Representations of Finite Groups

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1 Representations

Say, for example, that we have a finite group $G = \{g_1, \dots, g_n\}$. For $g \in G$, define $\sigma_g : G \to \mathrm{S}_n$ such that $gg_i = g_{\sigma_g(i)}$ for all $1 \le i \le n$. By Cayley, we know that $\varphi : G \to \mathrm{S}_n$ by $g \mapsto \sigma_g$ is an embedding. Now, let V be an n-dimensional vector spaces over $\mathbf{C}$ with basis $\{b_1, \dots, b_n\}$. Define $\psi : \mathrm{S}_n \to \mathrm{GL}(V)$ by $\sigma \mapsto T_\sigma$ where $T_\sigma(b_i) = b_{\sigma(i)}$, which we can also verify is an embedding. Therefore the composition $\psi \circ \varphi$ is an embedding of G into $\mathrm{GL}(V)$. We call the homomorphisms such as $\psi \circ \varphi$ a representation of G, and in this particular case, a faithful representation.

Definition 1.1 (Representation)**.** A **represetation** of a group G is a tuple (ρ, V) where $\rho : G \to \mathrm{GL}(V)$ is a homomorphism, and V is a vector space (usually over $\mathbf{C}$). Sometimes, by abuse of jargon, we also call ρ and V representations. We call $\dim V$ the **degree** of this representation. Further, if ρ is an embedding, then (ρ, V) is called a **faithful** representation.

For example, given a group G, define $\rho : G \to \mathrm{GL}(V)$ by $g \mapsto I$, and this is what we call the **trivial representation**. For another example, let $X = \{v_g \mid g \in G\}$ be a set of symbols and $V = \mathrm{Free}(X)$. Define $\rho : G \to \mathrm{GL}(V)$ by $\rho(g)(v_h) = v_{gh}$ for all $g, h \in G$. This is what we call a **regular representation**. Also, generalizing regular representations, let G act on the set X, let $V = \mathrm{Free}(X)$ where $X = \{v_x \mid x \in X\}$, and let $\rho : G \to \mathrm{GL}(V)$ by $g \mapsto T_g$ where $T_g(v_x) = v_{g \cdot x}$ for all $x \in X$, then ρ is what we would call a **permutation representation** associated with X.

Example 1.2. Here are some more examples of representations

1. Consider $G = \mathbf{Z}/n\mathbf{Z}$, then $\rho : G \to \mathbf{C}^\times$ defined by $1 \mapsto e^{2\pi i/n}$ is a representation.

2. Consider the group $\mathrm{D}_4 = \{1, r, r^2, r^3, s, rs, r^2 s, r^3 s\}$ of symmetries of a square and imagine the plane $\mathbf{R}^2$ where the square lives in. Each element of D_4 naturally corresponds to an isometry of the plane, that is, we define $\rho : \mathrm{D}_4 \to \mathrm{GL}(\mathbf{R}^2)$ by
$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad s \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
which we can verify is indeed a representation.

3. Let $G = \mathrm{SU}(2) = \{T \in \mathrm{GL}(\mathbf{C}^2) \mid T^* T = I, \det T = 1\}$ and $V = \mathbf{C}[x, y]/\langle x^{n+1}, y^{n+1} \rangle$. Define representation $\rho : G \to \mathrm{GL}(V)$ by $\rho(T)(P) = P \circ T^{-1}$. This is a classic example in the theory of Lie groups, and a very useful representation in QFT (Quantum Field Theory).

Going further, let $\rho : G \to \mathrm{GL}(V)$ be a representation of the group G and suppose that W is a subspace of V. Assume that W is $\rho(g)$-invariant for all $g \in G$, then define $\tau : G \to \mathrm{GL}(W)$ by $\tau(g)(v) = \rho(g)(v)$ for all $v \in W$ and $g \in G$. Obviously $\tau(g) \in \mathrm{GL}(W)$ for all $g \in G$, and $\tau(gh) = \tau(g)\tau(h)$ for all $g, h \in G$. Hence τ is also a representation.

Definition 1.3. Let (ρ, V) be a representation of the group G. If the subspace W of V is $\rho(g)$-invariant for all $g \in G$, then we say W is **G-stable** with respect to ρ. Supppse that W is G-stable and let the mapping $\tau : G \to \mathrm{GL}(W)$ be given by $\tau(g)(v) = \rho(g)(v)$ for all $v \in W$ and $g \in G$, then (τ, W) is called a **subrepresentation** of (ρ, V) and we write $\tau = \rho^W$. The subrepresentation (τ, W) is said to be **proper** if W is strictly smaller than V. Further, if a nonzero representation (ρ, V) has no nonzero proper subrepresentations, then it is said to be **irreducible**, and **reducible** elsewise.

