Homework 3

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1 Exercise 3.2

1.1 a)

We want to represent the specified recursion in the form:

$$\theta_{n+1} = \theta_n + \epsilon_n Y_n$$

,

where $\{Y_n\}$ is a stochastic process depending on the random variable $\xi_n \in \{A_{lose}, B_{lose}, A_{win}, B_{win}\}$. Values of the ξ_n represent one of the four possibilities for each step n.

In situations when $\xi_n \in \{A_{lose}, B_{lose}\}$ we do not update estimate of the θ, for $\xi_n = A_{win}$ we add $\epsilon_n \cdot (1-\theta_n)$ to our estimate, and for $\xi_n = B_{win}$ we subtract $\epsilon_n \theta_n$. This could be achieved with:

$$Y_n(\xi_n) = 1_{(\xi_n = A_{win})} \cdot (1 - \theta_n) - 1_{(\xi_n = B_{win})} \cdot \theta_n$$

Given initial value θ_0, recursively define the feedback process Y_n through

$$\theta_{n+1} = \theta_n + \epsilon_n Y_n$$

with either fixed step size ϵ or decreasing step size, where we typically assume that

$$\sum_{n=1}^{\infty} \epsilon_n = +\infty$$

$$\sum_{n=1}^{\infty} \epsilon_n^2 < \infty$$

and Y_n given via the feedback function

$$Y_n = \phi(\xi(\theta_n), \theta_n)$$

We assume that all random variables, that is, $\theta 0$ and $(\xi_n(\theta) : n >= 0, \theta \in \Theta)$, are defined on a probability space. Running the stochastic approximation algorithm, we observe the underlying sequence

$$\xi_0(\theta_0), \xi_1(\theta_1), ...$$

Here in the problem,

$$\xi_1(\theta_1) = (0_{initiallose}, (1-\theta_0)_{initialAwins}, (-\theta_0)_{initialBwins})$$

$$\xi_2(\theta_2) = (0_{1thlose}, (1-\theta_1)_{1thAwins}, (-\theta_1)_{1thBwins})$$

$$\xi_3(\theta_3) = (0_{2thlose}, (1-\theta_2)_{2thAwins}, (-\theta_2)_{2thBwins})$$

$$\xi_2(\theta_4) = (0_{3thlose}, (1-\theta_3)_{3thAwins}, (-\theta_3)_{3thBwins})$$

and so on, ...

1.2 b)

So we have the stochastic approximation of the form:

$$\theta_{n+1} = \theta_n + \epsilon_n \cdot (1_{(\xi_n = A_{win})} \cdot (1-\theta_n) - 1_{(\xi_n = B_{win})} \cdot \theta_n)$$

Assuming that there is no bias term the target field function we have target vector field (scalar in our case):

$$G(\theta_n) = 1_{(\xi_n = A_{win})} \cdot (1-\theta_n) - 1_{(\xi_n = B_{win})} \cdot \theta_n$$

because

$$\theta_{n+1} = \theta_n + \epsilon_n Y_n$$

set $Y_n(\xi_n(\theta_n))$ is the independent sequences of unbiased estimators of the target vector field, where

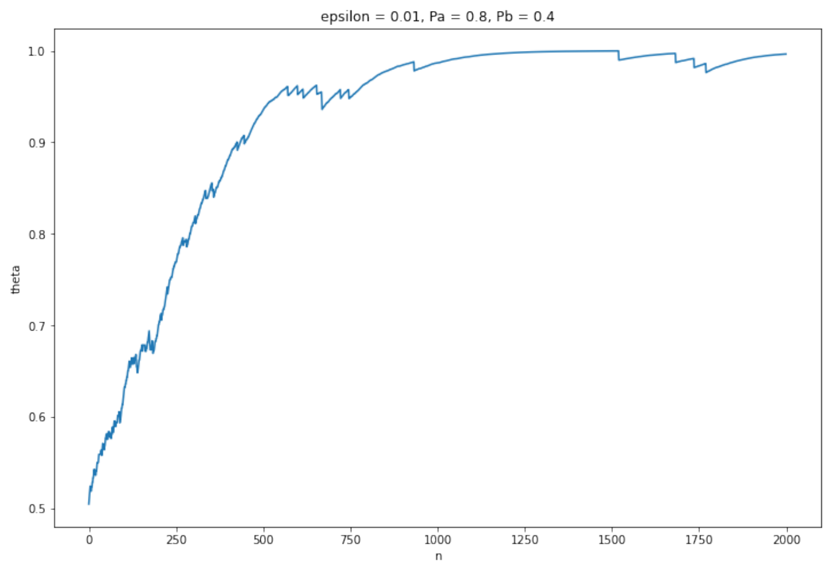
$$Y_n(\xi_n(\theta_n)) = (0_{n-1-thlose}, (1-\theta_{n-1})_{n-1-thAwins}, (-\theta_{n-1})_{n-1-thBwins})$$

