Practice Final 1 – Solutions

Q 1. Characteristic curves:

$$\frac{dy}{dx} = \frac{y}{1} \iff \int \frac{dy}{y} = \int dx$$

$$\iff \log|y| = x + C \iff |y| = e^{x+C}$$

$$\iff y = Ce^x, \quad C \text{ being any constant.}$$

($C=0$ corresponds to the special solution $y=0$)

Restricting u on characteristic curves:

Let $w(x) = u(x, Ce^x)$

then $w'(x) = u_x + Ce^x u_y = u_x + y u_y$

$$u_x + y u_y + yu = y$$

$$\iff w'(x) + Ce^x w(x) = Ce^x$$

$$\iff \left(e^{Ce^x} w(x)\right)' = e^{Ce^x} Ce^x$$

$$\iff e^{Ce^x} w(x) = \int e^{Ce^x} Ce^x \, dx$$

$$= e^{Ce^x} + f(C)$$

a constant depending on C

$$\iff w(x) = 1 + e^{-Ce^x} f(C)$$

$$\iff u(x, Ce^x) = 1 + e^{-Ce^x} f(C)$$

$$\iff u(x,y) = 1 + e^{-y} f(ye^{-x}) \quad (C = ye^{-x})$$

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<u>Last step</u> using $u(0,y)=e^y$

$$\Rightarrow u(0,y) = 1 + e^{-y} f(y) = e^y$$

$$\Rightarrow f(y) = e^y(e^y - 1)$$

<u>Conclusion</u> $u(x,y) = 1 + e^{-y} f(ye^{-x})$

$$= 1 + e^{-y} e^{ye^{-x}} \left(e^{ye^{-x}} - 1\right)$$

<u>Q2.</u> $u(0,t)=0 \Rightarrow$ odd extension of the initial data $\phi(x) = u(x,0) = 3,\ x>0$

$$\Rightarrow \phi_{ext}(x) = \begin{cases} 3, & x>0 \\ -3, & x<0 \end{cases}$$

$$\Rightarrow u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, \phi_{ext}(y)\, dy$$

$$= \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi_{ext}(y)\, dy$$

solves the problem.

<u>Last step</u> Simplify the solution formula:

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \left(\int_0^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot 3 \cdot dy + \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4kt}} (-3) \cdot dy \right)$$

$$= \frac{3}{\sqrt{4\pi k t}} \left(\int_0^{\infty} e^{-\frac{(x-y)^2}{4kt}} dy - \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4kt}} dy \right)$$

$$= \frac{3}{\sqrt{4\pi k t}} \left(\int_{-\frac{x}{\sqrt{4kt}}}^{\infty} e^{-s^2} \cdot \sqrt{4kt}\, ds - \int_{-\infty}^{-\frac{x}{\sqrt{4kt}}} e^{-s^2} \sqrt{4kt}\, ds \right)$$

$$= \frac{3}{\sqrt{\pi}}\left(\int_{-\frac{x}{\sqrt{4kt}}}^{\infty} e^{-s^2}ds - \int_{-\infty}^{-\frac{x}{\sqrt{4kt}}} e^{-s^2}ds\right)$$

$$= \frac{3}{\sqrt{\pi}}\left(\int_{-\frac{x}{\sqrt{4kt}}}^{0} e^{-s^2}ds + \underbrace{\int_0^{\infty} e^{-s^2}ds - \int_{-\infty}^{0} e^{-s^2}ds}_{0} - \int_0^{-\frac{x}{\sqrt{4kt}}} e^{-s^2}ds\right)$$

$$= \frac{6}{\sqrt{\pi}}\int_{-\frac{x}{\sqrt{4kt}}}^{0} e^{-s^2}ds$$

$$= \frac{6}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2}ds$$

Another way: The solution to the heat equation with initial data

$$\phi(x) = \begin{cases} A & x<0 \\ B & x>0 \end{cases}$$

is $u(x,t) = \frac{1}{\sqrt{\pi}}(B-A)\int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2}ds + \frac{1}{2}(A+B)$

In our problem, $A=-3$, $B=3$,

$$\Rightarrow u(x,t) = \frac{6}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2}ds$$

Lastly, note that $k=1$, thus $u(x,t) = \frac{6}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4t}}} e^{-s^2}ds$.