Example 1.4. Here are some examples of subrepresentations,

1. Let G be a finite group and define representation $\rho : G \to \mathrm{GL}(V)$ where $V = \mathrm{Free}(X)$ and $X = \{v_g \mid g \in G\}$. Let $W = \mathrm{Span}\{\sum_{g\in G} v_g\}$, then W is G-stable. Let $\tau = \rho^W$ then (τ, W) is a subrepresentation.

2. Let $\rho : \mathrm{S}_n \to \mathrm{GL}(V)$ be the regular representation of S_n and take $W = \mathrm{Span}\{\sum_{\sigma\in \mathrm{S}_n} \mathrm{sgn}(\sigma)v_\sigma\}$, then W is G-stable. Let $\tau = \rho^W$ then (τ, W) is a subrepresentation.

Example 1.5. Here are some examples of irreducible representations

1. Every representation of degree 1, which is quite obvious.

2. Every representation $\rho : G \to \mathrm{GL}(V)$ of degree 2 such that there does not exists a common eigenvector to all $\rho(g)$ for $g \in G$. This is also rather straightforward.

3. Consider $\mathrm{S}_3 = \langle(1\ 2), (1\ 2\ 3)\rangle$ and define representation $\rho : \mathrm{S}_3 \to \mathrm{GL}(\mathbf{C}^2)$ by

$$(1\ 2) \mapsto \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \qquad (1\ 2\ 3) \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

then we can verify that ρ is irreducible using (2).

Definition 1.6. Let $\rho : G \to \mathrm{GL}(V)$ be a representations of G such that $V = W_1 \oplus \cdots \oplus W_n$ where $W_1, \ldots, W_n$ are G-stable, and let $\tau_k = \rho^{W_k}$ for all $1 \le k \le n$, then define their **internal direct sum** $\tau_1 \oplus \cdots \oplus \tau_n : G \to \mathrm{GL}(W_1 \oplus \cdots \oplus W_n)$ given by

$$(\tau_1 \oplus \cdots \oplus \tau_n)(g)(v) = \sum_{k=1}^{n} \tau_k(g)(w_k)$$

for $g \in G$ and $v = w_1 + \cdots + w_n$ where $w_1 \in W_1, \ldots, w_n \in W_n$. A nontrivial representation is said to be **decomposable** if it is the direct sum of nontrivial representations. Moreover, a representation is said to be **completely reducible** if it can be written as the direct sum of a finite number of irreducible subrepresentations.

In terms of matrices, you can think of the decomposition of representations as decomposing the matrix of $\rho(g)$ into a block-diagonal matrix where each block is represented by $\tau_1(g), \tau_2(g), \ldots, \tau_n(g)$. Obviously, decomposable representations are reducible, and contrapositively we have irreducible representations are indecomposable. But is the converse true? Does reducible imply decomposable? The answer is no. As a counter example. Define a representation $(\rho, \mathbf{C}^2)$ of $\mathbf{Z}$ by

$$n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

Since $(1\ 0)^{\mathrm{T}}$ is an eigenvector of $\rho(n)$ for all n we have a $\mathbf{Z}$-stable subspace $\mathrm{Span}\{(1\ 0)^{\mathrm{T}}\}$. Therefore the representation is reducible. However since the minimal polynomial of $\rho(1)$ is $(1 - x)^2$, $\rho(1)$ is not diagonalizable. Since $\mathbf{C}^2$ has dimension 2, if $(\rho, \mathbf{C}^2)$ is decomposable then it can be written as the direct sum of two representations of degree 1. So $\rho(n)$ should be similar to a diagonal matrix for all n, contradiction. However, for a certain type of representations, reducibility does imply decomposability.

Definition 1.7. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G on the inner product space V, then ρ is said to be **unitary** if for all $g \in G$, the operator $\rho(g)$ is unitary, i.e.

$$\langle \rho(g)(v), \rho(g)(w) \rangle = \langle v, w \rangle$$

for all $g \in G$ and $v, w \in V$.

Proposition 1.8. Unitary representations are reducible iff decomposable.