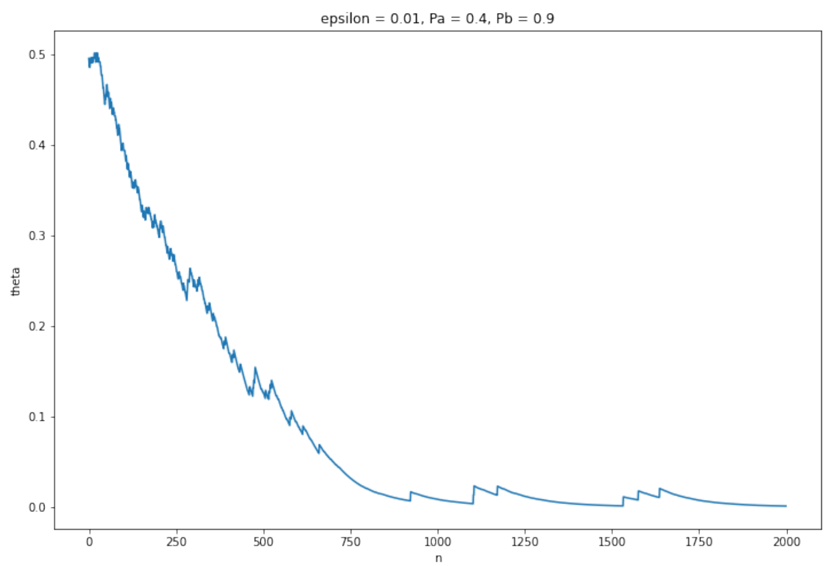
1.3 c)

Under strict monotonicity, if choose A win, $Y_n = \xi_n(\theta_n) = 1 - \theta_n$ the chosen direction the gradient is bigger than 0, which is always the grow direction;
And the probability that B win, $Y_n = \xi_n(\theta_n) = -\theta_n$ is always a descent direction, which is always the decent direction.
So this means that the field is coercive for the well-posed optimization problem.

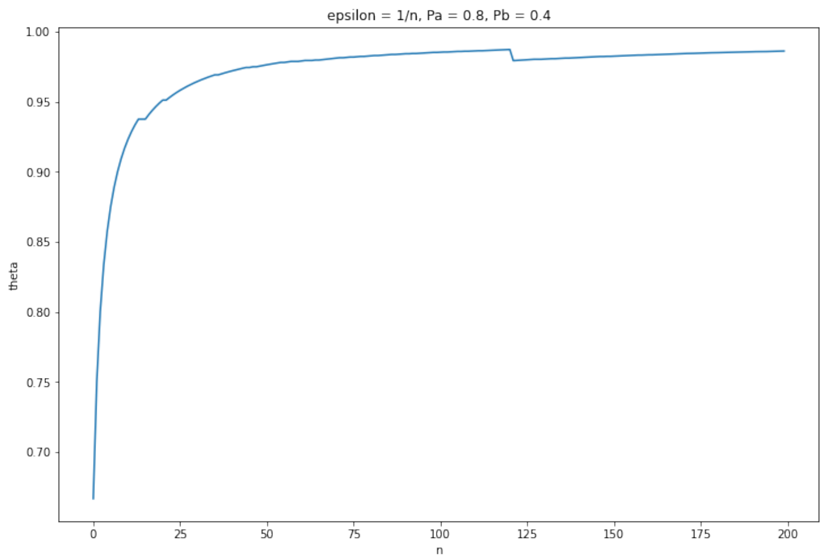
1.4 d)

