Vectors, Matrices and their Products

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Learning Target

- A system of linear equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Describe a system of linear equations by Matrix-Vector Products

Vectors, Matrices and their Products

Vector

Vectors

- A vector v is a set of numbers

$$v= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Column vector

$$v= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

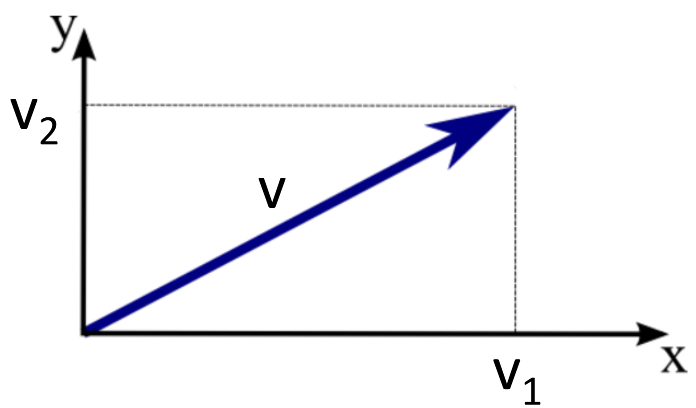
Row vector

In this course, the term **vector** refers to a **column vector** unless being explicitly mentioned otherwise.

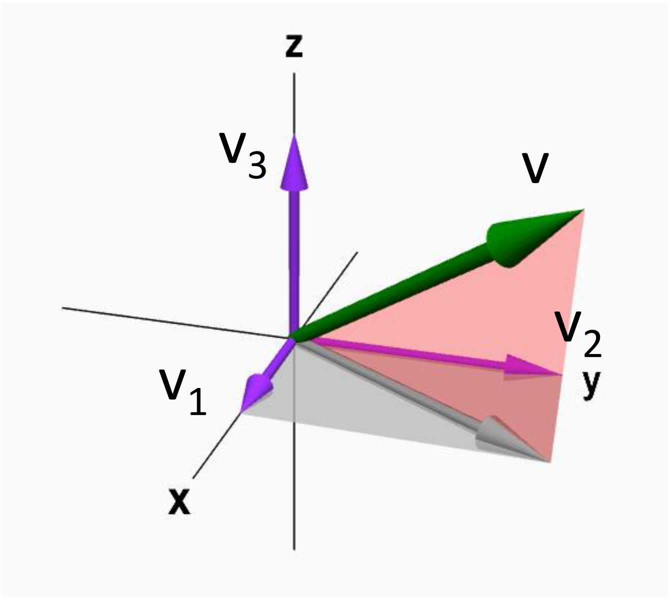
Vectors

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

- components: the entries of a vector.
  - The i-th component of vector v refers to v_i
  - $v_1=1$, $v_2=2$, $v_3=3$
- If a vector only has less than three components, you can visualize it.



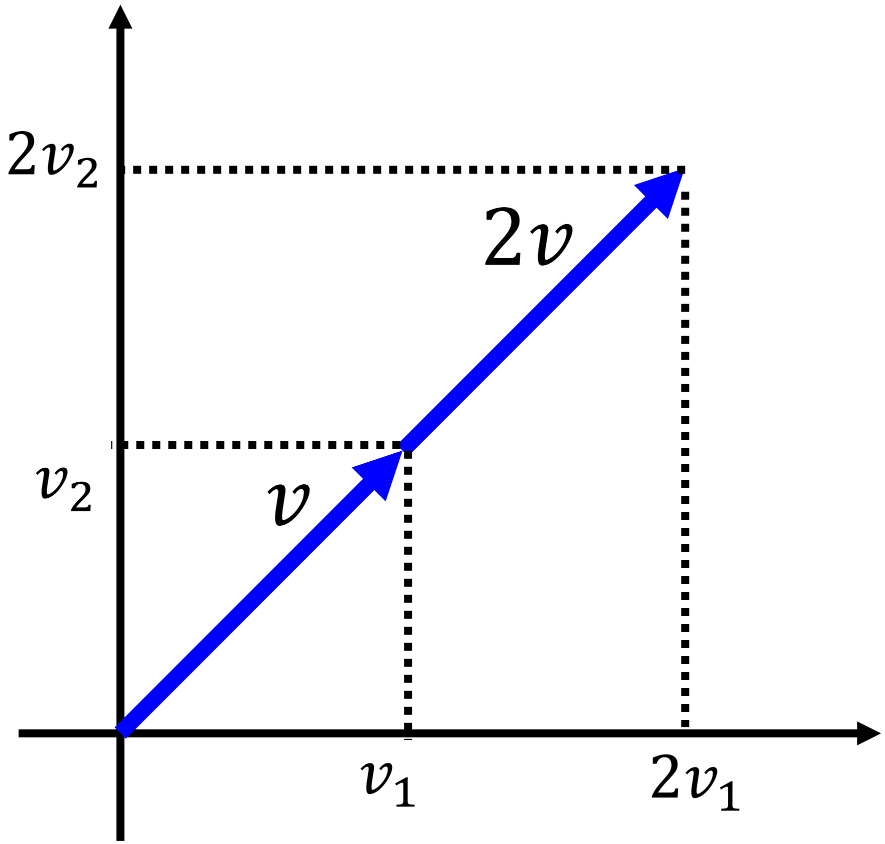
http://mathinsight.org/vectors_cartesian_coordinates_2d_3d#vector3D