Proof. Let G be group with unitary representation $\rho : G \to \mathrm{GL}(V)$ that has subrepresentation $\tau : G \to \mathrm{GL}(W)$. Then for $v \in W^{\perp}$, $w \in W$, and $g \in G$,

$$\langle \rho(g)(v), w \rangle = \langle v, \rho(g)^*(w) \rangle = \langle v, \rho(g)^{-1}(w) \rangle = \langle v, \rho(g^{-1})(w) \rangle = 0$$

therefore $\rho(g)(v) \in W^{\perp}$, so $W^{\perp}$ is G-stable. The rest is trivial. □

Theorem 1.9 (Weyl)**.** If G is a finite group with representation $\rho : G \to \mathrm{GL}(V)$, then there exists an inner product on V such that ρ is unitary.

Proof. For $x, y \in V$, define inner product $\langle x, y \rangle_\rho = \sum_{g \in G} \langle \rho(g)(x), \rho(g)(y) \rangle$ where $\langle \cdot, \cdot \rangle$ is an inner product. This is well-defined since G is finite. Since for $h \in G$

$$\begin{aligned}
\langle \rho(h)(x), \rho(h)(y) \rangle_\rho &= \sum_{g \in G} \langle \rho(g)(\rho(h)(x)), \rho(g)(\rho(h)(y)) \rangle \\
&= \sum_{g \in G} \langle \rho(gh)(x), \rho(gh)(y) \rangle \\
&= \sum_{g \in G} \langle \rho(g)(x), \rho(g)(y) \rangle \\
&= \langle x, y \rangle_\rho
\end{aligned}$$

we have $\rho(h)$ is unitary for all $h \in G$. □

Corollary 1.10. Representations of finite groups are (completely) reducible iff decomposable.

Proof. This directly follows from Theorem 1.9 and Proposition 1.8 and obviously a this representation is reducible iff completely reducible as we can break it down until it is irreducible. □

Now that we have defined the direct sum of representations, we can define the direct product.

Definition 1.11. Let (ρ, V), (τ, W) be representations of group G, then the **direct product** of the two representations is $\rho \otimes \tau : G \to \mathrm{GL}(V \otimes W)$ given by

$$(\rho \otimes \tau)(g) = (\rho(g)) \otimes (\tau(g))$$

where $V \otimes W$ is the tensor product of V and W.

The universal property guarentees that this definition is well defined.

Definition 1.12. Let V, W be vector spaces and let (ρ, V) and (τ, W) be representations of G, we say a linear map $T : V \to W$ **intertwines** (ρ, V) and (τ, W) if $T\rho(g) = \tau(g)T$ for all $g \in G$, or in other words, the diagram

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\tau(g)"'] & W \end{tikzcd}$$

commutes for all $g \in G$. If, further, that T is bijective, we say that T is an **isomorphism** and the two representations are **isomorphic**, and write $(\rho, V) \cong (\tau, W)$ or $\rho \cong \tau$.

Example 1.13. As an examples of representation isomorphisms. Let representations $\rho, \tau : \mathbf{Z}/n\mathbf{Z} \to \mathcal{M}_2(\mathbf{C})^\times$ be defined repectively by

$$\rho(1) = \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix} \qquad \tau(1) = \begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix}$$

then they are isomorphic, you can verify this by considering the matrix $\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$

Theorem 1.14 (Schur)**.** Let (ρ, V) and (τ, W) be irreducible representations of G where V, W are vector spaces over $\mathbf{C}$, and let $T : V \to W$ intertwine (ρ, V) and (τ, W). Then either $T = 0$ or T is an isomorphism. Moreover, if $(\rho, V) = (\tau, W)$ then $T = \lambda I$ where λ is a scalar.

Proof. Suppose $T \neq 0$. Let $v \in \ker T$, then for all $g \in G$ we have $T(\rho(g)(v)) = \tau(g)(T(v)) = \tau(g)(0) = 0$. Therefore $\rho(g)(v) \in \ker T$, so $\ker T$ is G-stable with respect to ρ. Since (ρ, V) is irreducible and $T \neq 0$ we have $\ker T = \{0\}$, so T is injective. Let $v \in \operatorname{im} T$ and say $v = T(x)$ for some $x \in V$. For $g \in G$, we have $\tau(g)(v) = \tau(g)(T(x)) = T(\rho(g)(x)) \in \operatorname{im} T$, therefore $\operatorname{im} T$ is G-stable with respect to τ. Since $T \neq 0$ and (τ, W) is irreducible we have $\operatorname{im} T = W$, therefore T is surjective. Now suppose $(\rho, V) = (\tau, W)$ and let λ be an eigenvalue of T, the existance of this eigenvalue is guarenteed because $\mathbf{C}$ is algebraically closed. Consider $Q = T - \lambda I$, we obviously have $\rho(g)Q = Q\rho(g)$ for all g. Since $\ker Q \neq \{0\}$, we have $Q = 0$, hence $T = \lambda I$. $\square$

Corollary 1.15. Irreducible representations of abelian groups has degree 1.

