Orthonormal Columns.

$Q^TQ = I$. where $Q^T = Q^{-1}$

$$\begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

but if $QQ^T = I$? yes when Q is square.

Q is "orthogonal" matrix

square

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ rotates points in xy plane counterclockwise through angle θ wrt x axis

Any x s.t. $\|Qx\| = \|x\|$ won't Δ length after muleiplication of Q.

$(Qx)^T Qx = x^T x$

$\| \qquad\qquad //$

$x^T Q^T Qx$

$\Downarrow$

because identity in the middle.

※ how does the rotation works,
e.g. multiply by $\begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

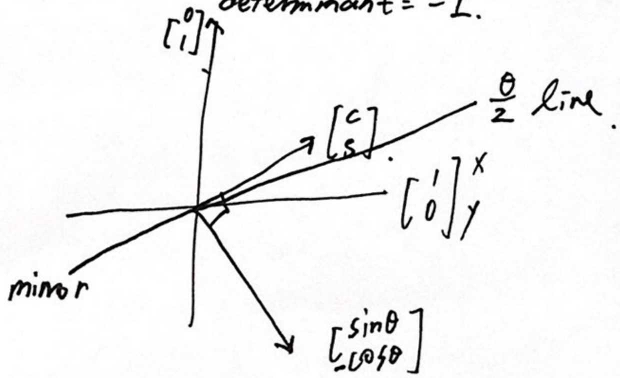
example 2.

$$Q = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

It's now not the rotation matrix

but the reflection matrix

determinant $= -1$.



Householder Reflection

start with $u^T u = I$.

$H = I - 2uu^T$

↳ becomes matrix

check $HH = I$.

$$I - 4uu^T + 4u\underbrace{(u^T u)}_{1}u^T = I$$

will use this in making things orthogonal.

Hadamard

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \rightarrow \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix}$$

$$\begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} \leftarrow$$

can see that columns vectors are orthogonal

H_{12}?? Yes Always possible if $\frac{N}{4}$ is a whole number.

where will we see orthogonal matrices automatically show up?

eigenvectors

Wavelet matrix

self-scaling.

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}$$

$$W_8 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

<u>Haar</u>, 1910.

squeeze down & rescale.

This ~~most~~ matrix of having quite sparse

Ingrid Daubechies, 1988

found a lot of wavelet

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eigenvectors of $S^T = S$ are orthogonal

$Q^TQ = I$

maybe the most important one is Fourier

discrete Fourier transform → orthogonal vectors

high speed → fast Fourier transform

e.g.

Eigenvector of $Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ are discrete Fourier Transform

permutation matrix

4

eigenvectors

① are all orthogonal

② heart of signal processing.

Just take a discrete fourier transform of a vector is split into its frequencies.

reordering identity matrix

eigenvalues & eigenvectors (con'd)

If S is a real symmetric matrix, its eigenvectors are real.

But if Q

, eigenvectors could be complex numbers.

denote F as the eigenvector matrix. of $Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$

could show that for every column vector, multiplied by Q, would get eigenvalues

means zero freq.

It's the discrete fourier, instead of e^i, e^{i^2}, e^{i^3}

dot product
col1·col2

$1+i+i^2+i^3=0$.

but if having complex number, should use conjugate

e.g. $\overline{col2}$

because if dot product col2, col4 in usual way

$1+i^4+i^8+i^{12}$ getting all ones

$\begin{bmatrix} 1 \\ -i \\ (-i)^2 \\ -i^3 \end{bmatrix} \begin{bmatrix} 1 \\ -i^3 \\ (-i)^6 \\ -i^9 \end{bmatrix}$ ~~$= 1+1+(-i)(-i)$~~

$\boxed{\begin{aligned} Q^TQ &= I \\ Qx &= \lambda x \\ Qy &= \mu y \end{aligned}}$

and $\bar{x}^T \cdot y = 0$

conjugate.