Assignment One: Solutions

Problem 1. Let A, B, C be three events.

(i) Show that

$$A \cup B \cup C = A \cup (A^c \cap B) \cup (A^c \cap B^c \cap C), \tag{1}$$

and

$$\mathbb{P}(A \cup B \cup C) \leqslant \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C).$$

(ii) By using the Addition Theorem (Lecture Slide 30, Property (9)) or otherwise, show that

$$\begin{aligned} \mathbb{P}(A \cup B \cup C) = &\ \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) \\ &- \mathbb{P}(A \cap C) + \mathbb{P}(A \cap B \cap C). \end{aligned}$$

(iii) Suppose that

$$\mathbb{P}(A) = \mathbb{P}(B) = 0.5,\ \mathbb{P}(C) = 0.2,\ \mathbb{P}(A \cup B \cup C) = 0.8,$$

and assume further that the events A, B, C are pairwisely independent. Are these three events mutually independent?

(iv) Suppose that the three events A, B, C are exhaustive, and they are mutually independent. Show that at least one of the following three statements must be true:

$$\mathbb{P}(A) = 1 \text{ or } \mathbb{P}(B) = 1 \text{ or } \mathbb{P}(C) = 1.$$

Solution. (i) *Method 1.* Since

$$A^c \cap B \subseteq B,\ A^c \cap B^c \cap C \subseteq C,$$

it is clear that

$$A \cup B \cup C \supseteq A \cup (A^c \cap B) \cup (A^c \cap B^c \cap C).$$

Conversely, let $\omega \in A \cup B \cup C$. Then ω belongs to A or B or C. If it belongs to A, then it belongs to the right hand side. If it does not belong to A but belongs to B, then it belongs to $A^c \cap B$ which implies that it belongs to the right hand side. If it does not belong to both of A and B, it must belong to C. In particular, it belongs to $A^c \cap B^c \cap C$, which also implies that it belongs to the right hand side. In all cases, ω belongs to the right hand side.

Method 2. One can also use event operation laws to show this. Firstly, we have

$$\begin{aligned} A \cup (A^c \cap B) &= (A \cup A^c) \cap (A \cup B) \\ &= \Omega \cap (A \cup B) \\ &= A \cup B. \end{aligned}$$

Therefore,

$$\begin{aligned} A \cup (A^c \cap B) \cup (A^c \cap B^c \cap C) &= (A \cup B) \cup (A^c \cap B^c \cap C) \\ &= (A \cup B) \cup \big((A \cup B)^c \cap C\big) \\ &= \Omega \cap (A \cup B \cup C) \\ &= A \cup B \cup C. \end{aligned}$$

For the next assertion, since the right hand side of (1) is a disjoint union, according to the finite additivity property, we have

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(A^c \cap B) + \mathbb{P}(A^c \cap B^c \cap C).$$

In addition, according to Lecture Slide 30, Property (7), we know that

$$\mathbb{P}(A^c \cap B) \leqslant \mathbb{P}(B), \quad \mathbb{P}(A^c \cap B^c \cap C) \leqslant \mathbb{P}(C).$$

Therefore,

$$\mathbb{P}(A \cup B \cup C) \leqslant \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C).$$

(ii) According to the Addition Theorem applied to $A \cup B$ and C, we have

$$\begin{aligned} \mathbb{P}(A \cup B \cup C) &= \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}((A \cup B) \cap C) \\ &= \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}((A \cap C) \cup (B \cap C)). \end{aligned}$$

Again using the Addition Theorem to the first and third terms on the right hand side, we obtain that

$$\begin{aligned} \mathbb{P}(A \cup B \cup C) &= (\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)) + \mathbb{P}(C) \\ &\quad - (\mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - \mathbb{P}(A \cap B \cap C)) \\ &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) \\ &\quad - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C). \end{aligned}$$

(iii) Since the events A, B, C are pairwise independent, to see whether they are mutually independent, it remains to compare $\mathbb{P}(A \cap B \cap C)$ with the product $\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$. To this end, by the result of Part (ii) and pairwise independence, we have

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B) \\ - \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(B)\mathbb{P}(C) + \mathbb{P}(A \cap B \cap C).\end{aligned}$$

By assumption,

$$\mathbb{P}(A) = \mathbb{P}(B) = 0.5,\ \mathbb{P}(C) = 0.2,\ \mathbb{P}(A \cup B \cup C) = 0.8.$$

