On "The Chow t-structure on the ∞-category of Motivic Spectra"

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Overview

Introduction

Motivic homotopy theory is a kind of homotopy theory of schemes, introduced by Morel and Voevodsky. Voevodsky's original motivation is tackling the problem of motivic cohomology. Or motives?
It has some successful applications. Such as solution of Bloch-Kato conjecture by Voevodsky, algebraic vector bundles by Asok and Fasel, enumerate geometry by Levin and Wickelgren, homotopy group of spheres by Wang, Xu and Isaksen.

Generalization

Theorem 1.1

Let $E \in SH(k)[1/e]$. Then there is a canonical isomorphism

$$\pi_{2w-s,w}E_{c=i} \cong \mathrm{Ext}^{s,2w}_{MU_{2*}MU}(MU_{2*}, MGL_{2*+i,*}E).$$

Construction

Let S be a Noetherian scheme of finite dimension and Sm_S the category of separated smooth of finite type schemes over S. Our stable motivic homotopy category $SH(S)$ is constructed over Sm_S.

Notations

The motivic spheres are defined by $S^{p,q} := (S^1)^{\wedge p-q} \wedge \mathbb{G}_m^{\wedge q}$.
For $E \in SH(S)$, $\underline{\pi}_{p,q}(E)$ is the Nisnevich sheaf associated with the presheaf

$$X \mapsto [\Sigma^{p,q}\Sigma^{\infty}_{\mathbb{P}^1}X_+, E],$$

E-cohomology $E^{p,q}(X) := [\Sigma^{\infty}_{\mathbb{P}^1}X_+, \Sigma^{p,q}E]$ and
E-homology $E_{p,q}(X) := [\Sigma^{p,q}\, 1_S, E \wedge \Sigma^{\infty}_{\mathbb{P}^1}X_+]$
For $X \in Sm_S$, $K(X)$ is the Thomason K-theory space, not spectrum.

Construction of Chow t-structure

Before we dive into construction, we recall some facts about algebraic cobordism MGL. MGL can be constructed as classical MU, replacing Grassmanian spaces by Grassmanian schemes. Bachmann and Hoyois give rise to an equivalent construction:

Theorem (Bachmann-Hoyois)

There is an equivalence

$$MGL_S \simeq colim_{X,\xi} Th_X(\xi)$$

where X range over Sm_S and $\xi \in K(X)$ has rank 0.

Construction of Chow t-structure

Definition (Chow t-structure)

Denote by $SH(S)_{c\geq 0}$ the subcategory generated under colimits and extensions by the Thom spectra $\mathrm{Th}_S(\xi)$ for X smooth and proper over S and $\xi \in K(X)$ arbitrary. According to 1.4.4.11 in HA, it's an non-negative part of a t-structure on $SH(S)$.

We denote the non-positive part by $SH(S)_{c\leq 0}$, the heart by $SH(S)_c^{\heartsuit}$.

Lemma

Let $\mathcal{C}$ be a presentable stable ∞-category. If $\mathcal{C}' \subset \mathcal{C}$ is a full subcategory which is presentable, closed under small colimits, and closed under extensions, then there exists a t-structure on $\mathcal{C}$ such that $\mathcal{C}' = \mathcal{C}_{\geq 0}$.

Some properties of Chow t-structure

Lemma

Given $E \in SH(S)$, $E \in SH(S)_{c\leq 0}$ if and only if $[\Sigma^i Th(\xi), E] = 0$ for all $i > 0$.

This lemma will be frequently used.

Theorem

- $SH(S)_{c\leq 0}, SH(S)_{c\geq 0}$ *and* $SH(S)_{c=0}$ *are closed under filtered colimits. Thus* $E \mapsto E_{c\leq 0}$ *preserves filtered colimits.*
- $SH(S)_{c\geq 0} \wedge SH(S)_{c\geq 0} \subset SH(S)_{c\geq 0}$.

