Naïve Set Theory

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Contents

1 Introduction

This note is a summary of some rudimentary set theory and logic that I believe one is supposed to be thoroughly familiar before taking courses like real analysis and abstract algebra. As the title entails, this summary is meant to be "naïve", which means that we will, for the most part, use informal language. If you are expecting to learn set theory in a more Bourbaki way, then this is not for you. For the most part, we will be sticking to ZF and ZFC set theory.

2 Statements, Formulas, and Propositions

Before we go into ZFC axioms, we first have to establish what statements and formulas are in a set theoretic sense. A **statement** is a string of **variables** and **logical symbols**. A variable is usually denoted by symbols such as A, B, C and etc. A logical symbol, on the other hand, is either a negation **not**($\neg$), or one of the **binary connectives**: **conjunction** ($\wedge$), **disjunction** ($\vee$), **implication** ($\rightarrow$), and **equivalence** ($\leftrightarrow$). We define a statement recursively by

1. a single variable is a statement
2. if F is a statement then so is $\neg F$, and
3. if A, B are statements then so is $A \wedge B$, $A \vee B$, $A \rightarrow B$, and $A \leftrightarrow B$.

and we often use the auxiliary bracket symbols () in statements to avoid ambiguity. A **valuation** is a a process where we give each variable in the statement a boolean value of either 1 (true) or 0 (false) by the following rules

A	$\neg A$
1	0
0	1

A	B	$A \wedge B$
1	1	1
1	0	0
0	1	0
0	0	0

A	B	$A \vee B$
1	1	1
1	0	1
0	1	1
0	0	0

and for implication and equivalence,

$$A \rightarrow B \equiv (\neg A) \vee B \quad \text{and} \quad A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

where the symbol $\equiv$ represents that the statements are sematically equivalent. We say that "A implies B" or "if A then B" if $A \rightarrow B$, and we say "A if and only if B" or "A is logically equivalent to B" if $A \leftrightarrow B$. Also, if $F \equiv A \rightarrow B$ then we say the **converse** of F is $B \rightarrow A$. Pay attention to why $A \rightarrow B$ is true if A is false, no matter what value B takes. This is called a **vacuous truth**: when the premise A is not met, it is but natural to set the value of $A \rightarrow B$ to true.

Example 2.1. To put you into perspective, consider the following three first order statements with two variables A and B

1. $F_1 \equiv ((A \vee B) \wedge A) \rightarrow \neg B$
2. $F_2 \equiv ((A \rightarrow B) \rightarrow A) \rightarrow A$
3. $F_3 \equiv (\neg A \rightarrow \neg B) \leftrightarrow \neg(A \rightarrow \neg B)$

Now, we can perform valuation to the three statements with both A and B as true. For F_1,

$$\begin{aligned}
&((A \vee B) \wedge A) \to \neg B \\
\Rightarrow\, &((1 \vee 1) \wedge 1) \to \neg 1 \\
\Rightarrow\, &(1 \wedge 1) \to 0 \\
\Rightarrow\, &1 \to 0 \\
\Rightarrow\, &\neg 1 \vee 0 \\
\Rightarrow\, &0 \vee 0 \\
\Rightarrow\, &0
\end{aligned}$$

so F_1 is false given A, B are true, in this case, we write $A, B \vDash \neg F_1$ where the symbol $\vDash$ denotes that "$\neg F_1$ is true given A and B are true". As another example, we perform valuation on F_2 with A as true and B as false,

$$\begin{aligned}
&((A \to B) \to A) \to A \\
\Rightarrow\, &((1 \to 0) \to 1) \to 1 \\
\Rightarrow\, &((\neg 1 \vee 0) \to 1) \to 1 \\
\Rightarrow\, &((0 \vee 0) \to 1) \to 1 \\
\Rightarrow\, &(0 \to 1) \to 1 \\
\Rightarrow\, &(\neg 0 \vee 1) \to 1 \\
\Rightarrow\, &(1 \vee 1) \to 1 \\
\Rightarrow\, &1 \to 1 \\
\Rightarrow\, &\neg 1 \vee 1 \\
\Rightarrow\, &0 \vee 1 \\
\Rightarrow\, &1
\end{aligned}$$

In fact, F_2 will always be evaluated to be true no matter what truth values A, B take on (verify this!). In this case, we write $\vDash F_2$ to indicate that F_2 will always be true. The statement F_2 is known as **Peirce's law**, and a statement such as F_2 that will always to evaluated to be true is called a **tautology**. If a statement F is such that $\vDash \neg F$ then we say F is a **contradiction**.