Therefore,

$$\begin{aligned}\mathbb{P}(A \cap B \cap C) &= \mathbb{P}(A \cup B \cup C) - \mathbb{P}(A) - \mathbb{P}(B) - \mathbb{P}(C) \\ &\quad + \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) + \mathbb{P}(B)\mathbb{P}(C) \\ &= 0.8 - 0.5 - 0.5 - 0.2 \\ &\quad + 0.5 \times 0.5 + 0.5 \times 0.2 + 0.5 \times 0.2 \\ &= 0.05.\end{aligned}$$

But we also have

$$\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = 0.5 \times 0.5 \times 0.2 = 0.05.$$

Therefore,

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C).$$

Together with pairwise independence, this implies that A, B, C are mutually independent.

(iv) *Method 1.* Since the events A, B, C are exhaustive, we have

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(\Omega) = 1.$$

Since they are also mutually independent, according to the result of Part (ii), we have

$$\begin{aligned}1 = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B) \\ - \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(B)\mathbb{P}(C) + \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C).\end{aligned}$$

The above equation can be rewritten as

$$(1 - \mathbb{P}(A)) \times (1 - \mathbb{P}(B)) \times (1 - \mathbb{P}(C)) = 0.$$

Therefore, at least one of the numbers $\mathbb{P}(A)$, $\mathbb{P}(B)$ and $\mathbb{P}(C)$ must be equal to one.

Method 2. Since

$$\mathbb{P}(A \cup B \cup C) = 1,$$

we know that

$$\mathbb{P}((A \cup B \cup C)^c) = \mathbb{P}(A^c \cap B^c \cap C^c) = 0.$$

By independence,

$$\mathbb{P}(A^c)\mathbb{P}(B^c)\mathbb{P}(C^c) = 0.$$

Therefore, at least one of the following identities

$$\mathbb{P}(A^c) = 0,\ \mathbb{P}(B^c) = 0,\ \mathbb{P}(C^c) = 0$$

is true.

Problem 2. A standard poker deck has 52 playing cards, which includes 13 ranks (2, 3, 4,$\cdots$, 10, J, Q, K, A) in each of the four suits: clubs (♣), diamonds (♦), hearts (♥) and spades (♠). A *full house* is a hand of five cards consisting of three cards of one rank and two cards of another rank. Suppose that 5 cards are randomly selected. Under each of the following two assumptions, describe the sample space and compute the probability that a full house is obtained.

(i) The 5 cards are selected one after another in an ordered manner.
(ii) The 5 cards are selected at the same time without orders.

Solution. (i) The sample space Ω consists of all possible ordered 5-tuples

$$(x_1, x_2, x_3, x_4, x_5)$$

where x_i denotes the result of the i-th selection. There is a total amount of $52 \times 51 \times 50 \times 49 \times 48$ elements in Ω. Let A be the event that a full house is obtained. We can count the total number of elements in A via the following steps:

Step 1. Choose three positions in the outcome $(x_1, x_2, x_3, x_4, x_5)$ to support the three cards of identical rank. There are $\binom{5}{3}$ ways to choose the positions.

Step 2. Place three cards of identical rank in the three positions chosen in Step 1. In the natural ordering of these three positions, there are 52 possibilities to place a card in the first position. Once this card is fixed (so is its rank), there are 3 possibilities to place a card in the second position, since its rank must be identical to the previously placed one. And then it remains 2 possibilities to fill in the third position. This completes the selection of the three cards with identical rank.

Step 3. Place two cards of another identical rank in the remaining two positions. Since we are left 12 ranks to choose, there are 48 possibilities for the first position and then 3 possibilities for the second. This completes the selection of a full house.

From the above counting procedure, we see that

$$\#A = \binom{5}{3} \times 52 \times 3 \times 2 \times 48 \times 3.$$

Since the problem corresponds to the classical probability model over Ω, we know that

$$\mathbb{P}(A) = \frac{\#A}{\#\Omega} = \frac{\binom{5}{3} \times 52 \times 3 \times 2 \times 48 \times 3}{52 \times 51 \times 50 \times 49 \times 48} = \frac{6}{4165} \approx 0.00144.$$

(ii) The sample space Ω consists of all possible unordered 5-tuples of different cards. As a result, we know that

$$\#\Omega = \binom{52}{5} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5!}.$$

To count the total number of elements in the event A in this case, we can consider the following counting steps.