Some properties of Chow t-structure

Definition

Denote by by $SH(S)^{pure} \subset SH(S)$ the smallest subcategory that is closed under filtered colimits and extensions and contains $\mathrm{Th}(\xi)$ for any K-theory point ξ on a smooth proper scheme X over S.

Obviously, $SH(S)^{pure} \subset SH(S)_{c\geq 0}$.

Theorem

- $SH(S)^{pure} \wedge SH(S)_{c\leq 0} \subset SH(S)_{c\leq 0}$.
- *For* $X \in SH(S)^{pure}$, $Y \in SH(S)$, *we have*

$$Y_{c\leq 0} \wedge X \simeq (Y \wedge X)_{c\leq 0}, Y_{c\geq 0} \wedge X \simeq (Y \wedge X)_{c\geq 0}.$$

- $Y_{c=0} \wedge X \simeq (Y \wedge X)_{c=0}$.

η-periodization

The algebraic Hopf map η is defined by $\eta : \mathbb{A}^2 - \{0\} \to \mathbb{P}^1, (x, y) \mapsto [x, y]$.
Note that $\mathbb{A}^2 - \{0\} \simeq S^{3,2}, \mathbb{P}^1 \simeq S^{2,1}$.
Write $SH(S)^{lisse}$ for the stable presentable subcategory generated by $SH(S)^{pure}$. Then $SH(S)_{c\geq 0} \subset SH(S)^{lisse}$ and $\cap_n SH(S)^{lisse}_{c\leq n} = 0$.

Definition

A spectra $E \in SH(S)^{lisse}$ is Chow ∞-connective is $E \in SH(S)_{c\geq n}$ for all n.

E is said to be η-periodic if $\mathrm{Hom}(\mathbb{P}^1, E) \to \mathrm{Hom}(\mathbb{A}^2 - \{0\}, E)$ is equivalent.

Theorem

If $E \in SH(S)^{lisse}$ is η-periodic, then E is Chow-∞-connective.

Cellular Chow t-structure

We denote by $SH(S)^{cell} \subset SH(S)$ the sub-category of cellular spectra, the subcategory generated by the spheres $S^{p,q}$, $p, q \in \mathbb{Z}$.

Definition

Denote by $SH(S)^{cell}_{c\geq 0}$ the subcategory generated under colimits and extensions by $S^{2n,n}$ for $n \in \mathbb{Z}$. This is the non-negative part of a t-structure on $SH(S)^{cell}$ called the cellular Chow t-structure. We denote the non-positive part by $SH(S)^{cell}_{c\leq 0}$.

By definition, $SH(S)^{cell}_{c\geq 0} \subset SH(S)_{c\geq 0} \cap SH(S)^{cell}$.

Example

Let k be a field of characteristic $\neq p$ with $k^\times/p \simeq \{1\}$, and containing a primitive p-th root of unity. Then by [MVW06], we have $\pi_{*,*}(H\mathbb{Z}/p) \cong K_*^M(k)/p[\tau] \simeq \mathbb{F}_p[\tau]$. This implies that $H\mathbb{Z}/(p,\tau) \in SH(k)^{cell,c\heartsuit}$. Indeed we have $MGL \in SH(k)^{cell}_{c\geq 0}$. By the Hopkins–Morel-Hoyois isomorphism, we have $H\mathbb{Z} \in SH(k)^{cell}_{c\geq 0}$. Therefore $H\mathbb{Z}/(p,\tau) \in SH(k)^{cell}_{c\geq 0}$. On the other hand, $H\mathbb{Z}/(p,\tau) \in SH(k)^{cell}_{c\leq 0}$ just means that $\pi_{2m+i,m}(H\mathbb{Z}/(p,\tau))$ vanishes for $m \in \mathbb{Z}$, $i > 0$, which is clear since $K_*^M(k)/p \simeq \mathbb{F}_p$.
Here τ is a generator of $H^{0,1}(k,\mathbb{Z}/p) = \mu_p(k)$.

