Question 1. Consider the PDE problem

$$\begin{cases} u_x + 2xy^2 u_y + u = 1, \\ u(0,y) = y \end{cases}$$

(1) (10pt) Solve the above problem for $y > 0$.
(2) (Extra credit) (5pt) Sketch the characteristic curves. In which region of the xy plane is the solution uniquely determined?

Question 2. (10pt) Suppose $u(x,t)$ satisfies the wave equation on a finite interval

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1, -\infty < t < \infty \\ u(x,0) = x^2(1-x), & 0 < x < 1, -\infty < t < \infty \\ u_t(x,0) = (1-x)^2, & 0 < x < 1, -\infty < t < \infty \\ u(0,t) = u(1,t) = 0, & -\infty < t < \infty \end{cases}$$

Find $u(\frac{1}{4}, \frac{7}{2})$.

Question 3. (10pt) Solve the wave equation on the half line

$$\begin{cases} u_{tt} - u_{xx} = 1, & x > 0, -\infty < t < \infty \\ u(x,0) = x^2, & x > 0 \\ u_t(x,0) = x^2, & x > 0 \\ u_x(0,t) = 0, & -\infty < t < \infty \end{cases}$$

(Hint: Extension method + Solution formula/Duhamel's principle)

Question 4. (10pt) Solve the heat equation on the half line

$$\begin{cases} u_t - 3u_{xx} = 0, & t > 0, x > 0 \\ u(x, 0) = -1, & x > 0 \\ u(0, t) = 0, & t > 0 \end{cases}$$

(Extra credit) (2pt) Then evaluate $\lim_{t \to \infty} u(x, t)$.

Question 5. (10pt) Let $\phi(x) \equiv 1$ for $0 \le x \le \pi$.
(1) Find the Fourier series of $\phi(x)$ with respect to the orthogonal system

$$\left\{ \cos\left[\left(n + \frac{1}{2}\right) x\right], x \in [0, \pi], n = 0, 1, \dots \right\}.$$

(2) Apply Parseval's equality to this series to calculate the infinite sum

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots .$$

Question 6. (10pt) Let (r, θ) be the polar coordinates.
(1) Find the harmonic function u in the region $\{1 < r < 2, 0 < \theta < \frac{\pi}{3}\}$ with the boundary conditions $u = 0$ on the two sides $\theta = 0$ and $\theta = \frac{\pi}{3}$, $u = \sin 3\theta$ on the arc $r = 1$, and $u = \sin 6\theta - \sin 9\theta$ on the arc $r = 2$.
(2) Find the value of u at the point $r = \frac{3}{2}$, $\theta = \frac{\pi}{6}$.

Question 7. (Extra credit) (6pt) Suppose u satisfies $u_{tt} - c^2 u_{xx} = 0$, $-\infty < x, t < \infty$. On the (x,t)-plane, apply the divergence theorem for the vector field $\mathbf{F} = \langle c^2 u_x u_t, -\frac{1}{2}(u_t^2 + c^2 u_x^2) \rangle$ on the trapozoid with vertices (a,t), (b,t), $(a-ct, 0)$, $(b+ct, 0)$ (assuming $a < b$, $t > 0$), to show that

$$\int_a^b \frac{1}{2}(u_t^2(x,t) + c^2 u_x^2(x,t))\ dx \leq \int_{a-ct}^{b+ct} \frac{1}{2}(u_t^2(x,0) + c^2 u_x^2(x,0))\ dx.$$

Question 8. (Extra credit) (6pt)
(1) State the mean value property for harmonic functions.
(2) State the maximum principle for the heat equation.
(3) State the definition of an approximation of the delta function and give an example.