1

1. $u(-x,t) = \int_{-\infty}^{\infty} S(-x-y,t)\phi(y)\,dy$

$$= \int_{-\infty}^{\infty} S(x+y,t)\phi(y)\,dy$$

(Here we used $S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} = S(-x,t)$)

$$= -\int_{-\infty}^{\infty} S(x+y,t)\phi(-y)\,dy \quad (\phi \text{ odd})$$

$$= -\int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy \quad (\text{change of variable})$$

$$= -u(x,t).$$

2. $u(-x,t) = \int_{-\infty}^{\infty} S(-x-y,t)\phi_{even}(y)\,dy$

$$= \int_{-\infty}^{\infty} S(x+y,t)\phi_{even}(y)\,dy \quad (S(x,t)=S(-x,t))$$

$$= \int_{-\infty}^{\infty} S(x+y,t)\phi_{even}(-y)\,dy \quad (\phi_{even} \text{ even})$$

$$= \int_{-\infty}^{\infty} S(x-y,t)\phi_{even}(y)\,dy \quad (\text{change of variable})$$

$$= u(x,t)$$

Now $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{even}(y)\,dy$ clearly satisfies $u_t - k u_{xx} = 0$ for $-\infty < x < \infty$, $t>0$ and $u(x,0) = \phi_{even}(x) = \phi(x)$ for $0<x<\infty$.

Then $u_x(x,t) = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} S(x-y,t)\phi_{even}(y)\,dy$

$\Rightarrow u_x(0,t) = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} S(-y,t)\phi_{even}(y)\,dy$

$$\left(\frac{\partial}{\partial x} S(x,t) = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}\right) = \frac{1}{\sqrt{4\pi k t}} \cdot \left(-\frac{x}{2kt}\right) \cdot e^{-\frac{x^2}{4kt}} \right)$$ ②

$$= \int_{-\infty}^{\infty} \underbrace{\underbrace{\frac{1}{\sqrt{4\pi k t}} \cdot \frac{y}{2kt} \cdot e^{-\frac{y^2}{4kt}}}_{\text{odd in } y} \cdot \underbrace{\phi_{even}(y)}_{\text{even in } y}}_{\text{odd in } y}\, dy$$

$$= 0. \quad \left(\int_{-\infty}^{\infty} \text{an odd function} = 0 \right)$$

We have checked that $u(x,t)$ is a sol. to $(*)$.

(2) $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, \phi_{even}(y)\, dy$

$$= \int_0^{\infty} S(x-y,t)\phi(y)\,dy + \int_{-\infty}^{0} S(x-y,t)\phi(-y)\,dy$$

$$= \int_0^{\infty} S(x-y,t)\phi(y)\,dy + \int_0^{\infty} S(x+y,t)\phi(y)\,dy$$

(change of variable)

$$= \int_0^{\infty} \left[S(x-y,t) + S(x+y,t)\right] \phi(y)\, dy.$$

3. $u(-x,t) = -u(x,t)$:

$$u(-x,t) = \frac{1}{2}\left[\phi_{ext}(-x-ct) + \phi_{ext}(-x+ct)\right] + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi_{ext}(y)\, dy$$

③

$$= \frac{1}{2}\left[-\phi_{ext}(x+ct) - \phi_{ext}(x-ct)\right]$$

$$- \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi_{ext}(-y)\, dy$$

(Using the oddness of ϕ_{ext} & ψ_{ext} at $x=0$)

$$= -\frac{1}{2}\left[\phi_{ext}(x-ct) + \phi_{ext}(x+ct)\right]$$

$$+ \frac{1}{2c}\int_{x+ct}^{x-ct} \psi_{ext}(y)\, dy \quad \text{(change of variable)}$$

$$= -\frac{1}{2}\left[\phi_{ext}(x-ct) + \phi_{ext}(x+ct)\right]$$

$$- \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(y)\, dy$$

$$= -u(x,t)$$

$u(2L-x,t) = -u(x,t)$:

$$u(2L-x,t) = \frac{1}{2}\left[\phi_{ext}(2L-x-ct) + \phi_{ext}(2L-x+ct)\right]$$

$$+ \frac{1}{2c}\int_{2L-x-ct}^{2L-x+ct} \psi_{ext}(y)\, dy$$

$$= \frac{1}{2}\left[-\phi_{ext}(x+ct) - \phi_{ext}(x-ct)\right]$$

$$- \frac{1}{2c}\int_{2L-x-ct}^{2L-x+ct} \psi_{ext}(2L-y)\, dy$$

(Using the oddness of ϕ_{ext} & ψ_{ext} at $x=L$)

$$= -\frac{1}{2}\left[\phi_{ext}(x-ct) + \phi_{ext}(x+ct)\right]$$

$$+\frac{1}{2c}\int_{x+ct}^{x-ct} \psi_{ext}(y)\,dy \quad (\text{change of variable})$$ ④

$$= -\frac{1}{2}\left[\phi_{ext}(x-ct)+\phi_{ext}(x+ct)\right]$$
$$-\frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(y)\,dy$$
$$= -u(x,t).$$