Proof. Let (ρ, V) be an irreducible representation of abelian group G. Let $T = \rho(h)$ for some fixed $h \in G$, then $T\rho(g) = \rho(h)\rho(g) = \rho(hg) = \rho(gh) = \rho(g)\rho(h) = \rho(g)T$. Therefore $T = \lambda I$ for some λ. Let $v \in V$ be nonzero then $\rho(h)(cv) = \lambda cv \in \operatorname{Span}\{v\}$. Since h is arbitrary we have $\operatorname{Span}\{v\}$ a G-stable subspace, and by irreducibility $\operatorname{Span}\{v\} = V$. $\square$

Proposition 1.16. Let $(\rho, V), (\tau, W)$ be two representations of G and $T : V \to W$ a nonzero linear transformation, we define

$$T^\star = \frac{1}{|G|} \sum_{g \in G} \tau(g^{-1}) T \rho(g)$$

then $T^\star$ intertwines (ρ, V) and (τ, W). Moreover, for any $T : V \to W$ that intertwines (ρ, V) and (τ, W), we have $T = T^\star$

Proof. First of all, $T^\star$ is linear as T is linear. Next, for $h \in G$, we use the substitution $u = gh^{-1}$

$$\begin{aligned}
\tau(h)T^\star &= \tau(h)\frac{1}{|G|}\sum_{g\in G}\tau(g^{-1})T\rho(g) \\
&= \frac{1}{|G|}\sum_{g\in G}\tau(h)\tau(g^{-1})T\rho(g) \\
&= \frac{1}{|G|}\sum_{g\in G}\tau(hg^{-1})T\rho(g) \\
&= \frac{1}{|G|}\sum_{u\in G}\tau(u^{-1})T\rho(uh) \\
&= \frac{1}{|G|}\sum_{u\in G}\tau(u^{-1})T\rho(u)\rho(h) \\
&= \rho(h)\frac{1}{|G|}\sum_{u\in G}\tau(u^{-1})T\rho(u) \\
&= \rho(h)T^\star
\end{aligned}$$

Moreover, if T intertwines (ρ, V) and (τ, W), then

$$T^\star = \frac{1}{|G|}\sum_{g\in G}\tau(g^{-1})T\rho(g) = \frac{1}{|G|}\sum_{g\in G}T\rho(g^{-1})\rho(g) = \frac{1}{|G|}\sum_{g\in G}T\rho(g^{-1}g) = \frac{1}{|G|}\sum_{g\in G}T = T$$

□

Proposition 1.17. Let $(\rho, V), (\tau, W)$ be two irreducible representations of G and $T : V \to W$ a linear transformation, then if $(\rho, V) \cong (\tau, W)$ then $T^\star$ is an isomorphism of $(\rho, V), (\tau, W)$, and if $(\rho, V) \ncong (\tau, W)$ then $T = 0$. Moreover, if $(\rho, V) = (\tau, W)$, we have $T^\star = \frac{\operatorname{tr} T}{\dim V} I$.

Proof. The first part is directly by Schur and Proposition 1.16. If $(\rho, V) = (\tau, W)$,

$$\begin{aligned}
\operatorname{tr} T^\star &= \operatorname{tr}\left(\frac{1}{|G|}\sum_{g\in G}\rho(g^{-1})T\rho(g)\right) \\
&= \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}(\rho(g^{-1})T\rho(g)) \\
&= \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}(\rho(g)\rho(g^{-1})T) \\
&= \frac{1}{|G|}\sum_{g\in G}\operatorname{tr} T \\
&= \operatorname{tr} T
\end{aligned}$$

And since $T^\star = \lambda I$ for some λ we have $\operatorname{tr} T^\star = \lambda \operatorname{tr} I = \lambda \dim V$, therefore we have $\lambda = \frac{\operatorname{tr} T}{\dim V}$. □

2 Characters and Orthogonality Relations

Definition 2.1. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of finite group G where V is a vector space over $\mathbf{C}$, then the **character** of ρ is a function $\chi_\rho : G \to F$ by $g \to \mathrm{tr}(\rho(g))$. We say that χ_ρ is **irreducible** if ρ is irreducible.

