int_{y<x-\delta} \eta_t(x-y)\left[\phi(y) - \phi(x-)\right] dy \right|$

$$\leq \int_{y<x-\delta} \eta_t(x-y)\,|\phi(y) - \phi(x-)|\, dy$$

$$\leq 2A \cdot \int_{y<x-\delta} \eta_t(x-y)\, dy$$

$$= 2A \cdot \int_{z>\delta} \eta_t(z)\, dz$$

Now $\lim_{t \downarrow 0} \int_{|z|>\delta} \eta_t(z)\, dz = 0$ by property 3.

since $|z| > \delta \iff z > \delta$ or $z < -\delta$.

$$\lim_{t\downarrow 0} \int_{z>\delta} \eta_t(z)\,dz = 0.$$

Thus $\left|\int_{z>\delta} \eta_t(z)\,dz\right| \leq \frac{\varepsilon}{4A}$ for t small enough.

Thus $\left|\int_{y<x-\delta} \eta_t(x-y)[\phi(y)-\phi(x_-)]\,dy\right|$

$$\leq 2A \cdot \frac{\varepsilon}{4A} = \frac{\varepsilon}{2} \quad \text{for } t \text{ small enough.}$$

In conclusion,

$$\left|\int_{y<x} \eta_t(x-y)[\phi(y)-\phi(x_-)]\right|$$

$$\leq \left|\int_{y<x-\delta}\right| + \left|\int_{x-\delta<y<x}\right|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \text{for } t \text{ small enough.}$$

The proof of $\lim_{t\downarrow 0} \int_{y>x} \eta_t(x-y)[\phi(y)-\phi(x_+)]\,dy = 0$

is completely analogous (by writing

$$\int_{y>x} = \int_{y>x+\delta} + \int_{x+\delta>y>x}), \text{ and}$$

we omit the details.

Then we may conclude that

$$\lim_{t\downarrow 0} u(x,t) - \frac{1}{2}[\phi(x_-)+\phi(x_+)] =$$

$$\lim_{t\downarrow 0} \int_{y<x} \eta_t(x-y)[\phi(y)-\phi(x_-)]\,dy + \lim_{t\downarrow 0} \int_{y>x} \eta_t(x-y)[\phi(y)-\phi(x_+)]\,dy$$

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$$= 0 + 0 = 0 .$$

thus $\lim_{t \downarrow 0} u(x,t) = \frac{1}{2}\left[\phi(x-) + \phi(x+)\right] .$