Basic Topology

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1 Topological Structure and Topological Space

Definition 1.1. Let X be a nonempty set, then a **base open neighborhood structure** on X is a mapping $\mathcal{N} : X \to \mathcal{P}(\mathcal{P}(X))$ such that

1. for all $x \in X$, $\mathcal{N}(x)$ is nonempty,
2. if $U \in \mathcal{N}(x)$, then $x \in U$,
3. if $U, V \in \mathcal{N}(x)$, then there exists $W \in \mathcal{N}(x)$ such that $W \subseteq U \cap V$.
4. if $U \in \mathcal{N}(x)$ and $y \in U$, then there exists $V \in \mathcal{N}(y)$ such that $V \subseteq U$

Further, if $U \in \mathcal{N}(x)$ then U is called a **base open neighborhood** of x. A superset of a **base open neighborhood** of x is called a **neighborhood** of x.

Example 1.2. Here are some examples of base open neighborhood structures,

1. Define $\mathcal{N} : \mathbf{R} \to \mathcal{P}(\mathcal{P}(\mathbf{R}))$ by $\mathcal{N}(x) = \{(x - \varepsilon, x + \varepsilon) \mid \varepsilon > 0\}$, then we can verify that $\mathcal{N}$ is a base open neighborhood structure on $\mathbf{R}$. We will see later that the real line whose continuity is defined using this base open neighborhood structure is called a **Euclidean line**.
2. Define $\mathcal{N} : \mathbf{R} \to \mathcal{P}(\mathcal{P}(\mathbf{R}))$ by $\mathcal{N}(x) = \{[x, x + \varepsilon) \mid \varepsilon > 0\}$, then we can verify that $\mathcal{N}$ is a base open neighborhood structure on $\mathbf{R}$. We will see later that the real line whose continuity is defined using this base open neighborhood structure is called a **Sorgenfrey line**.

Definition 1.3. Let $\mathcal{N}$ be a base open neighborhood structure on the nonempty set X, with neighborhood structure $\mathcal{M}$. We say that a set $U \subseteq X$ is **open** if U is a neighborhood of all of its elements. A set $U \subseteq X$ is **closed** if $X \setminus U$ is open. We use $\tau \subseteq \mathcal{P}(X)$ to denote the set of all open sets of X, which we call the **topological structure** (or simply the **topology**) generated by $\mathcal{N}$. The tuple (X, τ) is called a **topological space**.

Example 1.4. Here are some examples of topologies

1. Take the base open neighborhood structure $\mathcal{N}(x) = \{x\}$ then obviously $\tau = \mathcal{P}(X)$, this is called the **discrete topology** on X, which is the largest topology on X.
2. Let X be infinite, and define
$$\tau = \{\emptyset\} \cup \{A \subseteq X \mid |X \setminus A| < \aleph_0\}$$
then τ is called the **cofinite topology** on X.
3. Let X be uncountably infinite, and define
$$\tau = \{\emptyset\} \cup \{A \subseteq X \mid |X \setminus A| \leq \aleph_0\}$$
then τ is called the **cocountable topology**.
4. Let X be a matric space with metric d, take the base open neighborhood structure $\mathcal{N}(x) = \{B_\varepsilon(x) \mid \varepsilon > 0\}$ where $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ is the ε**-ball** centered at x, then the topology τ_d generated by $\mathcal{N}$ is called the **metric topology**.

Proposition 1.5. Let $\mathcal{N}$ be a base open neighborhood structure on the nonempty set X, then $U \subseteq X$ is open iff it is the union of base open neighborhoods.