Scalar Multiplication

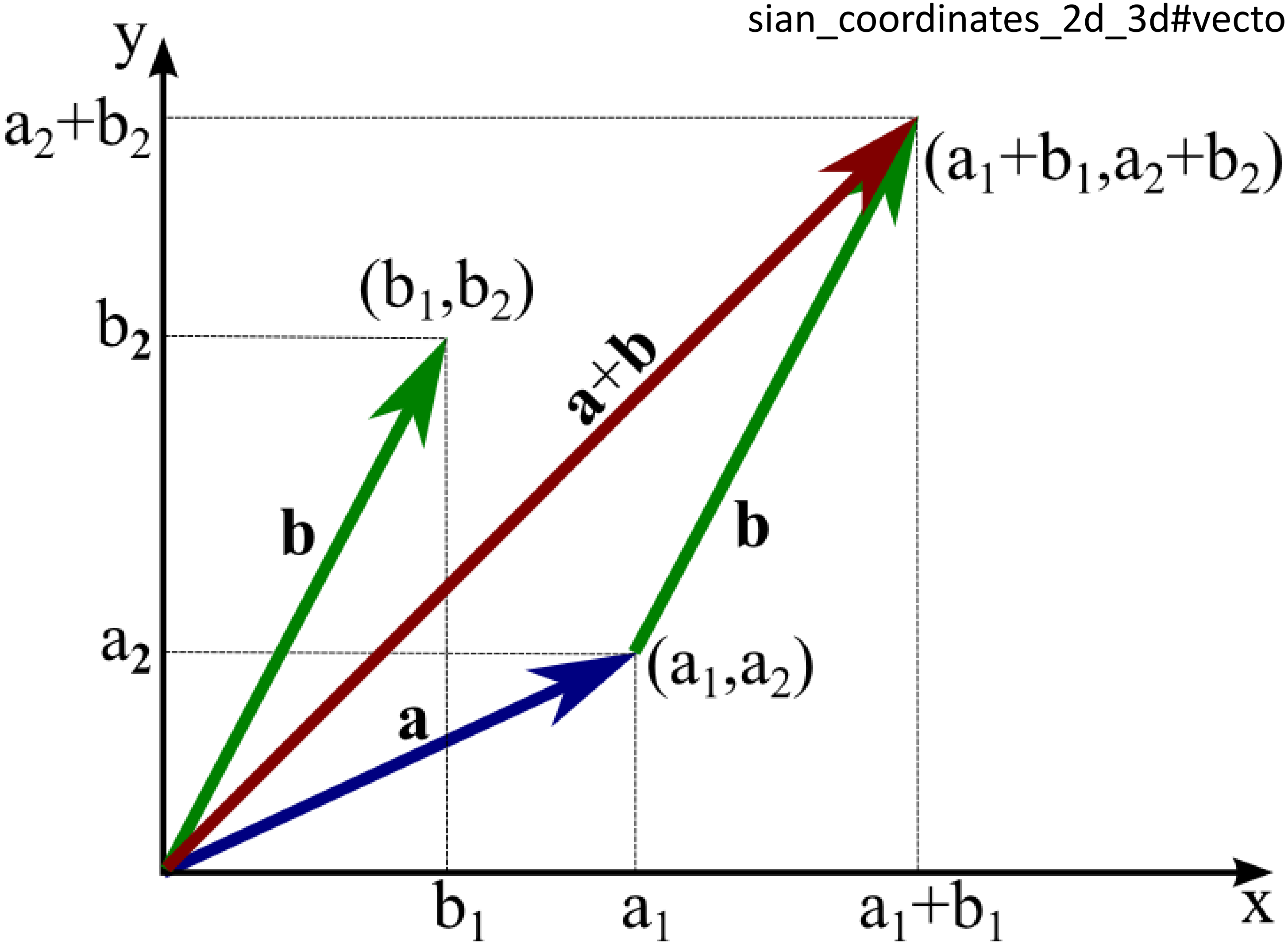
$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$cv$$