Q3. $u(0,t) = u(1,t) = 0 \Rightarrow$ odd extension at 0 & 1.

$$\phi_{ext}(x) = \begin{cases} \phi(x) = x^2(1-x), & 0<x<1 \\ -\phi(-x) = -x^2(1+x), & -1<x<0 \\ \text{Then extended to a 2-periodic function.} \end{cases}$$

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$$\psi_{ext}(x) = \begin{cases} \psi(x) = (1-x)^2, & 0<x<1. \\ -\psi(-x) = -(1+x)^2, & -1<x<0, \\ \text{Then extended to a 2-periodic function} \end{cases}$$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[\phi_{ext}(x-t) + \phi_{ext}(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \psi_{ext}(y)\,dy \qquad (c=1)$$

solves the PDE problem.

$$\Rightarrow u\left(\tfrac{2}{3}, 2\right) = \frac{1}{2}\left[\phi_{ext}\left(\tfrac{2}{3}-2\right) + \phi_{ext}\left(\tfrac{2}{3}+2\right)\right] + \frac{1}{2}\int_{\frac{2}{3}-2}^{\frac{2}{3}+2} \psi_{ext}(y)\,dy$$

$$= \frac{1}{2}\left[\phi_{ext}\left(\tfrac{2}{3}\right) + \phi_{ext}\left(\tfrac{2}{3}\right)\right] + \frac{1}{2}\int_{-2}^{2} \psi_{ext}(y)\,dy$$

(Here we used ϕ_{ext} being a periodic function with period 2, and ψ_{ext} being a periodic function with period 2, and thus also of period 4, and the integral of a periodic function over any interval of length equal to its period is the same, thus we may replace the interval $[\frac{2}{3}-2, \frac{2}{3}+2]$ by the interval $[-2, 2]$.)

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$$= \phi_{ext}\left(\tfrac{2}{3}\right) + 0$$

(Since ψ_{ext} is odd, $\int_{-2}^{2} \psi_{ext} = 0$)

$$= \left(\tfrac{2}{3}\right)^2\left(1-\tfrac{2}{3}\right) = \frac{4}{27}.$$

Q4. Solution formula for

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases} \qquad \begin{array}{l} -\infty < x < \infty \\ -\infty < t < \infty \end{array}$$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy$$

$$+ \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$$

Plug in $f(x,t) = \cos x$, $\phi(x) = \sin x$, $\psi(x) = 1+x$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[\sin(x-ct) + \sin(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} (1+y)\,dy$$

$$+ \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \cos y\,dy\,ds$$

$$= \frac{1}{2}\left[\sin(x-ct) + \sin(x+ct)\right] + \frac{1}{4c}\left[(1+x+ct)^2 - (1+x-ct)^2\right]$$

$$+ \frac{1}{2c}\int_0^t \left[\sin\left(x+c(t-s)\right) - \sin\left(x-c(t-s)\right)\right] ds$$

