MAST20004 Probability

Assignment Three: Solutions

Problem 1. Three boxes are labelled by $1, 2, 3$ respectively. Suppose that we randomly place two different balls into the three boxes. Let X and Y denote the minimal and maximal labels of the occupied boxes.

(i) Compute the joint probability mass function of (X, Y).
(ii) Compute the conditional probability mass function of X given $Y = 3$.
(iii) Are X and Y independent?
(iv) Compute $\mathbb{E}[XY]$.

Solution. (i) By definition, we have $S_X = S_Y = \{1, 2, 3\}$. For $x, y \in \{1, 2, 3\}$, if $y < x$ then

$$p_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y) = 0.$$

For the other cases, we have

$$p_{X,Y}(1,1) = \mathbb{P}(\text{both balls are placed in Box 1}) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9},$$

$$\begin{aligned} p_{X,Y}(1,2) &= \mathbb{P}(\text{1 ball placed in Box 1 and the other placed in Box 2}) \\ &= \frac{1}{3} \times \frac{1}{3} \times 2 = \frac{2}{9}, \end{aligned}$$

and similarly

$$p_{X,Y}(1,3) = \frac{1}{3} \times \frac{1}{3} \times 2 = \frac{2}{9},\ p_{X,Y}(2,2) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9},$$
$$p_{X,Y}(2,3) = \frac{1}{3} \times \frac{1}{3} \times 2 = \frac{2}{9},\ p_{X,Y}(3,3) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}.$$

We can express the joint pmf in the following table:

$$\begin{array}{cccc} {}_Y\backslash^X & 1 & 2 & 3 \\ 1 & \frac{1}{9} & 0 & 0 \\ 2 & \frac{2}{9} & \frac{1}{9} & 0 \\ 3 & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \end{array}.$$

(ii) From the joint pmf in Part (i), we see that

$$p_Y(3) = \mathbb{P}(Y = 3) = \frac{2}{9} + \frac{2}{9} + \frac{1}{9} = \frac{5}{9}.$$

Therefore, by using the formula

$$p_{X|Y}(x|3) = \mathbb{P}(X = x|Y = 3) = \frac{p_{X,Y}(x, 3)}{p_Y(3)},$$

we find the conditional pmf of X given $Y = 3$ as

$$\begin{array}{cccc} x & 1 & 2 & 3 \\ p_{X|Y}(x|3) & \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{array}.$$

(iii) X and Y are not independent, as seen from e.g.

$$p_X(2) = \frac{1}{9} + \frac{2}{9} = \frac{1}{3} \neq \frac{2}{5} = p_{X|Y}(2|3).$$

(iv) Using the joint pmf, we find

$$\begin{aligned} \mathbb{E}[XY] &= 1 \times 1 \times \frac{1}{9} + 1 \times 2 \times \frac{2}{9} + 1 \times 3 \times \frac{2}{9} + 2 \times 2 \times \frac{1}{9} \\ &\quad + 2 \times 3 \times \frac{2}{9} + 3 \times 3 \times \frac{1}{9} = 4. \end{aligned}$$

Problem 2. Let (X, Y) be a bivariate random variable whose joint probability density function is given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{Cy}{x^2}, & 0 < x < 1 \text{ and } 0 < y < x^2, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the constant C.
(ii) Compute the marginal probability density functions of X and Y.
(iii) Compute $f_{Y|X}(y|x)$ and deduce that

$$\mathbb{E}[Y|X] = \frac{2}{3}X^2.$$

(iv) Compute $f_{X|Y}(x|y)$ and $\mathbb{E}[X|Y]$.

Solution. (i) We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = \int_0^1 dx \int_0^{x^2} \frac{Cy}{x^2}dy$$
$$= \int_0^1 \frac{C}{x^2}\cdot\frac{x^4}{2}dx$$
$$= \frac{C}{6}.$$

Therefore, $C = 6$.
(ii)The marginal pdf's are given by:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \begin{cases} \int_0^{x^2} \frac{6y}{x^2}dy = 3x^2, & 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \begin{cases} \int_{\sqrt{y}}^1 \frac{6y}{x^2}dx = 6(\sqrt{y} - y), & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(iii) For each $0 < x < 1$, we have

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \frac{6y/x^2}{3x^2} = \frac{2y}{x^4}, & 0 < y < x^2, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\mathbb{E}[Y|X = x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy = \int_0^{x^2} y \cdot \frac{2y}{x^4}dy = \frac{2}{3}x^2.$$