Step 1. Choose a rank and three cards of this rank. There are $\binom{13}{1} \times \binom{4}{3}$ ways to do so.

Step 2. Choose another rank from the remaining twelve and two cards of this rank. There are $\binom{12}{1} \times \binom{4}{2}$ ways to do so.

Therefore, we have

$$\#A = \binom{13}{1} \times \binom{4}{3} \times \binom{12}{1} \times \binom{4}{2}.$$

Since the problem corresponds to the classical probability model over Ω, we conclude that

$$\mathbb{P}(A) = \frac{\#A}{\#\Omega} = \frac{\binom{13}{1} \times \binom{4}{3} \times \binom{12}{1} \times \binom{4}{2}}{\binom{52}{5}} = \frac{6}{4165},$$

which is exactly the same answer we find in Part (i).

Remark. It is not surprising that we should end up with the same result for the two viewpoints, as physically there is no difference between selecting 5 card one by one and grabbing 5 cards from a deck in one go.

Problem 3. A family dog is missing after a picnic in the park. Three hypotheses are suggested:

$$\begin{gathered} A : \text{it has gone home}, \\ B : \text{it is still enjoying the big bone in the picnic area}, \\ C : \text{it has gone into the woods in the park}. \end{gathered}$$

The a priori probabilities of the above hypotheses, which are assessed from the habits of the dog, are estimated to be $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ respectively. Two children are sent to look for the dog. The first child returns to the park to search the picnic area and the woods. If the dog is in the picnic area, there is 80% chance that it will be found. The chance drops down to 40% if the dog has gone into the woods. The other child goes back home to have a look.

(i) What is the probability that the dog will be found in the park?
(ii) What is the probability that the dog will be found at home?
(iii) Given that the dog is found in the park, what is the probability that it is indeed found in the picnic area?
(iv) Given that the dog is lost, what is the probability that it is lost in the woods?

Solution. Let us introduce the following events:

$$\begin{gathered} D_1 : \text{the dog is found in the park}, \\ D_2 : \text{the dog is found at home}, \\ D_3 : \text{the dog is lost}. \end{gathered}$$

(i) By the assumption, we have

$$\mathbb{P}(A) = \frac{1}{2},\ \mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{4},$$

and

$$\mathbb{P}(D_1|A) = 0,\ \mathbb{P}(D_1|B) = 80\% = \frac{4}{5},\ \mathbb{P}(D_1|C) = 40\% = \frac{2}{5}.$$

According to the law of total probability,

$$\begin{aligned} \mathbb{P}(D_1) &= \mathbb{P}(A) \cdot \mathbb{P}(D_1|A) + \mathbb{P}(B) \cdot \mathbb{P}(D_1|B) + \mathbb{P}(C) \cdot \mathbb{P}(D_1|C) \\ &= \frac{1}{2} \times 0 + \frac{1}{4} \times \frac{4}{5} + \frac{1}{4} \times \frac{2}{5} = \frac{3}{10}. \end{aligned}$$

(ii) Apparently, we have

$$\mathbb{P}(D_2|A) = 1,\ \mathbb{P}(D_2|B) = \mathbb{P}(D_2|C) = 0.$$

Therefore,

$$\begin{aligned}\mathbb{P}(D_2) &= \mathbb{P}(A)\cdot\mathbb{P}(D_2|A) + \mathbb{P}(B)\cdot\mathbb{P}(D_2|B) + \mathbb{P}(C)\cdot\mathbb{P}(D_2|C)\\ &= \frac{1}{2}\times 1 + \frac{1}{4}\times 0 + \frac{1}{4}\times 0 = \frac{1}{2}.\end{aligned}$$

(iii) According to Bayes' formula, we have

$$\mathbb{P}(B|D_1) = \frac{P(B)\cdot\mathbb{P}(D_1|B)}{\mathbb{P}(D_1)} = \frac{\frac{1}{4}\times\frac{4}{5}}{\frac{3}{10}} = \frac{2}{3}.$$

(iv) Firstly, the probability that the dog is lost is given by

$$\mathbb{P}(D_3) = 1 - \mathbb{P}(D_1) - \mathbb{P}(D_2) = 1 - \frac{3}{10} - \frac{1}{2} = \frac{1}{5}.$$