Vector Addition

http://mathinsight.org/vectors_cartesian_coordinates_2d_3d#vector3D



Special Vectors

- zero vector $\mathbf{0}$

$$\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Can be any size

$$\mathbf{0} + v = v$$

$$0v = \mathbf{0}$$

- Standard vectors

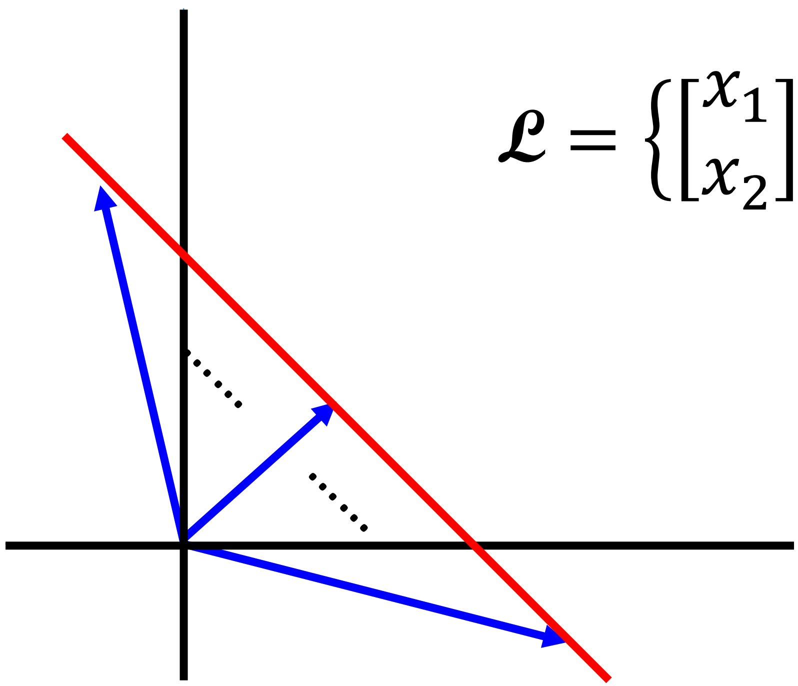
$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Vector Set

$$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}6\\8\\9\end{bmatrix},\begin{bmatrix}9\\0\\2\end{bmatrix}\right\}$$

A vector set with 4 elements

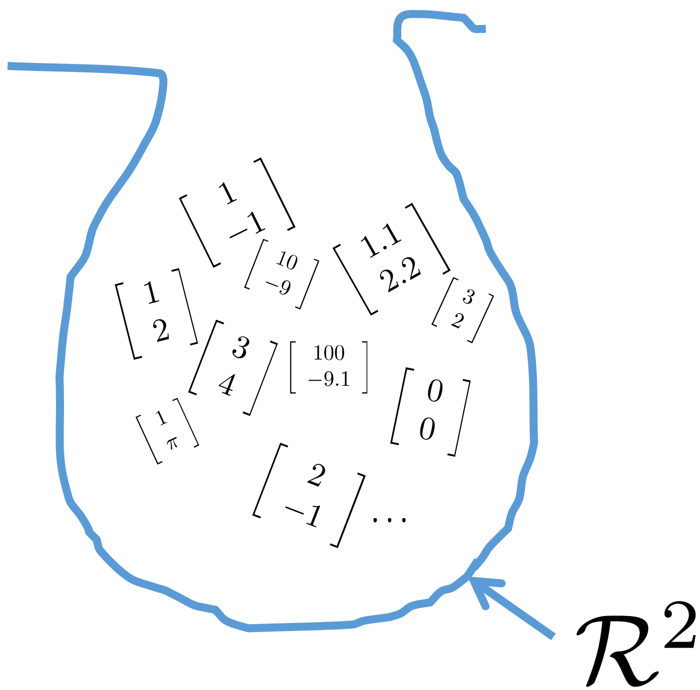
- A vector set can contain infinite elements

