PrivRM: A Framework for Range Mean Estimation under Local Differential Privacy

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ABSTRACT

The increasing collection and analysis of personal data driven by digital technologies has raised concerns about individual privacy. Local Differential Privacy (LDP) has emerged as a promising solution to provide rigorous privacy guarantee for users, without relying on a trusted data collector. In the context of LDP, range mean estimation over numerical values is an important yet challenging problem. Simply applying existing work may introduce overly large noise sensitivity, since all of them focus on statistical tasks (e.g., mean or distribution) across the entire domain. In this paper, we propose a novel framework for Private Range Mean (*PrivRM*) estimation under LDP. Three implementations of the framework, namely $PrivRM^I$, $PrivRM^O$ and $PrivRM^*$, are developed, which are adaptable to all existing numerical value perturbation mechanisms. As an optimization of the framework, we also propose an distribution-aware Adaptive Adjustment (AA) strategy to dynamically confine the perturbation space for skewed data distributions. Extensive experimental results show that under the same privacy guarantee and query range, our framework *PrivRM* significantly improve over existing solutions.

1 INTRODUCTION

With the increasing popularity of big data analytics, data collectors are becoming more interested in collecting and analyzing usage data from their customers in order to improve their services. Nevertheless, the practice of data sharing carries the risk of data leakage, which can affect not only users but also third parties susceptible to both internal and external data leakage. To safeguard personal data, Local Differential Privacy (LDP) [9, 16, 18] has been proposed to provide a robust privacy-preserving method for various data analysis tasks. By enabling users to perturb their data locally before sharing, LDP ensures that sensitive information remains confidential even in the presence of an untrusted data collector. This paradigm has far-reaching implications for the security and privacy of data in real-world applications, such as Apple [29], Google [20] and Microsoft [11].

In the context of LDP, mean estimation over numerical values is a fundamental query and attracts much attention from researchers [17, 19, 30]. In the literature, all the existing work focus on mean estimation over the entire domain. However, in practical applications, the query range is usually specific. Below are two examples.

- **Employee Salary Analysis**. Consider a substantial company with million of employees, the Human Resources (HR) department may seek to ascertain the average salary levels of employees falling within a specific salary range (e.g., annual salaries between \$50, 000 and \$100, 000) to facilitate compensation adjustments and budget planning.
- **House Price Analysis**. In extensive real estate datasets, the transaction price of each property is recorded. Real estate agents or potential purchasers may seek to know the transaction selling price of houses within a specific price range (e.g., from \$300, 000 to \$500, 000) to identify market trends and make informed purchasing decisions.

The two examples can be systematically formulated as a problem of range mean estimation. Without loss of generality, we assume each user possesses a value drawn from the domain $\mathcal{D}$. Given a specific range $[l, r] \subseteq \mathcal{D}$, the objective is to return the mean of values which fall within the range, while ensuring LDP guarantees.

In the literature, several studies focus on LDP-enabled numerical value perturbation, including the Laplace Mechanism (LM) [19], Stochastic Rounding (SR) [18], Piecewise Mechanism (PM), Hybrid Mechanism (HM) [30], and Square Wave (SW) mechanism [26]. These mechanisms are specifically designed for mean or distribution estimation. However, they all consider estimation over the entire domain, leaving a gap in range-specific mean estimation. Notably, directly applying these mechanisms for mean estimation over a specific range results in significant utility loss, as their noise sensitivity typically covers the entire domain. This issue is particularly pronounced when the query range $[l, r]$ is substantially smaller than the entire domain.

Intuitively, a range mean depends solely on the values within that range and is irrelevant to any values outside it. Therefore, an important way to improve the utility of range mean estimation is to concentrate on the query range to reduce the noise sensitivity. However, under LDP, a key challenge is how to perturb out-of-range values while eliminating their influence on the aggregation. Based on this idea, we propose a framework for locally differentially Private Range Mean (*PrivRM*) estimation. This framework reduces the sensitivity from the entire domain to the query range by dividing the task into two phases: estimating range count in Phase 1 and the range sum in Phase 2. We implement the framework *PrivRM* in three ways, namely $PrivRM^I$, $PrivRM^O$ and $PrivRM^*$. The first two eliminate the influence of the out-of-range values on the mean by employing randomization techniques to the input and output domains, respectively. To optimize both phases and enhance the accuracy, we devise an implementation called $PrivRM^*$, which exclusively employs in-range and out-of-range values for range mean estimation. Our observations indicate that compared to $PrivRM^I$ and $PrivRM^O$, $PrivRM^*$ consistently introduces less perturbation noise while ensuring the same LDP guarantee, resulting in superior data utility.

To further optimize the performance, we design a distribution-aware strategy that can be integrated into our framework. When estimating the range mean over a skewed distribution, where values are concentrated in an area much smaller than the query range, we can confine the perturbation space to a densely populated region to further reduce the noise sensitivity. Towards this idea, we introduce an Adaptive Adjustment (AA) strategy. AA strategy employs a binary search within the query range to dynamically confine the

perturbation space, ensuring that the perturbation mechanism is only applied to the dense region, thus preserving the utility of data while ensuring LDP guarantee.

Overall, the main contributions of this paper are as follows.

- To our knowledge, this is the first work to explore the problem of range mean estimation under LDP. We propose a novel framework *PrivRM*, which is capable of integrating all the existing numerical mechanisms.
- We implement the framework in three ways, namely $PrivRM^I$, $PrivRM^O$ and $PrivRM^*$, based on which a black-box guideline is provided to offer the best suggestion.
- We advance the framework with our Adaptive Adjustment (AA) strategy to optimize the utility, especially for skewed data distributions.
- Extensive experimental results on real-world datasets validate the effectiveness of *PrivRM* framework in supporting range mean query.

Roadmap. Section 2 introduces the preliminaries on LDP. Section 3 presents the problem definition and the existing solutions. Sections 4 and 5 introduce the *PrivRM* framework and its implementations. A framework optimization is proposed in Section 6. Experimental results are presented in Section 7. Finally, we discuss related work in Section 8 and conclude the paper in Section 9.

2 PRELIMINARIES

2.1 Local Differential Privacy

Differential privacy (DP) works in both centralized and local settings. Centralized DP requires the data curator to be fully trusted to collect all data [19], while local DP does not rely on this assumption [16]. In the local setting, each user locally perturbs her data before reporting them to an untrusted data collector, which makes it more secure and practical in real-world applications. The formal definition is as follows.

Definition 1. *(**Local Differential Privacy, LDP**) A randomized algorithm $\mathcal{A}$ satisfies ϵ-LDP if for any two input records w and w', and any output w^* of $\mathcal{A}$, the following inequality holds.*

$$\Pr[\mathcal{A}(w) = w^*] \leq e^{\epsilon} \times \Pr[\mathcal{A}(w') = w^*]. \quad (1)$$

In Equation 1, ϵ is called the privacy budget, which controls the deniability of a randomized algorithm taking w or w' as its input. As with centralized DP, LDP also has the property of sequential composition as below, which guarantees the overall privacy for a sequence of randomized algorithms.

Theorem 1. *(**Sequential Composition**) Given c randomized algorithms $\mathcal{A}_i (1 \leq i \leq c)$, each providing ϵ_i-local differential privacy. Then the sequence of algorithms $\mathcal{A}_i (1 \leq i \leq c)$ collectively provides $(\Sigma\epsilon_i)$-local differential privacy.*

2.2 Randomized Response

The technique of randomized response (RR) [33] serves as a paradigm to ensure ϵ-LDP, which has been widely adopted in LDP perturbation mechanisms. Specifically, RR enables respondents to answer a sensitive binary question while maintaining plausible deniability. In essence, each user reports a genuine answer with probability p and provides false answer with probability $1 - p$. To comply with ϵ-LDP, RR sets the probability $p = \frac{e^{\epsilon}}{1+e^{\epsilon}}$, so that $\frac{p}{1-p}$ is bounded by e^{ϵ}.

Various LDP mechanisms have been developed in accordance with RR, depending on whether the data type is categorical or numerical. This work primarily focuses on the LDP perturbation mechanisms for numerical values, which are detailed in the following section.

2.3 LDP Mechanisms for Numerical Values

We introduce five typical numerical values perturbation (NVP) mechanisms, namely Laplace Mechanism (LM) [19], Stochastic Rounding (SR) [18], Piecewise Mechanism (PM) [30], Hybrid Mechanism (HM) [30] and Square Wave Mechanism (SW) [26]. Without loss of generality, we assume the domain of an input value is $[-1, 1]$ in the sequel. Note that LM is an unbounded mechanism generating random noise from $[-\infty, +\infty]$, while the others are bounded mechanisms generating noise from a specific range.

2.3.1 Laplace Mechanism (LM). LM [19] adds noise randomly drawn from Laplace distribution to data, ensuring that individual privacy is preserved while allowing statistical analysis. Given an input $t_i \in [-1, 1]$ and privacy budget ϵ, the output of LM is defined as $\hat{t}_i = t_i + Lap(\frac{2}{\epsilon})$, where $Lap(\frac{2}{\epsilon})$ is a random variable that conforms to the Laplace distribution with the probability density function $pdf(x) = \frac{\epsilon}{4} \exp\left(-\frac{\epsilon|x|}{2}\right)$.

2.3.2 Stochastic Rounding (SR). The essence of SR [18] is the Bernoulli Distribution, so the output is bounded and discrete. Given an input value $t_i \in [-1, 1]$ and privacy budget ϵ, the output follows a Bernoulli Distribution, taking the value from $\{-C_{SR}(\epsilon), C_{SR}(\epsilon)\}$, where $C_{SR}(\epsilon) = \frac{e^{\epsilon}+1}{e^{\epsilon}-1}$:

$$\Pr\left[\hat{t}_i = y \mid t_i\right] = \begin{cases} \frac{e^{\epsilon}-1}{2e^{\epsilon}+2} \cdot t_i + \frac{1}{2}, & \text{if } y = C_{SR}(\epsilon), \\ -\frac{e^{\epsilon}-1}{2e^{\epsilon}+2} \cdot t_i + \frac{1}{2}, & \text{if } y = -C_{SR}(\epsilon). \end{cases}$$

2.3.3 Piecewise Mechanism (PM). Compared to SR, the output of PM is continuous [30]. Given an input value $t_i \in [-1, 1]$, PM outputs a value $\hat{t}_i \in [-C_{PM}(\epsilon), C_{PM}(\epsilon)]$, where $C_{PM}(\epsilon) = \frac{e^{\epsilon/2}+1}{e^{\epsilon/2}-1}$. Particularly, PM perturbs the data using a piecewise probability density function that consists of an interval and other intervals:

$$Pr\left[\hat{t}_i = y \mid t_i\right] = \begin{cases} p = \frac{(e^{\epsilon} - e^{\epsilon/2})}{2(e^{\epsilon/2}+1)}, & \text{if } y \in [l(t_i), r(t_i)], \\ q = \frac{e^{\epsilon/2}-1}{2(e^{\epsilon/2}+e^{\epsilon})}, & \text{otherwise}, \end{cases}$$

where $l(t_i) = \frac{e^{\epsilon/2}t_i - 1}{e^{\epsilon/2}-1}$ and $r(t_i) = \frac{e^{\epsilon/2}t_i + 1}{e^{\epsilon/2}-1}$.

2.3.4 Hybrid Mechanism (HM). When the data distribution is concentrated in the middle of the input domain, PM performs better than SR; otherwise, SR shows superior performance [30] . Here, HM combines both approaches by achieving lower worst-case variance of SR and PM [30]. Specifically, HM operates by invoking PM with probability α_{HM} while invoking SR with probability $1 - \alpha_{HM}$, where

$$\alpha_{HM} = \begin{cases} 1 - e^{-\epsilon/2}, & \text{if } \epsilon > 0.61, \\ 0, & \text{if } \epsilon \leq 0.61. \end{cases}$$

It has been proven that these four mechanisms are unbiased, i.e., $\mathbb{E}(\hat{t}_i) = t_i$.