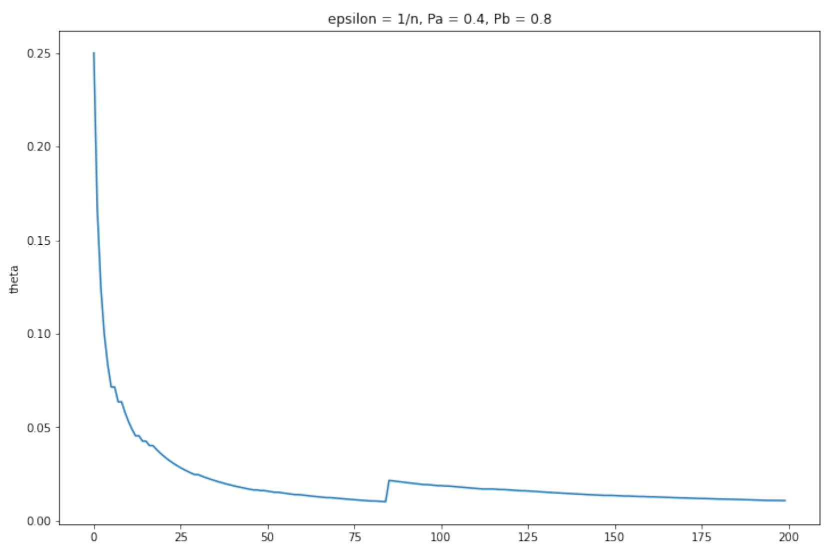
For the case of constant ϵ:





For the case of $\epsilon = \frac{1}{n}$:





It could be seen that procedure for $\epsilon = \frac{1}{7}n$ converges faster.
We use this python script to generate the plots:

Listing 1: Resolve contention

```
import os
import random
import matplotlib.pyplot as plt

random.seed(18)
```

```python
def bandit(pA, pB, thetan):
  if random.random() < thetan:
    # arm A
    if random.random() < pA:
      return 'Awin'
    else:
      return 'Alose'
  else:
    if random.random() < pB:
      return 'Bwin'
    else:
      return 'Blose'

probA = 0.8
probB = 0.4

theta = 0.5
eps = 0.01
thetas = []
ns = []
for n in range(2000):
  event = bandit(probA, probB, thetan=theta)
  if event == 'Awin':
    theta += eps*(1 - theta)
  if event == 'Bwin':
    theta -= eps*theta

  ns.append(n)
  thetas.append(theta)

plt.figure(figsize=(12, 8))
plt.title('epsilon = {}, Pa = {}, Pb = {}'.format(eps, probA, probB))
plt.plot(ns, thetas)
plt.xlabel('n')
plt.ylabel('theta')

probA = 0.4
probB = 0.9

theta = 0.5
eps = 0.01
thetas = []
ns = []
for n in range(2000):
  event = bandit(probA, probB, thetan=theta)
  if event == 'Awin':
    theta += eps*(1 - theta)
  if event == 'Bwin':
    theta -= eps*theta
```

```python
  ns.append(n)
  thetas.append(theta)

plt.figure(figsize=(12, 8))
plt.title('epsilon = {}, Pa = {}, Pb = {}'.format(eps, probA, probB))
plt.plot(ns, thetas)
plt.xlabel('n')
plt.ylabel('theta')


probA = 0.8
probB = 0.4

theta = 0.5
thetas = []
ns = []
for n in range(200):
  event = bandit(probA, probB, thetan=theta)
  if event == 'Awin':
    theta += (1/(n+3))*(1 - theta)
  if event == 'Bwin':
    theta -= (1/(n+3))*theta

  ns.append(n)
  thetas.append(theta)

plt.figure(figsize=(12, 8))
plt.title('epsilon = 1/n, Pa = {}, Pb = {}'.format(probA, probB))
plt.plot(ns, thetas)
plt.xlabel('n')
plt.ylabel('theta')

probA = 0.4
probB = 0.8

theta = 0.5
thetas = []
ns = []
for n in range(200):
  event = bandit(probA, probB, thetan=theta)
  if event == 'Awin':
    theta += (1/(n+2))*(1 - theta)
  if event == 'Bwin':
    theta -= (1/(n+2))*theta

  ns.append(n)
  thetas.append(theta)

plt.figure(figsize=(12, 8))
```

```
plt.title('epsilon␣=␣1/n,␣Pa␣=␣{},␣Pb␣=␣{}'.format(probA, probB))
plt.plot(ns, thetas)
plt.xlabel('n')
plt.ylabel('theta')
```

2 Exercise 4.4

Show that for a random variable x with finite variance

$$\nabla J(\theta) = (-E[Z(X) - \theta_1 - \theta_2 X], -E[XZ(X) - \theta_1 X - \theta_2 X^2])^\intercal \quad (1)$$

$$J(\theta) = \frac{1}{2}E[(Z(X) - (\theta_1 + \theta_2 X))^2] \quad (2)$$

Which we could get:

$$\frac{\partial J(\theta)}{\partial \theta_1} = -E[Z(X) - (\theta_1 - \theta_2 X)] \quad (3)$$

$$\frac{\partial J(\theta)}{\partial \theta_2} = -E[XZ(X) - \theta_1 X - \theta_2 X^2] \quad (4)$$

