MAST20004 Probability
Semester 2, 2020
Assignment One: Questions

Due 4:00 pm Monday 24 August

Important instructions:

(1) This assignment contains 5 questions, **three** of which will be randomly selected to be marked. Each question is worth 10 points. You are expected to provide solutions to all questions, for otherwise a penalty of 5 points per unanswered question will apply.

(2) Please either write on blank pages on a tablet or on blank A4 papers, clearly indicating the question numbers and pages. Please always start each question on a new page. When preparing your submission, if you write on blank A4 papers, it is vital that your scanned/photocopied solutions are **clearly readable**. A 10% penalty will be applied to unreadable submissions. Please also make sure that you obtain a **single combined PDF** file before submission. The submission page is linked to the Canvas assignment section so you can directly submit over there. When submitting the assignment, please make sure that you **identify the pages to each question** in Gradescope submission.

(3) The submission deadline is **strict**. Late submission within 48 hours after the deadline will be penalised by 20%. After that, the Gradescope submission channel will be closed, and your submission will no longer be accepted. You are strongly encouraged to submit the assignment a few days before the deadline just in case of unexpected technical issues. If you are facing a rather exceptional/extreme situation that prevents you from submitting on time, please contact the tutor coordinator **Robert Maillardet** with formal proofs such as medical certicate.

Problem 1. Let A, B, C be three events.

(i) Show that

$$A \cup B \cup C = A \cup (A^c \cap B) \cup (A^c \cap B^c \cap C)$$

and

$$\mathbb{P}(A \cup B \cup C) \leqslant \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C).$$

(ii) By using the Addition Theorem (Lecture Slide 30, Property (9)) or otherwise, show that

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) \\ - \mathbb{P}(A \cap C) + \mathbb{P}(A \cap B \cap C).\end{aligned}$$

(iii) Suppose that

$$\mathbb{P}(A) = \mathbb{P}(B) = 0.5,\ \mathbb{P}(C) = 0.2,\ \mathbb{P}(A \cup B \cup C) = 0.8,$$

and assume further that the events A, B, C are pairwisely independent. Are these three events mutually independent?

(iv) Suppose that the three events A, B, C are exhaustive, and they are mutually independent. Show that at least one of the following three statements must be true:

$$\mathbb{P}(A) = 1 \text{ or } \mathbb{P}(B) = 1 \text{ or } \mathbb{P}(C) = 1.$$

Problem 2. A standard poker deck has 52 playing cards, which includes 13 ranks (2, 3, 4, $\cdots$, 10, J, Q, K, A) in each of the four suits: clubs (♣), diamonds (♦), hearts (♥) and spades (♠). A *full house* is a hand of five cards consisting of three cards of one rank and two cards of another rank. Suppose that 5 cards are randomly selected. Under each of the following two assumptions, describe the sample space and compute the probability that a full house is obtained.

(i) The 5 cards are selected one after another in an ordered manner.

(ii) The 5 cards are selected at the same time without orders.

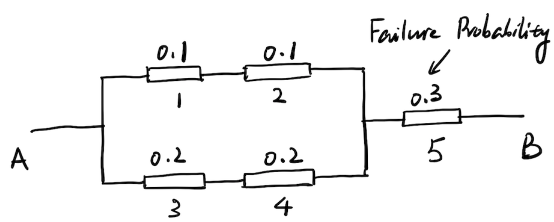
Problem 3. A family dog is missing after a picnic in the park. Three hypotheses are suggested:

$$A : \text{it has gone home,}$$
$$B : \text{it is still enjoying the big bone in the picnic area,}$$
$$C : \text{it has gone into the woods in the park.}$$

The a priori probabilities of the above hypotheses, which are assessed from the habits of the dog, are estimated to be $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ respectively. Two children are sent to look for the dog. The first child returns to the park to search the picnic area and the woods. If the dog is in the picnic area, there is 80% chance that it will be found. The chance drops down to 40% if the dog has gone into the woods. The other child goes back home to have a look.

(i) What is the probability that the dog will be found in the park?
(ii) What is the probability that the dog will be found at home?
(iii) Given that the dog is found in the park, what is the probability that it is indeed found in the picnic area?
(iv) Given that the dog is lost, what is the probability that it is lost in the woods?

Problem 4. A circuit contains 5 mutually independent components as shown in the figure below.



The probability of failure for each component is indicated in the figure respectively. The system will function normally if current can flow from point A to point B.

(i) What is the probability that the system will function normally?
(ii) Given that the system fails, what is the probability that Component 1 fails?

Problem 5. A fair coin is tossed repeatedly and independently for n times.

(i) Suppose that $n = 5$. Given the appearance of successive heads (namely, a run of heads appearing consecutively), what is the conditional probability that successive tails never appear?
(ii) Let p_n denote the probability that successive heads never appear in the n tosses. Find an explicit formula for p_n. [*Hint: Condition on the first toss.*]
(iii) Let q_n denote the conditional probability that successive heads appear in the n tosses, given that no successive heads are observed in the first $n - 1$ tosses. What is $\lim_{n\to\infty} q_n$?