Midterm 2

Math 3435-2, Spring 2019, UConn

Date: 04/12/2019, Friday
Duration: 1h

Instructions:

- No calculator is allowed.
- Unless it is indicated in the questions that only a final answer is required, you need to show your work to get credit.
- If you have work on the back of a page, please indicate this on the front of the page.

Name: ______________________________

Husky Card ID: ______________________________

Question	Points	Score
1	20	
2	10	
3	10	

Question 1. Consider the heat equation on the half line with homogeneous Neumann boundary condition:

$$(*)\begin{cases} u_t - ku_{xx} = 0, & 0 < x < \infty,\ 0 < t < \infty \\ u(x,0) = \phi(x), & 0 < x < \infty \\ u_x(0,t) = 0, & 0 < t < \infty \end{cases}$$

Here $\phi(x)$ is a bounded continuous function on the domain $[0,\infty)$. Let ϕ_{even} be the even extension of ϕ to the whole line, that is, $\phi_{\text{even}}(x) = \begin{cases} \phi(x), & x \geq 0 \\ \phi(-x), & x < 0 \end{cases}$. Then let

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi_{\text{even}}(y)\ dy.$$

(1) Check that $u(x,t)$ is an even function with respect to x, that is, $u(x,t) = u(-x,t)$.
(2) Check that $u_x(0,t) = 0$ for all $t > 0$. Explain why then $u(x,t)$ solves $(*)$.
(3) Check that

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}} \right) \phi(y)\ dy.$$

(4) Apply Duhamel's principle to find a solution to the inhomogeneous problem

$$(**)\begin{cases} u_t - ku_{xx} = f(x,t), & 0 < x < \infty,\ 0 < t < \infty \\ u(x,0) = \phi(x), & 0 < x < \infty \\ u_x(0,t) = 0, & 0 < t < \infty \end{cases}$$

(Empty page for continuing writing your solution to Question 1)

Choose from Question 2,3,4 **only two** questions to solve. Circle your two choices in the below. **If you fail to circle your choices, I will grade Question 2 and 3.**

Question 2 Question 3 Question 4

Question 2. Let $S(x,t) = \dfrac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$ be the heat kernel function. Show that $\{\eta_t(x) = S(x,t),\ t > 0\}$ is an *approximation of the delta function.*

Question 3. Consider the wave equation $u_{tt} - c^2 u_{xx} = 0$ on a finite interval $(0, l)$ with Dirichlet boundary conditions $u(0, t) = 0$, $u(l, t) = 0$. Please carry out the procedure of separation of variables, solve the corresponding eigenvalue problem, and write the series expansion for a solution $u(x, t)$.

Question 4. Show that $\left\{ \sin \dfrac{\left(n+\frac{1}{2}\right)\pi x}{l},\ x \in [0,l],\ n = 0, 1, \dots \right\}$ is an orthogonal system of functions. Then find the Fourier series for a general function $\phi(x)$ ($x \in [0,l]$) associated to this orthogonal system.
Hint: You may find the following trigonometric identities useful:

$$\sin\alpha \sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}, \quad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$