Relation with MGL

Definition

For $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, the chow degree is $c(p, q) = p - 2q$. Given a bigraded abelian group $M_{*,*}$, we say that $M_{*,*}$ is concentrated in Chow degrees $\geq d$ (respectively $\leq d, = d$) if $M_{p,q} = 0$ for $c(p, q) < d$ (respectively $> d, \neq d$).

Example

Let $E \in SH(k)$ with $\pi_{*,*}E$ concentrated in Chow degrees ≤ 0. Consider the cofiber sequence

$$\mathbb{G}_m^{\wedge n} \wedge E \xrightarrow{\eta^n} E \to E/\eta^n,$$

then $\pi_{p,q}(E/\eta^n) \simeq \pi_{p,q}E$ for any fixed (p, q) and n sufficiently large.

Let $I \subset SH(k)_{c\geq 0}$ be the subcategory generated by compact objects, closed under colimits and extensions. I defiens a t-structure. We denote its non-negative and non-positive parts by

$$SH(k)_{I\geq 0}, SH(k)_{I\leq 0}.$$

We denote the truncation by $\tau_{I\geq n}$, $\tau_{I\leq n}$ and $\tau_{I=n}$.

Theorem

Let $E \in SH(k)_{I\geq 0}$. Then $MGL \wedge E \in SH(k)_{I\geq 0}$ and $MGL_{*,*}E$ concentrated in Chow degree ≥ 0.

The proof relies some non-trivial results. We have the following corollary

Corollary

For arbitrary $E \in SH(k)$, we have $MGL_{p,q}\tau_{I=i}E = MGL_{p,q}(E)$ if $c(p,q) = i$ and 0 otherwise.

Using cofiber sequence of t-structure.

There is a *MGL* based motivic Adams-Novikov spectral sequence:

$$\mathrm{Ext}^{*,*,*}_{MGL_{*,*}MGL}(MGL_{*,*}, MGL_{*,*}X) \Rightarrow \pi_{*,*}X^{\wedge}_{MGL}.$$

One may see **localizations and completions of stable ∞-categories** for a general account. Here is a central result:

Theorem

Let $E \in SH(k)_{l \geq 0}$. Then
(1) the canonical map $\tau_{l \leq 0}E \to \tau_{l \leq 0}(E)^{\wedge}_{MGL}$ induces an isomorphism on $\pi_{*,*}$.
(2) $\pi_{2w-s,w}\tau_{l \leq 0}E \simeq Ext^{s,2w}_{MU_*MU}(MU_*, MGL_{2*,*}E)$.

Corollary

Let $E \in SH(k)$. Then E is Chow-∞-connective if and only if $E \wedge MGL \simeq 0$.

Let C be a quasi-category. The functor category $\mathrm{Fun}(C, C)$ has a monoidal structure. C is naturally a module over $\mathrm{Fun}(C, C)$. A monad on C is an associative algebra $A \in \mathrm{Fun}(C, C)$.
Given a monad Am we consider the category A-mod(C). Denote by $ob : A\text{-mod}(C) \to C$ the forgetful functor. ob has a left adjoint $in : C \to A\text{-mod}(C)$. The composite $ob \circ in : C \to C$ identifies with $c \mapsto A(c)$.
For any quasi-category D, $\mathrm{Fun}(D, C)$ is a module over $\mathrm{Fun}(C, C)$. Given a $G \in \mathrm{Fun}(D, C)$, a A-module structure on G is equivalent to factoring G as

$$D \to A\text{-mod}(C) \xrightarrow{ob} C.$$

Assume G admits a left adjoint F. $A := G \circ F \in \mathrm{Fun}(C, C)$ has a structure of associative algebra. Then G canonically factors as

$$D \xrightarrow{G^{enh}} A\text{-mod}(C) \xrightarrow{ob} C.$$

Reconstruction theorems

Suppose we have a prestably symmetric monoidal category $\mathcal{D}$ and $A \in \mathrm{CAlg}(\mathcal{D})$. We obtain a free-forgetful adjunction

$$F : \mathcal{D} \rightleftarrows A - Mod : U$$

and the endofunctor

$$C := FU = - \otimes A : A - Mod \to A - Mod.$$

We denote by $C - CoMod$ the category of comodules under C and a factorization

$$D \rightleftarrows C - CoMod \rightleftarrows A - Mod,$$

where $C - CoMod \to A - Mod$ is the forgetful functor that we denote by H and $D \to C - CoMod$ sends X to $X \otimes A$.