It follows that

$$\mathbb{E}[Y|X] = \frac{2}{3}X^2.$$

(iv) For each $0 < y < 1$, we have

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{6y/x^2}{6(\sqrt{y}-y)} = \frac{\sqrt{y}}{x^2(1-\sqrt{y})}, & \sqrt{y} < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\mathbb{E}[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx = \int_{\sqrt{y}}^1 x \cdot \frac{\sqrt{y}}{x^2(1-\sqrt{y})}dx = -\frac{\sqrt{y}\log y}{2(1-\sqrt{y})}.$$

It follows that

$$\mathbb{E}[X|Y] = \frac{-\sqrt{Y}\log Y}{2(1-\sqrt{Y})}.$$

Problem 3. Let $(X, Y) \sim N_2(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$ be a bivariate normal random variable.

(i) If $\mathrm{Cov}(X, Y) = 0$, show that X, Y are independent. [*Hint: Consider the expression of the joint pdf in this case*].
(ii) Suppose that $V[Y] = 4V[X]$. Explain why $2X+Y$ and $2X-Y$ are independent. [*Hint: Use Part (i)*]
(iii) Assume that $\mu_X = 0$, $\mu_Y = 1$, $\sigma_X^2 = 1$, $\sigma_Y^2 = 4$, $\rho = -1/2$. What is the joint distribution of $(X+Y, X-2Y)$?
(iv) Under the same assumption as in Part (iii), compute $\mathbb{P}(X+Y > 0)$ and $\mathbb{P}(X+Y > 0|X-2Y = -2)$.

Solution. (i) We first derive the joint pdf of $(X, Y) \sim N_2(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$. Let

$$X_s = \frac{X - \mu_X}{\sigma_X},\ Y_s = \frac{Y - \mu_Y}{\sigma_Y}.$$

By definition, $(X_s, Y_s) \sim N_2(\rho)$ and its joint pdf is given by

$$f_{X_s,Y_s}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\big(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\big).$$

To obtain the joint pdf of (X, Y), we think of

$$X = \mu_X + \sigma_X X_s,\ Y = \mu_Y + \sigma_Y Y_s$$

and use the general transformation formula. Here we consider (X_s, Y_s) as taking values in the xy-plane and (X, Y) as taking values in the uv-plane, so that the transformation is given by

$$u = \mu_X + \sigma_X x,\ v = \mu_Y + \sigma_Y y.$$

The inverse of this transformation is

$$x = \frac{u - \mu_X}{\sigma_X},\ y = \frac{v - \mu_Y}{\sigma_Y},$$

and its Jacobian matrix is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma_X} & 0 \\ 0 & \frac{1}{\sigma_Y} \end{pmatrix}.$$

According to the transformation formula, we obtain

$$
\begin{aligned}
f_{X,Y}(u,v) &= f_{X_s,Y_s}(x(u,v),y(u,v)) \times \big|\det\big(\frac{\partial(x,y)}{\partial(u,v)}\big)\big| \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\big(-\frac{1}{2(1-\rho^2)}\big((\frac{u-\mu_X}{\sigma_X})^2 \\
&\quad - 2\rho\big(\frac{u-\mu_X}{\sigma_X}\big)\big(\frac{v-\mu_Y}{\sigma_Y}\big) + \big(\frac{v-\mu_Y}{\sigma_Y}\big)^2\big)\big).
\end{aligned}
$$

In the current problem, since $\mathrm{Cov}(X,Y)=0$, we know that $\rho=0$. In this case, the joint pdf of (X,Y) has the form

$$
\begin{aligned}
f_{X,Y}(u,v) &= \frac{1}{2\pi\sigma_X\sigma_Y}\exp\big(-\frac{1}{2}\big(\frac{(u-\mu_X)^2}{\sigma_X^2}+\frac{(v-\mu_Y)^2}{\sigma_Y^2}\big)\big) \\
&= \frac{1}{\sqrt{2\pi}\sigma_X}\exp\big(-\frac{(u-\mu_X)^2}{2\sigma_X^2}\big) \times \frac{1}{\sqrt{2\pi}\sigma_Y}\exp\big(-\frac{(v-\mu_Y)^2}{2\sigma_Y^2}\big) \\
&= f_X(u)\cdot f_Y(v).
\end{aligned}
$$