Finally, $u(x,t)$ as given by the d'Alembert's formula clearly satisfies $u_{tt}-c^2u_{xx}=0$ for all x and t.
and $u(x,0)=\phi_{ext}(x)=\phi(x)$ for $0<x<L$
and $u_t(x,0)=\psi_{ext}(x)=\psi(x)$ for $0<x<L$.
Lastly, $u(x,t)=-u(-x,t) \Rightarrow u(0,t)=0$
and $u(x,t)=-u(2L-x,t) \Rightarrow u(L,t)=0$.
Thus $u(x,t)$ is a sol to $(*)$.

4. $$\phi_{ext}'(x)+a\phi_{ext}(x)=\begin{cases}\phi'(x)+a\phi(x), & x\geq 0\\ \phi_0'(x)+a\phi_0(x), & x\leq 0.\end{cases}$$

Oddness $\Rightarrow$ $$\phi_0'(x)+a\phi_0(x)=-\phi'(-x)-a\phi(-x), \quad x\leq 0$$

Solving it for $\phi_0(x)$ by integrating factor:
$$\left(e^{ax}\phi_0(x)\right)'=-e^{ax}\phi'(-x)-e^{ax}a\phi(-x)$$
$$\phi_0(x)=-e^{-ax}\left(\int_0^x e^{ay}\phi'(-y)+e^{ay}a\phi(-y)\,dy+C\right)$$
$$\phi_0(0)=\phi(0) \Rightarrow C=-\phi(0).$$

$$\Rightarrow \phi_0(x) = -e^{-ax}\left[\int_0^x e^{ay}\phi'(-y) + e^{ay}a\phi(-y)\,dy - \phi(0)\right]$$

(2) Since $u(x,t)$ is a sol. to the heat equation,

$u_x(x,t)$ is also a sol. to the heat equation

$$\left((u_x)_t - k(u_x)_{xx} = (u_t - ku_{xx})_x = 0 \right)$$

$\Rightarrow$ $w = u_x + au$ is also a sol. to the heat eq.

$$\left((u_x+au)_t - k(u_x+au)_{xx} = (u_x)_t - k(u_x)_{xx} + a(u_t - ku_{xx}) = 0 \right)$$

Now $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{ext}(y)\,dy$

satisfies $u(x,0) = \phi_{ext}(x)$

Then $w(x,0) = u_x(x,0) + au(x,0)$

$$= \phi'_{ext}(x) + a\phi_{ext}(x),$$

which is the initial condition for $w(x,t)$.

(There is a mathematical subtlety in the derivation of $u_x(x,0) = \phi_{ext}'(x)$ from $u(x,0) = \phi_{ext}(x)$. The continuity of $\phi_{ext}(x)$ becomes important here. This is the reason that we require $\phi_0(0) = \phi(0)$ in the definition of $\phi_{ext}(x)$ in order to make $\phi_{ext}(x)$ continuous. We will not explain this subtlety in detail here.)

(3) Now since $w(x,0) = \phi'_{ext}(x) + a\phi_{ext}(x)$ is an

odd function, by Problem 1 in HW7. $w(x,t)$ is odd w.r.t. x. thus in particular $w(0,t)=0$. That is. $u_x(0,t)+au(0,t)=0$, for all $0<t<\infty$. Now clearly $u(x,t)=\int_{-\infty}^{\infty} S(x-y,t)\phi_{ext}(y)\,dy$ satisfies the heat equation and that

$$u(x,0)=\phi_{ext}(x)=\phi(x) \quad \text{for } x\geq 0.$$

so in conclusion. $u(x,t)$ is a sol. to $(*)$.