$$= \frac{1}{2}\left[\sin(x-ct) + \sin(x+ct)\right] + (1+x)t$$

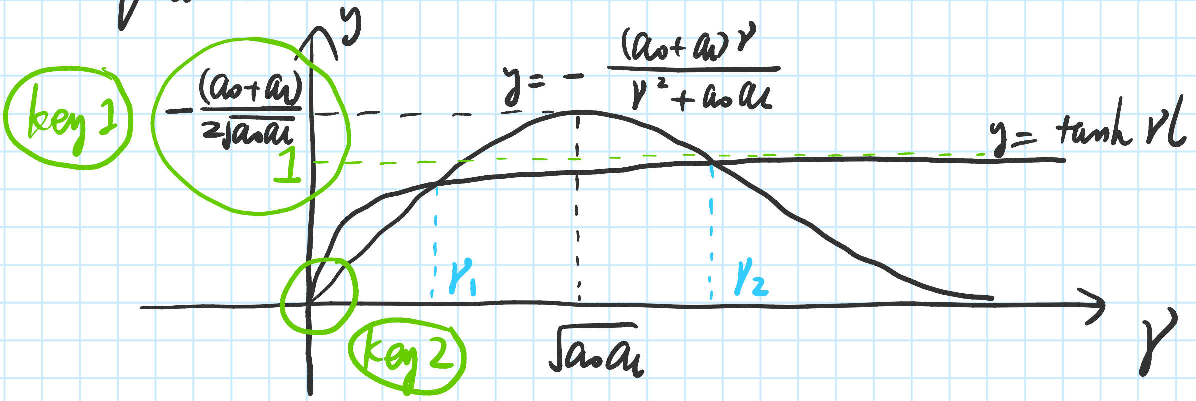
$$+ \frac{1}{2c^2}\left[2\cos x - \cos(x+ct) - \cos(x-ct)\right].$$

Q5. From lecture, $\lambda = -\gamma^2 < 0$ is an eigenvalue if and only if γ is a solution to

$$\tanh \gamma l = -\frac{(a_0 + a_1)\gamma}{\gamma^2 + a_0 a_1}$$

(Equation (16) in Page 97 of textbook.)

Provided $a_0, a_1 < 0$, $-a_0 - a_1 < a_0 a_1 l$, we sketch the graph of both sides of the above equation:



$$\frac{d}{d\gamma}\left(-\frac{(a_0+a_1)\gamma}{\gamma^2 + a_0 a_1}\right) = -\frac{(a_0+a_1)(a_0 a_1 - \gamma^2)}{(\gamma^2 + a_0 a_1)^2}$$

$$= \begin{cases} -\dfrac{a_0 + a_1}{a_0 a_1} & \text{if } \gamma = 0 \\ 0 & \text{if } \gamma = \sqrt{a_0 a_1} \end{cases}$$

In particular, the maximum of $-\frac{(a_0+a_1)\gamma}{\gamma^2 + a_0 a_1}$ is obtained at $\gamma = \sqrt{a_0 a_1}$, with the maximum being

$\boxed{-\frac{(a_0+a_1)}{2\sqrt{a_0 a_1}}, \text{ which is always } \geq 1.}$ key 1

Also $\left.\frac{d}{d\gamma}\right|_{\gamma=0}(\tanh \gamma L) = L$

Since $-a_0 - a_1 < a_0 a_1 L$, we have

$$\boxed{\left.\frac{d}{d\gamma}\right|_{\gamma=0}\left(-\frac{(a_0+a_1)\gamma}{\gamma^2+a_0a_1}\right) = -\frac{a_0+a_1}{a_0a_1} < L = \left.\frac{d}{d\gamma}\right|_{\gamma=0}(\tanh\gamma L)}$$

Key 2

Key 1 and Key 2 produce the shape of the graph, from which we observe that γ_1, γ_2 provide the two negative eigenvalues $\lambda_1 = -\gamma_1^2$, $\lambda_2 = -\gamma_2^2$.

<u>Q6.</u> We first compute the Fourier sine series of $f(x) = x$, $x \in [0, L]$.

$$x \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

with
$$b_n = \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}\,dx$$
$$= \frac{2}{L}\cdot\left(-\frac{L}{n\pi}\right)\left[\left. x\cos\frac{n\pi x}{L}\right|_0^L - \int_0^L \cos\frac{n\pi x}{L}\,dx\right]$$
$$= -\frac{2}{n\pi}\cdot(-1)^n L = (-1)^{n+1}\frac{2L}{n\pi}$$