Table 1: Notation

Symbol	Description
$\mathcal{U}$	the set of users
n	the total number of users, $n = \|\mathcal{U}\|$
u_i	the i-th user
$\mathcal{T}$	the set of values
t_i	the value possessed by u_i
$n_{(l,r)}$	the number of users with values in range$[l, r]$
$s_{(l,r)}$	the summation of values in range $[l, r]$
$m_{(l,r)}$	the mean of values in range $[l, r]$
$\mathbf{d}_{(l,r)}$	the distribution of values in range $[l, r]$
$\mathbb{M}_{\mathcal{A}}(\cdot)$	the output domain of NVP mechanism $\mathcal{A}$

2.3.5 Square Wave Mechanism (SW). Originally, SW [26] perturbs a value $t_i \in [0, 1]$ into a sanitized version $\hat{t}_i \in [-b, 1 + b]$ via Equation 2

$$\Pr\left[\hat{t}_i = y \mid t_i\right] = \begin{cases} p = \frac{e^\epsilon}{2be^\epsilon+1}, & \text{if } |t_i - y| \le b, \\ q = \frac{1}{2be^\epsilon+1}, & \text{otherwise}, \end{cases} \tag{2}$$

where $b = \frac{\epsilon e^\epsilon - e^\epsilon + 1}{2e^\epsilon(e^\epsilon - 1 - \epsilon)}$. To handle the unified domain $[-1, 1]$, we can normalize t_i into $[-1, 1]$, perturb it via Equation 2, and then denormalize the perturbed value. By merging normalization and denormalization with Equation 2, the perturbation of SW can be represented as

$$\Pr\left[\hat{t}_i = y \mid t_i\right] = \begin{cases} p = \frac{e^\epsilon}{2(2be^\epsilon+1)}, & \text{if } |t_i - y| \le 2b, \\ q = \frac{1}{2(2be^\epsilon+1)}, & \text{otherwise}, \end{cases} \tag{3}$$

which takes a value $t_i \in [-1, 1]$ as input and outputs $\hat{t}_i \in [-1 - 2b, 1 + 2b]$.

Note that, unlike LM, SR, PM and HM which ensure unbiasness, SW is not an unbiased estimator [15]. While this is to reduce the estimation variance, it can only cope with relatively symmetric distribution well. For the others, especially those skew distributions, SW may cause much estimation deviation from the ground truth. Fortunately, this bias can be corrected by applying a correction processing as

$$\hat{t}_i = \frac{y}{4b(p - q)} \tag{4}$$

The proof of unbiasedness of SW is presented in Appendix A.

After the perturbation (Equation 2 or 3), in order to reconstruct the histogram of the data distribution, the aggregator needs to divide both the input domain and the output domain into several bins, typically 1024 bins. And then collector reconstructs the distribution by using an Expectation Maximization algorithm (EM).

3 PROBLEM DEFINITION AND EXISTING SOLUTIONS

In this section, we first formulate the problem of range mean estimation under LDP, and then present two naive solutions that are directly adapted from existing works.

3.1 Problem Definition

We consider a system setting where there are a set of n users $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$, and a collector. Each use u_i holds a numerical value $t_i \in [-1, 1]$, where $[-1, 1]$ is the value domain. Given a specific range $[l, r] \subseteq [-1, 1]$, the collector's objective is to calculate the mean $m_{(l,r)}$ of users values which fall within $[l, r]$, while ensuring the data privacy of all users. Formally,

$$m_{(l,r)} = \frac{s_{(l,r)}}{n_{(l,r)}} = \frac{\sum_{i=1}^{n} t_i \cdot \mathbb{1}_{[l,r]}(t_i)}{\sum_{i=1}^{n} \mathbb{1}_{[l,r]}(t_i)}, \tag{5}$$

where $\mathbb{1}_{[l,r]}(t_i)$ is an indicator function that returns 1 if $t_i \in [l, r]$ and 0 otherwise.

3.2 Existing Solutions

To estimate the range mean as shown in Equation 5, there are two naive solutions adapted from existing works.

Solution 1: Direct Estimation. Each user perturbs her data t_i and reports a sanitized value $\hat{t}_i$ by a numerical value perturbation mechanism. Then the collector derives the mean $\hat{m}_{(l,r)}$ by calculating the sum and count of values within the range $[l, r]$. Specifically,

$$\hat{m}_{(l,r)} = \frac{\sum_{i=1}^{n} \hat{t}_i \cdot \mathbb{1}_{[l,r]}(\hat{t}_i)}{\sum_{i=1}^{n} \mathbb{1}_{[l,r]}(\hat{t}_i)}.$$

Note that only those NVP mechanisms whose output domains are continuous can be adopted, e.g., LM, PM and SW.

Solution 2: Distribution-based Estimation. Another idea is to approximate the distribution of user values, based on which we can then estimate the mean of a specific range. SW mechanism [26] supports such a solution. Assume the approximated distribution by SW is denoted by $\hat{\mathbf{d}}$. Given a specific range $[l, r]$, the solution works as follows. First, the bins of the specified range, represented by the set $\left\{t^1_{(l,r)}, \cdots, t^k_{(l,r)}\right\}$, are filtered, along with their corresponding probability densities, represented by the set $\left\{\hat{d}^1_{(l,r)}, \cdots, \hat{d}^k_{(l,r)}\right\}$. The mean of in-range data can be calculated by dividing the corresponding sum by the count of values. Formally,

$$\hat{m}_{(l,r)} = \frac{n \sum_{i=1}^{k} t^i_{(l,r)} \hat{d}^i_{(l,r)}}{n \sum_{i=1}^{k} \hat{d}^i_{(l,r)}} = \frac{\sum_{i=1}^{k} t^i_{(l,r)} \hat{d}^i_{(l,r)}}{\sum_{i=1}^{k} \hat{d}^i_{(l,r)}}.$$

Indeed, a range mean query primarily focuses on the values within a specified range. However, to prevent the disclosure of other values, both Solutions 1 and 2 treat all values equally, and consider the overall domain length 2 as the sensitivity when adding noise to a value. These broad-brush solutions do not differentiate between values within and outside the specified range, thus diverting attention from the specified range, which may result in significant utility loss.

More specifically, in Solution 1, the estimated mean is unbiased only if the effects of the data entering and exiting the range after the perturbation are cancelled out. As for Solution 2, SW needs to approximate the value distribution on the whole domain, dividing it into 1024 blocks. When given a small query range, such as 1/1024 of the domain, SW will treat that region as a single block, ignoring the distribution within that region and assuming a uniform distribution by default. This ultimately leads to large non-uniform estimation error.

In a nut shell, for range mean query, it is beneficial to have a more fine-grained solution that focuses on the values within the

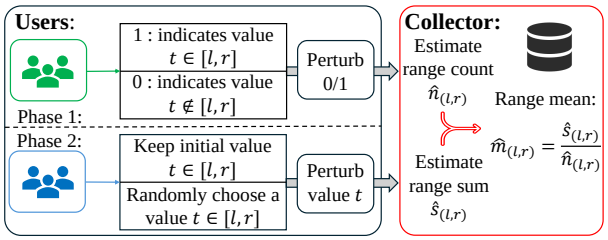


Figure 1: The overview of our *PrivRM* framework

specified range. This could potentially lead to a reduction in utility loss.

4 *PRIVRM*: PRIVATE RANGE MEAN ESTIMATION

In this section, we present a general framework for locally differentially Private Range Mean estimation (*PrivRM*), which differentiates values within and outside the specified range to enhance the estimation accuracy. In what follows, we will present an overview of *PrivRM* in Section 4.1 and then introduce two implementations in Sections 4.2 and 4.3, respectively.

4.1 Framework Overview

To concentrate on the target range rather than the entire domain, a sensible strategy is to confine the perturbed values to be within the range, while the impact of values initially outside this range can be eliminated from the aggregation. Inspired by this, we introduce the *PrivRM* framework for estimating the range mean, and Figure 1 shows an overview of it. Specifically, *PrivRM* first employs the RR mechanism to perturb whether each user's value falls within the range $[l,r]$, estimating the count of values within this range (a.k.a., range count), denoted by $\hat{n}_{(l,r)}$. Subsequently, it uses an NVP mechanism (see Section 2.3) to perturb the values, estimating the sum of values that are initially within the range (a.k.a., range sum). The range mean is then calculated by dividing this range sum by the range count. However, a challenge remains: how to perturb the values outside the range $[l,r]$ in a way that satisfies ϵ-LDP and produces an unbiased sum $\hat{s}_{(l,r)}$, free from the influence of those outside values.

Formally, we can calculate the range sum $\hat{s}_{(l,r)}$ as

$$\hat{s}_{(l,r)} = \sum_{i=1}^{n} \hat{t}_i - (n - \hat{n}_{(l,r)})\mathbb{E}[\hat{t}_{out}],$$

where $\mathbb{E}[\hat{t}_{out}]$ is the expectation of the perturbed values initially outside the range $[l,r]$. Then we can estimate the range mean as

$$\hat{m}_{(l,r)} = \frac{\hat{s}_{(l,r)}}{\hat{n}_{(l,r)}} = \frac{\sum_{i=1}^{n} \hat{t}_i - (n - \hat{n}_{(l,r)})\mathbb{E}[\hat{t}_{out}]}{\hat{n}_{(l,r)}} \tag{6}$$

Theorem 2. *In PrivRM, with an unbiased NVP mechanism, the estimated sum is also unbiased, i.e.,* $\mathbb{E}[\hat{s}_{(l,r)}] = s_{(l,r)}$. *And the variance of range mean is*

$$Var(\hat{m}_{(l,r)}) \approx \frac{\mathbb{E}^2[X]}{\mathbb{E}^2[Y]} \left(\frac{Var(X)}{\mathbb{E}^2[X]} + \frac{Var(Y)}{\mathbb{E}^2[Y]} \right) \tag{7}$$

where $X = \sum_{i=1}^{n} \hat{t}_i - n\mathbb{E}[\hat{t}_{out}]$ *and* $Y = \hat{n}_{(l,r)}$.

Proof. The proof appears in Appendix B. □

Algorithm 1: Workflow of *PrivRM*^{I}

Input: A set of user values $\{t_1, t_2, ..., t_n\}$, query range $[l,r]$, NVP mechanism $\mathcal{A}(\cdot)$, privacy budget ϵ
Output: The estimated range mean $\hat{m}_{(l,r)}$

1 **for** $i = 1$ **to** n **do** // Phase 1
2 Set $p = \frac{e^{\epsilon/2}}{1+e^{\epsilon/2}}$
3 **if** $t_i \in [l,r]$ **then**
4 Report bit 1 (resp. 0) with probability p (resp. $1-p$)
5 **else**
6 Report bit 0 (resp. 1) with probability p (resp. $1-p$)
7 Collector derives the range count $\hat{n}_{(l,r)} = \frac{n(p-1)+\#('1')}{2p-1}$
8 **for** $i = 1$ **to** n **do** // Phase 2
9 **if** $t_i \in [l,r]$ **then**
10 ***Normalization***: $t'_i = \frac{2(t_i - l)}{r-l} - 1$
11 ***Perturbation***: $y_i = \mathcal{A}(t'_i, \epsilon/2)$
12 ***Denormalization***: $\hat{t}_i = \frac{(y_i+1)(r-l)}{2} + l$
13 **else**
14 *Randomly draw a value* $t'_i \in [-1,1]$
15 ***Perturbation***: $y_i = \mathcal{A}(t'_i, \epsilon/2)$
16 ***Denormalization***: $\hat{t}_i = \frac{(y_i+1)(r-l)}{2} + l$
17 Calculate $\hat{m}_{(l,r)}$ by Equation 6, where $\mathbb{E}[\hat{t}_{out}] = \frac{l+r}{2}$
18 **return** $\hat{m}_{(l,r)}$

According to Equation 6, range mean estimation also relies on $\mathbb{E}[\hat{t}_{out}]$. To facilitate its estimation, we may apply input or output domain randomization to the values outside the range $[l,r]$. Therefore, we devise the following two implementations, namely *PrivRM*^{I} and *PrivRM*^{O}.

