Part I Discrete Mathematics

1. Answer the following questions (6% each)
   (i) What is the value of 3^{2003} mod 99?
   (ii) A regular graph is a simple graph that every vertex has the same degree. What is the number of vertices for a regular graph with degree 4 having 20 edges?
   (iii) Calculate the number of solutions of $x_1+x_2+x_3+x_4=10$ for (1) $x_i \in Z^+$ and (2) $x_i \in N=\{0,1,2,3,\ldots\}$ respectively, where $i \in \{1,2,3,4\}$.
   (iv) $\{a_n\}$, n=1,2,3,…, is a sequence of integers satisfying $a_n = 2a_{n-1}+f(n)$, with $a_1=1$. Please give a general solution for f(n)=1 and f(n)=n respectively.
   (v) In a binary tree, that is, every node could have at most 2 child nodes. What is the range of the number of nodes if the height of this tree is 4?

2. True or false (2% each for a correct answer, -1% for each wrong answer)
   (a) The set {1,2,3,12,18,36} under the "divide" relation is not a Poset..
   (b) Adjacency matrices can be used to represented directed pseudographs.
   (c) If both P and Q are propositions, then $P \rightarrow Q \equiv \neg P \vee Q$, where represents the logical equivalent relation.
   (d) Let P and Q be propositions. The implication of "$P \rightarrow Q$" is true if P is false and Q is true.
   (e) If a relation is antisymmetric, then it must also be symmetric.
   (f) Assume that the relation R is defined on {1,2,3,4} as R={(1,1),(1,2),(2,1),(2,2)}. Then, R is a reflexive relation.
   (g) The minimum number of numbers must be selected from {1,2,3,4,5,6} to guarantee that at least one pair of these numbers add up 7 is 5.
   (h) Hypercube Q_3 is a plannar graph.
   (i) Let m be a positive integer and let a and b be intergers. Then, (a+b) mod m = (a mod m) + (b mod m)
   (j) There is a vertex cut with n-1 vertices in a complete graph K_n, where n >2.

Part II Linear Algebra

1. Let A be an $m \times n$ matrix with rank r such that
$Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has no solution and $Ax = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ has exactly one solution. For each of the following statements, indicate whether the statement is true or false, respectively. (1 % each) If the statement is true, briefly state why it is true. If the statement is false, give a counterexample or explain why it is false. (4 % each)

   (a) $n = r$.

   (b) $det(A^T A) = det(AA^T)$.

   (c) $A^T A$ is invertible.

   (d) AA^T is positive definite.

   (e) $A^T y = c$ has at least one solution for every $c \in R^3$.

2. Find the closest straight line $y = cx + d$ to the following 5 points: (6 %)
(x, y) = (-2, 0), (-1, 0), (0, 1), (1, 1), (2, 1).

3. Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

   (a) Find the eigenvalues of A. (3 %)

   (b) Is A diagonalizable? Justify your answer. (6 %)

4. Let $u_1 = (3, 1, -1)$, $u_2 = (1, -2, 1)$, and the set $U = span\{u_1, u_2\}$.

   (a) Find the orthogonal complement of U. (5 %)

   (b) Find the point in U that is closest to the vector (2, 2, 8). (5 %)