Exercise 6.3

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1 6.3

1.1 a

a6.Show that $\theta_{n+1} = \theta_n + \epsilon Y_n$, $Y_n = D(\theta_n) - \xi_n$ satisfies the assumptions of Theorem 6.1.
Proof $E[\delta M_n^\epsilon (\delta M_n^\epsilon)^T 1_{||\theta_n^\epsilon - \theta^*||=<\rho} | \mathfrak{F}_{n-1}^\epsilon]$ **is symmetric matrix** Because:

$$\delta M_n^\epsilon = Y_n^\epsilon - g(\xi_{n-1}^\epsilon, \theta_n^\epsilon)$$

And:

$$g(\xi_{n-1}^\epsilon, \theta_n^\epsilon) = E[Y_n^\epsilon | \mathfrak{F}_{n-1}^\epsilon]$$

So:

$$\delta M_n^\epsilon = Y_n^\epsilon - E[Y_n^\epsilon | \mathfrak{F}_{n-1}^\epsilon]$$

Based on:

$$Y_n = D(\theta_n) - \xi_n$$

Then:

$$E[\delta M_n^\epsilon (\delta M_n^\epsilon)^T 1_{||\theta_n^\epsilon - \theta^*||=<\rho} | \mathfrak{F}_{n-1}^\epsilon] = E[(Y_n^\epsilon - E[Y_n^\epsilon | \mathfrak{F}_{n-1}^\epsilon])(Y_n^\epsilon - E[Y_n^\epsilon | \mathfrak{F}_{n-1}^\epsilon])^T 1_{||\theta_n^\epsilon - \theta^*||=<\rho} | \mathfrak{F}_{n-1}^\epsilon]$$

Because $D(\theta_n)$ and ξ_n are random variable, so Y_n is random variable also.

$$E[\delta M_n^\epsilon (\delta M_n^\epsilon)^T 1_{||\theta_n^\epsilon - \theta^*||=<\rho} | \mathfrak{F}_{n-1}^\epsilon] = E[(Y_n^\epsilon - E[Y_n^\epsilon])(Y_n^\epsilon - E[Y_n^\epsilon])^T 1_{||\theta_n^\epsilon - \theta^*||=<\rho} | \mathfrak{F}_{n-1}^\epsilon]$$

$$= E[(Y_n^\epsilon - E[Y_n^\epsilon])((Y_n^\epsilon)^T - (E[Y_n^\epsilon])^T) 1_{||\theta_n^\epsilon - \theta^*||=<\rho} | \mathfrak{F}_{n-1}^\epsilon]$$
$$= E[(Y_n^\epsilon - E[Y_n^\epsilon])((Y_n^\epsilon)^T - E[(Y_n^\epsilon)^T]) 1_{||\theta_n^\epsilon - \theta^*||=<\rho} | \mathfrak{F}_{n-1}^\epsilon]$$

For simplify, the following processes ignore ϵ for a second, but remember it is exist all the time.

$$= E[(Y_n Y_n^T - E[Y_n] Y_n^T - Y_n E[Y_n^T] + E[Y_n] E[Y_n^T]) 1_{||\theta_n^\epsilon - \theta^*||=<\rho} | \mathfrak{F}_{n-1}^\epsilon]$$

$$= E[Y_n Y_n^T - E[E[Y_n] Y_n^T] - E[-Y_n E[Y_n^T]] + E[E[Y_n] E[Y_n^T]]]]$$
$$= E[Y_n Y_n^T] - E[Y_n] E[Y_n^T] - E[Y_n] E[Y_n^T] + E[Y_n] E[Y_n^T]$$

Because $Y_n Y_n^T$ is symmetric matrix, $E[Y_n]E[Y_n^T]$is also symmetric matrix, so $E[\delta M_n^\epsilon (\delta M_n^\epsilon)^T 1_{||\theta_n^\epsilon - \theta^*||=<\rho} | \mathfrak{F}_{n-1}^\epsilon]$ is symmetric matrix.

