Math 3435, Spring 2019 – Practice Final 1

Question 1. Solve $u_x + yu_y + yu = y$ with $u(0, y) = e^y$.
Question 2. Solve the heat equation on the half line:

$$\begin{cases} u_t - u_{xx} = 0, & t > 0, x > 0 \\ u(x, 0) = 3, & x > 0 \\ u(0, t) = 0, & t > 0 \end{cases}$$

Question 3. Suppose $u(x, t)$ satisfies

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1, -\infty < t < \infty \\ u(x, 0) = x^2(1 - x), & 0 < x < 1, -\infty < t < \infty \\ u_t(x, 0) = (1 - x)^2, & 0 < x < 1, -\infty < t < \infty \\ u(0, t) = u(1, t) = 0, & -\infty < t < \infty \end{cases}$$

Find $u(\frac{2}{3}, 2)$.
Question 4. Solve $u_{tt} = c^2 u_{xx} + \cos x$, $u(x, 0) = \sin x$, $u_t(x, 0) = 1 + x$.
Question 5. Consider the Robin eigenvalue problem

$$\begin{cases} X''(x) + \lambda X(x) = 0, & 0 \le x \le l \\ X'(0) - a_0 X(0) = 0, & \\ X'(l) + a_l X(l) = 0. & \end{cases}$$

Assume that $a_0 < 0$, $a_l < 0$, and $-a_0 - a_l < a_0 a_l l$. Show that there are two negaive eigenvalues.

Question 6. Find the sum $\sum_{n=1}^{\infty} 1/n^2$ by applying Parseval's equality for the Fourier sine series of $f(x) = x$ on the interval $(0, l)$.
Question 7. Find the harmonic function in the square $\{0 < x < \pi, 0 < y < \pi\}$ with the boundary conditions $u_y(x, 0) = 0$, $u_y(x, \pi) = 0$, $u(0, y) = 0$, $u(\pi, y) = \cos^2 y = \frac{1}{2}(1 + \cos 2y)$.
Question 8. Solve $u_{xx} + u_{yy} = 0$ in the exterior $\{r > a\}$ of a disk, with the boundary condition $u = 1 + 3\sin\theta$ on $r = a$, and the condition at infinity that u be bounded as $r \to \infty$. Here (r, θ) are the polar coordinates.

Question 9. (Extra credit) Show that the Poisson kernel $P(r, \theta) = \dfrac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2}$ $(0 \le r < a, -\pi \le \theta \le \pi)$ is an approximation of the delta function as r approaches a in the following sense:
(1) $P(r, \theta) \ge 0$ for all $0 \le r < a$, $0 \le \theta \le 2\pi$.
(2) $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} P(r, \theta)\, d\theta = 1$ for all $0 \le r < a$.
(3) $\displaystyle\lim_{r \to a-} \frac{1}{2\pi} \int_{\varepsilon < |\theta| \le \pi} P(r, \theta)\, d\theta = 0$, for any $\varepsilon > 0$.