THE UNIVERSITY OF MELBOURNE

Student Number

Semester 2 Assessment, 2020

School of Mathematics and Statistics

MAST20004 Probability

This assignment consists of 13 pages (including this page)

Submission Deadline

- Submit (as instructed below) by **4:00 pm on Monday 14 September 2020**
- Please **print out** the assignment single-sided and write your answers in the **answer boxes** provided under each sub-question. For those who do not have a printer or cannot access a printing device, you can directly write on the electronic version of the assignment on a tablet with a stylus in the same way. If both options fail for you, please write on blank A4 papers, clearly indicating the question numbers and pages, always start each question on a new page, and identify the pages to each question when submitting to Gradescope.
- Extra answer boxes are provided at the end of the assignment if you need more space. In this case, you must **make a note** in the correct answer space or page for the question, warning the marker that you have appended additional remarks at the end.
- Scan your assignment submission to a single PDF file with a mobile phone or a scanner. Scan from directly above to avoid any excessive keystone effect. Scan the whole page, not just the portion where your answer is written. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned submissions may be impossible to mark.

Instructions to Students

- This assignment contains 5 questions, **three** of which will be randomly selected to be marked. Each question is worth 10 points. You are expected to provide solutions to all questions, for otherwise a penalty of 5 points per unanswered question will apply.
- The submission deadline is **strict**. Late submission within 24 hours after the deadline will be penalised by 10%, and submissions within 24-48 hours after the deadline will be penalised by 20%. After that, the Gradescope submission channel will be closed, and your submission will no longer be accepted. You are strongly encouraged to submit the assignment a few days before the deadline just in case of unexpected technical issues. If you are facing a rather exceptional/extreme situation that prevents you from submitting on time, please contact the tutor coordinator **Robert Maillardet** with formal proofs such as medical certificate.

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Problem 1. A randomly selected family has n children with probability αp^n ($n \geqslant 1$) where $p \in (0, 1)$ and $\alpha \leqslant (1-p)/p$.

(i) What is the probability that a randomly selected family does not have any children?

(ii) Suppose that each child is equally likely to be a boy or a girl (independently of each other), and the number of children is also independent of the sexes. What is the probability that a randomly selected family has one boy (and any number of girls)?
[Hint: *Given that the family has n children, what is the distribution of the number of boys? You don't need to justify this mathematically.*]

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(iii) Under the same assumption as in Part (ii), what is the probability that a randomly selected family has k boys (and any number of girls)?
[Hint: *You may use the following identity which is a direct corollary of the extended binomial theorem in the lecture:*

$$(1-x)^{-(k+1)} = \sum_{m=0}^{\infty} \binom{m+k}{k} x^k, \quad |x| < 1.]$$

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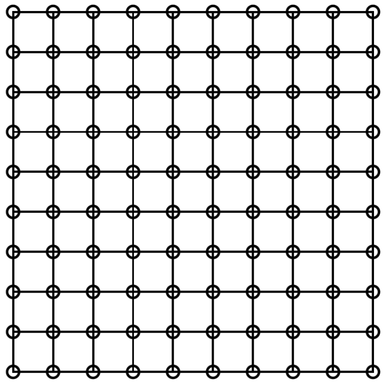
Problem 2. Suppose that a particular suburb of Melbourne is built on the following 10×10 grid.



At each intersection, a street light is placed (represented by a small circle). A street refers to a segment between two adjacent lights. We say that a street is *dark*, if the two lights at both ends are broken. Suppose that the lights on the boundary of the grid are lack of proper maintenance so that each has a probability of 0.2 being broken. All other lights have broken probability 0.1. All the lights are independent of each other. Let X be the number of dark streets in this suburb.

(i) How do you write X as a sum of Bernoulli random variables? State the corresponding Bernoulli trials precisely.

(ii) Use Poisson approximation to calculate the probability that there are at least three dark streets in the suburb. Explain heuristically why we can use Poisson approximation in this problem (you don't need to make any precise mathematical justification).

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Problem 3. Let X be the lifetime (measured in hours) of a particular type of electronic device, whose probability density function is given by

$$f_X(x) = \begin{cases} \frac{C}{x^3}, & x > 10, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the value of C.

(ii) Compute $\mathbb{P}(X > 20)$ and $\mathbb{E}[X]$. Does X have finite variance?

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(iii) In a batch of 6 such devices, what is the probability that at least 3 of them will function for at least 20 hours? We assume that all the 6 devices are independent.

Problem 4. Let X be a normal random variable with mean 3 and variance 4.

(i) Find the following probabilities:

$$\mathbb{P}(2 < X \leqslant 5),\ \mathbb{P}(X > 3),\ \mathbb{P}(|X| > 2).$$

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(ii) Find the values of a, b (where $a < b$) such that

$$\mathbb{P}(X < a) : \mathbb{P}(a < X < b) : \mathbb{P}(X > b) = 1 : 2 : 3.$$

You may use the fact that $\Phi(0.96741) \approx 0.83333$.

Problem 5. (1) A table tennis match is played between Horatio and Xi. The winner of the match is the one who first wins 4 games in total, and in any individual game the winner is the one who first scores 11 points. Note that in an individual game, if the score is 10 to 10, the game goes into extra play (called *deuce*) until one player has gained a lead of 2 points. Let p be the probability that Horatio wins a point in any single round of serve, and assume that different rounds in all games are independent.

(1-i) In an individual game, what is the probability that the game runs into the deuce stage?

(1-ii) Suppose that $p = 0.55$. What is the probability that Horatio wins the match?

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(2) Let $n \geqslant 1$ and $p \in (0,1)$ and $q = 1 - p$. By constructing suitable probability models or otherwise, show that

$$\binom{2n}{n} p^n q^n + \sum_{k=0}^{n-1} \binom{n+k}{k} (q^k p^{n+1} + p^k q^{n+1}) = 1.$$

[Hint: *There are at least two methods to show this. Method 1: Consider an artifical table tennis game which has the deuce rule when the score is n-n, and use two methods to calculate the probability of entering the deuce stage. Method 2: Consider the relation between binomial and negative binomial random variables.*]

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If you need additional answer space for any of the above 5 questions, please use the following boxes. Clearly indicate the question number before continuing your solutions.

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End of Assignment