For each x_n we obtain a corresponding random observation $\xi_n = Z(x_n)$

$$E(Z(x_n)) = h(x_n)$$

The feedback function is

$$Y_n = (\xi_n - \theta_n(1) - \theta_n(2)x_n)(1, x_n)^\intercal \quad (5)$$

Because x_n and $Z(x_n)$ are random, so Y_n is independent:

$$\begin{aligned} E[Y_n|\mathfrak{F}_{\mathfrak{n}-\mathfrak{1}}] &= E[(\xi_n - \theta_n(1) - \theta_n(2)x_n)(1, x_n)^\intercal] \\ &= E[(Z(x_n) - \theta_n(1) - \theta_n(2)x_n, x_n Z(x_n) - \theta_n(1)x_n - \theta_n(2)x_n^2)^\intercal] \\ &= (E[Z(x_n) - \theta_n(1) - \theta_n(2)x_n], E[x_n Z(x_n) - \theta_n(1)x_n - \theta_n(2)x_n^2])^\intercal \\ &= -\nabla J(\theta_n(1), \theta_n(2)) \\ &= -\nabla J(\theta_n) \end{aligned}$$

3 Exercise 4.3

$$\delta M_i = y_i - [y_i|\mathfrak{i} - \mathfrak{1}]$$

show $M_n = \sum_{i=0}^{n} \epsilon_i \delta M_i$ is a Martingale process on $(\Omega, \mathbb{P}, \mathfrak{F}_n)$ show that

$$\mathbb{E}[\delta M_n \delta M_m] = 0$$

To show that M_n is a Martingale, i.e. show $\mathbb{E}[M_{n+1}|\mathfrak{F}_n] = M_n$

$$M_{n+1} = \sum_{i=0}^{n+1} \epsilon_i \delta M_i = \sum_{i=0}^{n} \epsilon_i \delta M_i + \epsilon_{n+1}\delta M_{n+1} = M_n + \epsilon_{n+1}\delta M_{n+1}$$

Then:

$E[M_{n+1}|\mathfrak{F}_n]$
$= E[(M_n + \epsilon_{n+1}\delta M_{n+1})|\mathfrak{F}_n]$
$= E[M_n|\mathfrak{F}_n] + \epsilon_{n+1}E[\delta M_{n+1}|\mathfrak{F}_n]]$
$= E[M_n|\mathfrak{F}_n] + \epsilon_{n+1}E[(y_{n+1} - E[y_{n+1}|\mathfrak{F}_n])|\mathfrak{F}_n]$
$= E[M_n|\mathfrak{F}_n] + \epsilon_{n+1}E[y_{n+1}|\mathfrak{F}_n] - \epsilon_{n+1}E[E[(y_{n+1}|\mathfrak{F}_n)|\mathfrak{F}_n]]$
Because

$$\mathfrak{F}_{n-1} \in \mathfrak{F}_n$$

$= E[M_n|\mathfrak{F}_{n-1}] + \epsilon_{n+1}E[y_{n+1}|\mathfrak{F}_n] - \epsilon_{n+1}E[y_{n+1}|\mathfrak{F}_n]$
Here:

$$E[M_n|\mathfrak{F}_{n-1}] = M_n$$

And

$$\epsilon_{n+1}E[y_{n+1}|\mathfrak{F}_n] - \epsilon_{n+1}E[y_{n+1}|\mathfrak{F}_n] = 0$$

So:

$$E[M_{n+1}|\mathfrak{F}_n] = M_n$$

$$\delta M_i = y_i - \mathbb{E}[y_i|\mathfrak{F}_{i-1}]$$

$$\delta M_{n+1} = y_{n+1} - \mathbb{E}[y_{n+1}|\mathfrak{F}_n]$$

$$\delta M_n = y_n - \mathbb{E}[y_{n+1}|\mathfrak{F}_{n-1}]$$

$\mathbb{E}[\delta M_i|\mathfrak{n}] = \mathbb{E}[y_i - \mathbb{E}[y_i|\mathfrak{F}_{i-1}]|\mathfrak{F}_{i-1}] = \mathbb{E}[y_i] - \mathbb{E}[\mathbb{E}(y_i|\mathfrak{F}_i)|\mathfrak{F}_{i-1}] = \mathbb{E}[y_i] - \mathbb{E}[y_i] = 0$