Example 2.2. Here are some examples of tautologies,

1. (Law of Identity) $\vDash A \leftrightarrow A$
2. (Law of Excluded Middle) $\vDash A \vee \neg A$
3. (Law of Contradiction) $\vDash \neg(A \wedge \neg A)$
4. (Peirce's Law) $\vDash ((A \to B) \to A) \to A$
5. (Law of Double Negativity) $\vDash \neg\neg A \leftrightarrow A$
6. (De Morgan's Law) $\vDash \neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
7. (Law of Contrapositive) $\vDash (A \to B) \leftrightarrow (\neg B \to \neg A)$
8. (Proof by Contradiction) $\vDash (\neg A \to (B \wedge \neg B)) \to A$

It should not be hard for you to see that these are tautologies. All of these examples are very useful, so it would be helpful to remember them.

As an example to the application of the tautologies mentioned above, say we want to prove that there exists irrational numbers a, b such that a^b is rational, and we have to prove it from first principle (without assuming the irrationality of $\pi, e, \ln 2$ or any other constant you're familiar of). The first number that naturally comes to mind is $\sqrt{2}$. First, we use Proof by Contradiction. Assume that $\sqrt{2} = \frac{p}{q}$ for coprime p, q then $p^2 = 2q^2$ so $p = 2a$ for some integer a and therefore $4a^2 = 2q^2$ so $2a^2 = q^2$ so $q = 2b$ for some b, but then p, q are not coprime, a contradiction. Now, we apply the Law of Excluding Middle, Consider the number $\sqrt{2}^{\sqrt{2}}$, if it is rational then we are done, if it is irrational, then take

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2}\times\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2$$

which is rational, and we are done. Of course, here we usedd the fact that a rational number can be expressed as the quotient of a pair of coprime integers, which is not trivial, but we will see very soon how to prove this with the well-ordering principle.

Exercise 2.3. Convince yourself that the following are tautologies

1. $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
2. $\neg(A \to B) \leftrightarrow A \wedge \neg B$
3. $((A \wedge B) \vee C) \leftrightarrow ((A \vee C) \wedge (B \vee C))$

Now that we have a good understanding of statement, we can define formulas and propositions, and we do that by making changes to a statements. Say, for example, that P is a statement expressed in several variables. To change P into a proposition, for each variable A, we first have to substitude it with an **atomic formula** which is a string of the form $a \in b$ where a, b are unused variables and $\in$ is the **logical predicate**, and then we need use one of the two quantifiers, **exists** $\exists$ or **for all** $\forall$, to quantify the variables a, b by writing $\forall\, a$ (or $\exists\, a$) and $\forall\, b$ (or $\exists\, b$) at the beginning of the string. Note that the order we quantify the variables does matter. After we quantify a variable, it becomes a **bound variable**, and the variables not quantified are called **free variables**. Formally, a **formula** is defined by the following four rules:

1. if P is a statement, then substituding every variable with an atomic formula of the form $a \in b$, where a, b are unused variables in P and $\in$ is the logical predicate, gives you a formula,
2. if F is a formula with free variable a then $\forall\, a\, F$ and $\exists\, a\, F$ are formulas,
3. if F is a formula with free variable a then $\neg\,\forall\, a\, F \equiv \exists\, a\, \neg F$ and $\neg\,\exists\, a\, F \equiv \forall\, a\, \neg F$ are formulas,
4. if F is a formula with free variable a, G is a formula with free variable b, and F, G has no common variable symbols (if it does then substitute it with a different one), then

$$([\mathrm{Q}_1]a\, F)[\mathrm{B}]([\mathrm{Q}_2]b\, G) \equiv [\mathrm{Q}_1]a[\mathrm{Q}_2]b\,(F[\mathrm{B}]G)$$

   is a formula, where $[\mathrm{Q}_1], [\mathrm{Q}_2]$ are quantifiers and $[\mathrm{B}]$ is one of $\wedge, \vee, \to, \leftrightarrow$.

A formula with no free variables is called a **proposition**.

Example 2.4. For example, take the statement $A \wedge \neg(B \vee C)$, then the string $\exists\, y\, \forall\, x\, (x \in y) \wedge \neg(y \in x \vee z \in y)$ is a formula. And the free variable here is z, the bound variables are x, y. As an exercise, convince yourself that $\forall\, x\, ((\forall\, y\, \neg(x \in y) \wedge (y \in x)) \leftrightarrow (\exists\, z\, (z \in x) \to \neg(x \in x) \wedge (z \in x)))$ is a proposition.