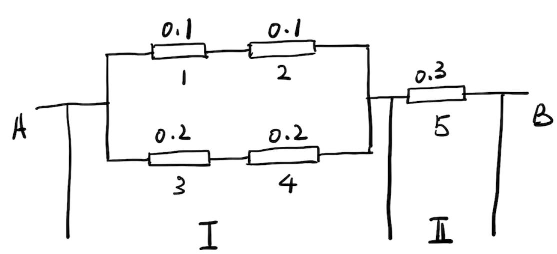
In addition, by the assumption we know that

$$\mathbb{P}(D_3|C) = 1 - 40\% = \frac{3}{5}.$$

Therefore,

$$\mathbb{P}(C|D_3) = \frac{\mathbb{P}(C)\cdot\mathbb{P}(D_3|C)}{\mathbb{P}(D_3)} = \frac{\frac{1}{4}\times\frac{3}{5}}{\frac{1}{5}} = \frac{3}{4}.$$

Problem 4. A circuit contains 5 mutually independent components as shown in the figure below.



The probability of failure for each component is indicated in the figure respectively. The system will function normally if current can flow from point A to point B.

(i) What is the probability that the system will function normally?
(ii) Given that the system fails, what is the probability that Component 1 fails?

Solution. (i) We divide the system into Part I and Part II as illustrated in the figure. The system functions if and only if both Part I and Part II function normally. Since the components are mutually independent, we see that

$$\mathbb{P}(\text{system functions}) = \mathbb{P}(\text{Part I functions}) \times \mathbb{P}(\text{Part II functions}).$$

Let us compute the above probabilities. Firstly, we have

$$\begin{aligned}
&\mathbb{P}(\text{Part I functions}) \\
&= 1 - \mathbb{P}(\text{Part I fails}) \\
&= 1 - \mathbb{P}(\text{Upper branch of I fails \& Lower branch of I fails}) \\
&= 1 - \mathbb{P}(\text{Upper branch of I fails}) \times \mathbb{P}(\text{Lower branch of I fails}) \\
&= 1 - (1 - 0.9 \times 0.9) \times (1 - 0.8 \times 0.8) \\
&= 0.9316.
\end{aligned}$$

In addition, we have

$$\mathbb{P}(\text{Part II functions}) = 0.7.$$

Therefore,

$$\mathbb{P}(\text{system functions}) = 0.9316 \times 0.7 = 0.65212.$$

(ii) Let F_i (respectively, S_i) be the event that "Component i fails" (respectively, "Component i functions"). Let F (respectively, S) be the event that "the system fails" (respectively, "the system functions"). Then

$$\mathbb{P}(F_1|F) = \frac{\mathbb{P}(F_1) \times \mathbb{P}(F|F_1)}{\mathbb{P}(F)} = \frac{\mathbb{P}(F_1)(1 - \mathbb{P}(S|F_1))}{1 - \mathbb{P}(S)}.$$

Note that given Component 1 fails, the system functions if and only if all the Components 3,4,5 function normally. Therefore,

$$\begin{aligned}
\mathbb{P}(S|F_1) &= \mathbb{P}(S_3 \cap S_4 \cap S_5|F_1) \\
&= \mathbb{P}(S_3) \times \mathbb{P}(S_4) \times \mathbb{P}(S_5). \quad \text{(by independence)} \\
&= 0.8 \times 0.8 \times 0.7
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{P}(F_1|F) &= \frac{\mathbb{P}(F_1) \times (1 - \mathbb{P}(S_3) \times \mathbb{P}(S_4) \times \mathbb{P}(S_5))}{1 - \mathbb{P}(S)} \\
&= \frac{0.1 \times (1 - 0.8 \times 0.8 \times 0.7)}{1 - 0.65212} \\
&= 0.15868.
\end{aligned}$$

Problem 5. A fair coin is tossed repeatedly and independently for n times.

(i) Suppose that $n = 5$. Given the appearance of successive heads (namely, a run of heads appearing consecutively), what is the conditional probability that successive tails never appear?
(ii) Let p_n denote the probability that successive heads never appear in the n tosses. Find an explicit formula for p_n. [*Hint: Condition on the first toss.*]
(iii) Let q_n denote the conditional probability that successive heads appear in the n tosses, given that no successive heads are observed in the first $n-1$ tosses. What is $\lim_{n\to\infty} q_n$?