Theorem 1.6 (Another Definition of Topology)**.** If X is nonempty and $\tau \subseteq \mathcal{P}(X)$ then τ is a topology generated by some base open neighborhood structure iff

1. $\emptyset, X \in \tau$,
2. if $U, V \in \tau$ then $U \cap V \in \tau$,
3. if $T \subseteq \tau$, then $\bigcup\limits_{U \in T} U \in \tau$

Corollary 1.7 (Properties of Open and Closed Sets)**.** Let (X, τ) be a topological space

1. $\emptyset, X$ are both open and closed,
2. Let $A, B \subseteq X$. If A, B are open then so is $A \cap B$, and if A, B are closed then so is $A \cup B$,
3. Let $A_\lambda \subseteq X$ where $\lambda \in \mathcal{I}$ for some index set $\mathcal{I}$. If A_λ is open for all λ then so is $\bigcup_{\lambda \in \mathcal{I}} A_\lambda$. If A_λ is closed for all λ then so is $\bigcap_{\lambda \in \mathcal{I}} A_\lambda$.

Definition 1.8. Let (X, τ) ba topological space and $A \subseteq X$. If A is a neighborhood of $x \in A$ then x is called an **interior point** of A. The set of all interior points of A is called the **interior**, denoted A° or $\operatorname{Int}(A)$.

Proposition 1.9. Let (X, τ) ba topological space and $A \subseteq X$, then A° is the largest open subset of A, which is the union of all open subsets of A.

Proposition 1.10 (Properties of Interior)**.** Let $A, B \subseteq X$ where (X, τ) is a topological space,

1. $(A^\circ)^\circ = A^\circ$
2. A is open iff $A^\circ = A$
3. $A \subseteq B$ implies $A^\circ \subseteq B^\circ$
4. $(A \cap B)^\circ = A^\circ \cap B^\circ$
5. Let $A_\lambda \subseteq X$ where $\lambda \in \mathcal{I}$ for some index set $\mathcal{I}$, then

$$\bigcup_{\lambda \in \mathcal{I}} A_\lambda^\circ \subseteq \left(\bigcup_{\lambda \in \mathcal{I}} A_\lambda \right)^\circ$$

Definition 1.11. Let (X, τ) ba topological space and $A \subseteq X$ then $x \in A$ is called a **accumulation point** if for every neighborhood S of x, we have $S \cap (A \setminus \{x\}) \neq \emptyset$. The set of all accumulation points of A is called the **derived set** of A, denoted A'. The **closure** of A is the set $A \cup A'$, denoted $\overline{A}$ or $\operatorname{Cl}(A)$.

Proposition 1.12. Let (X, τ) ba topological space and $A \subseteq X$, then $\overline{A}$ is the smallest closed superset of A, which is the intersection of all closed supersets of A.

Proposition 1.13 (Properties of Closure)**.** Let $A, B \subseteq X$ where (X, τ) is a topological space,

1. $\overline{\overline{A}} = \overline{A}$
2. A is closed iff $\overline{A} = A$,
3. $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$,
4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$,
5. Let $A_\lambda \subseteq X$ where $\lambda \in \mathcal{I}$ for some index set $\mathcal{I}$, then

$$\overline{\bigcup_{\lambda \in \mathcal{I}} A_\lambda} \subseteq \bigcup_{\lambda \in \mathcal{I}} \overline{A_\lambda}$$

Proof. The proof of this proposition can be derived directly by using De Morgan's law on 1.10. □

Proposition 1.14 (More Properties of Interiors and Closures)**.** Let (X, τ) be a topological space with $B \subseteq A \subseteq X$, then

1. $A \setminus B^\circ = \overline{A \setminus B}$
2. $(A \setminus B)^\circ = A^\circ \setminus \overline{B}$
3. $\overline{A} \setminus \overline{B} \subseteq \overline{A \setminus B}$

Definition 1.15. Let (X, τ) be a topological space with $A \subseteq X$, then the **boundary** of A, denoted ∂A is the set $\overline{A} \setminus A^\circ$, and an element of the boundary is called a **boundary point**.

Proposition 1.16. Let (X, τ) be a topological space with $A \subseteq X$, then $a \in \partial A$ iff for every neighborhood S of a, we have $S \cap A \neq \emptyset$ and $S \cap (X \setminus A) \neq \emptyset$.

Definition 1.17. Let (X, τ) be a topological space with $A \subseteq X$, then A is called **dense** if $\overline{A} = X$. If X has an at most countable number of dense subsets then it is said to be a **separable space**.

2 Continuity and Compactness

3 Strong and Weak Topologies

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