4.2 *PrivRM*^{I}: Input Domain Randomization

We first introduce input-randomization-based implementation of the framework, namely *PrivRM*^{I}. The idea is to randomize the values initially outside the query range $[l,r]$ into random values within this range. Therefore, the randomized values, which become the inputs of the NVP mechanism, satisfy a uniform distribution over the range $[l,r]$. After applying an NVP mechanism to these values which are initially outside the range, the expectation of their perturbed values is kept fixed, i.e., $\mathbb{E}[\hat{t}_{out}] = \frac{l+r}{2}$.

Algorithm 1 shows the pseudo-code of *PrivRM*^{I}. It involves two phases, each of which consumes half of the given privacy budget ϵ. In Phase 1, each user employ RR to report whether their value is within the range $[l,r]$ (Lines 3-6). Then in Line 7, the collector derives the range count $\hat{n}_{(l,r)}$ by taking noise calibration into account. In Phase 2, we focus on estimating the range sum. Since the value domain of the NVP mechanisms is unified as $[-1,1]$, before perturbing a value $t_i \in [l,r]$, we will first normalize it into $t'_i \in [-1,1]$ (Line 10). The perturbation is then applied to t'_i with the remaining privacy budget $\epsilon/2$ (Line 11). Subsequently, we will also denormalize the perturbed value y_i (Line 12). For a value outside the range $[l,r]$, we draw a random value from the domain, i.e., $t'_i \in [-1,1]$, perturb it and then apply denormalization (Lines 14-16). Here since the random value t'_i is drawn from the input domain $[-1,1]$, which is often different from the output domain of an NVP mechanism, t'_i will be then perturbed to ensure the indistinguishability of output values initially within and outside the range. Ultimately, we can estimate the range mean via Equation 6 (Line 17).

In *PrivRM*^{I}, all the involved perturbation mechanisms (including RR, LM, SR, PM, HM and Unbiased SW) are all unbiased. Thus we

Algorithm 2: Workflow of $PrivRM^O$

Input: A set of user values $\{t_1, t_2, ..., t_n\}$, query range $[l, r]$, NVP mechanism $\mathcal{A}(\cdot)$, output domain $\mathbb{M}_{\mathcal{A}}(\cdot)$, privacy budget ϵ
Output: The estimated range mean $\hat{m}_{(l,r)}$

1 Lines 1-7 of Algorithm 1 // Phase 1
2 **for** $i = 1$ **to** n **do** // Phase 2
3 **if** $t_i \in [l, r]$ **then**
4 ***Normalization, Perturbation, Denormalization*** with $\frac{\epsilon}{2}$
  /* Same as $PrivRM^I$, except TLM←LM */
5 **else**
6 *Randomly draw a value* $y_i \in \mathbb{M}_{\mathcal{A}}(\epsilon/2)$
7 ***Denormalization***
8 Calculate $\hat{m}_{(l,r)}$ by Equation 6, where $\mathbb{E}[\hat{t}_{out}] = \frac{l+r}{2}$
9 **return** $\hat{m}_{(l,r)}$

have

$$\mathbb{E}[X] = s_{(l,r)} + (n - n_{(l,r)})\frac{l+r}{2}, \mathbb{E}[Y] = n_{(l,r)}.$$

As for the variance,

$$Var(X) = n \cdot Var(\epsilon/2), Var(Y) = \frac{ne^{\epsilon/2}}{\left(e^{\epsilon/2}-1\right)^2},$$

where $Var(\epsilon/2)$ is the variance of a NVP mechanism with privacy budget $\epsilon/2$. We summarize the variance of each NVP mechanism in Appendix C. Subsequently, the variance of $PrivRM^I$ can be calculated through Equation 7.

4.3 $PrivRM^O$: Output Domain Randomization

In $PrivRM^I$, both randomization and perturbation are applied to the values initially outside the range $[l, r]$, since the randomization domain $[-1, 1]$ may be different from the output domain of an NVP mechanism $\mathcal{A}(\cdot)$. Inspired by this, we may alternatively draw a random value from the output domain $\mathbb{M}_{\mathcal{A}}(\cdot)$, so that we can remove the perturbation step in Phase 2. This directly leads to an output-randomization-based implementation of the framework, namely $PrivRM^O$.

As depicted in Algorithm 2, $PrivRM^O$ also starts by estimating the range count through RR with half the privacy budget (Line 1). In Phase 2, users whose values fall within the range, they normalize, perturb and then denormalize their values, similar to the $PrivRM^I$ procedure, also with half the privacy budget (Line 4). And others draw a random value from the output domain $\mathbb{M}_{\mathcal{A}}(\epsilon/2)$ of the NVP mechanism (Line 6).

However, when a perturbation mechanism produces unbounded noise (e.g., LM), it is not feasible to randomly select values from real number field $\mathbb{R}$. As such, we categorize them into two cases, namely bounded and unbounded LDP mechanisms, and discuss them as follows.

Case 1: Bounded LDP Mechanisms. As for the bounded NVP mechanisms, the details of Phase 2 remain consistent with those mentioned earlier. Here, we take SR and PM as examples to illustrate the process. When using SR, for users within the range, they directly perturb their values with the sensitivity $|r - l|$ through SR (Line 4). For users outside the range, each of their outputs satisfies the Bernoulli distribution with probability 0.5 in $\mathbb{M}_{SR}(\epsilon/2) = \{-C_{SR}(\epsilon/2), C_{SR}(\epsilon/2)\}$ (Line 6). When using PM, for users outside the range, each of their outputs satisfies the uniform distribution in $\mathbb{M}_{PM}(\epsilon/2) = [-C_{PM}(\epsilon/2), C_{PM}(\epsilon/2)]$ (Line 6).

Case 2: Unbounded LDP Mechanisms. As for unbounded LDP mechanisms, the idea is to truncate their output domains, which allows us to discuss the unbounded case by transforming it into the bounded case. Here, we take LM as an example. Without loss of generality, we truncate the output domain to the input domain, $[-1, 1]$. For users within the range, they perturb their values by LM and then truncate the perturbed data to $[-1, 1]$ (Line 4). As for out-of-range users, we take the input value of 0 as the reference standard. When the input is 0, the sum of probabilities at truncation points is $e^{-\epsilon/2}$, and the sum of probabilities within $(-1, 1)$ is $1 - e^{-\epsilon/2}$. Therefore, they should follow this reporting strategy: choosing a value uniformly in the range $(-1, 1)$ with probability $\alpha_{TLM}(\epsilon) = 1 - e^{-\epsilon/2}$, and randomly selecting a value from $\{-1, 1\}$ with probability $1 - \alpha_{TLM}(\epsilon) = e^{-\epsilon/2}$ (Line 6). This method is designated as Truncation-based Laplace Mechanism (TLM).

Theorem 3. *For TLM, if the truncation range is* $[-1, 1]$, *it achieves* ϵ*-LDP.*

Proof. The proof appears in Appendix D. □

After introducing these two cases, we can apply all NVP mechanisms to $PrivRM^O$ and estimate the range mean through Equation 6 (Line 8). It is clear that $\mathbb{E}[\hat{t}_{out}]$ remains constant, i.e., $\mathbb{E}[\hat{t}_{out}] = \frac{l+r}{2}$. Compared to $PrivRM^I$, in $PrivRM^O$, users outside the range do not undergo the perturbation step, thereby clearly demonstrating that $PrivRM^O$ satisfies ϵ-LDP.

When the NVP mechanism is TLM, truncation results in the equality of $\mathbb{E}[X]$ and $s_{(l,r)} + (n - n_{(l,r)})\frac{l+r}{2}$ only, when the data within the query range follows a symmetric distribution. When employing other NVP mechanisms, $PrivRM^O$ is unbiased in both phases. Thus,

$$\mathbb{E}[X] = s_{(l,r)} + (n - n_{(l,r)})\frac{l+r}{2}, \ \ \mathbb{E}[Y] = n_{(l,r)}.$$

In the Phase 1, all users perturb their data through RR mechanism. However, during the Phase 2, only in-range users report perturbed values through NVP mechanism and out-of-range users report random values from $\mathbb{M}_{\mathcal{A}}(\epsilon/2)$. Thus we have

$$Var(X) = n_{(l,r)}Var(\epsilon/2) + (n - n_{(l,r)})U(\epsilon/2),$$

$$Var(Y) = \frac{ne^{\epsilon/2}}{\left(e^{\epsilon/2}-1\right)^2},$$

where $U(\epsilon/2)$ is the variance of uniform distribution over $\mathbb{M}_{\mathcal{A}}(\epsilon/2)$. Subsequently, the variance of $PrivRM^O$ can be calculated through Equation 7.

5 *PRIVRM**: AN OPTIMIZED IMPLEMENTATION

The *PrivRM* framework enhances utility of the range mean estimation by reducing sensitivity from the entire domain to the target range. However, the two implementations of the framework, $PrivRM^I$ and $PrivRM^O$, completely separate the process into two distinct phases, allocating half of the privacy budget to each phase. This division inevitably leads to an increase in the scale of noise introduced. In this section, we present an optimized implementation, *PrivRM**, which allows us to leverage more privacy budget across

both phases while still providing LDP guarantee, thus achieving better utility.

5.1 Design Rationale of *PrivRM**

Note that *PrivRM*^{I} and *PrivRM*^{O} estimate both the range count and mean across all users, necessitating the division of the privacy budget ϵ into two parts. To avoid this division of the privacy budget, we could consider estimating the range count and mean using either in-range or out-of-range values exclusively. Specifically, given a range $[l, r]$, those in-range values could serve for estimating the range sum $s_{(l,r)}$, while out-of-range values could contribute to the range count $n_{(l,r)}$. For the later, it is feasible since the total number of in-range and out-of-range values always equals n. This idea directly leads to our optimized implementation, *PrivRM**, of the framework.

The details of *PrivRM** are shown in Algorithm 3. Firstly in Phase 1, the out-of-range users apply RR perturbation with probability p ($p > 0.5$) (Line 5), while others report 0 or 1 uniformly at random (Line 3). Then the collector aggregates the range count $\hat{n}_{(l,r)}$ (Line 6). Note that the aggregation in Line 6 is different from that in *PrivRM*^{I} and *PrivRM*^{O} (Line 7 of Algorithm 1), since we differentiate the perturbation between in-range and out-of-range values. The following Theorem 4 establishes the correctness of this estimation.

In Phase 2, users' reporting way is similar to *PrivRM*^{O}, but the difference is that the privacy budget ϵ' used in this phase depends on p. On the other hand, since the NVP mechanism needs to take a privacy budget as input, we empirically set it as $\epsilon' = \log(\frac{p}{1-p})$ (Line 7). The privacy guarantee will be further analyzed in Section 5.2. For values within the range, they will go through normalization, perturbation and denormalization with privacy budget ϵ', as with *PrivRM*^{O} (Lines 9-10). As for the values outside the range, they will be randomized into a value drawn from the output domain $\mathbb{M}_{\mathcal{A}}(\epsilon')$ (Line 12). Ultimately, the collector estimates the range mean $\hat{m}_{(l,r)}$ via Equation 6 (Line 14).