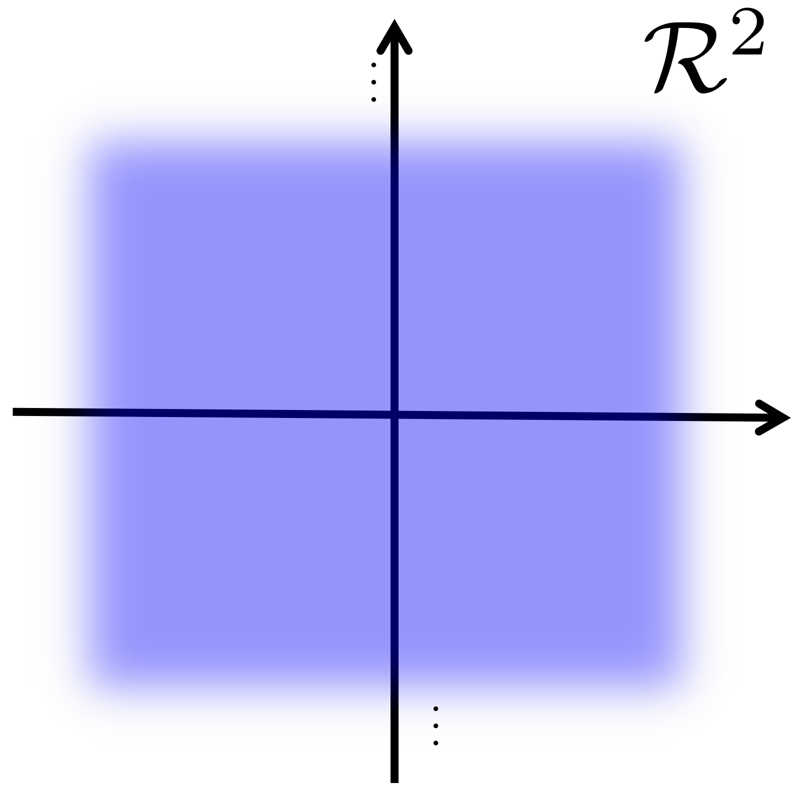


$$\mathcal{L}=\left\{\begin{bmatrix}x_1\\x_2\end{bmatrix}: x_1+x_2=1\right\}$$

Vector Set

- $\mathcal{R}^n$:We denote the set of all **vectors** with *n* entries by $\mathcal{R}^n$.





Vectors, Matrices and their Products

Matrix

Matrix

- A matrix is a set of vectors

$$a_1 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$$

$$A = [a_1 \quad a_2 \quad a_3] = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}$$

Matrix

- If the matrix has m rows and n columns, we say the size of the matrix is m by n, written m x n
  - The matrix is called square if m=n
  - We use $\mathcal{M}_{mxn}$ to denote the set that contains all matrices whose size is m x n

$$\begin{array}{cc} & \text{3 columns} \\ \text{2 rows} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \end{array} \in \mathcal{M}_{2x3}$$

2 X 3

$$\begin{array}{cc} & \text{2 columns} \\ \text{3 rows} & \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \end{array} \in \mathcal{M}_{3x2}$$

3 X 2

先 Row 再 Column

Matrix

- ***Index of component***: the scalar in the i-th row and j-th column is called (i,j)-entry of the matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

先 Row 再 Column

(1,2)-entry

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}$$

(3,1)-entry

(3,3)-entry

Matrix

- Two matrices with the same size can add or subtract.
- Matrix can multiply by a scalar

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 9 \\ 8 & 0 \\ 9 & 2 \end{bmatrix} \qquad 9B$$

$$A + B \qquad\qquad A - B$$

Zero Matrix

- zero matrix: matrix with all zero entries, denoted by O (any size) or $O_{m\times n}$.
  - For example, a 2-by-3 zero matrix can be denoted

$$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A + O = A$$
$$0A = O$$
$$A - A = O$$

- Identity matrix: must be square
  - 對角線是 1, 其它都是 0

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sometimes I_n is simply written as I (any size).