As a result, X and Y are independent.
(ii) From lecture we know that $(2X+Y, 2X-Y)$ is a bivariate normal random variable. According to Part (i), to see their independence it is sufficient to show that $\mathrm{Cov}(2X+Y,2X-Y)=0$. According to the bilinearity of the covariance function, we have

$$
\begin{aligned}
&\mathrm{Cov}(2X+Y,2X-Y) \\
&= \mathrm{Cov}(2X,2X) - \mathrm{Cov}(2X,Y) + \mathrm{Cov}(Y,2X) - \mathrm{Cov}(Y,Y) \\
&= 4V[X] - V[Y] \\
&= 0 \quad \text{(by assumption)}.
\end{aligned}
$$

Therefore, $2X+Y$ and $2X-Y$ are independent.
(iii) From the assumption, we have

$$
\mathrm{Cov}(X,Y) = \rho\sigma_X\sigma_Y = -\frac{1}{2}\cdot 1\cdot 2 = -1.
$$

Let $U = X+Y$, $W = X-2Y$. We first compute:

$$
\mathbb{E}[U] = \mathbb{E}[X] + \mathbb{E}[Y] = 1,\ \mathbb{E}[W] = \mathbb{E}[X] - 2\mathbb{E}[Y] = -2,
$$

and

$$
V[U] = \mathrm{Cov}(X+Y,X+Y) = V[X] + 2\mathrm{Cov}(X,Y) + V[Y] = 3,
$$

$$V[W] = \text{Cov}(X-2Y, X-2Y) = V[X] - 4\text{Cov}(X,Y) + 4V[Y] = 21,$$

$$\text{Cov}(U,W) = \text{Cov}(X+Y, X-2Y) = V[X] - \text{Cov}(X,Y) - 2V[Y] = -6,$$

$$\rho_{U,W} = \frac{\text{Cov}(U,W)}{\sqrt{V[U]\cdot V[W]}} = -\frac{2}{\sqrt{7}}(\approx -0.7559).$$

Therefore,

$$(U,W) \sim N_2(1,-2,3,21,-2/\sqrt{7}).$$

(iv) Since $U \sim N(1,3)$, we have

$$\mathbb{P}\big(U>0\big) = \mathbb{P}\big(\frac{U-1}{\sqrt{3}} > -\frac{1}{\sqrt{3}}\big) = \Phi\big(\frac{1}{\sqrt{3}}\big) \approx \Phi(0.58) \approx 0.71904.$$

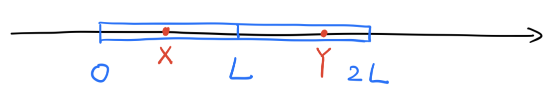
For the conditional probability, let $U_s = \frac{U-1}{\sqrt{3}}$ and $W_s = \frac{W+2}{\sqrt{21}}$ be the standardisations of U and W respectively. Then

$$U_s|_{W=-2} = U_s|_{W_s=0} \sim N\big(0, 1-\frac{4}{7} = \frac{3}{7}\big).$$

It follows that

$$\begin{aligned}
&\mathbb{P}(U>0|W=-2)\\
&= \mathbb{P}\big(U_s > -\frac{1}{\sqrt{3}}\big|W_s = 0\big)\\
&= \mathbb{P}\big(Z > -\frac{\sqrt{7}}{3}\big) = \Phi\big(\frac{\sqrt{7}}{3}\big) \approx \Phi(0.88) \approx 0.81057.
\end{aligned}$$

Problem 4. Consider a stick of length $2L$ which is placed on the real axis with its left end point coincide with the origin (as shown in the figure below).



Select two random points uniformly and independently on the stick, one from the first half and the other from the second half. Let X, Y be the coordinates of the first and second points respectively.