Proposition 2.2 (Properties of Characters)**.** Let $(\rho, V), (\tau, W)$ be representations of finite group G, and let χ_ρ, χ_τ be the corresponding characters. For $g, h \in G$

1. $\chi_\rho(1) = \dim V$
2. $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$
3. $\chi_\rho(ghg^{-1}) = \chi_\rho(h)$
4. $\rho \cong \tau$ iff $\chi_\rho = \chi_\tau$
5. $\chi_{\rho \oplus \tau} = \chi_\rho + \chi_\tau$
6. $\chi_{\rho \otimes \tau} = \chi_\rho \cdot \chi_\tau$

Proof.

1. $\chi_\rho(1) = \mathrm{tr}\,\rho(1) = \mathrm{tr}\, I = \dim V$.
2. Let $n = |G|$ then $g^n = 1$ for all $g \in G$, therefore $\rho(g)^n = \rho(g^n) = \rho(1) = I$, hence if λ is an eigenvalue of $\rho(g)$ then $|\lambda| = 1$. Therefore
$$\overline{\chi_\rho(g)} = \overline{\mathrm{tr}\,\rho(g)} = \overline{\sum_{\lambda \in \mathrm{Spec}\,\rho(g)} \lambda} = \sum_{\lambda \in \mathrm{Spec}\,\rho(g)} \overline{\lambda} = \sum_{\lambda \in \mathrm{Spec}\,\rho(g)} \lambda^{-1} = \mathrm{tr}\,\rho(g)^{-1} = \mathrm{tr}\,\rho(g^{-1}) = \chi_\rho(g^{-1})$$
3. $\chi_\rho(ghg^{-1}) = \mathrm{tr}\,\rho(ghg^{-1}) = \mathrm{tr}(\rho(g)\rho(h)\rho(g)^{-1}) = \mathrm{tr}(\rho(g)^{-1}\rho(g)\rho(h)) = \mathrm{tr}(\rho(h)) = \chi_\rho(h)$
4. If there exists a isomorphism T, then
$$\chi_\rho(g) = \mathrm{tr}\,\rho(g) = \mathrm{tr}(T^{-1}T\rho(g)) = \mathrm{tr}(T\rho(g)T^{-1}) = \mathrm{tr}\,\tau(g) = \chi_\tau(g)$$
We will prove the converse of 4 later in this chapter.
5. Let $\beta = (v_1, \ldots, v_n)$ be a basis for V and $\gamma = (w_1, \ldots, w_m)$ a basis for W, then
$$\alpha = ((v_1, 0), (v_2, 0), \ldots, (v_n, 0), (0, w_1), (0, w_2), \ldots, (0, w_m))$$
is a basis for $V \oplus W$. Then,
$$[(\rho \oplus \tau)(g)]_\alpha = \begin{pmatrix} [\rho(g)]_\beta & \\ & [\tau(g)]_\gamma \end{pmatrix}_{\text{block}}$$
Therefore $\mathrm{tr}((\rho \oplus \tau)(g)) = \mathrm{tr}(\rho(g)) + \mathrm{tr}(\tau(g))$, hence $\chi_{\rho \oplus \tau} = \chi_\rho + \chi_\tau$.

6. Let $\beta = (v_1, \ldots, v_n)$ be a basis for V and $\gamma = (w_1, \ldots, w_m)$ a basis for W, then

$$\alpha = (v_1 \otimes w_1, v_1 \otimes w_2, \ldots, v_2 \otimes w_1, v_2 \otimes w_2, \ldots, v_n \otimes w_m)$$

which is in lexicographic order, is a basis for $V \otimes W$, then let $[\rho(g)]_\beta = (a_{i,j})$ and $[\tau(g)]_\gamma = (b_{i,j})$, for $(v_i \otimes w_j) \in \alpha$, we have

$$\begin{aligned}(\rho \otimes \tau)(g)(v_i \otimes w_j) &= \rho(g)(v_i) \otimes \tau(g)(w_j) \\ &= (a_{1,i}v_i + \cdots + a_{n,i}v_i) \otimes (b_{1,j}w_j + \cdots + b_{m,j}w_j) \\ &= \cdots + (a_{i,i}b_{j,j})v_i \otimes w_j + \cdots\end{aligned}$$

Therefore,

$$\operatorname{tr}[(\rho \otimes \tau)(g)]_\alpha = \sum_{1 \le i \le n} \sum_{1 \le j \le m} a_{i,i}b_{j,j} = \left(\sum_{1 \le i \le n} a_{i,i}\right)\left(\sum_{1 \le j \le m} b_{j,j}\right) = \operatorname{tr}[\rho(g)]_\beta \operatorname{tr}[\tau(g)]_\gamma$$

and we conclude that $\chi_{\rho \otimes \tau} = \chi_\rho \cdot \chi_\tau$

□

Definition 2.3. Define the inner product space $\mathbf{C}^G$ over $\mathbf{C}$ to be the set of functions from G to C where G is a finite group, equipped with the inner product $\langle \cdot, \cdot \rangle$ where

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$$

where $f_1, f_2 \in \mathbf{C}^G$.