Properties

- A, B, C are mxn matrices, and s and t are scalars
  - $A + B = B + A$
  - $(A + B) + C = A + (B + C)$
  - $(st)A = s(tA)$
  - $s(A + B) = sA + sB$
  - $(s+t)A = sA + tA$

Transpose

- If A is an mxn matrix
- A^T (transpose of A) is an nxm matrix whose (i,j)-entry is the (j-i)-entry of A

$$A = \begin{bmatrix} 6 & \overset{(1,2)}{\boxed{9}} \\ 8 & 0 \\ 9 & \underset{(3,2)}{\boxed{2}} \end{bmatrix} \xrightarrow{\text{Transpose}} A^T = \begin{bmatrix} 6 & 8 & 9 \\ \underset{(2,1)}{\boxed{9}} & 0 & \underset{(2,3)}{\boxed{2}} \end{bmatrix}$$

以左上到右下的對角線為軸進行翻轉

Transpose

$$A = \begin{bmatrix} 5 & 5 \\ 6 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & 7 \\ 8 & 8 \end{bmatrix}$$

- A and B are mxn matrices, and s is a scalar
- $(A^T)^T = A$
- $(sA)^T = sA^T$
- $(A + B)^T = A^T + B^T$

$$2A = \begin{bmatrix} 10 & 10 \\ 12 & 12 \end{bmatrix} \quad (2A)^T = \begin{bmatrix} 10 & 12 \\ 10 & 12 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} \qquad 2A^T = \begin{bmatrix} 10 & 12 \\ 10 & 12 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 12 & 12 \\ 14 & 14 \end{bmatrix} \qquad (A + B)^T = \begin{bmatrix} 12 & 14 \\ 12 & 14 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} \quad B^T = \begin{bmatrix} 7 & 8 \\ 7 & 8 \end{bmatrix} \qquad A^T + B^T = \begin{bmatrix} 12 & 14 \\ 12 & 14 \end{bmatrix}$$

Vectors, Matrices and their Products

Matrix-Vector Products

Matrix-Vector Product

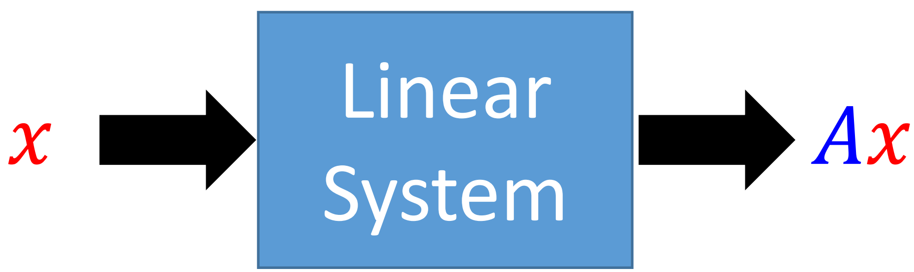
$$\underset{m \times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Matrix-Vector Product

$$
\boxed{\begin{array}{c}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{array}}
\begin{array}{c} = \\ = \\ \\ = \end{array}
\boxed{\begin{array}{c} b_1 \\ b_2 \\ \\ b_m \end{array}}
$$

$$Ax = b$$



Coefficients are A

Row Aspect

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Column Aspect

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

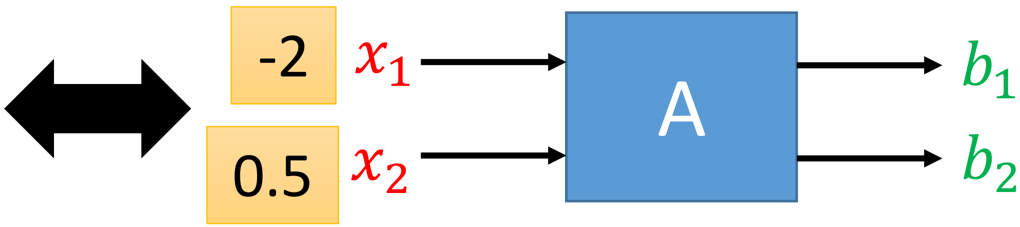
$$Ax =$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