Solution. (i) The event that "successive heads appear" consists of the following outcomes:

$$\begin{gathered} HHHHH \\ HHHHT, THHHH \\ HHHTT, HHHTH, THHHT, TTHHH, HTHHH \\ HHTHH, HHTTT, HHTTH, HHTHT, \\ THHTT, THHTH, TTHHT, HTHHT \\ TTTHH, HTTHH, THTHH. \end{gathered}$$

There are 19 outcomes in the event. In this event, there are 11 outcomes for which successive tails do not appear. These are marked with red color. Therefore,

$$\mathbb{P}(\text{no successive tails}|\text{successive heads}) = \frac{11}{19}.$$

(ii) Let A_n denote the event that "successive heads does not appear". To compute $\mathbb{P}(A_n)$, we condition on the result of the first toss. Let H be the event that "the first toss gives a head", and T the event that "the first toss gives a tail". By the law of total probability,

$$\begin{aligned} p_n &= \mathbb{P}(H)\mathbb{P}(A_n|H) + \mathbb{P}(T)\mathbb{P}(A_n|T) \\ &= \frac{1}{2}\mathbb{P}(A_n|H) + \frac{1}{2}\mathbb{P}(A_n|T). \end{aligned}$$

To compute the conditional probabilities on the right hand side, if the first toss gives a head, in order that no successive heads appear in the n tosses, it is equivalent to saying that the second toss gives a tail and there are no successive heads in the following $n-2$ tosses. Since different tosses are independent, we arrive at

$$\mathbb{P}(A_n|H) = \frac{1}{2} \times p_{n-2}.$$

Similarly, if the first toss gives a tail, in order to guarantee that no successive heads appear in the n tosses, it is equivalent to saying that no successive heads appear in the following $n-1$ tosses. Therefore,

$$\mathbb{P}(A_n|T) = p_{n-1}.$$

It follows that

$$p_n = \frac{1}{2} \times \frac{1}{2} \times p_{n-2} + \frac{1}{2} \times p_{n-1},$$

that is

$$p_n = \frac{1}{2}p_{n-1} + \frac{1}{4}p_{n-2}. \tag{2}$$

This is a recursive relation for the sequence $\{p_n : n \geqslant 1\}$, which can be solved explicitly if we know p_1 and p_2. Indeed, we have $p_1 = 1$ since there are no successive heads if we only perform one toss. If $n = 2$, we have $p_2 = 3/4$, since resulting in 2 heads is the only way to produce successive heads in this case.

There are at least three enlightening ways of finding the explicit formula for p_n from the relation (2).

Method 1. We start by guessing that p_n has the form $p_n = c^n$ where c is a constant. Substituting this into (2), we get

$$c^n = \frac{1}{2}c^{n-1} + \frac{1}{4}c^{n-2}.$$

Therefore, c must be a solution of the quadratic equation

$$x^2 = \frac{1}{2}x + \frac{1}{4}. \tag{3}$$

Conversely, if c is a solution of the above equation, then $p_n = c^n$ satisfies (2). The next crucial observation is that, the relation (2) is a linear relation. That means, if $\{p_n\}$, $\{q_n\}$ both satisfy the relation (2) and a, b are two constants, then $\{ap_n + bq_n\}$ also satisfy (2). As a result, it is natural to expect that, the general form of p_n is given by

$$p_n = C_1\mu_1^n + C_2\mu_2^n,$$

where μ_1, μ_2 are the two solutions to the quadratic equation (3), and C_1, C_2 are two arbitrary constants. Using the initial conditions $p_1 = 1$, $p_2 = 3/4$, we can determine C_1, C_2 uniquely, thus obtaining the explicit formula for p_n.

In my biased opinion, this method is the least inspiring, so we will not provide finer details for the above argument. Instead, we give two other approaches which

I find much more enlightening. The first one is based on linear algebra, while the second one is based on calculus.

Method 2. We can rewrite the relation (2) by using matrix notation:

$$\begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} p_{n-1} \\ p_{n-2} \end{pmatrix}.$$

Recursively, we further have

$$\begin{aligned} \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} p_{n-2} \\ p_{n-3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{pmatrix}^2 \cdot \begin{pmatrix} p_{n-2} \\ p_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{pmatrix}^2 \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} p_{n-3} \\ p_{n-4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{pmatrix}^3 \cdot \begin{pmatrix} p_{n-3} \\ p_{n-4} \end{pmatrix} \\ &\dots \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{pmatrix}^{n-2} \cdot \begin{pmatrix} p_2 \\ p_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{pmatrix}^{n-2} \cdot \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix}. \end{aligned}$$

