HD·EDUCATION

MAST20004 Probability

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HD · EDUCATION是由数位悉尼大学学生创办于2014年3月的**学业辅导机构**，创办伊始秉承着 “**让年轻人成为知识的生产者、传播者、受惠者**” 的理念，从学生的角度出发，量身制定符合他们需求的课程。“成为最受年轻人喜爱的学业辅导机构”一直是我们的不懈追求。

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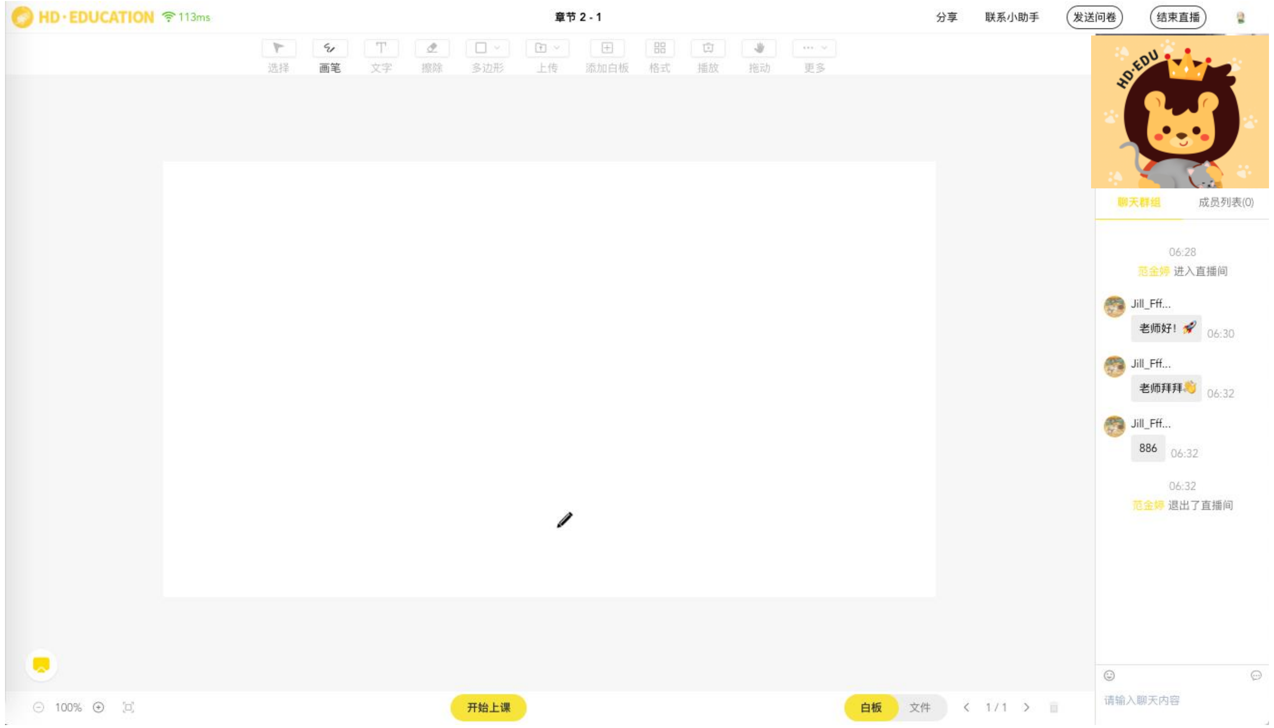

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Before We Start…

- Something you really need when you are doing math subjects:
  - Symbolab Website: https://www.symbolab.com/#

Revision: Normal Distribution

- $X \stackrel{d}{=} \mathrm{N}(\mu, \sigma^2)$.
- pdf: $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad (-\infty < x < \infty)$ $\qquad f_Z(z) = \varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad (-\infty < z < \infty)$
- Standardisation: $Z = \frac{X-\mu}{\sigma} \stackrel{d}{=} \mathrm{N}(0,1)$.
- Expectation and Variance:
  - How to find E(X^2) ?
  - Higher Moments: $\mathbb{E}(Z^n) = (n-1)\mathbb{E}(Z^{n-2})$. $\quad \mathbb{E}(Z^{2k+1}) = 0$
    $\mathbb{E}(Z^{2k}) = (2k-1)(2k-3)\ldots.1 = (2k)!/(2^k k!)$.
  - Approximation: Binomial? Poisson?

Assignment 2, Question 4

Problem 4. Let X be a normal random variable with mean 3 and variance 4.

(i) Find the following probabilities:

$$\mathbb{P}(2 < X \leqslant 5), \ \mathbb{P}(X > 3), \ \mathbb{P}(|X| > 2).$$

Assignment 2, Question 4

(ii) Find the values of a, b (where $a < b$) such that

$$\mathbb{P}(X < a) : \mathbb{P}(a < X < b) : \mathbb{P}(X > b) = 1 : 2 : 3.$$

You may use the fact that $\Phi(0.96741) \approx 0.83333$.

Revision: Expectation And Variance from pdf

- Expectation:
    - Method 1: $\mathbb{E}(X) = \int_{x \in S_X} x f_X(x)dx$
    - Method 2:
        - Condition?
    - Method 3?
- Variance:
- Find cdf from pdf:

Revision: Binomial Distribution

Assignment 2, Question 3

Problem 3. Let X be the lifetime (measured in hours) of a particular type of electronic device, whose probability density function is given by

$$f_X(x) = \begin{cases} \dfrac{C}{x^3}, & x > 10, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the value of C.