(i) Find the joint pdf of (X,Y).
(ii) What is the probability that the distance between the two selected points is

greater than $L/2$?
(iii) Observe that the two random points divide the stick into three segments. What is the probability that these three segments can form a triangle?
[*Hint: You may observe that (X,Y) is a "uniform" bivariate random variable on some region S. In this case, for any sub-region $R \subseteq S$, you can use the simple formula*

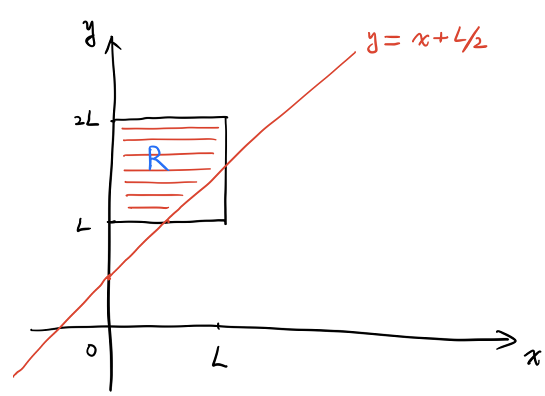
$$\mathbb{P}((X,Y)\in R) = \frac{\textit{area of } R}{\textit{area of } S}$$

to evaluate probabilities. Although you can always perform double integrals, it will be more complicated than just using the above geometric intuition.]

Solution. (i) By the assumption, we know that $X \sim R(0,L)$, $Y \sim R(L,2L)$ and X,Y are independent. Therefore, the joint pdf of (X,Y) is the product of the marginal pdf's, namely

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{L^2}, & 0<x<L,\ L<y<2L; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) In terms of X,Y, the desired event is $\{Y-X>L/2\}$. Let S be the square $[0,L]\times[L,2L]$ in the xy-plane. The intersection of the region $\{(x,y): y-x>L/2\}$ with the square S is indicated in the shaded part (called R) of the following figure.



Simple Euclidean geometry shows that its area is given by

$$\begin{aligned} \text{area of } R &= \text{area of square} - \text{area of triangle} \\ &= L^2 - \frac{1}{2}\times\frac{L}{2}\times\frac{L}{2} = \frac{7}{8}L^2. \end{aligned}$$

Since (X, Y) is uniformly distributed on the square, we have

$$\mathbb{P}(Y - X > L/2) = \frac{\text{area of } R}{\text{area of } S} = \frac{7L^2/8}{L^2} = \frac{7}{8}.$$

(iii) Three positive numbers a, b, c can form the sides of a triangle if

$$a + b > c, \ \ b + c > a, \ \ a + c > b.$$

In our context, the required inequalities are

$$X + (Y - X) > 2L - Y,$$

$$(Y - X) + (2L - Y) > X$$

and

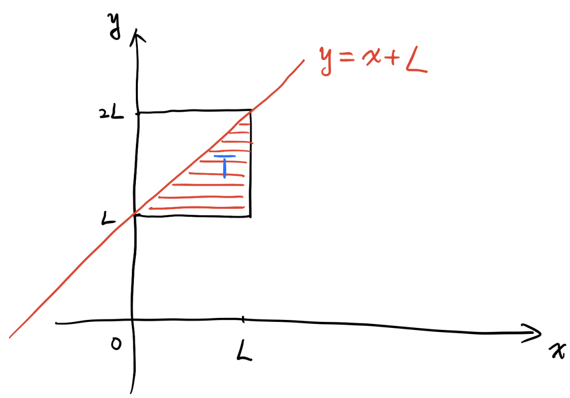
$$X + (2L - Y) > Y - X.$$

The first inequality is equivalent to $Y > L$ which is always true. The second inequality is equivalent to $X < L$ which is also always true. Therefore, we only need the third inequality which is simplified to

$$Y - X < L.$$

The resulting region T in the square S is indicated by the figure below, from which we see that

$$\mathbb{P}(Y - X < L) = \frac{\text{area of } T}{\text{area of } S} = \frac{1}{2}.$$



Problem 5. Let X, Y be independent random variables, each following the uniform distribution on the interval $(0,1)$.

(i) Find the probability density functions of $Z = \max(X, Y)$ and $W = \min(X, Y)$.
(ii) What is the joint pdf of (Z, W)? [*Hint: Try to proceed by geometric intuition rather than explicit calculation. More specifically, the nature of the joint pdf $f_{Z,W}(z,w)$ is revealed by the following equation:*

$$\mathbb{P}((Z,W) \in A) = \iint_A f_{Z,W}(z,w)dzdw \quad \textit{for any arbitrary region } A \textit{ in the plane.}$$

Given an arbitrary region A contained in the domain of (Z,W), try to understand what the probability of "$(Z,W) \in A$" should be. Deduce the joint distribution of (Z,W) directly from this observation.]

