mit math lectures 22 and 23

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1 Lecture 24 Large Deviations

1.1 Markov's Thms

If R is a non-negative r.v., then $\forall x > 0$,

$$P_r(R) \geq \frac{E_x(R)}{x}$$

Cor: If R is a non-negative r.v., then $\forall c > 0$,

$$P_r(R \geq cE_x(R)) \leq \frac{1}{c}$$

Cor: If $R \leq u$ for some $u \in IR$, then $\forall x < u$,

$$P_r(R \leq x) \leq \frac{u - E_x(R)}{u - x}$$

1.2 Chebyshev's Thms(Another version of Markov's thm)

$\forall x > 0$, $\exists$ any r.v R,

$$P_r(|R - E_x(R)| \geq x) \leq \frac{Var(R)}{x^2}$$

Cor: $P_r(|R - E_x(R)| \geq c\sigma(R)) \leq \frac{Var(R)}{c^2\sigma(R)} = \frac{1}{c^2}$

1.3 Thm

For any rv R,

$$P_r(R - E_x(R) \geq c\sigma(R)) \leq \frac{1}{c^2 + 1}$$

$$P_r(R - E_x(R) \leq -c\sigma(R)) \leq \frac{1}{c^2 + 1}$$

1.4 Cherrioff Bound Thm

Let $T_1, T_2, ..., T_n$ be any mutually independent rv, such that $\forall j, 0 \leq T_j \leq 1$. Let $T = \sum_{j=1}^{n} T_i$, then for any $c > 1$. Then for any $c > 1$,

$$P_r(T \geq cE_x(T)) \leq e^{-zE_x(T)}$$

, where $z = c\log(c) + 1 - c > 0$

2 Lecture 25 Random Walks

2.1 Gambler's Ruin: Same with our Monte Carlo hw description

Start with 1 dollar and total n dollars, each bet win 1 dollar with probability p or lose a dollar with probability $1 - p$, play until win m more dollars or lose n dollars(go broke)

2.2 Martingale

Probability of up move is p, probability of down move is 1-p, they are mutually independent of past moves.
If p is not half, random walk is not biased.
If p is half, ... is unbiased.

2.2.1 Win the 2.1 bet

Def: W^* is the event hits $T = n + m$ before it hits 0
D is the number of dollars at start
$x_n = P_r(W^*|D = n)$
claim:

$$x_n = \begin{cases} 0 & \text{if} \quad n = 0 \\ 1 & \text{if} \quad n = T \\ p^{X_{n-1}+(1-p)X_{n+1}} & \text{if} \quad 0 < n < T \end{cases}$$

When proof that, with this Def: E is the event win the 1st bet, $\overline{E}$ is the event lose the 1st bet.
If $p \neq i/2$, $X_n = A(1-p/p)^n + B(1)^n$, use boundery conditions to figure out A and B.

2.3 Thm

If $p < 1/2$, then

$$P_r(winmdollarsbeforelosendollars) \leq (1-p)/p$$

2.4 Thm

If $p = 1/2$, then

$$P_r(win m dollars before lose n dollars) = n/n + m$$

Def: S is the steps that we hit the bdry, just as our hw how many times we play, $E_n = E_x(S|D = n)$
Claim:
$$x_n = \begin{cases} 0 & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n = T \\ 1 + pE_{n+1} + (1-p)E_{n-1} & \text{if} \quad 0 < n < T \end{cases}$$

2.5 Thm: Quit while you are ahead

If you start wiht n dollars and the prob is 1/2, you play until you broke, then the prob of you broke is 1.