Algorithm 3: Workflow of *PrivRM**

Input: A set of user values $\{t_1, t_2, ..., t_n\}$, query range $[l, r]$, NVP mechanism $\mathcal{A}(\cdot)$, output domain $\mathbb{M}_{\mathcal{A}}(\cdot)$, perturbation probability p
Output: The estimated range mean $\hat{m}_{(l,r)}$

1 **for** $i = 1$ **to** n **do** // Phase 1
2 **if** $t_i \in [l, r]$ **then**
3 Report bit 1 or 0 uniformly at random
4 **else** // RR Perturbation
5 Report bit 0 (resp. 1) with probability p (resp. $1 - p$)
6 Aggregator derives the count $\hat{n}_{(l,r)} = \frac{2\#(\text{'1'})+2(p-1)n}{2p-1}$
7 Set $\epsilon' = \log(\frac{p}{1-p})$
8 **for** $i = 1$ **to** n **do** // Phase 2
9 **if** $t_i \in [l, r]$ **then**
10 ***Normalization, Perturbation, Denormalization*** with ϵ'
11 **else**
12 *Randomly draw a value* $y_i \in \mathbb{M}_{\mathcal{A}}(\epsilon')$
13 ***Denormalization***
14 Calculate $\hat{m}_{(l,r)}$ by Equation 6, where $\mathbb{E}[\hat{t}_{out}] = \frac{l+r}{2}$
15 **return** $\hat{m}_{(l,r)}$

Theorem 4. *In Algorithm 3, we have* $\mathbb{E}[\frac{2\#(\text{'1'})+2(p-1)n}{2p-1}] = n_{(l,r)}$.

Proof. Bit 1 will be reported from out-of-range values with probability $1 - p$ or in-range values with probability 0.5. Thus, the expected number of observed bit 1 is

$$\mathbb{E}[\#(\text{'1'})] = (1-p)(n - n_{(l,r)}) + 0.5n_{(l,r)}. \tag{8}$$

Through Equation 8, we have,

$$\mathbb{E}[\frac{2\#(\text{'1'}) + 2(p-1)n}{2p-1}] = n_{(l,r)}.$$

□

5.2 Privacy Analysis of *PrivRM**

Given a perturbation probability p, the following Theorem 5 demonstrates that the we can apply various NVP mechanisms to *PrivRM** under LDP setting.

Theorem 5. *Given a probability p, PrivRM** *satisfies* ϵ*-LDP, where* $\epsilon = \log(max\left\{\frac{p}{1-p}, \frac{p\cdot q}{0.5\cdot q'}, \frac{0.5\cdot p'}{(1-p)\cdot q}\right\})$, p' *and* q' *are the minimum and maximum in the perturbation probability density function (probability mass function for SR) of the NVP mechanism, and q is the probability of a random output on* $\mathbb{M}_{\mathcal{A}}(\cdot)$.

Proof. For an input value t_i, let $s_i = \left\langle \hat{k}_i, \hat{t}_i \right\rangle$ be the output of *PrivRM**, where $\hat{k}_i$ and $\hat{t}_i$ correspond to the outputs of Phase 1 and Phase 2, respectively. According to Definition 1, for any pair of distinct input values, the probability ratio for producing the same output must fall between $e^{-\epsilon}$ and e^{ϵ}. Here we consider two distinct cases: the first is when both users with differing inputs are within the query range, or alternatively, both are outside this range. The second case occurs when one user is inside the range and another is outside.

Case 1: If both values t_1, t_2 come from outside the query range, then in Phase 2, output randomization occurs for both. For any output s we have,

$$\frac{\Pr[s \mid t_1]}{\Pr[s \mid t_2]} = \frac{p \cdot q}{p \cdot q} = \frac{(1-p)\cdot q}{(1-p)\cdot q} = 1.$$

If both values t_1, t_2 are inside the query range $[l, r]$, they both report 0/1 randomly during Phase 1. Similarly they both use NVP perturbations with a privacy budget of $\epsilon' = \log(\frac{p}{1-p})$ in Phase 2. For any output s we have,

$$\frac{\Pr[s \mid t_1]}{\Pr[s \mid t_2]} \leq \frac{0.5 \cdot p'}{0.5 \cdot q'} = e^{\epsilon'} = \frac{p}{1-p}.$$

Case 2: Assume that input $t_1 \in [l, r]$ and $t_2 \notin [l, r]$, it is easy to see that

$$\frac{0.5 \cdot q'}{p \cdot q} \leq \frac{Pr[s \mid t_1]}{Pr[s \mid t_2]} \leq \frac{0.5 \cdot p'}{(1-p)\cdot q}.$$

Therefore, according to Definition 1, *PrivRM** satisfies ϵ-LDP, where $\epsilon = \log(max\left\{\frac{p}{1-p}, \frac{p\cdot q}{0.5\cdot q'}, \frac{0.5\cdot p'}{(1-p)\cdot q}\right\})$. □

In what follows, we further analyze the privacy guarantee when integrating TLM, SR, PM, HM and Unbiased SW into *PrivRM**, as shown in Theorems 6-10. For ease of reference, in Table 2, we summarize the relationship between the perturbation probability p and the privacy budget ϵ achieved by *PrivRM**.

Table 2: Relationship between p and ϵ for different NVP mechanisms

Mechanism $\mathcal{A}(\cdot)$	Privacy budget achieved by *PrivRM** with $\mathcal{A}(\cdot)$
TLM	$\epsilon = \begin{cases} \log\left(\frac{4p\left(\sqrt{\frac{p}{1-p}}p+p-\sqrt{\frac{p}{1-p}}\right)}{(1-p)\log\left(\frac{p}{1-p}\right)}\right), & \text{if } p < 0.75, \\ \log\left(\frac{1}{2-2p}\sqrt{\frac{p}{1-p}}\right), & \text{otherwise}. \end{cases}$
SR	$\epsilon = \log\left(\frac{p}{1-p}\right)$
PM	$\epsilon = \log\left(\frac{1}{2-2p}\sqrt{\frac{p}{1-p}}\right)$
HM	$\epsilon = \begin{cases} \log\left(\frac{p}{1-p}\right), & \text{if } p < 0.65, \\ \log\left(\frac{1}{2-2p}\sqrt{\frac{p}{1-p}}\right), & \text{otherwise}. \end{cases}$
Unbiased SW	$\epsilon = \log\left(\frac{2p-1}{2(p-1)^2\log\left(\frac{p}{1-p}\right)}\right)$

Theorem 6. *Applying TLM to PrivRM** *satisfies ϵ-LDP, where*

$$\epsilon = \begin{cases} \log\left(\frac{4p\left(\sqrt{\frac{p}{1-p}}p+p-\sqrt{\frac{p}{1-p}}\right)}{(1-p)\log\left(\frac{p}{1-p}\right)}\right), & \text{if } p < 0.75, \\ \log\left(\frac{1}{2-2p}\sqrt{\frac{p}{1-p}}\right), & \text{otherwise}. \end{cases}$$

Proof. The proof appears in Appendix E. □

Theorem 7. *Applying SR to PrivRM** *satisfies ϵ-LDP, where* $\epsilon = \log\left(\frac{p}{1-p}\right)$.

Proof. The proof appears in Appendix F. □

Theorem 8. *Applying PM to PrivRM** *satisfies ϵ-LDP, where* $\epsilon = \log\left(\frac{1}{2-2p}\sqrt{\frac{p}{1-p}}\right)$.

Proof. The proof appears in Appendix G. □

Theorem 9. *Applying HM to PrivRM** *satisfies ϵ-LDP, where*

$$\epsilon = \begin{cases} \log\left(\frac{p}{1-p}\right), & \text{if } p < 0.65, \\ \log\left(\frac{1}{2-2p}\sqrt{\frac{p}{1-p}}\right), & \text{otherwise}. \end{cases}$$

Proof. The proof appears in Appendix H. □

Theorem 10. *Applying Unbiased SW to PrivRM** *satisfies ϵ-LDP, where* $\epsilon = \log\left(\frac{2p-1}{2(p-1)^2\log\left(\frac{p}{1-p}\right)}\right)$.

Proof. The proof appears in Appendix I. □

Now given a privacy budget ϵ, we can compute the privacy parameters used in the two phases of *PrivRM**, including perturbation probability p and the privacy budget ϵ' used in Phase 2. With varying privacy budgets eventually achieved by *PrivRM**, Figure 2 shows the corresponding privacy budget actually used in each phase, when integrating each NVP mechanism. For comparison purpose, we also plot the cyan curve, the privacy budget used in each phase of *PrivRM*^{I} and *PrivRM*^{O} (i.e., $\epsilon/2$). Obviously, we can observe that regardless of the NVP mechanism used, the privacy budget used by *PrivRM** is always larger than that used by the other two implementations. This means that, to achieve a same privacy level, *PrivRM** can use a larger privacy budget, leading to more accurate results. we also observe that when $\epsilon > 1.24$, the privacy parameter in TLM, PM and HM are the same. The reason is that these three mechanisms satisfies $\log\left(\frac{1}{2-2p}\sqrt{\frac{p}{1-p}}\right)$-LDP in this case.

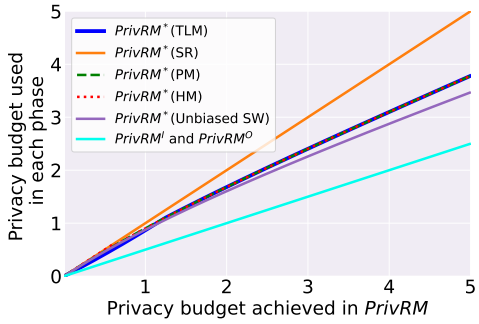


Figure 2: Corresponding privacy parameter used in each phase - Privacy budget achieved in *PrivRM*

5.3 Which Implementation to Use?

Note that all the five NVP mechanisms can be integrated into our implementations, namely *PrivRM*^{I}, *PrivRM*^{O} and *PrivRM**. We may need to answer a question: which implementation to use in a given setting? Fortunately, Theorem 2 enables us to theoretically evaluate these three implementations through estimation variance.

First we need to calculate the variance of *PrivRM**. With the above Theorems 6-10 and Theorem 2, given the privacy budget ϵ, we can compute the corresponding the variance of *PrivRM**. Let $X = \sum_{i=1}^{n} \hat{t}_i - n\mathbb{E}[\hat{t}_{out}]$ and $Y = \hat{n}_{(l,r)}$ When the NVP mechanism is TLM, truncation similarly results in the equality of $\mathbb{E}[X]$ and $s_{(l,r)} + (n - n_{(l,r)})\frac{l+r}{2}$ only if the data within the query range follows a symmetric distribution. When employing other NVP mechanisms, *PrivRM** is also unbiased in both phases. Thus,

$$\mathbb{E}[X] = s_{(l,r)} + (n - n_{(l,r)})\frac{l+r}{2}, \quad \mathbb{E}[Y] = n_{(l,r)}.$$

In *PrivRM**, in-range and out-of-range users report randomly in Phase 1 and Phase 2 respectively, so we have

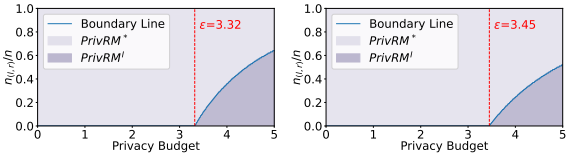
$$Var(X) = n_{(l,r)}Var(\epsilon') + (n - n_{(l,r)})U(\epsilon'),$$

$$Var(Y) = \frac{n_{(l,r)}p(1-p) + 0.25(n - n_{(l,r)})}{(2p-1)^2}.$$

Subsequently, the variance of *PrivRM** can be calculated through Equation 7.

