Review Solutions

1) $\left[\begin{array}{cccc|c} 1 & 4 & -2 & 8 & 12 \\ 0 & 1 & -7 & 2 & -4 \\ 0 & 0 & 5 & -1 & 7 \\ 0 & 0 & 1 & 3 & -5 \end{array}\right] \begin{array}{c} R1-4R2 \\ = \\ R3\cdot\frac{1}{5} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 26 & 0 & 28 \\ 0 & 1 & -7 & 2 & -4 \\ 0 & 0 & 1 & -1/5 & 7/5 \\ 0 & 0 & 1 & 3 & -5 \end{array}\right]$

$\begin{array}{c} R1-26R3 \\ R2+7R3 \\ R4-R3 \end{array} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{26}{5} & -42/5 \\ 0 & 1 & 0 & 3/5 & 29/5 \\ 0 & 0 & 1 & -1/5 & 7/5 \\ 0 & 0 & 0 & 16/5 & -32/5 \end{array}\right] \begin{array}{c} R1-\frac{26}{5}R4 \\ R2-\frac{3}{5}R4 \\ = R3+1/5R4 \\ R4\cdot\frac{5}{16} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$

$\boxed{\text{sol} = (2,7,1,-2)}$

2) Yes, we see a pivot in all 3 rows and can use $5x_3 = 0 \Rightarrow x_3 = 0$ to solve for x_1 & x_2.

3) Note $3h = 3(2x_1 - x_2) = 6x_1 - 3x_2$

The second eq. can be written as $0 = k + 6x_1 - 3x_2$

$0 = k + 3h$

so if $k + 3h \neq 0$, the system has no solution.

It is consistent for any h & k such that $3h + k = 0$.

4) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \begin{matrix} \\ R2-4R1 \\ R3-6R1 \end{matrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix} \begin{matrix} R1-2R2 \\ R2\cdot\frac{-1}{3} \\ R3+5R2 \end{matrix} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ pivot cols. 1 & 2

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5) $\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right] R3+R1 = \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{array}\right] R3+4R2 = \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

x_2, x_4, free

Gen. solution = $\begin{cases} x_1 = 5 + 7x_2 - 6x_4 \\ x_2 \text{ free} \\ x_3 = -3 + 2x_4 \\ x_4 \text{ free} \end{cases}$

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6) $\left[\begin{array}{ccc|c} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{array}\right] R3+2R1 = \left[\begin{array}{ccc|c} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{array}\right]$

R3 $\Rightarrow$ $0 = 3$ so this is inconsistent. So $\vec{b}$ is not.

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7) $\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{array}\right] \begin{matrix} R2-4R1 \\ R3+2R1 \end{matrix} = \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & h+8 \end{array}\right] = \begin{matrix} R2\cdot\frac{1}{5} \\ R3-3R2 \end{matrix} = \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & h+17 \end{array}\right]$

$\vec{b} \in \text{Span}\{\vec{a}_1, \vec{a}_2\}$ when $h + 17 = 0 \Rightarrow \boxed{h = -17}$

8. $\left[\begin{array}{ccc|c} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{array}\right] \xrightarrow{R3+2R1} = \left[\begin{array}{ccc|c} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 5 & 5 \end{array}\right] \begin{array}{l} R1-2R2 \\ \\ R3\cdot\frac{1}{5} \end{array} = \left[\begin{array}{ccc|c} 1 & 0 & -6 & -6 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$\begin{array}{l} R1+6R3 \\ R2-5R3 \end{array} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$ sol. is $\begin{cases} x_1 = 0 \\ x_2 = -3 \\ x_3 = 1 \end{cases}$

as a vector: $\vec{x} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$

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9) $\left[\begin{array}{cccc|c} 1 & -2 & 9 & 5 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{array}\right] \overset{R1+2R2}{=} \left[\begin{array}{cccc|c} 1 & 0 & -5 & -7 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{array}\right]$

so $x_1 = 5x_3 + 7x_4$
$x_2 = -2x_3 + 6x_4$

$\boxed{\vec{x} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}}$

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10) $x_1 = 4x_3$, $x_2 = -2-7x_3$, x_7 free.