Parseval's identity:

$$(x, x) = \sum_{n=1}^{\infty} |b_n|^2 \left(\sin\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\right)$$

inner product

$$\Rightarrow \int_0^l x^2\,dx = \sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{2l}{n\pi} \right|^2 \int_0^l \sin^2 \frac{n\pi x}{l}\,dx$$

$$\Rightarrow \frac{l^3}{3} = \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2} \cdot \frac{l}{2}$$

$$= \frac{2l^3}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Q7. Separation of variables: $u(x,y) = X(x)Y(y)$

$$u_{xx} + u_{yy} = 0 \Rightarrow \begin{cases} Y'' + \lambda Y = 0 \\ X'' - \lambda X = 0 \end{cases}$$

(You may also write $\begin{cases} Y'' - \lambda Y = 0 \\ X'' + \lambda X = 0 \end{cases}$)

$$u_y(x,0) = 0 \Rightarrow Y'(0) = 0$$

$$u_y(x,\pi) = 0 \Rightarrow Y'(\pi) = 0$$

$\Rightarrow$ The eigenvalue problem:

$$\begin{cases} Y'' + \lambda Y = 0 \\ Y'(0) = 0 \\ Y'(\pi) = 0 \end{cases}$$

Solution $\Rightarrow$ $\lambda_n = n^2$, $n = 0, 1, \cdots$

$$Y_n = \cos ny.$$

Now $X'' - \lambda X = 0 \Rightarrow X_n'' - \lambda_n X_n = X_n'' - n^2 X_n = 0$

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$$\Rightarrow \quad X_0 = A_0 x + B_0$$

$$X_n = A_n e^{nx} + B_n e^{-nx}$$

$$u(0,y)=0 \quad \Rightarrow \quad X(0) = 0$$

$$\Rightarrow \quad B_0 = 0 \quad \Rightarrow \quad X_0 = A_0 x$$

$$A_n + B_n = 0 \quad \Rightarrow \quad X_n = A_n\left(e^{nx} - e^{-nx}\right)$$

$\Rightarrow$ <u>Series expansion:</u>

$$u = \sum_{n=0}^{\infty} X_n(x) Y_n(y)$$

$$= A_0 x + \sum_{n=1}^{\infty} A_n\left(e^{nx} - e^{-nx}\right) \cos ny$$

$$\Rightarrow u(\pi, y) = A_0 \pi + \sum_{n=1}^{\infty} A_n\left(e^{n\pi} - e^{-n\pi}\right) \cos ny$$

$$= \frac{1}{2}\left(1 + \cos 2y\right) = \frac{1}{2} + \frac{1}{2}\cos 2y$$

$$\Rightarrow \begin{cases} A_0 \pi = \frac{1}{2} \\ A_2\left(e^{2\pi} - e^{-2\pi}\right) = \frac{1}{2} \\ A_n\left(e^{n\pi} - e^{-n\pi}\right) = 0 \quad \text{for } n \neq 0, 2. \end{cases}$$

$$\Rightarrow \begin{cases} A_0 = \frac{1}{2\pi} \\ A_2 = \frac{1}{2\left(e^{2\pi} - e^{-2\pi}\right)} \\ A_n = 0 \quad \text{for } n \neq 0, 2. \end{cases}$$

Thus $u(x,y) = \frac{1}{2\pi} \cdot x + \frac{e^{2x} - e^{-2x}}{2\left(e^{2\pi} - e^{-2\pi}\right)} \cos 2y$

Q8. $u_{xx} + u_{yy} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$

Separation of variables: $u = R(r)\Theta(\theta)$

$$\Rightarrow \begin{cases} \Theta'' + \lambda\Theta = 0 \\ r^2 R'' + rR' - \lambda R = 0 \end{cases}$$

$\Theta(\theta)$ satisfies the periodic boundary condition on $\theta \in [0, 2\pi]$. thus we get an eigenvalue problem