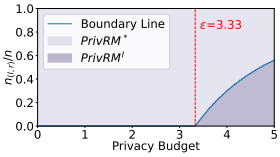
After obtaining the variance of *PrivRM**, we can choose the optimal implementation. When the data within the range is not symmetrically distributed, TLM is biased in both the *PrivRM** and *PrivRM*^{O}. Therefore, under these conditions, *PrivRM*^{I} should be used. When other mechanisms are employed, the sum and count estimation of these three implementations are unbiased, that is, $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ are the same. Besides, since n is much larger than $Var(Y)$, we have $\frac{Var(Y)}{\mathbb{E}^2[Y]} \approx 0$. This means that the final variance depends on $Var(X)$. First we compare *PrivRM*^{I} and *PrivRM*^{O}. It is evident that, regardless of the value of $n_{(l,r)}$, we always have

$$nVar(\epsilon) \leq n_{(l,r)}Var(\epsilon/2) + (n - n_{(l,r)})U(\epsilon/2),$$

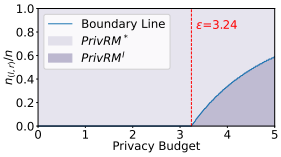


(a) SR

(b) PM



(c) HM



(d) Unbiased SW

Figure 3: Comparison between *PrivRM** and *PrivRM*^{I}

which means that *PrivRM*^{I} is always better than *PrivRM*^{O}. Then, we compare *PrivRM*^{I} and *PrivRM**. As we mentioned above, given the privacy budget ϵ, we can obtain the corresponding ϵ'. The variance of *PrivRM** is less than that of *PrivRM*^{I} when the following conditions are satisfied:

$$\Delta = nVar(\epsilon) - n_{(l,r)}Var(\epsilon') - (n - n_{(l,r)})U(\epsilon') > 0.$$

A Black-box Guidelines. Given privacy budget ϵ, $n_{(l,r)}$ and NVP mechanism we used $\mathcal{A}(\cdot)$ as inputs, our analysis gives the following guidelines for choosing optimal implementation.

- When employing the LM, it is advisable to use *PrivRM*^{I}, as it remains unbiased under all circumstances.
- When using other mechanisms, if $\Delta > 0$, we should employ the *PrivRM**; otherwise, *PrivRM*^{I} should be used.

Discussion. We can change inputs of black-box guidelines to get the implementation with the lowest variance [1], as shown in Figure 3. It is evident that when the privacy budget is relatively small ($\epsilon < 3.32$ for SR, $\epsilon < 3.45$ for PM, $\epsilon < 3.33$ for HM, $\epsilon < 3.24$ for Unbiased SW), the variance of *PrivRM** is consistently minimized. Intuitively, when the privacy budget is small, the variance of any mechanism decreases exponentially as the privacy budget increases. This is why the *PrivRM** performs better when the privacy budget is small.

6 DISTRIBUTION-AWARE OPTIMIZATION ON FRAMEWORK

Given a specific range, the *PrivRM* framework reduces the sensitivity of estimating the range mean from the entire domain to the query range. Upon looking into the range, it's observed that for a skewed data distribution, where the majority of values are concentrated in an area much smaller than the query range, the noise sensitivity can be further reduced by confining the perturbation space to a densely populated region. Inspired by this, we propose a strategy of Adaptive Adjustment (AA) on perturbation space based on a binary search.

[1]Here to maximize the difference between the variance of NVP mechanism and that of uniform distribution, we use the best-case variance of the NVP mechanism.

6.1 Adaptive Adjustment Strategy

When the majority of values are concentrated in a small range, confining the perturbation space to this dense region can reduce the estimation variance, enhancing the utility of the estimation result. Considering the lack of prior information on the data distribution, a feasible solution is to allocate a portion of users to estimate this distribution, based on which we can traverse the query range to explore the optimal densely populated region. However, since this strategy requires traversing all sub-intervals within the query range, it suffers from low efficiency. To address this issue, we can then use a binary search to identify the densely populated region. Based on this intuition, we design a strategy of Adaptive Adjustment (AA) on perturbation space which can be integrated into *PrivRM* framework to enhance the accuracy of range mean estimation. In the AA strategy, we divide the perturbation space equally until we identify the most suitable dense region.

6.2 Iteration Condition

The essence of the AA strategy lies in the iteration condition, which can be constructed by jointly considering the variance and the data loss from confining the perturbation space, using the total squared error as the objective.

Given the query range $[l, r]$, privacy budget ϵ and the number of iterations k, let Ω_k be the current dense region with a width of $\frac{r-l}{2^k}$. In our framework *PrivRM*, the total variance for values perturbation is equal to $Var(\sum_{i=1}^{n}\hat{t}_i) = Var(X_k) = Var(\sum_{i=1}^{n}\hat{t}_i - n\mathbb{E}[\hat{t}_{out}])$. So the final variance is $Var(X)\left(\frac{r-l}{2^k}\right)^2$. Beside, the bias error of data loss caused by confining the perturbation space is $\left(\sum_{t_i \notin \Omega_k \cap t_i \in [l,r]} t_i\right)^2$. So the total squared error is as follows:

$$\mathbb{E}_k^1 = Var(X_k)\left(\frac{r-l}{2^k}\right)^2 + \left(\sum_{t_i \notin \Omega_k \cap t_i \in [l,r]} t_i\right)^2. \tag{9}$$

Here we assume that the dense region after the k-th round of binary division is Ω_{k+1}, from which we can easily infer that its width is $\frac{r-l}{2^k}$. Thus, we have,

$$\mathbb{E}_k^2 = Var(X_{k+1})\left(\frac{r-l}{2^{k+1}}\right)^2 + \left(\sum_{t_i \notin \Omega_{k+1} \cap t_i \in [l,r]} t_i\right)^2. \tag{10}$$

Through Equation 9 and Equation 10, we can derive the reduction of total squared error as,

$$\delta_k = \mathbb{E}_k^1 - \mathbb{E}_k^2, \tag{11}$$

When $\delta_k > 0$, it indicates that the total squared error has decreased after the binary division, thereby justifying the progression to the $(k+1)$-th round of binary search.

Intuitively, in the k-th round of searching, if most users are concentrated in region Ω_{k+1}, we have $Var(X_k) \approx Var(X_{k+1})$ and $\left(\sum_{t_i \notin \Omega_k \cap t_i \in [l,r]} t_i\right)^2 \approx \left(\sum_{t_i \notin \Omega_{k+1} \cap t_i \in [l,r]} t_i\right)^2$. Under these circumstances, it becomes clear that $\mathbb{E}_k^1 \approx 4\mathbb{E}_k^2$, i.e., $\delta_k > 0$ always holds. This means that we can designate Ω_{k+1} as the current dense region and proceed to the next round of the search.

Algorithm 4: Workflow of AA

Input: A set of user values $\{t_1, t_2, ..., t_n\}$, query range $[l, r]$, NVP mechanism $\mathcal{A}(\cdot)$, privacy budget ϵ
Output: The dense region Ω

1 Randomly sample 10% of the values from $\{t_1, t_2, ..., t_n\}$
2 Estimate the distribution of the sampled data by SW mechanism
3 Initialize $\Omega_1 = [l, r]$
4 **for** $k = 1$ **to** 10 **do** // $2^{10} = 1024$
5 Divide Ω_k equally into two sub-ranges Ω_k^1 and Ω_k^2
6 **if** $\sum \hat{d}^i_{\Omega_k^1} > \sum \hat{d}^i_{\Omega_k^2}$ **then**
7 $\Omega_{k+1} = \Omega_k^1$
8 **else**
9 $\Omega_{k+1} = \Omega_k^2$
10 Calculate the total squared error gap δ_k by Equation 11
11 **if** $\delta > 0$ **then** // Iteration condition
12 $\Omega_k = \Omega_{k+1}$
13 **else**
14 **return** Ω_k

6.3 Putting Things Together

By putting things together, Algorithm 4 shows the workflow of AA strategy. We start by randomly sampling 10% of values for estimating the data distribution (Lines 1 and 2) and then divide the perturbation space. Here we take the first round as an example. In the first round, we first divide perturbation space $[l, r]$ equally into two sub-ranges $\Omega_1^1 = [l, (l + r)/2]$ and $\Omega_1^2 = [(l + r)/2, r]$ (Line 5). We then calculate the sum of the probability densities of these two sub-ranges separately, i.e., $\sum \hat{d}^i_{\Omega_1^1}$ and $\sum \hat{d}^i_{\Omega_1^2}$. We then determine which sub-range has a higher density and label it as the dense region (Lines 6-9). Subsequently, we calculate the total squared error gap δ_k (Line 10), and then assess whether the iteration condition holds (Line 11). If it proves to be favorable, binary search is further applied to the dense sub-range (Line 12). Otherwise, we return the current perturbation space and designate it as the final dense region (Line 16).

Beside, with the prior distribution, we can estimate the range count $n_{(l,r)}$. In turn, we can choose the optimal implementation through the black box in Section 5.4 based on our chosen NVP mechanism and the privacy budget. Eventually, we can utilize $Var(X)$ of our chosen implementation to confine the perturbation space.

7 EXPERIMENTAL EVALUATION

In this section, we evaluate the performance of our proposed framework *PrivRM* for range mean estimation.

7.1 Experiment Setup

Datasets. We conduct experiment over four real-world datasets. Note that the original value domains of them are different, and we normalize all the values into $[-1, 1]$.

- Kosarak [3] is click-stream data , where the categories of clicks are considered representative values for the users. The value ranges from 1 to 41, 270, with a total of 990, 002 samples.
- House [4] contains real estate listings in the US. The value ranges from 0 to approximately 2.15×10^9, with a total of 2, 226, 382 samples.
- Fare [2] contains fare data for taxi travel in New York City, specifically the Yellow Taxi Trip Records from January 2019. The value ranges from −362 to 623, 259, with a total of 7, 696, 617 samples.
- Salary [1] is salary data from workers. The value ranges from 0 to 3, 421, 512, with a total of 1, 079, 289 samples.

Competitors. In our experiments, we first implement two baseline methods introduced in Section 3.2, namely Direct Estimation and Distribution-based Estimation. As for our methods, we focus on two versions. The first is *Optimal-PrivRM*, the best choice from *PrivRM*^{I}, *PrivRM*^{O} and *PrivRM**, according to the black-box guideline described in Section 5.3. The second is *Optimal-PrivRM-AA*, the *Optimal-PrivRM* coupled with AA strategy.

Experiment Design. We evaluate the performance of the four competitors with a fixed query range or with a fixed privacy budget, evaluate Distribution-based Estimation with different bin sizes, verify the correctness of our black-box selection scheme, and compare the performance of Original SW and Unbiased SW for mean estimation.

Metrics. As for the result evaluation, we employ the metric of Mean Squared Error (MSE)[27], which quantifies the average squared difference between the estimated values $\hat{m}_{(l,r)}$ and the actual values $m_{(l,r)}$. Formally,

$$MSE(l, r) = \frac{1}{N} \sum\nolimits_N [\hat{m}_{(l,r)} - m_{(l,r)}]^2, \tag{12}$$

where N represents the number of repetitions for each experiment. In our study, N is set to 50.

We conduct experiments using Python 3.11.5 and the Numpy 1.24.3 library on a desktop equipped with an Intel Core i5-13400F 1.50 GHz CPU and 64GB of RAM, running Windows 11.

7.2 Impact of Privacy Budgets

In this section, we evaluate the performance of four methods across four datasets and different NVP mechanisms with varying privacy budgets. As for the query range $[l, r]$, we set it to the first half, i.e. $[-1, 0]$. It is worth noting that only those NVP mechanisms whose output domains are continuous can be applied to Direct Estimation. Therefore, Direct Estimation in SR and HM is not feasible in Figure 4.

Overall, we observe that as the privacy budget increases, the MSE of all methods decreases. Besides, *Optimal-PrivRM-AA* performs the best, followed by *Optimal-PrivRM*, and finally two baseline methods. It is obvious that the gap between *Optimal-PrivRM-AA* and *Optimal-PrivRM* widens as the privacy budget increases. This is because, when the query range is fixed, the sensitivity of *Optimal-PrivRM* remains constant. As the privacy budget increases, the AA strategy's prior distribution estimate becomes more accurate, reducing the adjusted perturbation space and resulting in less noise in *Optimal-PrivRM-AA*. Additionally, the performance of Direct Estimation does not significantly differ when using LM, PM, and Unbiased SW, regardless of the privacy budget. This is because, for a given query range, the query bias is fixed. As the privacy budget increases, the variance of the estimated result decreases, but this change is negligible compared to the bias.

