3.3: Cramer's Rule

Theorem 7: (Cramer's Rule) Let A be an invertible matrix. For any $\vec{b} \in \mathbb{R}^n$, the unique solution $\vec{x}$ of $A\vec{x} = \vec{b}$ has entries given by

$$x_i = \frac{\det A_i(\vec{b})}{\det A}, \quad i = 1, \dots, n$$

Ex 1: Use Cramer's Rule to solve

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8$$

$$\vec{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$$

Note: $A_i(\vec{b})$ is the matrix obtained by replacing column i by vector $\vec{b}$.

$$A_1(\vec{b}) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}$$

$$A_2(\vec{b}) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

$$\det A = 3(4) - (-2)(-5) = 2$$
$$\det A_1(\vec{b}) = 6(4) - (-2)(8) = 40$$
$$\det A_2(\vec{b}) = 3(8) - (6)(-5) = 54$$

$$x_1 = \frac{40}{2} = 20 \qquad x_2 = \frac{54}{2} = 27 \qquad \boxed{\vec{x} = \begin{bmatrix} 20 \\ 27 \end{bmatrix}}$$

Ex 2: Use Cramer's rule to solve

$$4x_1 + x_2 = 6$$
$$3x_1 + 2x_2 = 7$$

$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

$\vec{b} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$

$A_1(\vec{b}) = \begin{bmatrix} 6 & 1 \\ 7 & 2 \end{bmatrix}$

$A_2(\vec{b}) = \begin{bmatrix} 4 & 6 \\ 3 & 7 \end{bmatrix}$

$\det A = 4(2) - 1(3) = 5$

$\det A_1(\vec{b}) = 6(2) - 1(7) = 5$

$\det A_2(\vec{b}) = 4(7) - 6(3) = 10$

$x_1 = \frac{5}{5} = 1 \quad x_2 = \frac{10}{5} = 2$

$$\boxed{\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

Ex 3: Consider the following system where s is an unspecified parameter. Determine the values of s for which the system has a unique solution.

$$3s\,x_1 - 2x_2 = 4$$
$$-6x_1 + s\,x_2 = 1$$

$\vec{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}$

$A_1(\vec{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix}$

$A_2(\vec{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}$

$\det A = 3s^2 - (-2)(-6) = 3s^2 - 12$

$\det A_1(\vec{b}) = 4s - (-2)(1) = 4s + 2$

$\det A_2(\vec{b}) = 3s - (4)(-6) = 3s + 24$

$$x_1 = \frac{4s+2}{3s^2-12} = \frac{4s+2}{3(s^2-4)} = \frac{2(2s+1)}{3(s-2)(s+2)}$$

$$x_2 = \frac{3s+24}{3s^2-12} = \frac{\cancel{3}(s+8)}{\cancel{3}(s-2)(s+2)} = \frac{s+8}{(s-2)(s+2)}$$

$$\boxed{\vec{x} = \begin{bmatrix} \frac{2(2s+1)}{3(s-2)(s+2)} \\ \frac{s+8}{(s-2)(s+2)} \end{bmatrix} \quad \text{for every } s \text{ that satisfies this or } s \neq \pm 2}$$

Proof of Cramer's Rule: Denote the columns of A by $\vec{a}_1, \dots, \vec{a}_n$ and the columns of I_n by $\vec{e}_1, \dots \vec{e}_n$.

If $A\vec{x} = \vec{b}$, we know, via matrix mult.,

$$A \cdot I_i(\vec{x}) = A\begin{bmatrix} \vec{e}_1 & \cdots & \underset{\substack{\uparrow \\ i^{th}\text{ column}}}{\vec{x}} & \cdots & \vec{e}_n \end{bmatrix}$$

$$= \begin{bmatrix} A\vec{e}_1 & \cdots & A\vec{x} & \cdots & A\vec{e}_n \end{bmatrix}$$

$$= \begin{bmatrix} \vec{a}_1 & \cdots & \underset{\substack{\uparrow \\ i^{th}\text{column}}}{\vec{b}} & \cdots & \vec{a}_n \end{bmatrix} = A_i(\vec{b})$$

Since determinants are multiplicative

$$(\det A)(\det I_i(x)) = \det A_i(b)$$

$$\Rightarrow \frac{(\det A)(x_i)}{\det A} = \frac{\det A_i(b)}{\det A} \Rightarrow x_i = \frac{\det A_i(b)}{\det A} \quad ■$$

Def: The matrix of cofactors of A given below is called the Adjugate (or Adjoint) of A, denoted by $\operatorname{adj} A$.

$$\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

* $C_{ij} = (-1)^{i+j} \det A_{ij}$

Theorem 8: Let A be an $n \times n$ invertible matrix.

Then $A^{-1} = \frac{1}{\det A} \text{adj } A$

Ex 3: Find A^{-1} for $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$

$C_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2$ $\quad C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -(-3) = 3$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = +(5) = 5$ $\quad C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = (-)(-14) = 14$

$C_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = +(-7) = -7$ $\quad C_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = (-)(7) = -7$

$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = +(4) = 4$ $\quad C_{32} (-1)^{3+2} \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = (-)(-1) = 1$

$C_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = +(-3) = -3$

$\cancel{\text{adj } A = \begin{bmatrix} -2 & 14 & 5 \\ 3 & & \\ 5 & & \end{bmatrix}}$

$$\text{adj } A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

$\det A = 2 \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}$

$= 2(2-4) - 1(-2-12) + 1(1-(-3)) = -4 + 14 + 4 = 14$

$$A^{-1} = \frac{1}{14} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \boxed{\begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix}}$$

Ex: Find the volume of the para. with one vertex at the origin and adjacent vertices at $(1,0,-3)$, $(1,2,4)$ and $(5,1,0)$.

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 2 & 1 \\ -3 & 4 & 0 \end{bmatrix}$$

$$\text{Volume} = |\det A| = \left| +1\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} - 0\begin{vmatrix} 1 & 5 \\ 4 & 0 \end{vmatrix} + -3\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} \right|$$

$$= \left| 1(2(0)-4(1)) - 0 + -3(1-10) \right|$$

$$= \left| -4 - 0 + 27 \right| = |23| = \boxed{23}$$

Thm 10: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a lin. trans. determined by A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

If $T: \mathbb{R}^3 \to \mathbb{R}^3$ and S is a parallelepiped, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}.$$

Theorem 9: If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, then the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Corollary: Let $\vec{a}_1$ and $\vec{a}_2$ be nonzero vectors. Then for any scalar c, the area of the parallelogram determined by $\vec{a}_1$ and $\vec{a}_2$ equals the area of the parallelogram determined by $\vec{a}_1$ and $\vec{a}_2 + c\vec{a}_1$.

Ex 4: Calculate the area of the parallelogram determined by $(-2,-2)$, $(0,3)$, $(4,-1)$ and $(6,4)$

First, let's shift one vertex as the origin. Let's subtract $(-2,-2)$ from each vertex so we have $(0,0)$, $(2,5)$, $(6,1)$, and $(8,6)$

From our geometric discussion in chapter 1,

$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$$

$$\text{Area} = |\det A| = |2(1) - 6(5)| = \boxed{28}$$