Exercise 2.4

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September 23, 2020

(a) because

$$x_n(t) = \vartheta^\epsilon(t_n + t)$$

and

$$\vartheta^\epsilon(t) = \theta_{m(t)}$$

so

$$x_n(t) = \theta_{m(t+t_n)}$$

$$x_n(t+s) = \theta_{m(t+s+t_n)}$$

then

$$x_n(t+s) - x_n(t) = \theta_{m(t+s+t_n)} - \theta_{m(t+t_n)}$$

which

$$= \sum_{i=m(t_n+t)}^{m(t_n+t+s)-1} \epsilon_i G(\theta_i)$$

Because $X_\epsilon(.)$ is piecewise point, $G(X_\epsilon(.))$ is also piecewise constant and its jump times are given by $t_n = \sum_{k=1}^{n} \epsilon_k$. Thus the definite integral on $[t_n+t, t_n+t+s]$ of $G(X_\epsilon(.))$ is a sum that can be approximation expressed as

$$\int_{t_n+t}^{t_n+t+s} G[x_\epsilon(u)]du$$

together

$$\int_{t_n+t}^{t_n+t+s} G[x_\epsilon(u)]du \approx \sum_{i=m(t_n+t)}^{m(t_n+t+s)-1} \epsilon_i G(\theta_i)$$

$$\int_{t_n+t}^{t_n+t+s} G[x_\epsilon(u)]du = \sum_{i=m(t_n+t)}^{m(t_n+t+s)-1} \epsilon_i G(\theta_i) + \rho_(\epsilon), (2.1)$$

(b) formula

$$x_n(t+s) - x_n(t) = \theta_{m(t+s+t_n)} - \theta_{m(t+t_n)} = \sum_{i=m(t_n+t)}^{m(t_n+t+s)-1} \epsilon_i G(\theta_i)$$

contains $m(q)-m(r)-1$ terms. For ϵ sufficiently small, set the ϵ_b is the biggest ϵ and the ϵ_s is the smallest ϵ in interval (r,q) so that the number of terms is bounded by $(\frac{q-r}{\epsilon_b}, \frac{q-r}{\epsilon_s})$. This yields, for small ϵ,

$$\|(x_\epsilon(q) - x_\epsilon(r))\|_\infty = \sum_{i=m(r)}^{m(q)-1} \epsilon_i G(\theta_i)$$

Because G is bounded, let use L to represent G's bounder, so

$$\|x_\epsilon(q) - x_\epsilon(r)\|_\infty = L \sum_{i=m(r)}^{m(q)-1} \epsilon_i =< \epsilon_b L(q-r)/\epsilon_s, (2.2)$$

To summarize, for ϵ sufficiently small, we have shown that for any $\eta > 0$, we may let $\delta_\eta = \frac{\eta}{L(\epsilon_b/\epsilon_s)}/$ so that it follows that $\|x_\epsilon(q) - x_\epsilon(r)\|_\infty =< \eta$ wherever $\|q-r\| =< \delta_\eta(\epsilon_b/\epsilon_s)$. This establishes equicontinuity in the extended sense.

(c) Let $a < t$ and $b > t+s$ and consider $x_{\epsilon_k}(.)$ on (a,b). Set $x_n(0) = \theta_0$ for all k. Therefor, for ϵ sufficiently small, by (b) formula 2.2,

$$|x_{\epsilon_k}(r)|_\infty =< |\theta_0|_\infty + rL\frac{\epsilon_b}{\epsilon_s}$$

for all $r > 0$, which suffices to show that x_ϵ is uniformly bounder in (a,b). This together with equicontinuity of x_{ϵ_k} implies by the Ascoli-Arzela Theorem 2.2 that any infinite subsequence of x_{ϵ_k} has a convergent subsequence with a continuous limit on (a, b). Consider a convergent subsequence along $\epsilon_r \to 0$, so that $\hat{x}(.) = \lim_{\epsilon_r\to 0} x_{\epsilon_r}(.)$ (in the sup norm) and continuous. Then

$$\begin{aligned}\lim_{\epsilon_r\to 0}(x_{\epsilon_r}(t+s) - x_{\epsilon_r}(t)) &= \lim_{\epsilon_r\to 0}\int_t^{t+s} G(x_{\epsilon_r}(u))du \\ &= \int_t^{t+s} \lim_{\epsilon_r\to 0} G(x_{\epsilon_r}(u))du \\ &= \int_t^{t+s} G(\hat{x}(u))du\end{aligned}$$

Where the first formula follows from the fact that $\rho(\epsilon)$ in (2.1) is bounded by $L(\epsilon_b + \epsilon_s)$ and thus of order $\mathcal{O}(\epsilon)$, the second formula follows from Lebesgue Dominated Convergence Theorem, and the third formula is a consequence of the continuity of $G(\hat{x}(.))$ on (a,b). We arrive for $s > 0$ at

$$\frac{\hat{x}(t+s) - \hat{x}(t)}{s} = \frac{1}{s}\int_t^{t+s} G(\hat{x}(u))du$$

By continuity of $G(\hat{x}(.))$, taking the limit as s goes to zero, the above right-hand side converges to $G(\hat{x}(t))$, which establishes the ODE in the question for $\hat{x}(.)$. Because G is continuous and bounded on the trajectory $\hat{x}$, it follows from Theorem 2.1 that the ODE has a unique solution for each initial condition,establishing that all accumulation points have the same limit, proving the claim for the unbiased case.