3.2.6. We look for solutions given by the d'Alembert's

$$u(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{ext}(y)\,dy,$$

where ψ_{ext} is the extension of the constant function V defined for $x>0$ to the whole line. Then $u(x,t)$ already satisfies

$$\begin{cases} u_{tt}-c^2u_{xx}=0, \\ u(x,0)=0 \\ u_t(x,0)=\psi_{ext}(x)=V, \quad \text{when } x>0. \end{cases}$$

It suffices to check for the boundary condition

$$u_t(0,t)+au_x(0,t)=0.$$

Computation: $u_t(x,t)=\frac{1}{2c}\left[\psi_{ext}(x+ct)\cdot c-\psi_{ext}(x-ct)\cdot(-c)\right]$

$$u_x(x,t)=\frac{1}{2c}\left(\psi_{ext}(x+ct)-\psi_{ext}(x-ct)\right)$$

$$\Rightarrow u_t(0,t)+au_x(0,t)=\frac{1}{2c}\left[(c+a)\psi_{ext}(ct)+(c-a)\psi_{ext}(-ct)\right]$$
$$=0$$

$$\Rightarrow \psi_{ext}(-ct)=\frac{a+c}{a-c}\psi_{ext}(ct)=\frac{a-c}{a+c}\cdot V,\quad t>0.$$

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Thus $\psi_{ext}(x) = \begin{cases} V, & x > 0 \\ \frac{a+c}{a-c} V, & x < 0 \end{cases}$

Then $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(y)\, dy$

$$= \begin{cases} \frac{1}{2c}\int_{x-ct}^{x+ct} V\, dy = V \cdot t, & \text{if } x-ct > 0 \\ \frac{1}{2c}\left(\int_{x-ct}^{0} \frac{a+c}{a-c} V\, dy + \int_{0}^{x+ct} V\, dy\right) & \text{if } x-ct < 0. \\ = \frac{1}{2c}\left(\frac{a+c}{a-c} V \cdot (ct-x) + V \cdot (x+ct)\right). & \end{cases}$$

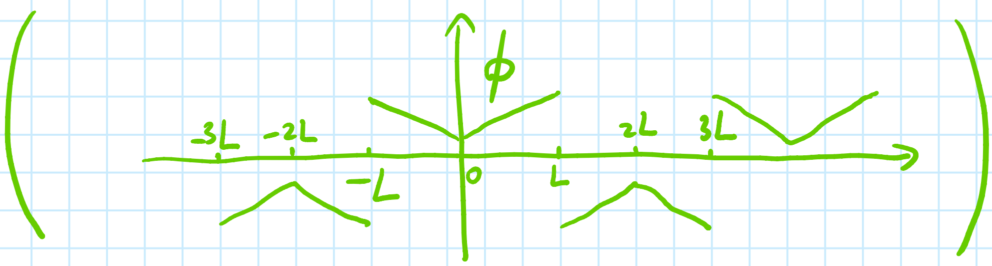
$$(\, x > 0, \quad t \geq 0 \,)$$

6. To make sure $u_x(0,t) = 0$, we require that ϕ_{ext} be even at $x = 0$, i.e.

$$\phi_{ext}(x) = \phi_{ext}(-x) \;;$$

To make sure $u(L,t) = 0$, we require that ϕ_{ext} be odd at $x = L$, i.e.

$$\phi_{ext}(x) = -\phi_{ext}(2L - x).$$



Thus ϕ_{ext} is $4L$-periodic, and

$$\phi_{ext}(x) = \begin{cases} \phi(x-4Ln), & 4Ln \le x < 4Ln+L \\ -\phi(2L-(x-4Ln)), & 4Ln+L \le x < 4Ln+2L \\ -\phi(x-4Ln-2L), & 4Ln+2L \le x < 4Ln+3L \\ \phi(4L-(x-4Ln)) & 4Ln+3L \le x < 4Ln+4L \end{cases}$$

$(n \in \mathbb{Z}.)$

Then let $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi_{ext}(y)\,dy$,

then $u(x,t)$ is a sol. to the heat equation

satisfying $u(x,0) = \phi_{ext}(x) = \phi(x)$, $0<x<L$

and the evenness of ϕ_{ext} at $x=0$

would imply that $u_x(0,t)=0$ and

the oddness of ϕ_{ext} at $x=L$

would imply that $u(L,t)=0$.

(Verification details omitted. Compare with Problem 1 & 2 in HW7.)

thus $u(x,t)$ is a sol. to $(*)$.