Reconstruction thorems

Form the cobar resolution:

$$CB^*(A) - Mod := (A - Mod \to (A \otimes A) - Mod \to \dots).$$

It follows [HA,4.7.6.2] that $C - CoMod \simeq \lim_{\Delta} CB^*(A) - Mod$.
If $\mathcal{D}$ has a t-structure and $- \otimes A : \mathcal{D} \to \mathcal{D}$ is t-exact. Then the categories $A^{\otimes n} - Mod$ has a t-structure detected by the forgetful functor to $\mathcal{D}$. We denote by $A^{\otimes p} - Mod_{[m,n]}$ the subcategory bounded in the t-structure.
Define

$$C - CoMod_{[m,n]} := \lim_{\Delta_s} CB^*(A) - Mod_{[m,n]}$$

where Δ_s is the subcategory of Δ with the same objetcs, but morphisms are given by injective order preserving maps.
Define the t-structure on $C - CoMod$:

$$C - CoMod_{\geq 0} := C - CoMod_{[0,\infty]}, C - CoMod_{\leq 0} := C - CoMod_{[-\infty,0]}.$$

Reconstruction theorems

Recall that an object $X \in \mathcal{C}$ is compact if $\mathrm{Hom}_{\mathcal{C}}(X, -)$ preserves filtered colimits. A set $\mathcal{G}$ of $\mathcal{C}$ is called a generator if given $f_1, f_2 : X \to Y$, $f_1 \neq f_2$, there exists a $G \in \mathcal{G}$, $g : G \to X$ such that $f_1 g \neq f_2 g$.
Suppose $C - CoMod^{\heartsuit}$ is compactly generated. Consider the category $Hov(C) := \mathrm{Ind}(\mathrm{Thick}(C - CoMod^{\heartsuit\omega})$.
By [HTT,5.3.10,5.3.5.13], there is an adjunction

$$Hov(C) \rightleftarrows C - CoMod.$$

Now, we consider the category $D = 1_{c=0} - Mod$ and $A = MGL \wedge 1_{c=0} \simeq MGL_{c=0}$. We obtain

$$1_{c=0} - Mod \rightleftarrows C - CoMod \rightleftarrows MGL_{c=0} - Mod.$$

We consider the t-structure on $1_{c=0} - Mod$ induced by the Chow t-strcuture: its non-negative part is the subcategory generated under colimits and extensions by the free $1_{c=0}$-modules $1_{c=0} \wedge \mathrm{Th}(\xi)$ for X smooth and proper over k and $\xi \in K(X)$.

Reconstruction theorems

Here are some properties

- the free functor $\bar{F} : 1_{c=0} - Mod \to C - CoMod$ is t-exact and symmetric monoidal.
- $1_{c=0} - Mod_{[m,n]} \simeq C - CoMod_{[m,n]}$.
- $\bar{F}$ induces a symmetric monoidal equivalence $1_{c=0} - Mod \simeq Hov(C)$.

  We have the following description of Chow heart:

Corollary

There are canonical symmetric monoidal equivalences

$$SH(k)^{c\heartsuit} \simeq 1_{c=0} - Mod^{c\heartsuit} \simeq C^{c\heartsuit} - CoMod.$$

where $C^{c\heartsuit}$ is the comonad on $MGL_{c=0} - Mod^{c\heartsuit}$ obtained by restricting C.