Solution. (i) We know that Y takes values in $(0,1)$. Given $y \in (0,1)$, we have

$$\begin{aligned} F_Y(y) &= \mathbb{P}(\max(X_1, X_2) \leqslant y) = \mathbb{P}(X_1 \leqslant y,\ X_2 \leqslant y) \\ &= \mathbb{P}(X_1 \leqslant y) \cdot \mathbb{P}(X_2 \leqslant y) = y^2. \end{aligned}$$

By differentiation, the pdf of Y is

$$f_Y(y) = \begin{cases} 2y, & y \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

For the random variable Z, it is more convenient to look at the tail probability. Given $z \in (0,1)$, we have

$$\begin{aligned} \mathbb{P}(Z > z) &= \mathbb{P}(\min(X_1, X_2) > z) = \mathbb{P}(X_1 > z,\ X_2 > z) \\ &= \mathbb{P}(X_1 > z)\mathbb{P}(X_2 > z) = (1-z)^2. \end{aligned}$$

Therefore,

$$F_Z(z) = 1 - (1-z)^2 = 2z - z^2.$$

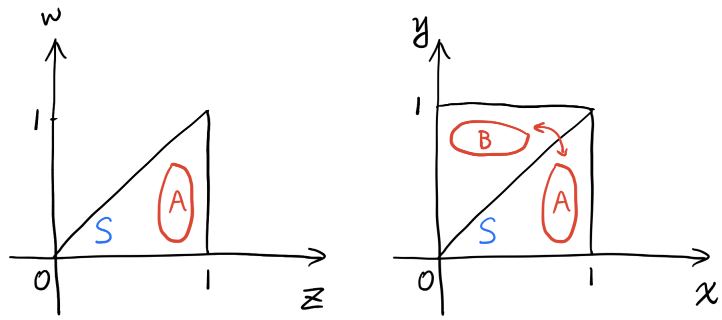
By differentiation, the pdf of Z is

$$f_Z(z) = \begin{cases} 2(1-z), & z \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

(ii) First of all, the set of possible values for (Z,W) is the triangular region given by

$$S = \{(z,w) : 0 < w < z < 1\}.$$

Let A be an arbitrary region contained in S and we try to figure out $\mathbb{P}((Z,W) \in A)$ (see the left part of the figure below).



To this end, we can first express the event $\{(Z,W) \in A\}$ in terms of X, Y as

$$\begin{aligned}
&\{(Z,W) \in A\} \\
&= \{(Z,W) \in A,\ X > Y\} \cup \{(Z,W) \in A,\ X < Y\} \\
&= \{(X,Y) \in A\} \cup \{(Y,X) \in A\}. \qquad (1)
\end{aligned}$$

To compute the probability of the right hand side, note that (X,Y) is uniformly distributed on the unit square $\{(x,y) : 0 < x, y < 1\}$. Therefore, we only need to figure out which geometric region is the right hand side of (1) representing within the unit square. The first part $\{(X,Y) \in A\}$ just corresponds to the region A. To understand the second part $\{(Y,X) \in A\}$, let us draw the region B which is the mirror image of A in the other triangle with respect to the diagonal line of the square (see the right part of the above figure). Then we know that

$$(Y,X) \in A \iff (X,Y) \in B.$$

In other words,

$$\{(Z,W) \in A\} = \{(X,Y) \in A\} \cup \{(X,Y) \in B\} = \{(X,Y) \in A \cup B\}.$$

Since (X,Y) is uniformly distributed on the square, we have

$$\mathbb{P}((X,Y) \in A \cup B) = \frac{\text{area of } A \cup B}{\text{area of square}}.$$

But by symmetry we know that A and B are identical in shape, so that

$$\frac{\text{area of } A \cup B}{\text{area of square}} = \frac{2 \times \text{area of } A}{\text{area of square}} = \frac{2 \times \text{area of A}}{2 \times \text{area of the triangle } S} = \frac{\text{area of } A}{\text{area of } S}.$$

As a result, we conclude that

$$\mathbb{P}((Z,W) \in A) = \frac{\text{area of } A}{\text{area of } S}.$$

Since (Z, W) takes values in S, the above equation suggests that (Z, W) is uniformly distributed on the triangle S. Its joint pdf is thus given by

$$f_{Z,W}(z,w) = \begin{cases} \frac{1}{\text{area of } S} = 2, & (z,w) \in S, \\ 0, & \text{otherwise.} \end{cases}$$