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$$

$$\xRightarrow{\text{Solution}} \quad \lambda_n = n^2, \quad n = 0, 1, \dots$$

$$\Theta_0 = 1, \quad \Theta_n = A_n \cos n\theta + B_n \sin n\theta, \quad n \geq 1.$$

$$\Rightarrow r^2 R_n'' + r R_n' - n^2 R_n = 0$$

$$\Rightarrow \begin{cases} n = 0 : & R_0 = C_0 + D_0 \log r \\ n \geq 1 : & R_n = C_n r^{-n} + D_n r^n \end{cases}$$

Since the question requires that u be bounded as $r \to \infty$, we thus cannot have the terms

$$D_0 \log r \quad \text{and} \quad D_n r^n \quad (n \geq 1).$$

Thus $R_0 = C_0$, $R_n = C_n r^{-n}$ $(n \geq 1)$.

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Series expansion:

$$u(r,\theta) = \sum_{n=0}^{\infty} R_n(r)\,\Theta_n(\theta)$$

$$= C_0 + \sum_{n=1}^{\infty} r^{-n}\left(A_n \cos n\theta + B_n \sin n\theta\right)$$

Now $u(a,\theta) = C_0 + \sum_{n=1}^{\infty} a^{-n}\left(A_n \cos n\theta + B_n \sin n\theta\right)$

$$= 1 + 3\sin\theta$$

$$\Rightarrow \begin{cases} C_0 = 1 \\ a^{-1}B_1 = 3 \Rightarrow B_1 = 3a, \quad a^{-1}A_1 = 0 \Rightarrow A_1 = 0 \\ A_n, B_n = 0, \quad \text{for } n \neq 1. \end{cases}$$

Thus $u(r,\theta) = 1 + \frac{3a}{r}\sin\theta$

Q9. (1) $P(r,\theta) \geq 0$: Since $r < a$, $a^2 - r^2 > 0$.

Then $a^2 - 2ar\cos\theta + r^2 = \underbrace{(a - r\cos\theta)^2}_{>0} + \underbrace{r^2(1-\cos^2\theta)}_{\geq 0}$

Thus $P(r,\theta) = \dfrac{\overset{>0}{a^2 - r^2}}{a^2 - 2ar\cos\theta + r^2} > 0$.

(2). From lecture,

$$P(r,\theta) = 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n\theta$$

(See Formula (17) on Page 170 of textbook)

Then $\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta)\,d\theta$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n\theta\right) d\theta$$

$$= \frac{1}{2\pi}\left[\underbrace{\int_{-\pi}^{\pi} 1\, d\theta}_{2\pi} + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \underbrace{\int_{-\pi}^{\pi}\cos n\theta\, d\theta}_{=0}\right]$$

$$= 1.$$

(3) $\left|\frac{1}{2\pi}\int_{\varepsilon<|\theta|\le\pi} P(r,\theta)\,d\theta\right|$

$$= \left|\frac{1}{2\pi}\int_{\varepsilon<|\theta|\le\pi}\frac{a^2-r^2}{a^2-2ar\cos\theta+r^2}\,d\theta\right|$$

$$= \frac{a^2-r^2}{2\pi}\left|\int_{\varepsilon<|\theta|\le\pi}\frac{1}{a^2-2ar\cos\theta+r^2}\,d\theta\right|$$

$$\le \frac{a^2-r^2}{2\pi}\int_{\varepsilon<|\theta|\le\pi}\frac{1}{a^2-2ar\cos\varepsilon+r^2}\,d\theta$$

($\cos\theta \le \cos\varepsilon$ for $\varepsilon < |\theta| \le \pi$)

$$= \frac{a^2-r^2}{2\pi}\cdot\frac{1}{a^2-2ar\cos\varepsilon+r^2}\cdot\int_{\varepsilon<|\theta|\le\pi} d\theta$$

$$\longrightarrow 0, \quad \text{as } r\to a$$

We have proved $\lim_{r\to a-}\frac{1}{2\pi}\int_{\varepsilon<|\theta|\le\pi} P(r,\theta)\,d\theta = 0.$