Then:

$\mathbb{E}[\delta M_n \delta M_m] = \mathbb{E}[\mathbb{E}[\delta M_n \delta M_m|\mathfrak{F}_{n-1}]|\mathfrak{F}_{m-1}] = \mathbb{E}[\delta M_n \mathbb{E}[\delta M_m|\mathfrak{F}_{m-1}]|\mathfrak{F}_{m-1}] = \mathbb{E}[0] = 0$

4 Exercise 4.7

4.1 a)

For the given vector of times per route:

$$T(\theta) = \begin{pmatrix} 3 + \theta_1 + \theta_2 \\ 2.25 + \theta_1 + 2\theta_2 + \theta_3 \\ 3 + \theta_2 + \theta_3 \end{pmatrix}$$

and the total amount of traffic $\sum_{i=1}^{3} \theta_i = 1.0$, we want to show that, all T_i are equal to some constant, we get this system of equations:

$$\begin{cases} 3 + \theta_1 + \theta_2 = c \\ 2.25 + \theta_1 + 2\theta_2 + \theta_3 = c \\ 3 + \theta_2 + \theta_3 = c \\ \theta_1 + \theta_2 + \theta_3 = 1.0 \end{cases}$$

where $c \geq 0$ is some constant. The standard form:

$$\begin{cases} -c + \theta_1 + \theta_2 + 0 \cdot \theta_3 = -3 \\ -c + \theta_1 + 2 \cdot \theta_2 + \theta_3 = -2.25 \\ -c + 0 \cdot \theta_1 + \theta_2 + \theta_3 = -3 \\ 0 \cdot +\theta_1 + \theta_2 + \theta_3 = 1 \end{cases}$$

Rewrite the system in the matrix form, to use the Gauss elimination method:

$$\left(\begin{array}{cccc|c} -1 & 1 & 1 & 0 & -3 \\ -1 & 1 & 2 & 1 & -2.25 \\ -1 & 0 & 1 & 1 & -3 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

Divide the first row by -1:

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 3 \\ -1 & 1 & 2 & 1 & -2.25 \\ -1 & 0 & 1 & 1 & -3 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

Add first row to the second row, add the first row to the third row:

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0.75 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

Interchange the rows 2 and 3:

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 3 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0.75 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

Divide the second row by -1:

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0.75 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

Add the second row to the first one, subtract the second row from the fourth:

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0.75 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right)$$

Add the row 3 to the first row, subtract the row 3 from the 4th row:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3.75 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0.75 \\ 0 & 0 & 0 & 1 & 0.25 \end{array}\right)$$

Add the row 4 to the row 2, subtract the row 4 from the row 3:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3.75 \\ 0 & 1 & 0 & 0 & 0.25 \\ 0 & 0 & 1 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0.25 \end{array}\right)$$

So:

$$\begin{cases} c = 3.75 \\ \theta_1 = 0.25 \\ \theta_2 = 0.5 \\ \theta_3 = 0.25 \end{cases}$$

Therefore such constant c indepentent of i exists.

4.2 b)

The $\hat{T}(\theta)$ is an unbiased estimator,

$$\sum_i \theta_{0,i} = 1$$

and update rule for θ is:

$$\theta_{n+1,i} = \theta_{n,i} - \epsilon_n(\hat{T}_i(\theta_n) - \frac{1}{3} \cdot \sum_k \hat{T}_k(\theta_n))$$

Because of $\epsilon_n \neq 0$,

$$\begin{aligned} \hat{T}_i(\theta_n) - \frac{1}{3} \cdot \sum_k \hat{T}_k(\theta_n) &= \\ \hat{T}_1(\theta_n) - \frac{1}{3} \cdot (\hat{T}_1(\theta_n) + \hat{T}_2(\theta_n) + \hat{T}_3(\theta_n)) + \hat{T}_2(\theta_n) - \frac{1}{3} \cdot (\hat{T}_1(\theta_n) + \hat{T}_2(\theta_n) + \\ \hat{T}_3(\theta_n)) + \hat{T}_3(\theta_n) - \frac{1}{3} \cdot (\hat{T}_1(\theta_n) + \hat{T}_2(\theta_n) + \hat{T}_3(\theta_n)) &= \\ \hat{T}_1(\theta_n) + \hat{T}_2(\theta_n) + \hat{T}_3(\theta_n) - \hat{T}_1(\theta_n) - \hat{T}_2(\theta_n) - \hat{T}_3(\theta_n) = 0 \end{aligned}$$

c)