Assignment 2, Question 3

(ii) Compute $\mathbb{P}(X > 20)$ and $\mathbb{E}[X]$. Does X have finite variance?

Assignment 2, Question 3

(iii) In a batch of 6 such devices, what is the probability that at least 3 of them will function for at least 20 hours? We assume that all the 6 devices are independent.

Revision: Bernoulli Distribution & Poisson Distribution

Bernoulli Distribution

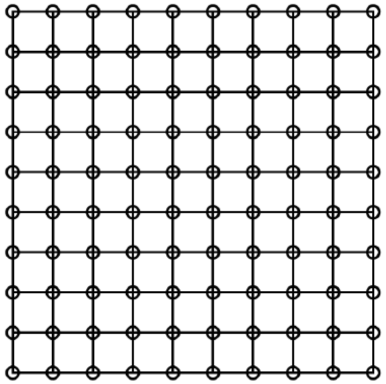
- pdf:
- cdf:
- Expectation and Variance:

Poisson Distribution

- pdf: $$p_N(k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad (k = 0, 1, 2, \ldots).$$
- Taylor?
- Expectation and Variance
- Approximation:

Assignment 2, Question 2

Problem 2. Suppose that a particular suburb of Melbourne is built on the following 10×10 grid.



At each intersection, a street light is placed (represented by a small circle). A street refers to a segment between two adjacent lights. We say that a street is *dark*, if the two lights at both ends are broken. Suppose that the lights on the boundary of the grid are lack of proper maintenance so that each has a probability of 0.2 being broken. All other lights have broken probability 0.1. All the lights are independent of each other. Let X be the number of dark streets in this suburb.

(i) How do you write X as a sum of Bernoulli random variables? State the corresponding Bernoulli trials precisely.

Assignment 2, Question 2

(ii) Use Poisson approximation to calculate the probability that there are at least three dark streets in the suburb. Explain heuristically why we can use Poisson approximation in this problem (you don't need to make any precise mathematical justification).

Revision: Conditional Probability

- Independence?

- Law of Total Probability: $$\mathbb{P}(H) = \sum_i \mathbb{P}(H|A_i)\mathbb{P}(A_i).$$

Assignment 2, Question 1

Problem 1. A randomly selected family has n children with probability αp^n $(n \geqslant 1)$ where $p \in (0,1)$ and $\alpha \leqslant (1-p)/p$.

(i) What is the probability that a randomly selected family does not have any children?

Assignment 2, Question 1

(ii) Suppose that each child is equally likely to be a boy or a girl (independently of each other), and the number of children is also independent of the sexes. What is the probability that a randomly selected family has one boy (and any number of girls)?
[Hint: *Given that the family has n children, what is the distribution of the number of boys? You don't need to justify this mathematically.*]

Assignment 2, Question 1

(iii) Under the same assumption as in Part (ii), what is the probability that a randomly selected family has k boys (and any number of girls)?
[Hint: *You may use the following identity which is a direct corollary of the extended binomial theorem in the lecture:*

$$(1-x)^{-(k+1)} = \sum_{m=0}^{\infty} \binom{m+k}{k} x^k, \quad |x| < 1.]$$

Typo: it should be x^m, the result from Symbolab:

The Taylor Series of $\left(\frac{1}{1-x}\right)^{1+k}$ with center 0

Answer: $1+(1+k)x-\frac{(1+k)(-k-2)}{2}x^2+\frac{(1+k)(-k-2)(-k-3)}{6}x^3-\frac{(1+k)(-k-2)(-k-3)(-k-4)}{24}x^4+\ldots$

Assignment 2, Question 5

Problem 5. (1) A table tennis match is played between Horatio and Xi. The winner of the match is the one who first wins 4 games in total, and in any individual game the winner is the one who first scores 11 points. Note that in an individual game, if the score is 10 to 10, the game goes into extra play (called *deuce*) until one player has gained a lead of 2 points. Let p be the probability that Horatio wins a point in any single round of serve, and assume that different rounds in all games are independent.

(1-i) In an individual game, what is the probability that the game runs into the deuce stage?

Assignment 2, Question 5

(1-ii) Suppose that $p = 0.55$. What is the probability that Horatio wins the match?

Assignment 2, Question 5

(2) Let $n \geqslant 1$ and $p \in (0,1)$ and $q = 1 - p$. By constructing suitable probability models or otherwise, show that

$$\binom{2n}{n} p^n q^n + \sum_{k=0}^{n-1} \binom{n+k}{k} (q^k p^{n+1} + p^k q^{n+1}) = 1.$$

[Hint: *There are at least two methods to show this. Method 1: Consider an artifical table tennis game which has the deuce rule when the score is n-n, and use two methods to calculate the probability of entering the deuce stage. Method 2: Consider the relation between binomial and negative binomial random variables.*]

Selected Problems

Assignment 2, Sem 1 2020

1. Let X be a random variable with $S_X = \{0, 1, 2, \ldots\}$ and probability mass function of the form

$$p_X(k) = c\,2^{-k}.$$

   (a) Determine c.

Selected Problems

(b) Is X more likely to take values that are divisible by 4, or values that are not divisible by 4? Justify your answer.

Selected Problems

(c) If they exist, calculate $\mathbb{E}(X)$ and $\mathrm{sd}(X)$ from first principles.

Selected Sample Answers

Selected Sample Answers

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