THE UNIVERSITY OF MELBOURNE

Student Number

Semester 2 Assessment, 2020

School of Mathematics and Statistics

MAST20004 Probability

This assignment consists of 15 pages (including this page)

Submission Deadline

- Submit (as instructed below) by **4:00 pm on Thursday 01 October 2020**
- Please **print out** the assignment single-sided and write your answers in the **answer boxes** provided under each sub-question. For those who do not have a printer or cannot access a printing device, you can directly write on the electronic version of the assignment on a tablet with a stylus in the same way. If both options fail for you, please write on blank A4 papers, clearly indicating the question numbers and pages, always start each question on a new page, and identify the pages to each question when submitting to Gradescope.
- Extra answer boxes are provided at the end of the assignment if you need more space. In this case, you must **make a note** in the correct answer space or page for the question, warning the marker that you have appended additional remarks at the end.
- Scan your assignment submission to a single PDF file with a mobile phone or a scanner. Scan from directly above to avoid any excessive keystone effect. Scan the whole page, not just the portion where your answer is written. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned submissions may be impossible to mark.

Instructions to Students

- This assignment contains 5 questions, **three** of which will be randomly selected to be marked. Each question is worth 10 points. You are expected to provide solutions to all questions. For the two questions that are not selected to be marked, a penalty of 2 points will apply to very poor / no attempts.
- The submission deadline is **strict**. Late submission within 24 hours after the deadline will be penalised by 10%, and submissions within 24-48 hours after the deadline will be penalised by 20%. After that, the Gradescope submission channel will be closed, and your submission will no longer be accepted. You are strongly encouraged to submit the assignment a few days before the deadline just in case of unexpected technical issues. If you are facing a rather exceptional/extreme situation that prevents you from submitting on time, please contact the tutor coordinator **Robert Maillardet** with formal proofs such as medical certificate.

Problem 1. Three boxes are labelled by $1, 2, 3$ respectively. Suppose that we randomly place two different balls into the three boxes. Let X and Y denote the minimal and maximal labels of the occupied boxes.

(i) Compute the joint probability mass function of (X, Y).

(ii) Compute the conditional probability mass function of X given $Y = 3$.

(iii) Are X and Y independent?

(iv) Compute $\mathbb{E}[XY]$.

Problem 2. Let (X,Y) be a bivariate random variable whose joint probability density function is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{Cy}{x^2}, & 0 < x < 1 \text{ and } 0 < y < x^2, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the constant C.

(ii) Compute the marginal probability density functions of X and Y.

(iii) Compute $f_{Y|X}(y|x)$ and deduce that

$$\mathbb{E}[Y|X] = \frac{2}{3}X^2.$$

(iv) Compute $f_{X|Y}(x|y)$ and $\mathbb{E}[X|Y]$.

Problem 3. Let $(X, Y) \sim N_2(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$ be a bivariate normal random variable.

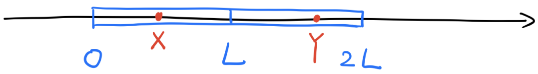
(i) If $\mathrm{Cov}(X, Y) = 0$, show that X, Y are independent. [*Hint: Consider the expression of the joint pdf in this case.*]

(ii) Suppose that $V[Y] = 4V[X]$. Explain why $2X + Y$ and $2X - Y$ are independent. [*Hint: Use Part (i).*]

(iii) Assume that $\mu_X = 0$, $\mu_Y = 1$, $\sigma_X^2 = 1$, $\sigma_Y^2 = 4$, $\rho = -1/2$. What is the joint distribution of $(X + Y, X - 2Y)$?

(iv) Under the same assumption as in Part (iii), compute $\mathbb{P}(X + Y > 0)$ and $\mathbb{P}(X + Y > 0 | X - 2Y = -2)$.

Problem 4. Consider a stick of length $2L$ which is placed on the real axis with its left end point coincide with the origin (as shown in the figure below).



Select two random points uniformly and independently on the stick, one from the first half and the other from the second half. Let X, Y be the coordinates of the first and second points respectively.

(i) Find the joint pdf of (X, Y).

(ii) What is the probability that the distance between the two selected points is greater than $L/2$?

(iii) Observe that the two random points divide the stick into three segments. What is the probability that these three segments can form a triangle?
[*Hint: You may observe that (X, Y) is a "uniform" bivariate random variable on some region S. In this case, for any sub-region $R \subseteq S$, you can use the simple formula*

$$\mathbb{P}((X,Y) \in R) = \frac{\textit{area of } R}{\textit{area of } S}$$

to evaluate probabilities. Although you can always perform double integrals, it will be more complicated than just using the above geometric intuition.]

Problem 5. Let X, Y be independent random variables, each following the uniform distribution on the interval $(0, 1)$.

(i) Find the probability density functions of $Z = \max(X, Y)$ and $W = \min(X, Y)$.

(ii) What is the joint pdf of (Z, W)? [*Hint: Try to proceed by geometric intuition rather than explicit calculation. More specifically, the nature of the joint pdf* $f_{Z,W}(z, w)$ *is revealed by the following equation:*

$$\mathbb{P}((Z, W) \in A) = \iint_A f_{Z,W}(z, w)dzdw \quad \textit{for any arbitrary region } A \textit{ in the plane.}$$

Given an arbitrary region A *contained in the domain of* (Z, W)*, try to understand what the probability of "*$(Z, W) \in A$*" should be. Deduce the joint distribution of* (Z, W) *directly from this observation.*]

If you need additional answer space for any of the above 5 questions, please use the following boxes. Clearly indicate the question number before continuing your solutions.

End of Assignment