In order to compute p_n, it suffices to compute $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{pmatrix}^{n-2}$. In linear algebra, the standard way to do so is to diagonalise the matrix. That is, to find an invertible matrix P and a diagonal matrix D, such that

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{pmatrix} = P \cdot D \cdot P^{-1}.$$

In this case, after the diagonalisation procedure (which we will omit since this is standard linear algebra), we get

$$P = \begin{pmatrix} \frac{1+\sqrt{5}}{4} & \frac{1-\sqrt{5}}{4} \\ 1 & 1 \end{pmatrix}, \; D = \begin{pmatrix} \frac{1+\sqrt{5}}{4} & 0 \\ 0 & \frac{1-\sqrt{5}}{4} \end{pmatrix}, \; P^{-1} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{\sqrt{5}-1}{2\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{\sqrt{5}+1}{2\sqrt{5}} \end{pmatrix}.$$

Therefore,

$$
\begin{aligned}
\begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & 0 \end{pmatrix}^{n-2} \cdot \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix} \\
&= \underbrace{PDP^{-1} \cdot PDP^{-1} \cdot \cdots \cdot PDP^{-1}}_{n-2 \text{ times}} \cdot \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix} \\
&= PD^{n-2}P^{-1} \cdot \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix} \\
&= P \cdot \begin{pmatrix} \left(\frac{1+\sqrt{5}}{4}\right)^{n-2} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{4}\right)^{n-2} \end{pmatrix} \cdot P^{-1} \cdot \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix}.
\end{aligned}
$$

By performing the matrix multiplication explicitly and extracting the first entry, we obtain that

$$
p_n = \frac{\sqrt{5}+3}{2\sqrt{5}} \times \left(\frac{1+\sqrt{5}}{4}\right)^n + \frac{\sqrt{5}-3}{2\sqrt{5}} \times \left(\frac{1-\sqrt{5}}{4}\right)^n . \tag{4}
$$

Method 3. We now use a calculus-based method to compute p_n. Consider the following function defined by a power series whose coefficients are p_n:

$$
F(x) = p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4 + p_5 x^5 + \cdots . \tag{5}
$$

We multiply $F(x)$ by $\frac{1}{2}x$ and $\frac{1}{4}x^2$ respectively to get

$$
\frac{1}{2}xF(x) = \frac{1}{2}p_1 x^2 + \frac{1}{2}p_2 x^3 + \frac{1}{2}p_3 x^4 + \frac{1}{2}p_4 x^5 + \cdots , \tag{6}
$$

and

$$
\frac{1}{4}x^2F(x) = \frac{1}{4}p_1 x^3 + \frac{1}{4}p_2 x^4 + \frac{1}{4}p_3 x^5 + \cdots . \tag{7}
$$

The reason that we perform such multiplication is to make use of the relation (2). Indeed, by subtracting (5) from (6) and (7), we obtain that

$$
\begin{aligned}
&F(x) - \frac{1}{2}xF(x) - \frac{1}{4}x^2F(x) \\
&= p_1 x + \left(p_2 - \frac{1}{2}p_1\right) x^2 + \left(p_3 - \frac{1}{2}p_2 - \frac{1}{4}p_1\right) x^3 \\
&\quad + \left(p_4 - \frac{1}{2}p_3 - \frac{1}{4}p_2\right) x^4 + \left(p_5 - \frac{1}{2}p_4 - \frac{1}{4}p_3\right) x^5 + \cdots .
\end{aligned}
$$

According to the relation (2), the coefficients of $x^3, x^4, x^5, \cdots$ all vanishes. In addition, by using $p_1 = 1, p_2 = 3/4$, we obtain that

$$F(x) - \frac{1}{2}xF(x) - \frac{1}{4}x^2F(x) = x + \frac{1}{4}x^2,$$

or equivalently,

$$F(x) = -\frac{x^2 + 4x}{x^2 + 2x - 4}.$$

Recall that $F(x)$ is the power series defined by the coefficients $\{p_n\}$. The problem of computing p_n therefore reduces to computing the Taylor expansion of $F(x)$. For this purpose, we first expand the function $\frac{1}{x^2+2x-4}$. Let μ_1, μ_2 be the two roots of the polynomial $x^2 + 2x - 4$. Explicitly, we have

$$\mu_1 = \sqrt{5} - 1, \ \mu_2 = -(\sqrt{5} + 1).$$

Next, we write

$$\begin{aligned} \frac{1}{x^2 + 2x - 4} &= \frac{1}{(\mu_1 - x)(\mu_2 - x)} = \frac{1}{\mu_2 - \mu_1} \cdot \left(\frac{1}{\mu_1 - x} - \frac{1}{\mu_2 - x}\right) \\ &= \frac{1}{\mu_2 - \mu_1} \cdot \left(\frac{1}{\mu_1} \cdot \frac{1}{1 - x/\mu_1} - \frac{1}{\mu_2} \cdot \frac{1}{1 - x/\mu_2}\right). \end{aligned}$$

