Part I Discrete Mathematics

1. Answer the following questions. (4 % each)
   (i) What is the general solution for the linear congruence $4x \equiv 5 \bmod 7$?
   (ii) How many solutions are there to the equality $y_1+ y_2+ y_3=13$, where y_1, y_2, y_3 are positive integers?
   (iii) What are the minimal number of colors needed for a coloring of the graphs: C_4 and W_4, which represent cycle and wheel respectively?
   (iv) How many edges does a full binary tree with 99 internal vertices have?
   (v) How many bit strings of length 8 either start with a 0-bit or end with 01?
   (vi) The following arithmetic expression is written in prefix notation. Please re-write it using infix notation. */ 9 3 + * 2 4 – 7 6

2. True or false (2 % for each correct answer and -1 % for each wrong answer)
   (a) Assume that a and p are positive integer greater than 1. If p is a prime, then $a^{p-1} \equiv 1 \pmod{p}$.
   (b) There exists an Euler path in the hypercube Q_3.
   (c) The number of rationals in (0,1) is in infinite but is countable.
   (d) $5^{222} \equiv 3 \pmod{11}$.
   (e) "$\neg (\exists x\ f(x))$" $\equiv$ "$\forall x \neg f(x)$" , where $\neg$ stands for "not", $\equiv$ for logical equivalence.
   (f) Let P(x) and Q(x) be propositional functions. $\exists x ((P(x) \wedge Q(x)) \equiv (\exists x\ P(x) \wedge \exists x\ Q(x))$.
   (g) The minimal number of colors needed for a coloring of plannar graphs is no more than 3.
   (h) The relation R is not antisymmetric on S, where R={(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)} and S={1,2,3,4}.

3. Pleae draw the Hasse Diagram of the poset ({1,2,4,5,10,15,20,30},\) . (8%)
   Determine and explain whether this diagram is a lattice or not. (4%)

背面有題，請繼續作答。

Part II Linear Algebra

1. We say that two matrices A and B are similar if $A = SBS^{-1}$ for some invertible matrix S. For each of the following statements, indicate whether the statement is true or false, respectively (1 % each). If the statement is true, briefly state why. If the statement is false, give a counterexample or explain why (4 % each).

   (a) If A and B are similar, then A and B have the same eigenvalues.

   (b) If A and B are similar, then A - 5I and B - 5I are similar where I is the identity matrix.

   (c) If A and B are similar, then A^T and B^T are similar.

   (d) If A and B are similar, then AB and BA are similar.

   (e) If A and B are similar, then A^2 and B^2 are similar.

   (f) If A and A^{-1} are similar, then all the eigenvalues of A equal 1 or -1.

2. Let A be a 3×3 matrix having the characteristic equation $x^3 - 3x + 2 = 0$.

   (a) True or false: A is diagonalizable. (1 %)
   Justify your answer. (4 %)

   (b) True or false: A is noninvertible. (1 %)
   Justify your answer. (4 %)

3. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 1 & 0 \\ 2 & 9 & 11 & 16 \end{bmatrix}$.

   (a) Give a condition on $b = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ such that Ax = b is solvable. (5 %)

   (b) Let $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find the least square solution of the system Ax = b. (5 %)