Finally, we write $u(x,t)$ as an infinite sum of integrals :

$$u(x,t) = \sum_{n\in\mathbb{Z}} \int_{4Ln}^{4Ln+4L} S(x-y,t)\, \phi_{ext}(y)\, dy$$

$$= \sum_{n\in\mathbb{Z}} \left(\int_{4Ln}^{4Ln+L} + \int_{4Ln+L}^{4Ln+2L} + \int_{4Ln+2L}^{4Ln+3L} + \int_{4Ln+3L}^{4Ln+4L} \right)$$

where

$$\int_{4Ln}^{4Ln+L} = \int_{4Ln}^{4Ln+L} S(x-y,t)\, \phi(y-4Ln)\, dy$$

$$= \int_0^L S(x-4Ln-y,\, t)\, \phi(y)\, dy$$

$$\int_{4Ln+L}^{4Ln+2L} = -\int_{4Ln+L}^{4Ln+2L} S(x-y,t)\, \phi(2L-(y-4Ln))\, dy$$

$$= -\int_0^L S(x-4Ln-2L+y,\, t)\, \phi(y)\, dy$$

$$\int_{4Ln+2L}^{4Ln+3L} = -\int_{4Ln+2L}^{4Ln+3L} S(x-y,t)\, \phi(y-4Ln-2L)\, dy$$

$$= -\int_0^L S(x-4Ln-2L-y,\, t)\, \phi(y)\, dy$$

$$\int_{4Ln+3L}^{4Ln+4L} = \int_{4Ln+3L}^{4Ln+4L} S(x-y,t)\, \phi(4L-(x-4Ln))\, dy$$

$$= \int_0^L S(x-4Ln-4L+y,\, t)\, \phi(y)\, dy$$

Thus

$$u(x,t) = \sum_{n\in\mathbb{Z}} \int_0^L \begin{pmatrix} S(x-4Ln-y,t) - S(x-4Ln-2L+y,t) \\ - S(x-4Ln-2L-y,t) + S(x-4Ln-4L+y,t) \end{pmatrix} \phi(y)\, dy$$

7. By Exercise 2 in HW7, a sol. to

$$\begin{cases} u_t - k u_{xx} = 0, & x > 0, \quad t > 0 \\ u(x,0) = \phi(x), & x > 0 \\ u_x(0,t) = 0 & t > 0 \end{cases}$$

is $u(x,t) = \int_0^\infty \left(S(x-y,t) + S(x+y,t) \right) \phi(y)\, dy$.

Denote this formula by $\mathcal{S}(t)\phi$.

Then

$$\int_0^t \mathcal{S}(t-s) f(\cdot, s)\, ds$$

$$= \int_0^t \int_0^\infty \left(S(x-y, t-s) + S(x+y, t+s) \right) f(y,s)\, dy\, ds.$$

Thus by the Duhamel's principle,

$$u(x,t) = \mathcal{S}(t)\phi + \int_0^t \mathcal{S}(t-s) f(\cdot, s)\, ds$$

$$= \int_0^\infty \left(S(x-y,t) + S(x+y,t) \right) \phi(y)\, dy \; +$$

$$\int_0^t \int_0^\infty \left(S(x-y, t-s) + S(x+y, t-s) \right) f(y,s)\, dy\, ds$$

is a sol. to the inhomogeneous problem $(*)$.

8. Provided u is a sol. to $(*)$,

$w(x,t) = u(x,t) - x g(t)$ is a sol. to

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$$(**)\quad \begin{cases} w_t - k w_{xx} = f(x,t) - x g'(t) \\ w(x,0) = \phi(x) - x g(0) \\ w_x(0,t) = g(t) - g(t) = 0 \end{cases}$$

and vice versa (that is, provided $w(x,t) = u(x,t) - x g(t)$ is a sol. to the above problem $(**)$, $u(x,t)$ is a sol. to $(*)$.)

By the previous exercise,

$$w(x,t) = \int_0^\infty \big(S(x-y,t) + S(x+y,t)\big)\big(\phi(y) - y g(0)\big)\,dy$$
$$+ \int_0^t \int_0^\infty \big(S(x-y,t) + S(x+y,t)\big)\cdot\big(f(y,s) - y g'(s)\big)\,dy\,ds$$

is a sol. to $(**)$.

Then $u(x,t) = w(x,t) + x g(t)$

$$= \int_0^\infty \big(S(x-y,t) + S(x+y,t)\big)\big(\phi(y) - y g(0)\big)\,dy +$$
$$\int_0^t \int_0^\infty \big(S(x-y,t) + S(x+y,t)\big)\cdot\big(f(y,s) - y g'(s)\big)\,dy\,ds + x g(t)$$

is a sol. to $(*)$.