By using the expansion

$$\frac{1}{1 - z} = 1 + z + z^2 + z^3 + \cdots ,$$

we arrive at

$$\begin{aligned} F(x) &= -\frac{x^2 + 4x}{x^2 + 2x - 4} \\ &= \frac{x^2 + 4x}{\mu_1 - \mu_2} \cdot \left(\frac{1}{\mu_1}\left(1 + \frac{x}{\mu_1} + \frac{x^2}{\mu_1^2} + \frac{x^3}{\mu_1^3} + \cdots\right)\right. \\ &\quad \left. - \frac{1}{\mu_2} \cdot \left(1 + \frac{x}{\mu_2} + \frac{x^2}{\mu_2^2} + \frac{x^3}{\mu_2^3} + \cdots\right)\right). \end{aligned} \tag{8}$$

By simplifying the expression in (8) and extracting the x^n-coefficient, we obtain that

$$p_n = \frac{1}{\mu_1 - \mu_2} \cdot \left(\frac{1}{\mu_1^{n-1}} - \frac{1}{\mu_2^{n-1}}\right) + \frac{4}{\mu_1 - \mu_2} \cdot \left(\frac{1}{\mu_1^n} - \frac{1}{\mu_2^n}\right).$$

We get the same answer as in Method 2 after substituting the values of μ_1, μ_2 into the above expression.

Remark. There is an elegant relationship between p_n and the Fibonacci sequence. In the experiment of tossing n coins, let F_n be the total number of outcomes which contain no successive heads. By conditioning on the first toss, it is not hard to see that

$$F_n = F_{n-1} + F_{n-2},$$

where $F_1 = 2$, $F_2 = 3$. The probability p_n is then given by $p_n = F_n/2^n$.

(iii) The desired conditional probability is given by

$$\begin{aligned} q_n &= \mathbb{P}(A_n^c|A_{n-1}) = 1 - \mathbb{P}(A_n|A_{n-1}) \\ &= 1 - \frac{\mathbb{P}(A_n \cap A_{n-1})}{\mathbb{P}(A_{n-1})} = 1 - \frac{\mathbb{P}(A_n)}{\mathbb{P}(A_{n-1})} \\ &= 1 - \frac{p_n}{p_{n-1}}, \end{aligned}$$

where we have used the fact that $A_n \subseteq A_{n-1}$ so that $A_n \cap A_{n-1} = A_n$. In explicit formula (4) of p_n, to simply the notation let us set

$$c_1 = \frac{\sqrt{5}+3}{2\sqrt{5}}, \ c_2 = \frac{\sqrt{5}-3}{2\sqrt{5}}, \ \lambda_1 = \frac{1+\sqrt{5}}{4}, \ \lambda_2 = \frac{1-\sqrt{5}}{4}.$$

It is apparent that $\lambda_1 > |\lambda_2|$. Therefore,

$$\begin{aligned} \frac{p_n}{p_{n-1}} &= \frac{c_1\lambda_1^n + c_2\lambda_2^n}{c_1\lambda_1^{n-1} + c_2\lambda_2^{n-1}} \quad \text{(dividing both sides by } \lambda_1^{n-1}) \\ &= \frac{c_1\lambda_1 + c_2\lambda_2 \times (\lambda_2/\lambda_1)^{n-1}}{c_1 + c_2(\lambda_2/\lambda_1)^{n-1}.} \end{aligned}$$

Since $|\lambda_2| < \lambda_1$, we know that $\lim_{n\to\infty}(\lambda_2/\lambda_1)^{n-1} = 0$. Therefore,

$$\lim_{n\to\infty} \frac{p_n}{p_{n-1}} = \lambda_1,$$

and thus

$$\lim_{n\to\infty} q_n = 1 - \lambda_1 = \frac{3-\sqrt{5}}{4}.$$