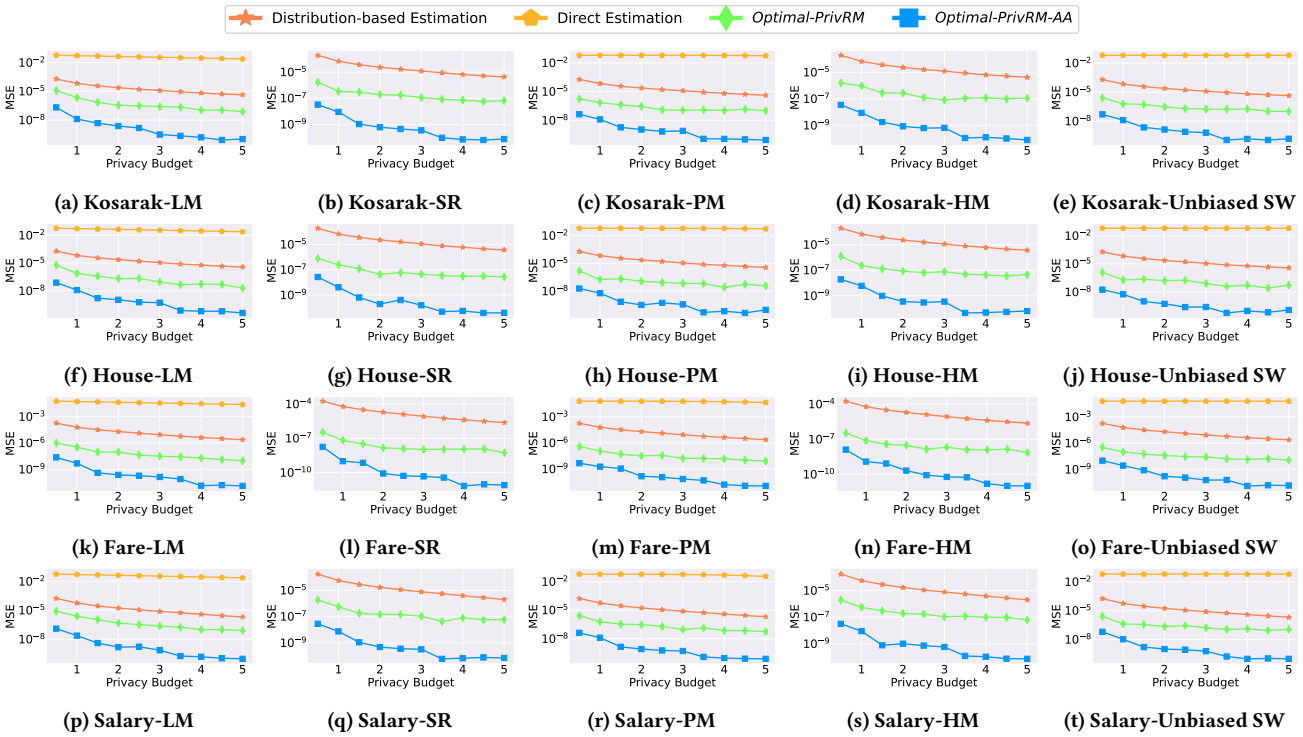


Figure 4: Performance of different methods with varying privacy budgets.

7.3 Impact of Range Size

This subsection studies the impact of the range size. The results are shown in Figure 5, where the range size refers to the span length starting from -1, and the privacy budget is fixed at $\epsilon = 2.5$. Likewise, results of Direct Estimation in SR and HM are not feasible.

We observe that *Optimal-PrivRM-AA*always yields the highest accuracy, followed by *Optimal-PrivRM*. Notably, with an increasing range size, the MSE of *Optimal-PrivRM* also increases, whereas MSE of *Optimal-PrivRM-AA* remains almost unchanged. This can be attributed to the lower noise sensitivity of *Optimal-PrivRM-AA* than *Optimal-PrivRM*. The sensitivity of *Optimal-PrivRM* increases with the query range size, whereas *Optimal-PrivRM-AA* always tunes the query range to the densely distributed region, resulting in a relatively constant sensitivity.

7.4 Correctness of Black-box Guidelines

In this subsection, we validate the correctness of the black-box guideline proposed in Section 5.3. Given a privacy budget and an NVP mechanism, it guides us to choose the optimal implementation of the *PrivRM* framework from $PrivRM^I$, $PrivRM^O$, and $PrivRM^*$.

We conduct experiment on Salary dataset, and show the results in Figure 6. First, we analyze the implementations without AA strategy, namely $PrivRM^I$, $PrivRM^O$ and $PrivRM^*$, and *Optimal-PrivRM*. We observe that *Optimal-PrivRM* achieves the lowest MSE in most cases.

In Figure 6(a), the lines for *Optimal-PrivRM* and $PrivRM^I$ overlap. This is because when the NVP mechanism is LM, $PrivRM^I$ is optimal according to the black box. When applying other mechanisms, black box indicates that $PrivRM^*$ is optimal, so the line for *Optimal-PrivRM* overlaps with that of $PrivRM^*$. However, when the privacy budget exceeds 3.5, we observe that *Optimal-PrivRM* may not always be the best. This is because when $\epsilon > 3.5$, the optimal implementation starts to depend on the range count according to the prior distribution estimation. However, the prior distribution estimation involves noise, leading to inaccurate range counts. For a detailed theoretical analysis, please refer to Section 5.3.

Integrating our black-box guideline into *PrivRM-AA* leads to a solution *Optimal-PrivRM-AA*, which achieves the best performance in most cases. *Optimal-PrivRM-AA* overlaps with $PrivRM^I$*-AA* in Figure 6(a), and overlaps with $PrivRM^*$*-AA* in Figures 6(b)-(d), which is consistent with the non-AA versions. This is because, for a given dataset and query range, the range counts of the adjusted perturbation space derived from the AA strategy are highly consistent with those of the original ranges.

7.5 Impact of the Number of Bins on Distribution-based Estimation

For baseline method Distribution-based Estimation, more bins ensures more accurate results, but at the cost of more time consumption. To show some empirical analysis, we set the number of bins to 256, 512, 1024, 2048, and 4096 in our evaluations, and utilize PM as the NVP mechanism for *Optimal-PrivRM* and *Optimal-PrivRM-AA*. As shown in Figure 7, the MSE of Distribution-based Estimation decreases as the number of bins increases. This is because more

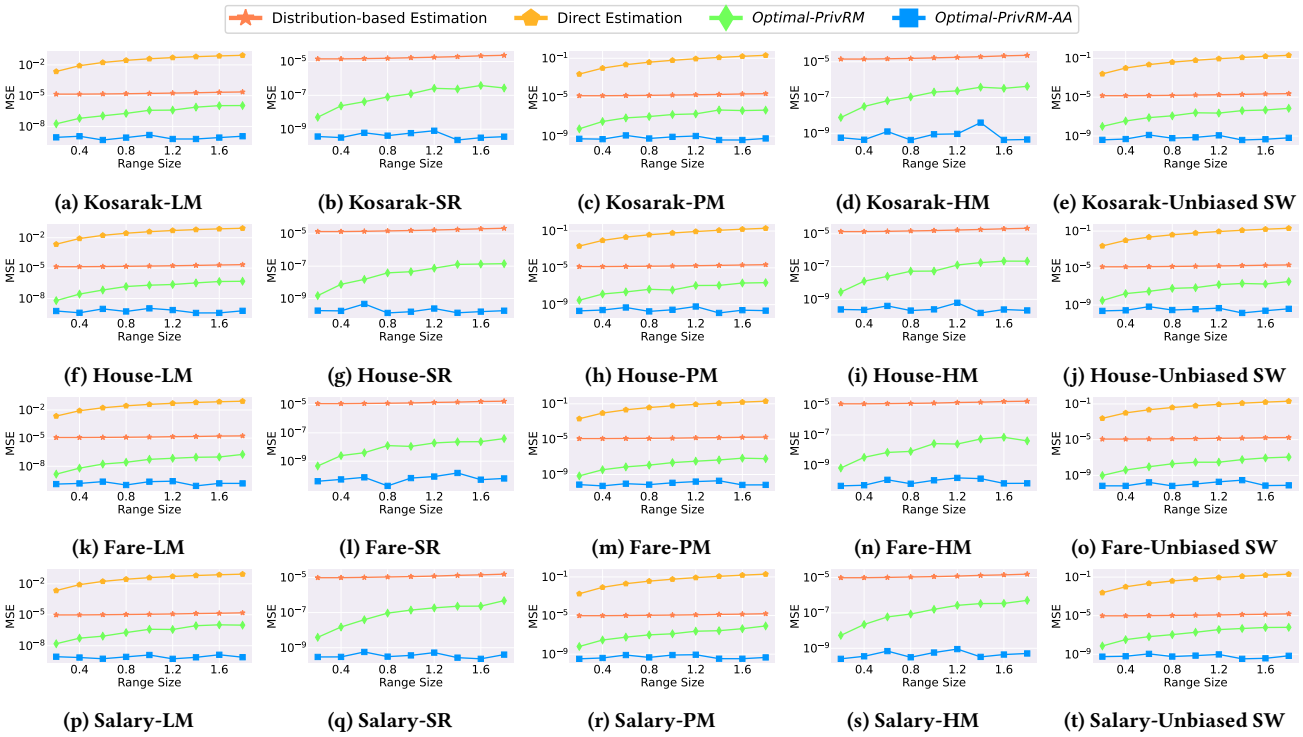


Figure 5: Performance on real-world dataset with varying size of query range. Query range starts from -1.

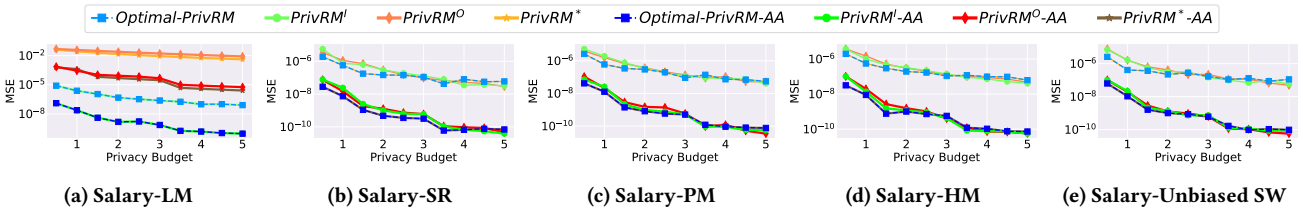


Figure 6: Results of different implementations on Salary Dataset. The query range is $[-1, 0]$.

bins allows SW to estimate the changes in data distribution more accurately, which in turn leads to a performance improvement. Nevertheless, even when the number of bins is as large as 4096 [2], Distribution-based Estimation still causes significantly higher MSE than that of *Optimal-PrivRM* and *Optimal-PrivRM-AA*.

7.6 Comparison between Original SW and Unbiased SW

As previously stated, the original SW mechanism is unbiased only if the data is symmetrically distributed over its value domain. Therefore, when the data is symmetrical, the mean estimation performance of the Original SW and the Unbiased SW should be similar. However, when the data is asymmetrical, the Original SW is expected to perform significantly worse than the Unbiased SW. To verify this hypothesis, we conduct experiments using both the Original SW and the Unbiased SW on Gaussian distributions with varying parameters.

In Figure 8(a), we use a Gaussian distribution centered at 100 with a domain of $(0, 300)$, discarding any data point outside this range. This makes the data distribution asymmetric. We conduct experiments using both the Original SW and the Unbiased SW with varying privacy budgets. As demonstrated, the Original SW is less effective than the Unbiased SW for any privacy budget, as the Original SW introduces bias when estimating the mean.

Next, we fix the privacy budget and change the degree of asymmetry of the Gaussian distribution in the range by moving the symmetry axis from 100 to 200. As illustrated in Figure 8(b), the Original SW outperforms the Unbiased SW only when symmetry axis is 150, where the Gaussian distribution is symmetric over the domain $(0, 300)$. Additionally, in this case, the variance of the Original SW is lower than that of the Unbiased SW, as shown in Appendix C.

