mit math lectures 22 and 23

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1 Chapter 22 Expectation I

1.1 Expectation Definition

The expected value(also known as average or mean) of a r.v. R over a Probability space S is denoted by: $E_x(R) = \sum^{w \in S} R(w)P_r(w)$
Def median: the median of a r.v. R is the $x \in Range(R)$, such that

$$P_r(R < x) \leqslant 1/2$$

$$P_r(R > x) < 1/2$$

Thm: $E_x(R) = \sum_{min}^{x \in Range(R)} xP_r(R = x)$

Note: think about what you want, max return or max probability
Cor: If the random variable has a range of naturals, then

$$E_x(R) = \sum_{i=1}^{\infty} iP_r(R == i)$$

Thm: If R the random variable has a range of naturals, then

$$E_x(R) = \sum_{i=1}^{\infty} P_r(R > i)$$

1.2 Linearity of Expectation

Thm: For any random variables R_1 and R_2 on the probability space S,

$$E_x(R_1 + R_2) = E_x(R_1) + E_x(R_2)$$

Cor: $\forall k \in n$ (n means natural number), k is random variables R_1, R_2,... on the probability space S,

$$E_x(R_1 + R_2 + ... + R_k) = E_x(R_1) + E_x(R_2) + ... + E_x(R_k)$$

Note: No Independence needed

2 Chapter 23 Expectation II

2.1 Thm 1. Another way to calculate expectation

Thm 1: Given a Probability space S, events A_1, A_2, ... $A_n \subseteq S$, then the expected number of these events to occur is $\sum_{i=1}^{n} P_r(A_i)$, where $P_r(A_i)$ is the probability A_i occurs

Note: Some time use theorem 1 is much easier than expectation original definition

2.2 Thm 2. bound of the probability of at least one event occurs

Thm 2: $P_r(T \geq 1) \leq E_x(T)$, means the probability of at least one event occurs is bounded by the expectation of events occur.

Cor: $P_r(T \geq 1) \leq \sum_{i=1}^{n} P_r(A_i)$

2.3 Thm 3. Murphy's law

Given mutual independent events, $A_1, ..., A_n$, then the prob that none of them occurs is $P_r(T = 0) \leq e^{-E_x(T)}$

when proof uses $\forall x$, $1 - x \leq e^{-x}$

Cor: If we expect 10 or more mutually independent events to occur, the prob that no event occurs is $\leq e^{-10} < 1/22000$

2.4 Thm 4. Product Rule for Expectation

For any **independent** r.v.'s R_1, R_2,

$$E_x(R_1R_2) = E_x(R_1)E_x(R_2)$$

Noe ex: 6 sizes dice $E_x(R_1R_1) = E_x(R_1^2) = \sum_{i=1}^{6} i^2 P_r(R_1 = 1) = 1/6(1 + 4 + 9 + 16 + 25 + 36) = 15(1/6)\neg(31/2)^2 = E_x^2(R_1)$

Cor: If R_1,..., R_n are mutually independent, then

$$E_x(R_1...R_n) = E_x(R_1)...E_x(R_n)$$

Cor: For any constants a,b and any rv R,

$$E_x(aR + b) = aE_x(R) + b$$

Cor?? $E_x(1/R) = 1/E_x(R)$ **No, because if** $R \in (1, -1)$**, then** $E_x(1/R) = 0$**, so false**

Cor?? Given independent rv's R and T, if $E_x(R/T) > 1$, then $E_x(R) > E_x(T)$, **No No No**

2.5 Deviation

Def: The variance of a rv R is denoted by

$$Var(R) = E_x((R - E_x(R))^2)$$

deviation from mean: $R - E_x(R)$
square deviation $(R - E_x(R))^2$
var = Expectation of square deviation