a7.Where the error term satisfies $E[\rho_1(\theta, \xi_n^\epsilon)] = \mathcal{O}(||\theta - \theta^*||^2)$, as $n \to \infty$ $\epsilon \to 0$

$$Y_n = D(\theta_n) - \xi_n$$

because Y_n is random value, so

$$E[Y_n^\epsilon | \mathfrak{F}_{n-1}^\epsilon] = E[Y_n^\epsilon]$$

$$= E[D(\theta_n) - \xi_n] = E[D(\theta_n)] - E[\xi_n | \theta_n] = E[D(\theta_n)] - S(\theta_n)$$

Then

$$\nabla_\theta^2 g(\theta^*, \xi) = E[\nabla^2 D(\theta^*)] - \nabla^2 S(\theta^*)$$

$$g(\theta, \xi) = g(\theta^*, \xi) + \nabla_\theta g(\theta^*, \xi)^T (\theta - \theta^*) + \rho_1(\theta, \xi)$$

and

$$E[Y_n^\epsilon | \mathfrak{F}_{n-1}^\epsilon] = g(\xi_{n-1}^\epsilon, \theta_n^\epsilon)$$

So

$$g(\theta, \xi) = g(\theta^*, \xi) + \nabla_\theta g(\theta^*, \xi)^T (\theta - \theta^*) + \frac{1}{2} \nabla_\theta^2 g(\theta^*, \xi)(\theta - \theta^*)^2 + \rho_2(\theta, \xi)$$

$$E[\rho_1(\theta, \xi_n^\epsilon)] = E[\frac{1}{2} \nabla_\theta^2 g(\theta^*, \xi)(\theta - \theta^*)^2 + \rho_2(\theta, \xi)]$$

Because $E[\rho_2(\theta, \xi)] \to 0$ and $\nabla_\theta^2 g(\theta^*, \xi) = E[\nabla^2 D(\theta^*)] - \nabla^2 S(\theta^*) > 0$ but limited, so $E[\rho_1(\theta, \xi_n^\epsilon)] = \mathcal{O}(||\theta - \theta^*||^2)$

a8.There is a Hurwitz matrix A (i.e. a matrix where all the eigenvalues have a negative real part) such that $\lim_{m \to \infty} \frac{1}{m} \sum_{i=n}^{n+m-1} E[\nabla_\theta g(\xi_{n-1}^\epsilon, \theta^*)^T - A] = 0$

$$\nabla_\theta g(\theta^*, \xi) = E[\nabla D(\theta^*)] - \nabla S(\theta^*)$$

$$E[\nabla_\theta g(\xi_{n-1}^\epsilon, \theta^*)^T] = E[E[\nabla_\theta D(\theta_n^*)]^T] - E[\nabla_\theta S(\theta_n^*)^T] = E[\nabla_\theta D(\theta_n^*)]^T - E[\nabla_\theta S(\theta_n^*)^T] = E[\nabla_\theta D(\theta_n^*)^T - \nabla_\theta S(\theta_n^*$$

1.2 b

Use $d = 5$ for the demand function. Your economics guru has estimated that $\theta^* \approx 1$ and $S'(\theta^*) \approx 4.5$. With this information, apply Theorem 6.1 to identify the values of a, σ^2 for the (approximate) limit Orstein Uhlenbeck process $U(t)$, and find T such that $e^{-aT} \approx 0.0001$

$$D(\theta) = \theta^{-5}$$

$$D(\theta^* \approx 1) \approx S(\theta^* \approx 1)$$

$$S'(\theta^*) \approx 4.5$$
$$D'(\theta^*) < S'(\theta^*)$$
$$D'(\theta^*) < 4.5$$
$$U_n^\epsilon = \frac{\theta_n^\epsilon - 1}{\sqrt{\epsilon}}$$
$$U^\epsilon(t) = U_n^\epsilon; t \in [n\epsilon, (n+1)\epsilon]$$

1.3 c

Show that $\epsilon \approx 0.0005$ yields a precision of 0.01(half width of the approximate confidence interval after T/ϵ iterations, with confidence level $\alpha = 0.05$).