[2]In existing work [26], the number of bins is set to 1024.

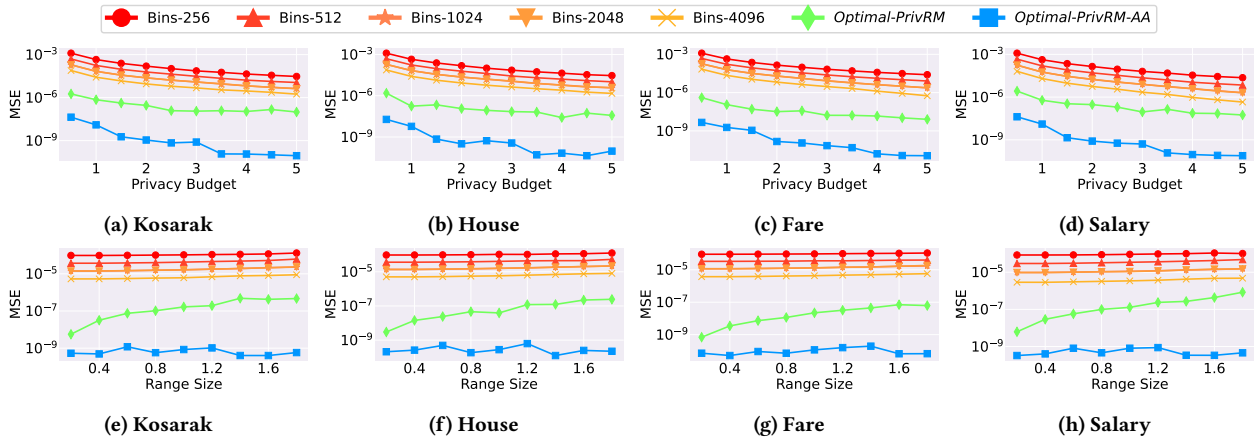


Figure 7: Performance with varying bins of distribution. The NVP mechanism used is PM. The query range and privacy budget are the same as those set in Figures 4 and 5.

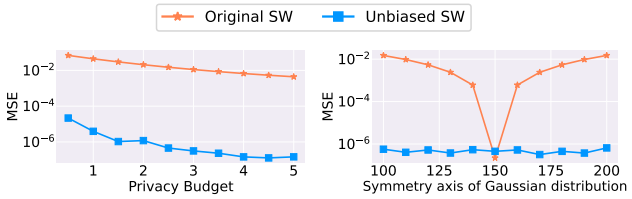


Figure 8: Comparison of Original SW and Unbiased SW

8 RELATED WORK

Differential privacy was initially introduced within a centralized framework [19]. To eliminate the need for a trusted data collector, the concept of local differential privacy (LDP) emerged, allowing individuals to independently apply perturbations to their data [16]. Over time, a variety of LDP solutions have been developed to address different statistical data collection challenges, including frequency estimation over categorical data [7, 23, 31] and mean estimation over numerical data [11, 30]. More recently, research in LDP has expanded to tackle more intricate tasks. These advanced applications include key-value data analysis [21, 40], heavy hitters identification [6, 8], trajectory data analysis [14, 41], marginal release [10, 42], graph data mining [28, 37], time series data collection [5, 38, 39] and range query [12, 22, 24, 32, 35]. Additionally, within the LDP framework, there are existing studies that separately provide strategies for attacking and defending these systems [13, 25, 34]. Moving forward, we will provide an overview of existing LDP research pertinent to our work, with a specific emphasis on mechanisms for handling categorical and numerical data.

LDP Mechanisms for categorical data A variety of LDP mechanisms have been designed specifically for perturbing categorical data. For binary data, the Randomized Response (RR) [33] serves as the simplest mechanism, while its extension, the Generalized Randomized Response (GRR) [23], addresses categories with a domain size greater than 2. To reduce the increasing perturbation noise associated with larger domain sizes in GRR, Wang et al. introduced the Optimized Unary Encoding (OUE) [31], which offers improved utility. Additionally, some other perturbation protocols such as RAPPOR [20], SHist [7], and subset selection [36] have been proposed to improve either utility or communication issue.

LDP Mechanisms for numerical data Similar to centralized DP, the Laplace Mechanism [19] can be adapted for local applications. Alternatively, Duchi et al. introduced a method for estimating mean of numerical values [18]. To overcome the computational and storage complexities associated with this method, an enhanced technique [17] was later developed to convert numerical inputs into binary outputs based on specific probabilities. More recently, Wang et al. [30] introduced the Piecewise Mechanism (PM) to enhance estimation accuracy, while Li et al. [26] developed the Square Wave (SW) mechanism to facilitate the estimation of numerical distributions. Subsequently, Duan et al. [15] proposed a unified framework to evaluate these mechanisms.

9 CONCLUSION

In this work, we study the problem of range mean estimation under Local Differential Privacy. We design a novel framework *PrivRM* for enhancing estimation accuracy by enabling mechanisms to concentrate on the target range. Three implementations are also developed, namely $PrivRM^I$, $PrivRM^O$ and $PrivRM^*$, which is adaptable to all the existing LDP mechanisms for numerical value perturbation. We also provide a black-box guideline to suggest which implementation to use in different settings. Moreover, we further design a distribution-aware strategy on perturbation space adjustment, which improves estimation accuracy especially for skewed data distributions. Finally we validate the effectiveness of our methods through extensive experimental evaluations.

As for future work, we plan to extend our framework to handle more complicated range query, such as range distribution estimation. We also plan to explore efficient solutions to answer successive range queries with arbitrary query ranges.

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A UNBIASED SW MECHANISM UNDER MEAN VALUE ESTIMATION

In SW, half of the square wave width is $2b = \frac{\epsilon e^\epsilon - e^\epsilon + 1}{e^\epsilon(e^\epsilon - 1 - \epsilon)}$. Additionally, the maximum of probability density p is set to $\frac{e^\epsilon}{2(2be^\epsilon + 1)}$, while the minimum q is set to $\frac{1}{2(2be^\epsilon + 1)}$.

Theorem 11. *Suppose the input value is $t \in [-1, 1]$, and after perturbation by SW, the output value is y. Then $\frac{y}{4b(p-q)}$ is an unbiased estimation of t, that is, $\mathbb{E}[\frac{y}{4b(p-q)}] = t$.*

Proof. According to SW mechanism, the expected output is as follow,

$$\begin{aligned}\mathbb{E}[y] &= \int_{-1-2b}^{1+2b} yPr\left[\mathcal{A}_{SW}(t)=y\right]\mathrm{d}y \\ &= \int_{-1-2b}^{t-2b} yq\mathrm{d}y + \int_{t-2b}^{t+2b} yp\mathrm{d}y + \int_{t+2b}^{1+2b} yq\mathrm{d}y \\ &= q\frac{y^2}{2}\Big|_{-1-2b}^{t-2b} + p\frac{y^2}{2}\Big|_{t-2b}^{t+2b} + q\frac{y^2}{2}\Big|_{t+2b}^{1+2b} \\ &= t(4b(p-q)).\end{aligned}$$

Given the privacy budget ϵ, coefficient p, q, b is constant. So the equation $\mathbb{E}[\frac{y}{4b(p-q)}] = t$ is proofed. □

B PROOF OF THEOREM 2

Proof. Since NVP mechanisms we utilized are unbiased, we have,

$$\begin{aligned}\mathbb{E}[\hat{s}_{(l,r)}] &= \sum_{i=1}^{n}\mathbb{E}[\hat{t}_i] - (n - \mathbb{E}[\hat{n}_{(l,r)}])\mathbb{E}[\hat{t}_{out}] \\ &= \sum_{i=1}^{n}\mathbb{E}[\hat{t}_i]\cdot\mathbb{1}_{[l,r]}(t_i) \\ &+ \mathbb{E}[\hat{t}_{out}]\left(\sum_{i=1}^{n}(1-\mathbb{1}_{[l,r]}(t_i)) - n + n_{(l,r)}\right) \\ &= \sum_{i=1}^{n}\mathbb{E}[\hat{t}_i]\cdot\mathbb{1}_{[l,r]}(t_i) = s_{(l,r)}.\end{aligned}$$

which means that the sum estimation is unbiased.

Based on Equation 6, we can calculate the variance of the mean within the range (l, r), as follows:

$$\begin{aligned}Var(\hat{m}_{(l,r)}) &= Var\left(\frac{\sum_{i=1}^{n}\hat{t}_i - (n-\hat{n}_{(l,r)})\mathbb{E}[\hat{t}_{out}]}{\hat{n}_{(l,r)}}\right) \\ &= Var\left(\frac{\sum_{i=1}^{n}\hat{t}_i - n\mathbb{E}[\hat{t}_{out}]}{\hat{n}_{(l,r)}}\right).\end{aligned}$$

Let the random variable X be $\sum_{i=1}^{n}\hat{t}_i - n\mathbb{E}[\hat{t}_{out}]$, and let the random variable Y be $\hat{n}_{(l,r)}$.

For any $f(x, y)$, the bivariate first order Taylor expansion about any $\boldsymbol{\theta} = (\theta_x, \theta_y)$ is

$$f(x,y) = f(\boldsymbol{\theta}) + f'_x(\boldsymbol{\theta})(x-\theta_x) + f'_y(\boldsymbol{\theta})(y-\theta_y) + \boldsymbol{R},$$

where $\boldsymbol{R}$ is a remainder of smaller order than the terms in this Equation. Here we assume that $\mathbb{E}[X] = \mu_x$ and $\mathbb{E}[Y] = \mu_y$. We can choose the expansion point to be $\boldsymbol{\theta} = (\mu_x, \mu_y)$. The approximation for $E(f(X, Y))$ is

$$\begin{aligned}&E(f(X,Y)) \\ &= E\left[f(\boldsymbol{\theta}) + f'_x(\boldsymbol{\theta})(X-\mu_x) + f'_y(\boldsymbol{\theta})(Y-\mu_y) + R\right] \\ &\approx E[f(\boldsymbol{\theta})] + E\left[f'_x(\boldsymbol{\theta})(X-\mu_x)\right] + E\left[f'_y(\boldsymbol{\theta})(Y-\mu_y)\right] \\ &= E[f(\boldsymbol{\theta})] + f'_x(\boldsymbol{\theta})E\left[(X-\mu_x)\right] + f'_y(\boldsymbol{\theta})E\left[(Y-\mu_y)\right] \\ &= E[f(\boldsymbol{\theta})] + 0 + 0 \\ &= f\left(\mu_x, \mu_y\right).\end{aligned} \tag{13}$$

Table 3: Variances of NVP mechanisms

LM	$\frac{8}{\epsilon^2}$
SR	$\left(\frac{e^\epsilon+1}{e^\epsilon-1}\right)^2 - t_i^2$
PM	$\frac{t_i^2}{e^{\epsilon/2}-1} + \frac{e^{\epsilon/2}+3}{3\left(e^{\epsilon/2}-1\right)^2}$
HM	$\begin{cases}\frac{e^{\epsilon/2}+3}{3e^{\epsilon/2}\left(e^{\epsilon/2}-1\right)} + \frac{\left(e^\epsilon+1\right)^2}{e^{\epsilon/2}\left(e^\epsilon-1\right)^2}, & \text{for } \epsilon > 0.61, \\ \left(\frac{e^\epsilon+1}{e^\epsilon-1}\right)^2 - t_i^2, & \text{for } \epsilon \le 0.61.\end{cases}$
Original SW	$4(k_1+k_2-k_3)$
Unbiased SW	$4(k_1+k_2-k_3)/k_4$

By the definition of variance, the variance of $f(X, Y)$ is

$$Var(f(X,Y)) = E\left\{\left[f(X,Y) - E(f(X,Y))\right]^2\right\}.$$

Through Equation 13, we have

$$Var(f(X,Y)) \approx E\left\{\left[f(X,Y) - f(\boldsymbol{\theta})\right]^2\right\}.$$