Reconstruction theorems

For a smooth proper variety X and $i \in \mathbb{Z}$, denote by $X\{i\} \in MGL_{c=0} - Mod$ the object $(\Sigma^{2i,i} X_+ \wedge MGL)_{c=0} \simeq \Sigma^{2i,i}(X_+)_{c=0} \wedge MGL$. By Thom isomorphism, these are generators of $MGL_{c=0} - Mod$.

Theorem (Thom isomorphism)

Let $V \to X$ be a n dimensional vector bundle over X. Then

$$E^{i-2n,j-n}(X) \cong \tilde{E}^{i,j}(Th(V))$$

and

$$\tilde{E}_{i+2n,j+n}(Th(V)) \cong E_{i,j}(X).$$

Reconstruction theorems

An additive category A is idempotent complete if any idempotent $e : A \to A$, $e^2 = e$ arise from a splitting of A: $A = \mathrm{Im}(e) \oplus \mathrm{Ker}(e)$.

Definition

Write $PM_{MGL}(k) \subset MGL_{c=0} - Mod$ for the smallest idempotent complete additive subcategory containing the objects $X\{i\}$.

The hom set can be described as

$$\mathrm{Hom}_{PM_{MGL}(k)}(X\{i\}, Y\{j\}) \simeq MGL_{2(i+d_X-j),(i+d_X-j)}(X \times Y)$$

$$\simeq MGL^{2(d_Y+j-i),(d_Y+j-i)}(X \times Y).$$

Let E be a motivic ring spectra with unit $\eta : 1 \to E$. An orientation of E is a class c in the reduced cohomology $\tilde{E}^{2,1}(\mathbb{P}^\infty)$ such that $\ell^*(c) = \eta$. Where $\ell : \mathbb{P}^1 \to \mathbb{P}^\infty$.

Definition

More generally. If $B \in SH(k)$ is any oriented ring spectrum, the category PM_B of pure B-motives is the idempotent complete additive 1-category generated by objects $X\{i\}_B$ and morphisms given by

$$\mathrm{Hom}_{PM_B}(X\{i\}_B, Y\{j\}_B) := [\Sigma^{2i,i}\Sigma^\infty_+ X \wedge B, \Sigma^{2i,i}\Sigma^\infty_+ Y \wedge B]_{B-Mod}.$$

Using pure MGL-motive, we can gives a description of $MGL_{c=0} - Mod$.

Theorem

We have a t-exact, symmetric monoidal equivalence

$$MGL_{c=0} - Mod \simeq Fun^{\times}(PM_{MGL}(k)^{op}, SP),$$

where $Fun^{\times}$ denotes the category of product-preserving functors.

Here is a corollary

Corollary

Given $E \in MGL_{c=0} - Mod$, denote by $\pi_i^c \in Fun^{\times}(PM_{MGL}(k)^{op}, Ab)$ the presheaf $\pi_i^c(E)(X) := \pi_i Map_{MGL_{c=0}-Mod}(X, E)$ for all $X \in PM_{MGL}(k)$. Then π_0^c indces an equivalence

$$MGL_{c=0} - Mod_c^{\heartsuit} \xrightarrow{\sim} Fun^{\times}(PM_{MGL}(k)^{op}, Ab).$$

An object $F \in MGL_{c=0} - Mod_c^{\heartsuit}$ consists of the following data:

- for every smooth proper variety X a graded group $F(X)_* = \pi_0^c(F)(X\{*\})$.
- for every MGL-correspondence $\alpha \in MGL^{2*,*}(X \times Y)$ a homomorphism $\alpha^* : F(Y)_* \to F(X)_*$.

subject to the conditions that

- for composable MGL-correspondences α, β we have $\alpha^*\beta^* = (\beta\alpha)^*$.
- $id^* = id$ and $0^* = 0$
- for parallel MGL-correspondences α, β, we have $\alpha^* + \beta^* = (\alpha + \beta)^*$.

Since $MGL_{2*,*} \cong MU_{2*}$, each $F(X)_*$ is an MU_*-module.

The End