4.2.1 Solve

$$\begin{cases} u_t - k u_{xx} = 0, & 0 < x < L \\ u(x,0) = \phi(x) \\ u(0,t) = u_x(L,t) = 0 \end{cases}$$

by the method of separation of variables.

Solution Let $u = X(x)T(t)$

Then $u_t - k u_{xx} = X(x)T'(t) - kX''(x)T(t) = 0$

$$\Rightarrow \quad \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$\Rightarrow \quad T'(t) + \lambda k T(t) = 0$$

$$X''(x) + \lambda X(x) = 0$$

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u_x(L,t) = X'(L)T(t) = 0 \Rightarrow X'(L) = 0$$

$\Rightarrow$ the eigenvalue problem:

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \\ X'(L) = 0 \end{cases}$$

②

$$X'' + \lambda X = 0 \Rightarrow \begin{cases} \lambda = 0 & X = C + Dx \\ \lambda \neq 0 & X = Ce^{\gamma x} + De^{-\gamma x} \\ & \gamma^2 = -\lambda \end{cases}$$

$\lambda = 0.$ $X = C + Dx$

$X(0) = C = 0$ $\quad X'(L) = D = 0$

$\Rightarrow X = 0$ trivial !

$\lambda \neq 0.$ $X = Ce^{\gamma x} + De^{-\gamma x}$

$X(0) = C + D = 0$

$X'(L) = C\gamma e^{\gamma L} - D\gamma e^{-\gamma L} = 0$

$\Rightarrow D = -C$

$\Rightarrow C\gamma e^{\gamma L} + C\gamma e^{-\gamma L} = 0$

$\Rightarrow e^{\gamma L} + e^{-\gamma L} = 0$

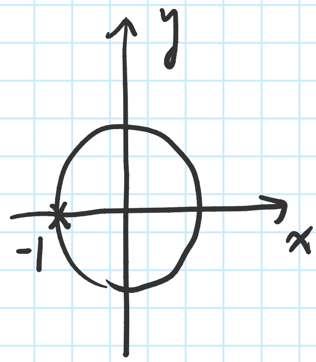
($C \neq 0$. otherwise $D = -C$ would also be zero. which would yield a trivial solution.)

$\Rightarrow e^{2\gamma L} = -1$

let $\gamma = x + iy$

$\Rightarrow e^{2xL + i2yL} = -1$

$\Rightarrow e^{2xL} \cdot e^{i2yL} = -1$



$$\Rightarrow e^{2xL} = \cancel{0}\, 1 \Rightarrow x = 0$$

$$2yL = (2n+1)\pi, \quad n \in \mathbb{Z}$$

the set of integers

$$\Rightarrow y = (n+\tfrac{1}{2})\pi/L$$

$$\Rightarrow \gamma = x + iy = i(n+\tfrac{1}{2})\pi/L$$

$$\Rightarrow \lambda = -\gamma^2 = (n+\tfrac{1}{2})^2\pi^2/L^2 \qquad (i^2 = -1)$$

$$X = Ce^{\gamma x} - Ce^{-\gamma x} \qquad (D = -C)$$

$$= C\left(e^{i(n+\frac{1}{2})\pi x/L} - e^{-i(n+\frac{1}{2})\pi x/L}\right)$$

$$= C \cdot 2i \sin(n+\tfrac{1}{2})\pi x/L$$

$$\left(e^{i\theta} - e^{-i\theta} = 2i \sin\theta \right)$$

Thus we get the eigenvalues and eigenfunctions

$$\begin{cases} \lambda_n = (n+\frac{1}{2})^2\pi^2/L^2, \\ X_n = \sin(n+\frac{1}{2})\pi x/L. \quad n \in \mathbb{Z} \end{cases}$$

Note that when n is negative, $-n-1 \geq 0$.

and
$$\begin{cases} \lambda_{-n-1} = \lambda_n \\ X_{-n-1} = -X_n. \end{cases}$$

Thus we may restrict n to be nonnegative integers, without losing any eigenvalues and eigenfunctions. Final solution to the eigenvalue problem:

$$\begin{cases} \lambda_n = (n+\frac{1}{2})^2\pi^2/L^2 . \\ X_n = \sin(n+\frac{1}{2})\pi x/L \qquad n = 0, 1, 2, \ldots \end{cases}$$

Going back to $T(t)$:

$$T_n'(t) + \lambda_n k T(t) = 0$$

$$\Rightarrow \quad T_n(t) = A_n e^{-\lambda_n k t} = A_n e^{-\frac{(n+\frac{1}{2})^2\pi^2 kt}{L}}$$

$\Rightarrow$ Formal series solutions:

$$u = \sum_{n=0}^{\infty} X_n(x) T_n(t)$$

$$= \sum_{n=0}^{\infty} A_n e^{-\frac{(n+\frac{1}{2})^2\pi^2 kt}{L}} \sin(n+\tfrac{1}{2})\pi x/L .$$

with initial data

$$\phi(x) = u(x,0) = \sum_{n=0}^{\infty} A_n \sin(n+\tfrac{1}{2})\pi x/L .$$