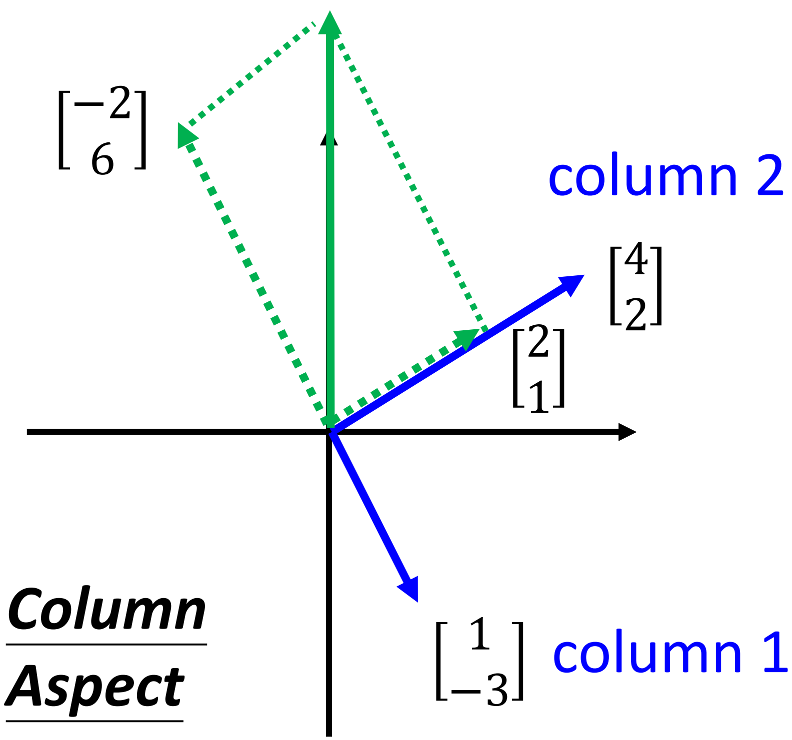
Example

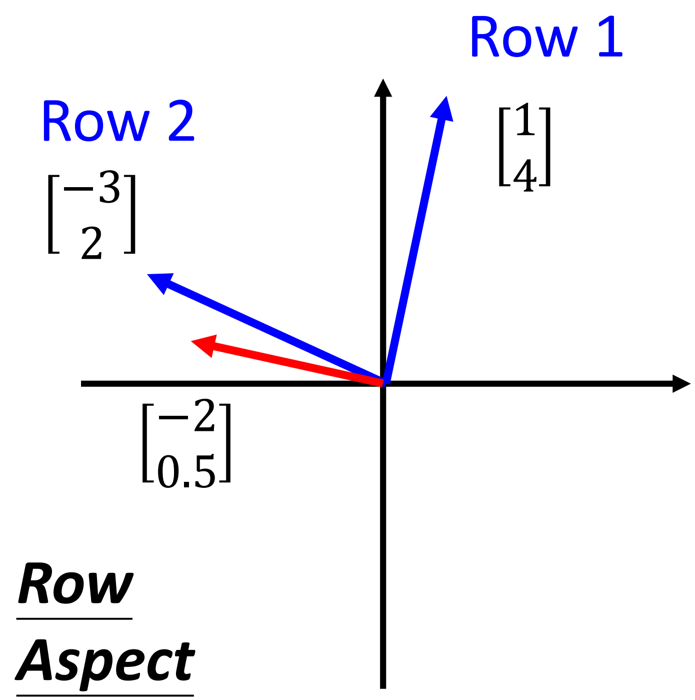
$$A = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$$

$$x_1 + 4x_2 = b_1$$
$$-3x_1 + 2x_2 = b_2$$



$= Ax$





Matrix-vector Product

- The size of matrix and vector should be matched.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 4 \end{bmatrix} \qquad A'' = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 0 & -1 \\ 1 & -3 \end{bmatrix}$$

Properties of Matrix-vector Product

- A and B are mxn matrices, u and v are vectors in R^n, and c is a scalar.
- $A(u+v) = Au + Av$
- $A(cu) = c(Au) = (cA)u$
- $(A+B)u = Au + Bu$
- $A\mathbf{0}$ is the mx1 zero vector
- $\boldsymbol{O}v$ is also the mx1 zero vector
- $I_n v = v$

Properties of Matrix-vector Product

- A and B are mxn matrices. If $Aw = Bw$ for all w in R^n. Is it true that $A = B$?

$Ae_j = a_j$ for $j = 1,2,\cdots,n$, where e_j is the j-th standard vector in R^n

$$e_1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} \qquad Ae_1 = [a_1 \quad \cdots \quad a_n]\begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} = 1 \cdot a_1 + 0 \cdot a_2 + \cdots + 0 \cdot a_n$$
$$= a_1$$

$$Ae_1 = Be_1 \Downarrow a_1 = b_1 \qquad Ae_2 = Be_2 \Downarrow a_2 = b_2 \qquad \cdots\cdots \qquad Ae_n = Be_n \Downarrow a_n = b_n \qquad \Rightarrow A = B$$

Concluding Remarks

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Row Aspect

$$= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Column Aspect