$\vec{x} = \begin{bmatrix} 5+4x_3 \\ -2-7x_3 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}}_{\vec{p}} + x_3 \underbrace{\begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}}_{\vec{q}} = \vec{p} + x_3 \vec{q}$

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11)

a) $\left[\begin{array}{cc|c} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & 6 & h \end{array}\right] \begin{matrix} R2+3R1 \\ R3-2R1 \end{matrix} = \left[\begin{array}{cc|c} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & h-10 \end{array}\right]$ $0=8$ shows inconsistent.

So $v_3 \in \text{Span}\{v_1, v_2\}$ for no values of h.

b) $\left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ -3 & 9 & -7 & 0 \\ 2 & 6 & h & 0 \end{array}\right] \begin{matrix} R2+3R1 \\ R3-2R1 \end{matrix} = \left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & h-10 & 0 \end{array}\right] \begin{matrix} = R2 \cdot \frac{1}{8} \\ R3-(h-10)R2 \end{matrix} \left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

x_2 is free for all values of h so the homogenous equation has a non-trivial solution. Thus $\{v_1, v_2, v_3\}$ are linearly dependent for all h.

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12) Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is lin. dep. Then there exist scalars c_1, c_2, c_3 so that (not all 0):

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}.$$

Then $T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = T(\vec{0}) = \vec{0}$. Since T is linear $c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) = 0$. Since not all the c_i are 0, $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ are linearly dependent. ∎

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13) $\left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 3 & -5 & -9 & -9 \end{array}\right] \xrightarrow{R3-3R1} = \left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{array}\right] \begin{array}{c} R1+3R2 \\ = \\ R3-4R2 \end{array} = \left[\begin{array}{ccc|c} 1 & 0 & -10 & -15 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$\begin{array}{c} R1+10R3 \\ = R2+4R3 \end{array} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\boxed{\text{unique } \vec{x} = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}}$

14) $T(\vec{x}) = \begin{bmatrix} x_1 + x_2 \\ 4x_1 + 5x_2 \end{bmatrix} = [A]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 4 & 5 & 8 \end{array}\right] \xrightarrow{R2-4R1} = \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -4 \end{array}\right] \xrightarrow{R1-R2} = \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -4 \end{array}\right]$

$\boxed{\vec{x} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}}$

13 correction: $R1+10R3$, $R2+4R3$ $= \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$ so $\boxed{\text{unique } \vec{x} = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}}$

15) $AB = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$

$BA =$ undefined

$A - 2B =$ undefined

16) $A^T = \begin{bmatrix} 1 & 2 & -.5 \\ 4 & 7 & 8 \\ 3 & -8 & -1 \end{bmatrix}$

17) $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}^{-1} = \frac{1}{32-30}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix} = A^{-1}}$

$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ $R2+3R1$, $R3-2R1$ $= \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] =$ $R3+3R2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right]$

$R1+2R3$, $R2+2R3$, $\frac{1}{2}R3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$ so $\boxed{A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}}$

18) $\det A = 1\begin{vmatrix} 3 & 4 \\ 6 & 0 \end{vmatrix} - 0\begin{vmatrix} -5 & -4 \\ 6 & 0 \end{vmatrix} + -3\begin{vmatrix} -5 & -4 \\ 3 & 4 \end{vmatrix}$

$= 1(0-24) - 0 - 3((-5\cdot4)-(-4\cdot3)) = 0$ so $\boxed{\text{Not Invertible}}$

Monster Thm.

19) $\det A = -4\begin{vmatrix} 3 & -3 \\ 1 & 5 \end{vmatrix} + 0\begin{vmatrix} 2 & -3 \\ 6 & 5 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 6 & 1 \end{vmatrix}$

using Row 2

$= -4(18) - 3(-16) = \boxed{-24}$

20) $\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix} \underset{\substack{R2+2R1 \\ R3-3R1 \\ R4-R1}}{=} \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix}$ No change to det.

$\underset{\substack{R3+4R2 \\ R4+4R2}}{=} \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 30 & 27 \end{vmatrix}$ No change to det.

$\underset{R4-R3}{=} \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 0 & 0 \end{vmatrix}$ No change to det.

$= (1)(1)(30)(0) = \boxed{0}$