3 Zermelo-Fraenkel Axioms

Before we go into axioms, bear in mind that everything we're covered up to now, propositions, formulas, variables, logical symbols, does not have any meaning: they are no more than a string of meaningless symbols. None of them is yet to be considered true or false, because we have yet to accept a set of axioms. In mathematics, an **axiom** is a proposition taken for granted to be true. And the set of axioms we choose will be the fundamentals of all mathematics we do.

The most common choice for axioms is the **Zermelo-Fraekel**(ZF) axioms with **the axiom of choice**(C). In ZF and ZFC set theory, the primitive mathematical object is called a **class**. And using the concept of a class, we can define **sets**, which is the core subject of study in this note. The introduction of classes and sets enables us to turn a bound variable into a free variable. We say that a formula $F(a_1, \ldots, a_n)$ with free variables $a_1, \ldots, a_n$ is a **well formed formula about** $a_1, \ldots, a_n$ if all of its free variables except $a_1, \ldots, a_n$ (if any) are pre-defined classes. Formally, classes and sets is defined as follows: Let $\exists\, a\, F(a)$ be a formula where $F(a)$ is a well formed formula about a, and let C be an unused variable symbol, we say

1. the class C is defined by $F(C)$ and write $\exists\, a\, F(a) \equiv F(C)$, and
2. the class C defined by $F(C)$ is called a set if $\exists\, a\, F(a)$ is true under the ZFC axioms.

Obviously all sets are classes, but not all classes are sets. To make life easier for us, we introduce the **class builder notation**. The class $C \equiv \{x \mid P(x)\}$, where $P(x)$ is a well formed formula about x, is defined by $\forall\, x\, (x \in C \leftrightarrow P(x))$. Finally, we can now introduce the ZF axioms.

Axiom 3.1 (Axiom of Empty Set)**.** The empty set denoted with the symbol $\emptyset$ is a set.

$$\exists\, a\, \forall x\, \neg(x \in a)$$

In other words, the symbol $\emptyset$ is a set defined by $\forall x\, \neg(x \in \emptyset)$

Axiom 3.2 (Axiom of Extensionality)**.** Two sets are **equal** if they have the same elements

$$\forall x\, \forall y\, ((x = y) \leftrightarrow (\forall\, z\, ((z \in x) \leftrightarrow (z \in y)))$$

Note that here we introduced a new binary connective $=$ so we can further define the notation

$$\{a_1, \ldots, a_n\} \equiv \{x \mid x = a_1 \vee x = a_2 \vee \cdots \vee x = a_n\}$$

where $a_1, \ldots, a_n$ are pre-defined classes.

Axiom 3.3 (Axiom of Pairing)**.** If x, y are sets then so is $\{x, y\}$

$$\forall\, x\, \forall\, y\, \exists\, a\, \forall\, z\, ((z \in a) \leftrightarrow (z = x \vee z = y))$$

This axiom allows us to create ordered pairs. Define the notation $(a, b) \equiv \{a, \{a, b\}\}$ which we call the **ordered pair** of a and b. There are many definitions for ordered pairs, and the definition we use is Kuratowski's definition, which guarentees that under ZFC we have

$$\forall\, a\, \forall\, b\, \forall\, x\, \forall\, y\, (((a, b) = (x, y)) \leftrightarrow ((a = x) \wedge (b = y)))$$

Axiom 3.4 (Axiom of Union)**.** If x is a set then the union set $\cup\, x$ is also a set

$$\forall\, x\, \exists\, a\, \forall\, y\, ((y \in a) \leftrightarrow (\exists\, z\, ((z \in x) \wedge (y \in z))))$$

Axiom 3.5 (Axiom of Power Set)**.** If x is a set then the power set $\mathcal{P}(x)$ is a set

$$\forall\, x\, \exists\, a\, \forall\, y\, ((y \in a) \leftrightarrow (\forall\, z\, ((z \in y) \rightarrow (z \in x))))$$

Axiom 3.6 (Axiom Schema of Separation)**.**

4 Axiom of Choice

5 Relations and Order

6 Functions and Cardinality

A **function** or a **mapping** is a

7 Construction of Real Numbers

8 Sigma-Algebra and Measures

9 Category Theory