Exercise 3.2+4.4

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3.2

a) Given initial value $\theta 0$, recursively define the feedback process Y_n through

$$\theta_{n+1} = \theta_n + \epsilon_n Y_n$$

with either fixed step size ϵ or decreasing step size, where we typically assume that

$$\sum_{n=1}^{\infty} \epsilon_n = +\infty$$

$$\sum_{n=1}^{\infty} \epsilon_n^2 < \infty$$

and Y_n given via the feedback function

$$Y_n = \phi(\xi(\theta_n), \theta_n)$$

We assume that all random variables, that is, $\theta 0$ and $(\xi_n(\theta) : n >= 0, \theta \in \Theta)$, are defined on a probability space. Running the stochastic approximation algorithm, we observe the underlying sequence

$$\xi_0(\theta_0), \xi_1(\theta_1), ...$$

Here in the problem,

$$\xi_1(\theta_1) = (0_{initiallose}, (1-\theta_0)_{initialAwins}, (-\theta_0)_{initialBwins})$$

$$\xi_2(\theta_2) = (0_{1thlose}, (1-\theta_1)_{1thAwins}, (-\theta_1)_{1thBwins})$$
$$\xi_3(\theta_3) = (0_{2thlose}, (1-\theta_2)_{2thAwins}, (-\theta_2)_{2thBwins})$$
$$\xi_2(\theta_4) = (0_{3thlose}, (1-\theta_3)_{3thAwins}, (-\theta_3)_{3thBwins})$$

and so on, ...

b) because

$$\theta_{n+1} = \theta_n + \epsilon_n Y_n$$

set $Y_n(\xi_n(\theta_n))$ is the independent sequences of unbiased estimators of the target vector field, where

$$Y_n(\xi_n(\theta_n)) = (0_{n-1-thlose}, (1-\theta_{n-1})_{n-1-thAwins}, (-\theta_{n-1})_{n-1-thBwins})$$

c) **Under** strict monotonicity, if choose A win, $Y_n = \xi_n(\theta_n) = 1 - \theta_n$ the chosen direction the gradient is bigger than 0, which is always the grow direction;
And the probability that B win, $Y_n = \xi_n(\theta_n) = -\theta_n$ is always a descent direction, which is always the decent direction.
So this means that the field is coercive for the well-posed optimization problem.

Mohamed4.4
Show that for a random variable x with finite variance

$$\nabla J(\theta) = (-E[Z(X) - \theta_1 - \theta_2 X], -E[XZ(X) - \theta_1 X - \theta_2 X^2])^\intercal \quad (1)$$

$$J(\theta) = \frac{1}{2}E[(Z(X) - (\theta_1 + \theta_2 X))^2] \quad (2)$$

Which we could get:

$$\frac{\partial J(\theta)}{\partial \theta_1} = -E[Z(X) - (\theta_1 - \theta_2 X)] \quad (3)$$

$$\frac{\partial J(\theta)}{\partial \theta_2} = -E[XZ(X) - \theta_1 X - \theta_2 X^2] \quad (4)$$

For each x_n we obtain a corresponding random observation $\xi_n = Z(x_n)$

$$E(Z(x_n)) = h(x_n)$$

The feedback function is

$$Y_n = (\xi_n - \theta_n(1) - \theta_n(2)x_n)(1, x_n)^\intercal \quad (5)$$

Because x_n and $Z(x_n)$ are random, so Y_n is independent:

$$\begin{aligned} E[Y_n|\mathfrak{F}_{\mathfrak{n}-\mathfrak{1}}] &= E[(\xi_n - \theta_n(1) - \theta_n(2)x_n)(1, x_n)^\intercal] \\ &= E[(Z(x_n) - \theta_n(1) - \theta_n(2)x_n, x_n Z(x_n) - \theta_n(1)x_n - \theta_n(2)x_n^2)^\intercal] \\ &= (E[Z(x_n) - \theta_n(1) - \theta_n(2)x_n], E[x_n Z(x_n) - \theta_n(1)x_n - \theta_n(2)x_n^2])^\intercal \\ &= -\nabla J(\theta_n(1), \theta_n(2)) \\ &= -\nabla J(\theta_n) \end{aligned}$$

Mohamed4.3

$$\begin{aligned} &E[M_{n+1}|\mathfrak{F}_n] \\ &= E[(M_n + \epsilon_{n+1}\delta M_{n+1})|\mathfrak{F}_n] \\ &= E[M_n|\mathfrak{F}_n] + \epsilon_{n+1}E[\delta M_{n+1}|\mathfrak{F}_n]] \\ &= E[M_n|\mathfrak{F}_n] + \epsilon_{n+1}E[(y_{n+1} - E[y_{n+1}|\mathfrak{F}_n])|\mathfrak{F}_n] \end{aligned}$$

$= E[M_n|\mathfrak{F}_n] + \epsilon_{n+1}E[y_{n+1}|\mathfrak{F}_n] - \epsilon_{n+1}E[E[(y_{n+1}|\mathfrak{F}_n)|\mathfrak{F}_n]]$
Because

$$\mathfrak{F}_{n-1} \in \mathfrak{F}_n$$

$= E[M_n|\mathfrak{F}_{n-1}] + \epsilon_{n+1}E[y_{n+1}|\mathfrak{F}_n] - \epsilon_{n+1}E[y_{n+1}|\mathfrak{F}_n]$
Here:

$$E[M_n|\mathfrak{F}_{n-1}] = M_n$$

And

$$\epsilon_{n+1}E[y_{n+1}|\mathfrak{F}_n] - \epsilon_{n+1}E[y_{n+1}|\mathfrak{F}_n] = 0$$

So:

$$E[M_{n+1}|\mathfrak{F}_n] = M_n$$