Then using the first order Taylor expansion for $f(X, Y)$ expanded around $\boldsymbol{\theta}$:

$$\begin{aligned}&Var(f(X,Y)) \\ &\approx E\left\{\left[f(\boldsymbol{\theta}) + f'_x(\boldsymbol{\theta})\left(X-\theta_x\right) + f'_y(\boldsymbol{\theta})\left(Y-\theta_y\right) - f(\boldsymbol{\theta})\right]^2\right\} \\ &= E\left\{\left[f'_x(\boldsymbol{\theta})\left(X-\theta_x\right) + f'_y(\boldsymbol{\theta})\left(Y-\theta_y\right)\right)\right]^2\right\} \\ &= E\left\{f'^2_x(\boldsymbol{\theta})\left(X-\theta_x\right)^2 + 2f'_x(\boldsymbol{\theta})\left(X-\theta_x\right)f'_y(\boldsymbol{\theta})\left(Y-\theta_y\right)\right. \\ &\quad \left. + f'^2_y(\boldsymbol{\theta})\left(Y-\theta_y\right)^2\right\} \\ &= f'^2_x(\boldsymbol{\theta})Var(X) + 2f'_x(\boldsymbol{\theta})f'_y(\boldsymbol{\theta})Cov(X,Y) + f'^2_y(\boldsymbol{\theta})Var(Y).\end{aligned}$$

Now we set: $f(X, Y) = X/Y$ expanded around $\boldsymbol{\theta} = (\mu_x, \mu_y)$. Since $f'_X = Y^{-1}$, $f'_Y = \frac{-X}{Y^2}$ and $\boldsymbol{\theta} = \left(\mu_x, \mu_y\right)$, we now have $f'^2_X(\boldsymbol{\theta}) = \frac{1}{(\mu_y)^2}$, $f'_X(\boldsymbol{\theta})f'_Y(\boldsymbol{\theta}) = \frac{-\mu_x}{(\mu_y)^3}$, $f'^2_Y(\boldsymbol{\theta}) = \frac{(\mu_x)^2}{(\mu_y)^4}$. Thus, we have

$$\begin{aligned}&Var(X/Y) \\ &\approx \frac{1}{(\mu_y)^2}Var(X) + 2\frac{-\mu_x}{(\mu_y)^3}Cov(X,Y) + \frac{(\mu_x)^2}{(\mu_y)^4}Var(Y) \\ &= \frac{(\mu_x)^2}{(\mu_y)^2}\left[\frac{Var(X)}{(\mu_x)^2} - 2\frac{Cov(X,Y)}{\mu_x\mu_y} + \frac{Var(Y)}{(\mu_y)^2}\right]\end{aligned}$$

Our analysis indicates that both X and Y are associated with the number of individuals within the range, $n_{(l,r)}$. Therefore. given range (l, r), X and Y are independent, that is, $Cov(X, Y) = 0$. So we have,

$$Var(\hat{m}_{(l,r)}) \approx \frac{\mathbb{E}^2[X]}{\mathbb{E}^2[Y]}\left(\frac{Var(X)}{\mathbb{E}^2[X]} + \frac{Var(Y)}{\mathbb{E}^2[Y]}\right).$$

□

C VARIANCES OF NVP MECHANISMS

Variances of NVP mechanisms are shown in Table 3. In Table 3, the total domain of t_i is $[-1, 1]$. As for Original SW and Unbiased SW,

$$k_1 = q\left(\frac{1+3b+3b^2-6t_i^2b}{3}\right),$$

$$k_3 = \left(2t_i b(p-q) + q\left(b+\frac{1}{2}\right)\right)^2,$$

$$k_4 = (2b\,(p-q))^2,$$

where $p = \frac{e^\epsilon}{2be^\epsilon+1}$, $q = \frac{1}{2be^\epsilon+1}$ and $b = \frac{\epsilon e^\epsilon - e^\epsilon + 1}{2e^\epsilon(e^\epsilon-1-\epsilon)}$.

D PROOF OF THEOREM 3

Proof. Assume the truncation range of $[-1,1]$. When the output value $y \in (-1,1)$, for any input, the maximum of probability density is $p_{high} = \epsilon/4$, and the minimum is $p_{low} = \frac{\epsilon}{4}e^{-\epsilon}$. For any two different inputs t_1 and t_2, we have,

$$\frac{\Pr[y \in (-1,1) \mid t_1]}{\Pr[y \in (-1,1) \mid t_2]} \le \frac{p_{high}}{p_{low}} = \frac{\epsilon/4}{\frac{\epsilon}{4}e^{-\epsilon}} = e^\epsilon.$$

When the output value $y \in \{-1,1\}$, since the output range is symmetrical with respect to the input range, we only discuss the case where the output $y = 1$. When the input t is 1 within the specific range, the probability of outputting 1 is maximized as $p_{high} = 1/2$. Similarly, when the input t is -1 outside the range, the probability is minimized to $p_{low} = \frac{e^{-\epsilon}}{2}$. Thus, for any two distinct inputs t_1 and t_2, we have,

$$\frac{\Pr[y = 1 \mid t_1]}{\Pr[y = 1 \mid t_2]} \le \frac{p_{high}}{p_{low}} = \frac{1/2}{e^{-\epsilon}/2} = e^\epsilon.$$

In summary, TLM satisfies ϵ-LDP. □

E PROOF OF THEOREM 6

Proof. In TLM, due to the truncation process, the randomized outputs can be divided into two parts: truncation points and the truncation range. Therefore, the privacy analysis can be conducted in two different cases.

Case 1 (Truncation Points $\{-1,1\}$): If output falls within truncation points $\{-1,1\}$, considering that these two truncation points are symmetrical, we only need to analyze the case where $\hat{t}_i = 1$. When the input $t_i = 1$, the output probability is the highest, $p' = 0.5$. Conversely, when the input $t_i = -1$, $q' = \frac{1-p}{2p}$. As for q, we have, $q = (1 - \alpha_{TLM}(\frac{p}{1-p}))/2 = 0.5\sqrt{\frac{1-p}{p}}$. We have,

$$\frac{p\cdot q}{0.5\cdot q'} = 2p\sqrt{\frac{p}{1-p}}, \frac{0.5\cdot p'}{(1-p)\cdot q} = \frac{1}{2-2p}\sqrt{\frac{p}{1-p}}.$$

Case 2 (Truncation Range $(-1,1)$): If output falls within $(-1,1)$, we have $p' = \frac{1}{4}\log\left(\frac{p}{1-p}\right), q' = \frac{(1-p)\log\left(\frac{p}{1-p}\right)}{4p}$ and $q = \frac{1}{2}\left(1-\sqrt{\frac{1-p}{p}}\right)$, thus

$$\frac{p\cdot q}{0.5\cdot q'} = \frac{4p\left(\sqrt{\frac{p}{1-p}}p + p - \sqrt{\frac{p}{1-p}}\right)}{(1-p)\log\left(\frac{p}{1-p}\right)},$$

$$\frac{0.5\cdot p'}{(1-p)\cdot q} = \frac{\sqrt{\frac{p}{1-p}}\log\left(\frac{p}{1-p}\right)}{4(1-p)\left(\sqrt{\frac{p}{1-p}}-1\right)}.$$

According to Theorem 5, applying TLM to *PrivRM** satisfies ϵ-LDP, where

$$\epsilon = \begin{cases} \log(\frac{4p\left(\sqrt{\frac{p}{1-p}}p+p-\sqrt{\frac{p}{1-p}}\right)}{(1-p)\log\left(\frac{p}{1-p}\right)}), & \text{if } p < 0.75, \\ \log(\frac{1}{2-2p}\sqrt{\frac{p}{1-p}}), & \text{otherwise .} \end{cases}$$

□

F PROOF OF THEOREM 7

Proof. When the input $t_i = 1$ and $\hat{t}_i = C_{SR}(\log\left(\frac{p}{1-p}\right))$, the probability is highest, $p' = p$. On the contrary, the lowest probability is $q' = 1-p$. Meanwhile, When the NVP mechanism is set to SR, the output randomization is a random selection of points on $\mathbb{M}_{SR}(\log\left(\frac{p}{1-p}\right))$, akin to a Bernoulli distribution with a probability of 0.5, that is, $q = 0.5$. We have

$$\frac{p\cdot q}{0.5\cdot q'} = \frac{p}{1-p}, \frac{0.5\cdot p'}{(1-p)\cdot q} = \frac{p}{1-p}.$$

According to Theorem 5, applying SR to *PrivRM** satisfies ϵ-LDP, where $\epsilon = \log(\frac{p}{1-p})$. □

G PROOF OF THEOREM 8

Proof. In PM, $p' = \frac{\sqrt{\frac{p}{1-p}}-2p}{4p-2}$, with $q' = \frac{2p-1}{2\left(2p+\sqrt{\frac{p}{1-p}}\right)}$. PM outputs on a continuous domain. Therefore, the probability density function of the output randomization is the same as that of the uniform distribution on the output domain $\mathbb{M}_{PM}(\log\left(\frac{p}{1-p}\right))$. So the probability of uniform distribution is $q = \frac{\sqrt{\frac{p}{1-p}}-1}{2\sqrt{\frac{p}{1-p}}+2}$. We have,

$$\frac{p\cdot q}{0.5\cdot q'} = 2p\sqrt{\frac{p}{1-p}}, \frac{0.5\cdot p'}{(1-p)\cdot q} = \frac{1}{2-2p}\sqrt{\frac{p}{1-p}}.$$

According to Theorem 5, applying PM to *PrivRM** satisfies ϵ-LDP, where $\epsilon = \log(\frac{1}{2-2p}\sqrt{\frac{p}{1-p}})$.

□

H PROOF OF THEOREM 9

Proof. When the privacy budget is less than 0.61 ($p < 0.65$), HM is SR. When the privacy budget is larger than 0.61, HM is a hybrid of SR and PM. Besides, given the probability p, it is clear that $\epsilon_{SR} > \epsilon_{PM}$. Thus, to ensure compliance with ϵ-LDP we use $\epsilon = min\{\epsilon_{SR}, \epsilon_{PM}\} = \epsilon_{PM}$ under this circumstance. In a word, applying HM to *PrivRM** satisfies ϵ-LDP, where

$$\epsilon = \begin{cases} \log(\frac{p}{1-p}), & \text{if } p < 0.65, \\ \log(\frac{1}{2-2p}\sqrt{\frac{p}{1-p}}), & \text{otherwise .} \end{cases}$$

This completes the proof. □

I PROOF OF THEOREM 10

Proof. Unbiased SW differs from the original SW only in the post-processing. Because both PM and SW report data through segmented probability density functions, Unbiased SW can be analyzed in a way similar to PM. In Unbiased SW, with a given privacy budget $\log\left(\frac{p}{1-p}\right)$, the high probability interval has a probability density p' delineated by $\frac{p}{2bp-p+1}$, and the lowest probability density $q' = \frac{1-p}{2bp-p+1}$. Considering the output domain of SW spans a width of $1+2b$, the probability density associated with the uniform distribution is calculated as $q = \frac{1}{1+2b}$, where $b = \frac{(1-p)\left(-2p+p\log\left(\frac{p}{1-p}\right)+1\right)}{2p\left(2p+(p-1)\log\left(\frac{p}{1-p}\right)-1\right)}$.

Thus, we obtain the following:

$$\frac{p \cdot q}{0.5 \cdot q'} = \frac{2p^2 \log\left(\frac{p}{1-p}\right)}{2p-1}, \frac{0.5 \cdot p'}{(1-p) \cdot q} = \frac{2p-1}{2(p-1)^2 \log\left(\frac{p}{1-p}\right)}.$$

According to Theorem 5, applying Unbiased SW to *PrivRM** satisfies ϵ-LDP, where $\epsilon = \log(\frac{2p-1}{2(p-1)^2 \log\left(\frac{p}{1-p}\right)})$. □