Exercise 8.15 and 8.20

Yunhua Zhao

November 19, 2020

0.1 8.15

a. Show that the reliability $\varnothing$ of Figure 8.2 can be expressed as:

$$\varnothing(X_1, ..., X_n) = X_1 X_5 max(X_2, X_3 X_4)$$

X_i : The indicator that component i is working, then X_i is either the value

$$X_i = \begin{cases} 0 & \text{not working} \\ 1 & \text{working} \end{cases}$$

As shown in the Figure 8.2, in order to make the system to work, component 1 and component 5 must work as well; Between component 1 and 5, the system depends on components 2,3 and 4, here there are three paths:

$$\begin{cases} 2 & \\ 2 & \text{and4} \\ 3 & \text{and4} \end{cases}$$

Because $max(X_2, X_2 X_4)$ is same with X_2
then

$$\varnothing(X_1, ..., X_5) = X_1 max(X_2, X_3 X_4) X_5 = X_1 X_5 max(X_2, X_3 X_4)$$

b. Consider the representation $X_3 = 1_{\{U \leq \theta\}}$ and show that IPA is biased for $\varnothing$ (8.28)
Show IPA is biased:

$$E[X_3(\theta)] = P(U \leq \theta) = \theta$$

$$\Rightarrow \frac{\partial}{\partial \theta}[E(X_3(\theta))] = 1$$

However $X_3(\theta, v)$ is piecewise constant function, jump at $v = \theta$
So for $\forall w$, such that $U(w) \neq \theta$
we have $\frac{\partial}{\partial \theta} X_3(\theta, v) = 0$
As $P(V(w) = \theta) = 0$ (when $U(w) \neq \theta$)
then $E[X_3'(\theta, v)] = 0$,
then $1 = \frac{\partial}{\partial \theta}[E(X_3(\theta))] \neq E[\frac{\partial}{\partial \theta} X_3(\theta, v)] = 0$
then IPA is biased function X_3, and consequently is biased as well for $\varnothing$ (8.28)

c. Calculate the SF estimator for (8.28) and show that it is unbiased for (8.28)
X_i is Bernoulli (p_i), eg:

$$\begin{cases} P(X_i = 1) = p_i \\ P(X_i = 0) = 1 - p_i \end{cases}$$

the pdf of X_i is $f(X_i, p_i) = p_i^i (1-p_i)^{1-X_i}$, $X_i \in 0, 1$
The density function is given by $f_n\ \theta = P_3$
$L(\theta|x) = f(x, \theta) = \theta^{x_3}(1-\theta)^{1-x_3}\pi_{i\neq 3}p_i^{x_i}(1-p_i)^{1-x_i}$, $x_i \in \{0, 1\}$
$\varnothing(X_1, ..., X_5) = X_1 X_5 max(X_2, X_3 X_4)$
then $\varnothing(X_1, ..., X_5) = 1$ only if one of the following

$$\begin{cases} X = (1,1,1,1,1) \\ X = (1,1,0,1,1) \\ X = (1,1,1,0,1) \\ X = (1,1,0,0,1) \\ X = (1,0,1,1,1) \end{cases}$$

$L(\theta|X) = \theta[p_1p_2p_4p_5 + p_1p_2p_5 + p_1p_2p_4p_5] + (1-\theta)[p_1p_2p_4p_5 + p_1p_2p_5]$
$\log L(\theta|X) = \log[\theta[p_1p_2p_4p_5 + p_1p_2p_5 + p_1p_2p_4p_5] + (1-\theta)[p_1p_2p_4p_5 + p_1p_2p_5]]$
$= \log\theta + \log(1-\theta) + \log[p_1p_2p_4p_5 + p_1p_2p_5 + p_1p_2p_4p_5] + \log[p_1p_2p_4p_5 + p_1p_2p_5]$
The score function will be

$$S(\theta|X) = \frac{\partial \log(L(\theta|X))}{\partial\theta} = \frac{1}{\theta} - \frac{1}{1-\theta}$$

unbiased $L(\theta|X) = f(x, \theta) = \theta C_1 + (1-\theta)C_2$, where
$C_1 = p_1p_2p_4p_5 + p_1p_2p_5 + p_1p_2p_4p_5$
$C_2 = p_1p_2p_4p_5 + p_1p_2p_5$
(i) Easy to see that $f(X, \theta)$ is differentiable in $\theta \in (0, 1)$, using the bounding condition theorem

$$\frac{\partial}{\partial\theta} f(X, \theta) = C_1 - C_2$$

let $k(n) = \begin{cases} C_1 - C_2 & n = 0 | n = 1 \\ 0 & \text{else} \end{cases}$
$\to \sum_{n\in v} V(n)K(n) = (V(0) + V(1))(C_1 - C_2) < \infty$
then (ii) is satisfied
Using theorem 8.3 then SF and MVD estimate function 8.28 are unbiased.

d. Calculate the MVD estimator for (8.28) and show that it is unbiased for (8.28)
$\frac{d}{d\theta}|_{\theta=\theta_0} \int h(x) f_\theta(x)\,dx = \int h(x) S(\theta_0, x) f_{\theta_0}(x)\,dx = C_{\theta_0}(\int h(x) f^+_{\theta_0}(x)\,dx - \int h(x) f^-_{\theta_0}(x)\,dx)$
In particular when

$$S(\theta_0, x) f_{\theta_0}(x) = C_{\theta_0}(f^+_{\theta_0}(x) - f^-_{\theta_0}(x))$$

From tabel 7.1 $C_{\theta_0} = 1$

$$f^+_{\theta_0}(x) = Dirac(0) = S_0$$

$$f^-_{\theta_0}(x) = Dirac(1) = S_1$$

then

$$\frac{\partial}{\partial\theta}\int h(x) f_\theta(x)\, dx = \frac{\partial}{\partial\theta}[\theta S(0) + (1-\theta)S(1)] = h(0) - h(1)$$

while yields the unbiased MVD estimator

$$D^{MVD}(\theta) = h(0) - h(1)$$

Same proof in c, as the MVD was derived from SF

0.2 8.20

(a) Calculate the Score Function for this distribution.
Because $f_\theta(x) = (x^2/2\theta^3)e^{-x/\theta}$
And $S(\theta, x) = \frac{\partial}{\partial\theta}\log(f_\theta(x))$
So:

$$S(\theta, x) = \frac{\partial}{\partial\theta}\log((x^2/2\theta^3)e^{-x/\theta}) = \frac{\frac{\partial}{\partial\theta}f_\theta(x)}{f_\theta(x)}$$

$$= \frac{x^2}{2\theta^3}e^{-x/\theta}\frac{x}{\theta^2} - e^{-x/\theta}\frac{3x^2}{2\theta^4} = e^{-x/\theta}\frac{x^2}{2\theta^4}(\frac{x}{\theta} - 3)$$

(b) Verify the conditions of Theorem 8.3. What family of functions $v(.)$ can be used? Is h in the family?
Given $h(x) = max(0, x - s)$ and $f_\theta(x) = (x^2/2\theta^3)e^{-x/\theta}$
(i) It is obvious that $f_\theta(x)$ is differentiable with respect to θ for all $x \in S$
(ii)