Lemma 2.4. Let $(\rho, V), (\tau, W)$ be two irreducible representations of G and $T : V \to W$ a linear transformation. Let $[\rho(g)]_\beta = (\rho_{i,j}(g))_{i,j}$ and $[\tau(g)]_\gamma = (\tau_{m,n}(g))_{m,n}$ where β is a basis for V and γ is a basis for W. Then

Proof. Let $[T]_\beta^\gamma = (t_{m,i})_{m,i}$ and $[T^\star]_\beta^\gamma = (t^\star_{m,i})_{m,i}$, then by matrix multiplication, □

Theorem 2.5 (Orthogonality Relations I)**.** Let G be a finite group and $(\rho, V), (\tau, W)$ two irreducible representations of G. Then

$$\langle \chi_\rho, \chi_\tau \rangle = \begin{cases} 1 & \rho \cong \tau \\ 0 & \rho \ncong \tau \end{cases}$$

Proposition 2.6. Let (ρ, V) be a representation of finite group G and let

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

an irreducible decomposition. Let (τ, W) be an irreducible representation of G then the multiplicity of (τ, W) in (ρ, V) is given by $\langle \chi_\rho, \chi_\tau \rangle$.

Corollary 2.7. Let (ρ, V) be a representation of finite group G then $\langle \chi_\rho, \chi_\rho \rangle \in \mathbf{N}$ and $\langle \chi_\rho, \chi_\rho \rangle = 1$ iff (ρ, V) is irreducible.

The above proposition tells us that the irreducible decomposition of a representation is unique up to isomorphism of irreducible components and reordering. Moreover, if two representations have the same character then they have the same irreducible decomposition hence they are isomorphic. This completes the proof of Part 4 of Proposition 2.2.

Proposition 2.8. Let (ρ, V) be the regular representation of finite group G and $(\tau_1, W_1), \dots, (\tau_n, W_n)$ the complete set of irreducible representations of G, then the irreducible decomposition of V is

$$V = W_1^{\oplus \dim W_1} \oplus \cdots \oplus W_n^{\oplus \dim W_n}$$

Corollary 2.9. Let $\chi_1, \dots, \chi_k$ be all distinct irreducible characters of G with degree $n_1, \dots, n_k$,

$$|G| = n_1^2 + \cdots + n_k^2$$

Corollary 2.10. Let $\chi_1, \dots, \chi_k$ be all distinct irreducible characters of G with degree $n_1, \dots, n_k$. Let χ be the character of the regular representation, then if $g \neq 1$

$$n_1\chi_1(g) + \cdots + n_k\chi_k(g) = 0$$

Definition 2.11. A function $f \in \mathbf{C}^G$ where G is a finite group is called a **class function** if for all $g, h \in G$, we have $f(ghg^{-1}) = f(h)$.

Lemma 2.12. Let (ρ, V) be a representation of finite group G with character χ. Define

$$\rho_f = \sum_{g \in G} f(g)\rho(g)$$

then ρ_f is linear on V. Moreover, if ρ is irreducible then $\rho_f = \lambda I$ where

$$\lambda = \frac{|G|}{\dim V} \langle f, \overline{\chi} \rangle$$

Proposition 2.13. The irreducible characters form an orthonormal basis for

Corollary 2.14. The number of nonisomorphic irreducible representations of a group is the number of conjugacy classes of the group.

Proposition 2.15 (Orthogonality Relations II)**.** Let G be a finite group with $\chi_1, \dots, \chi_k$ as all the distinct irreducible characters, then for $g \in G$, let $\mathrm{Cl}(g)$ be the conjugacy class that contains g,

$$\sum_{i=1}^{k} |\chi_i(g)|^2 = \frac{|G|}{|\operatorname{Cl}(g)|}$$

and if $h \notin \mathrm{Cl}(g)$,

$$\sum_{i=1}^{k} \chi_i(g)\overline{\chi_i(h)} = 0$$